Pach's Animal Problem Within the Bounding Box

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Abstract

A collection of unit cubes with integer coordinates in $\mathbb{R}^3$ is an *animal* if its union is homeomorphic to the 3-ball. Pach's animal problem asks whether any animal can be transformed to a single cube by adding or removing cubes one by one in such a way that any intermediate step is an animal as well. Here we provide an example of an animal that cannot be transformed to a single cube this way within its bounding box.

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1 Introduction

Pach's animal problem. A *grid cube* is a subset of $\mathbb{R}^3$ that can be written as $[a, a+1] \times [b, b+1] \times [c, c+1]$ where a, b and c are integers. A *grid complex* is a 3-dimensional polytopal complex[1] formed by a finite collection of grid cubes and the faces of the cubes in this collection. A grid complex is an animal if the union of cubes in the complex (i. e., the *polyhedron* of the complex) is homeomorphic to a 3-ball. In 1988 Pach asked whether any animal can be transformed to a single cube by adding or removing cubes one by one in such a way that any intermediate step is an animal as well [14]. This question is known as *Pach's animal problem* and has been reproduced in several other venues (including SoCG); see, e.g., notes in Chapter 8 of [20] or [6, 13]. In the following text, when we consider cube removals or additions, we always mean that each intermediate step is an animal.

Surprisingly, this innocent-looking question is actually very complex and resistant. On the one hand, there are examples of animals that cannot be transformed to a single cube by removals only: The first one (the author is aware of) is Furch's "knotted hole ball" from 1924 [8] (see also [19]). Another one from 1964 is a 3-dimensional variant of famous Bing's house with two rooms [4].[2] After Pach asked about the animal problem, Shermer obtained particularly small such animals independently of the earlier results [16]. (See also [13].) On the other hand, allowing also cube additions adds much more flexibility how to transform animals. If we replace "cube removals" and "cube additions" with closely related "collapses" and "anticollapses", it follows from a classical result of Whitehead [18] that any animal can be reduced to a point (or a cube) by collapses and anticollapses. But the integer

[1] By a *polytopal complex* we mean a collection of polytopes in $\mathbb{R}^d$ for some d (in our case $d = 3$) such that (i) every face of any polytope in the collection belongs to the collection and (ii) an intersection of two polytopes in the collection is a face of both.

[2] The aim of the constructions of Furch and Bing is to obtain so called non-shellable balls. But for an animal non-shellable exactly means that the animal cannot be transformed to a single cube by removals only.

grid does not seem to be flexible enough to emulate all possible anticollapses. Altogether, adding geometric restrictions coming from the integer grid to classical setting in topology makes the question interesting.

In 2010, Nakamura wrote a technical report claiming a solution of Pach's animal problem [12]. This technical report neither appeared in a print nor it has been verified by the community. The author of this work believes that Nakamura's proof contains significant gaps. Thus Pach's animal problem should be still regarded as open. We comment on this in more detail in the appendix of the full version [17].

Pach's animal problem within the bounding box. For all the aforementioned examples, even if they cannot be transformed to a single cube by removals, it is extremely easy to transform them to a single cube if we also allow additions of cubes. All the aforementioned examples can be built by gradual removals of cubes from the *bounding box* (i. e., the smallest grid-aligned box containing the animal) while each intermediate step is an animal. If we revert this process, each of the aforementioned examples can be transformed to the bounding box (by cube additions) and then to a single cube (by cube removals).

Dumitrescu and Hilscher [5] provided an example of an animal which cannot be transformed to the bounding box by cube additions but this example is essentially the complement of Shermer's construction. Thus the cost is that this animal can be easily transformed to a single cube by removals.

In principle it should be possible to combine (and possibly iterate) two types of the aforementioned constructions which would require alternating cube additions and cube removals in order to transform the animal into a single cube within the bounding box. But this would still leave the hope that there is an algorithm for Pach's animal problem which gradually simplifies the "innermost" part of the animal (or its complement) eventually reaching a single cube.

We provide a new significantly stronger construction showing that this hope is vain.

▶ **Theorem 1.** *There is an animal A such that it cannot be transformed to a single cube by additions or removals of cubes which are inside the bounding box of A. In fact, if we remove a cube from A or add a cube to A contained inside the bounding box, we never obtain an animal.*

Part of our motivation for proving Theorem 1 is also that we find it realistic that this construction would be a part of a construction of a counterexample to original Pach's animal problem (without any restriction coming from the bounding box), of course, only if such a counterexample exists.[3]

Another (independent) part of the motivation is that staying within the bounding box is a natural restriction from point of view of digital geometry. Here Pach's animal problem fits into a framework of deformations studied in digital geometry; see [10, Chapter 16]. (This is also a language used in Nakamura's technical report [12].) If cubes correspond to "voxels" stored in a computer, then providing a bounding box is natural. We may also be interested in an algorithm (possibly a brute force algorithm) transforming one animal into another (if

[3] The construction of A shows a way to block the "interior" of an animal. (In this context the interior should be understood loosely as a collection of cubes that are in the interior of the convex hull of the animal.) The "boundary" cubes cannot be fully blocked. For example, a cube can be always added to a top of a topmost cube. However it could be realistic to control the "expansion" of the boundary using some "undilatable patterns" (discussed in [12]) so that the cubes in the "interior" stay blocked no matter what is the expansion of the boundary.

this is possible). Our example shows that we have to leave the bounding box and it would be interesting to know whether there is any (computable) bound on the space required for such a transformation.[4]

Finally our problem is loosely related to reconfiguration problems of cubical or crystalline robots [1, 3, 7, 11]. The general aim of such reconfiguration problems is to transform one configuration of a robot (formed by modules which correspond to grid cubes in our language) to another configuration following certain prescribed rules. Stated in this generality, our problem fits into this framework; however additions or removals of cubes are perhaps less natural conditions in the setting of reconfigurations of robots.

2 Sketch of the construction (mostly in dimension 2)

We start with a sketch of the construction, then we provide details.

Furch's construction. The starting point is Furch's construction. Here the idea is to take a thickening of a non-trivial knot made by grid cubes inside a box; see Figure 1. More precisely, the knot here is an image of an interval but together with a curve on the boundary it is a nontrivial knot in the standard sense. All the cubes forming the knot are removed from the box except a single one at one of the boundaries. This way we obtain an animal because we can remove the cubes of the knot one by one while keeping the homeomorphism type. In this example, we can remove a few more cubes by keeping animality (for example those that are at the corners) but at some point we get a non-trivial animal from which no other cubes can be removed (not proved here – we are only sketching an idea).

A 2-dimensional example. One of the key ingredients in our construction will be to redesign the knot so that it fills all "available" space. In order to explain this idea (as well as few other ones), it is beneficial to describe a simplified example in the plane (but not fully reaching our goals). During the construction we specify the dimensions of the intermediate objects because these dimensions will be analogous in the final construction in the 3-space.

First, we consider a square S of dimensions 4×4 subdivided into 16 smaller (unit) squares together with a certain (red) piecewise linear curve; see Figure 2, left. The segments of the red curve connect the centers of pairs of adjacent squares, or possibly centers of the edges of the squares on the boundary. Next we expand this picture as follows (see Figure 2, right): Each point p in S with coordinates which are integer multiples of $1/2$ will correspond to a

[4] The author is strongly persuaded that the animal constructed in this paper can be transformed into a single cube if we allow leaving the bounding box. Thus our A should not be a counterexample to the original Pach's animal problem. On the other hand, it is beyond the targets of this paper to provide a sequence of additions and removals transforming A to a cube possibly outside the bounding box. Thus we provide only a sketch of an idea how to do so without any guarantee of correctness: Step 1. It seems possible to magnify A to an animal A' obtained by replacing each cube of A with k^3 cubes in A' (for arbitrarily chosen $k > 0$); see also Figure 17 in the appendix of the full version [17] for a 2-dimensional analogy. In our case, A does not contain "undilatable patterns" thus the approach sketched in Nakamura's report [12] seems to work in our case. Step 2. Once we have A', it is essentially a subdivision of A. Here we would use that a sufficiently deep barycentric subdivision of any traingulated/polytopal ball is shellable; see [2, Cor. I.3.10]. Then it seems possible to emulate a shelling of an iterated barycentric subdivision via removals of cubes in A' provided that k is sufficiently large.

The approach suggested here is, however, probably far from being optimal if we want to minimize the space required for a transformation of A to a single cube. It is, for example, not known to the author whether it is possible to transform A to a cube within a box obtained by extending one of the dimensions of the bounding box by 1.

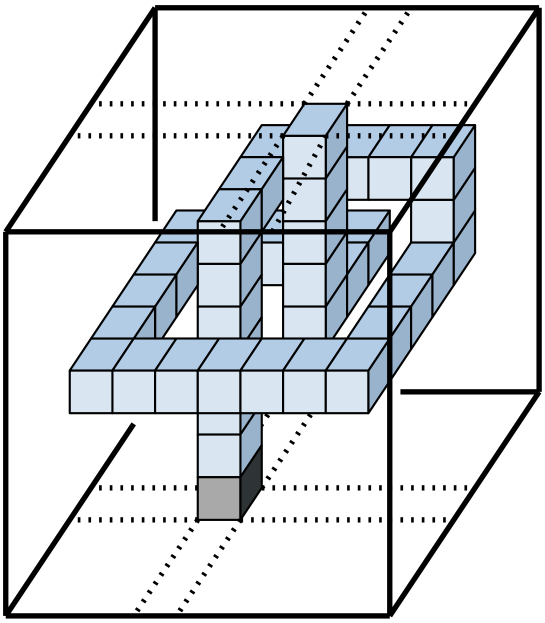

Figure 1 Furch's knotted ball. All displayed cubes are removed from the box except the dark one. The picture we provide here is very similar to a picture in [19].

unit square $S(p)$ in the expanded picture. A square $S(p)$ in the expanded picture is red, if p is on the red curve, otherwise it is black. This way we obtain a square S' of dimensions 9×9 where some of the squares are black and other ones are red. The reader is encouraged to think about this construction as an analogy of Furch's construction in dimension 2 where the red squares correspond to the cubes of the knot. (The aim of the introduction of the red curve and the expansion is to describe the construction efficiently.)

Next we extend S' to a rectangle of dimensions 9×14 as shown in Figure 3, left. The boundary squares are black, while the remaining squares are white. We again draw a red curve through the red and white squares as depicted. Inside the black squares, we connect the centers of two black squares by a segment if they share an edge (essentially constructing the dual graph). Next we perform an analogous expansion as in the previous case; see

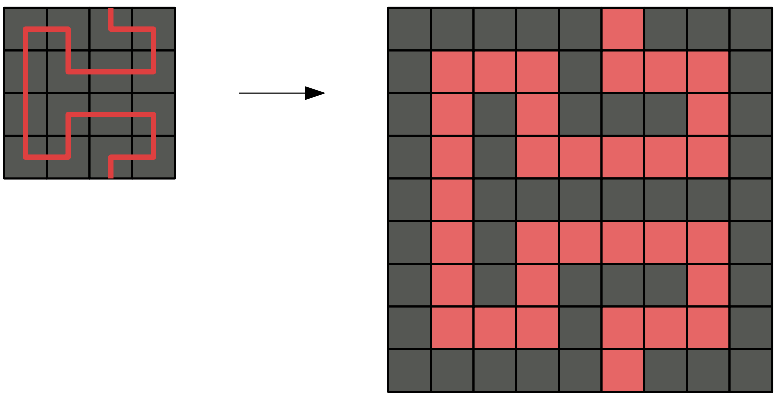

Figure 2 The first expansion of a 2-dimensional example.

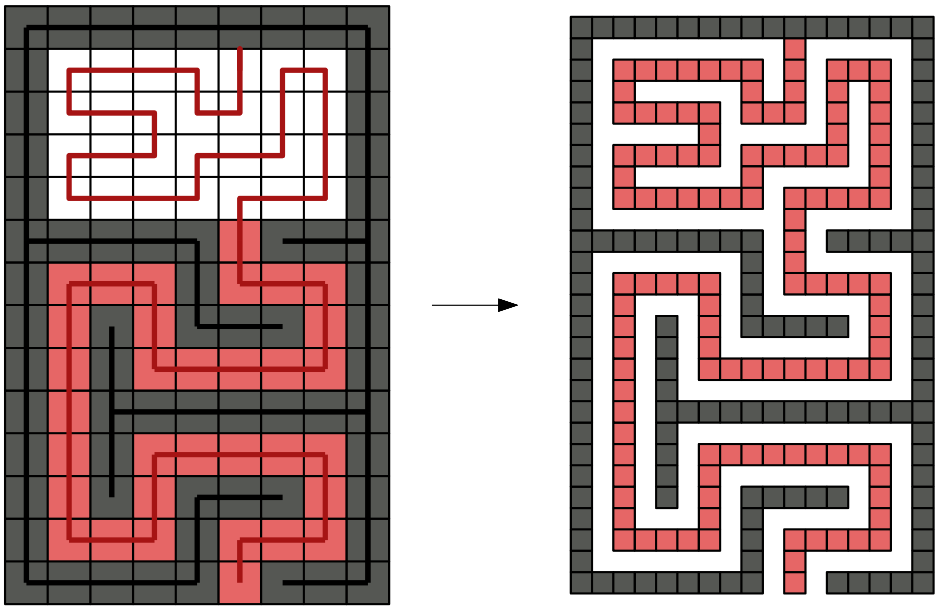

Figure 3 The second expansion of a 2-dimensional example. The squares on both sides of the picture should be understood as unit squares. The dimensions of the right right picture are 17×27 but it is shrunk due to space constraints.

Figure 3, right. (For purposes of this sketch, we mostly rely on the picture; we only keep the squares that correspond to the points on the red curve or black segments. Because of this, the dimensions of the box after the second expansion are 17×27.)

Finally, we double the construction by taking the mirror copy of the collection obtained so far, putting this mirror copy from below and connecting the red squares of the two copies; see Figure 4, left.

Now, it is easy to check that the black and the red squares in the final construction form a 2-dimensional animal. In dimension 2, it is easily possible to remove or add squares within the bounding box so that we preserve animality. Examples of squares that can be added or removed are depicted in Figure 4, right. However, all such squares either correspond to U-turns of the red curve (before the first expansion or before the second expansion), or to the spots where the red curve leaves the original square S.

The main idea for the proof of Theorem 1 is that it is possible to perform an analogous construction in dimension 3 so that U-turns are avoided and the exceptional cases corresponding to the spots where the red curve leaves the original square S can be set up in a way that no addition or removal of cubes is possible while preserving animality. This will prove Theorem 1.

3 Full construction

Now we provide the details for the 3-dimensional construction analogous to the example in the previous section.

Box filling curves. We start with a construction of curves in a box that avoid U-turns. More precisely by a *curve filling a box* we mean a piecewise linear simple curve inside a box B of dimensions $a \times b \times c$ formed by abc grid cubes, satisfying the following properties: (i) The

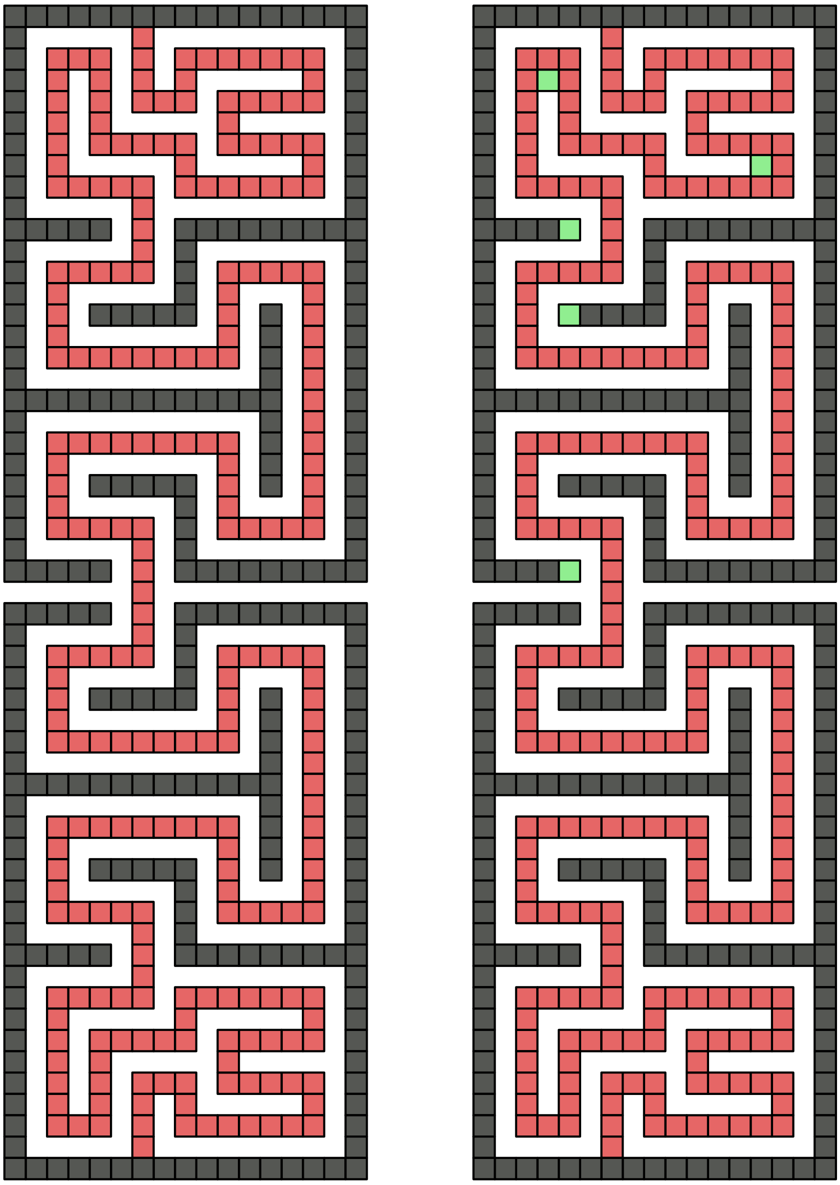

Figure 4 Left: Joining the construction from Figure 3 with its mirror copy. Now the dimensions are 17×55. Right: After adding or removing the squares in green we still have a 2-dimensional animal.

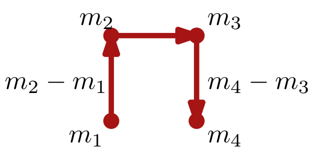

Figure 5 U-turn.

curve passes through the center of every grid cube inside the box; (ii) the curve is composed of segments connecting the centers of pairs of grid cubes sharing a square (that is, the pairs that are adjacent in the strongest possible sense); and (iii) the curve does not contain any *U-turn*; that is, the curve does not pass through four different centers of grid cubes m_1, m_2, m_3, m_4 consecutive in this order along the curve so that $m_2 - m_1 = -(m_4 - m_3)$; see Figure 5. (If a curve contains a U-turn, then it can be easily locally shortened within its isotopy class making some space empty. For purposes of this paper we do not regard this as honestly filling the box.)

Surprisingly, box filling curves can already be found in quite small boxes (ignoring the trivial case of a box $a \times 1 \times 1$). In our construction, we will need a specific curve γ_{444} in a box $4 \times 4 \times 4$ and another one γ_{774} in a box $7 \times 7 \times 4$. (It should not be a surprise that these curves were found via a computer search.) These curves are drawn in Figure 6 layer by layer. (By a *layer* we mean a collection of cubes whose centers have the same z-coordinate.)

Throughout the rest of the construction, a grid cube $[a-1, a] \times [b-1, b] \times [c-1, c]$ will also be called the (a, b, c) cube. The curve γ_{444} fills the box $[0, 4]^3$ and it connects the cube $(3, 2, 1)$ with the cube $(3, 2, 4)$. It will also be important later on that the second cube along the curve is $(3, 2, 2)$ and the last but one cube is $(3, 2, 3)$. The curve γ_{774} fills the box $[0, 7]^2 \times [0, 4]$ and it connects the cube $(5, 3, 1)$ with the cube $(2, 6, 4)$; the second cube is $(5, 3, 2)$ and the last but one cube is $(2, 6, 3)$. It is routine (but slightly tedious) to check that Figure 6 indeed depicts curves (there is no hidden cycle). It is again routine to check that there are no U-turns. (It is immediately visible that there is no U-turn in a layer orthogonal to the z-axis. Other U-turns would either consist of a segment in such a layer where both endpoints continue up or both endpoints continue down – there is no such a segment – or they would appear at the end of the path in a layer – there is no such U-turn either but this requires a bit more effort to check because it is necessary to check how the curve continues in the next layer.)

The first expansion. Now we proceed analogously as in the 2-dimensional case. We take the box $B_1 = [0, 4]^3$ and we consider γ_{444} inside this box. Because γ_{444} connects the centers of cubes $(3, 2, 1)$ and $(3, 2, 4)$ it connects the points $p_1 := (2.5, 1.5, .5)$ and $q_1 := (2.5, 1.5, 3.5)$. We extend this curve to the boundary by adding segments $p_1'p_1$ and q_1q_1' where $p_1' := (2.5, 1.5, 0)$ and $q_1' := (2.5, 1.5, 4)$. Then we declare the resulting curve as the *first red curve.* Altogether, the red curve connects the points p_1' and q_1'.

Next we perform the expansion. We consider a mapping ϕ from points in B_1 with coordinates divisible by $1/2$ to grid cubes defined so that it maps a point $p = (a/2, b/2, c/2)$ in $[0, 4]^3$ to the grid cube (a, b, c). With a slight abuse of the notation, we extend this map to arbitrary subsets of $\mathbb{R}^3$: If $X \subset \mathbb{R}^3$, then by $\phi(X)$ we mean a grid complex formed by cubes (a, b, c) where $(a/2, b/2, c/2)$ belongs to X. (We also allow X to be a grid complex, and then we mean that ϕ is applied to the polyhedron of this complex.) We remark that $\phi(B_1)$ consists of cubes filling the box $B_2 = [-1, 8]^3$. (Note that the lexicographically smallest cube in this box is $(0, 0, 0)$.) We color a cube in $[-1, 8]^3$ *red* if the corresponding point belongs to the red curve, otherwise it is colored *black.* Note that the dual graph of the red cubes is a path, denoted π, connecting the centers of cubes $\phi(p_1') = (5, 3, 0)$ and $\phi(q_1') = (5, 3, 8)$. (Here we again use a slight abuse of the notation identifying a cube with its coordinates.) Here by the *dual graph* of a collection of cubes we mean the graph defined so that its vertices are centers of the cubes in the collection and the edges, realized as straight line segments, connect the centers of those pairs of cubes which share a square.

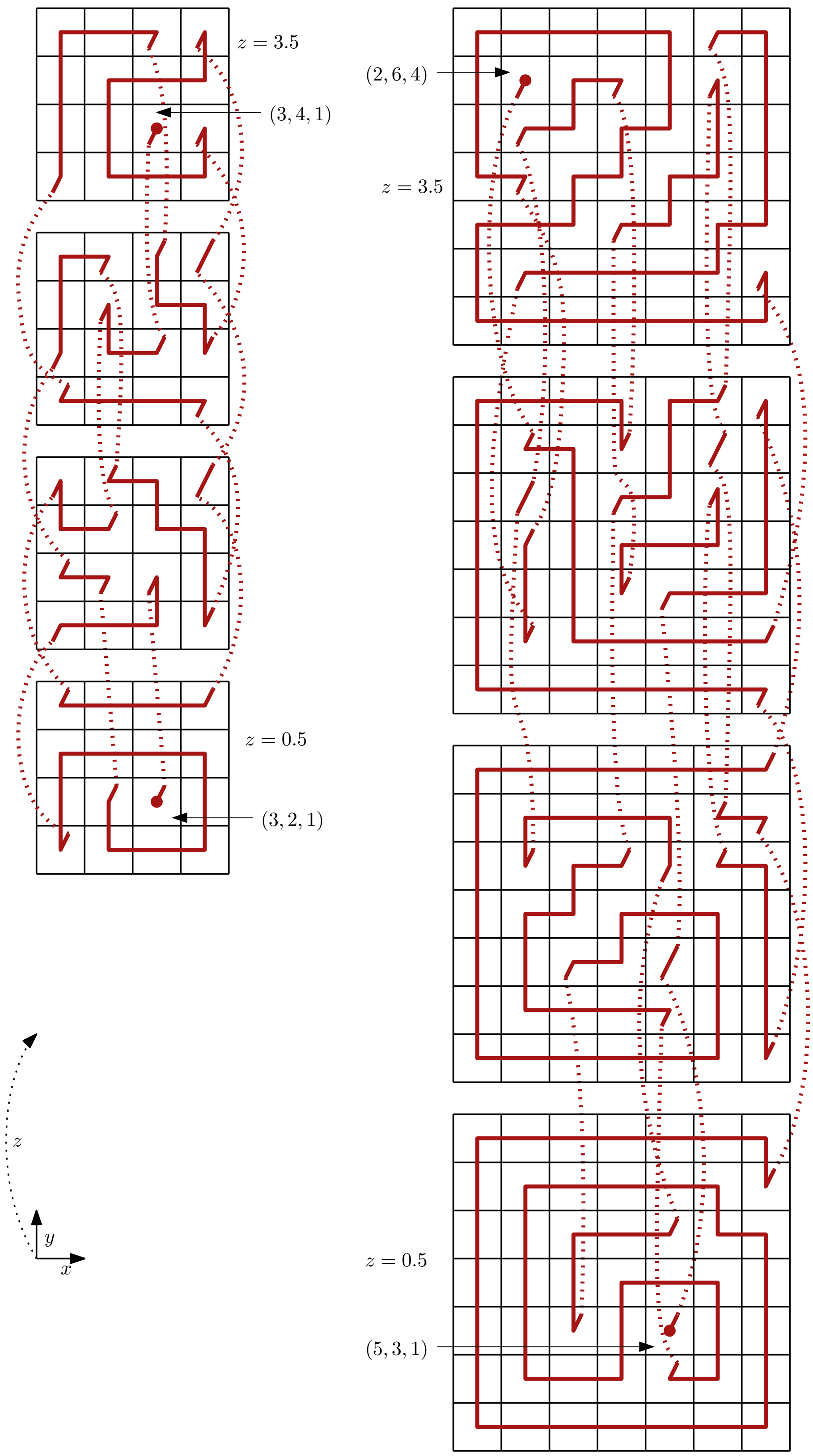

Figure 6 Box filling curves.

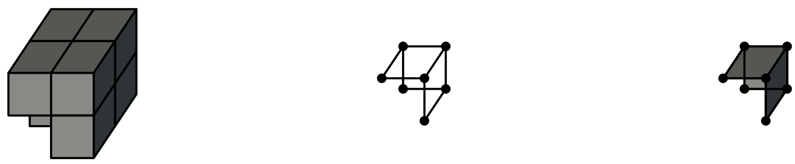

Figure 7 A simplified example of a construction of the black dual complex. Left: A collection of seven cubes for which we construct an analogy of the black dual complex. Middle: The dual graph of these cubes. Right: The resulting complex for these seven cubes.

The second expansion. Now we extend B_2 to the box $B_2' = [-1,8]^2 \times [-1,13]$. Inside the "subbox" $[-1,8]^2 \times [8,13]$ we declare a cube *black* if it is on the boundary of $[-1,8]^2 \times [-1,13]$; that is the (new) black cubes are of the form $(a,b,13)$, $(a,0,c)$, $(a,8,c)$, $(0,b,c)$ or $(8,b,c)$ for $a,b \in \{0,\dots,8\}$ and $c \in \{9,\cdots,13\}$. All remaining cubes of the subbox are white – these are cubes inside *white box* $W := [0,7]^2 \times [8,12]$. Note that the dimensions of the white box are $7 \times 7 \times 4$ suitable for putting γ_{774} inside it.

We obtain the *second red curve* as follows. First, we take the path π considered as a curve connecting the centers of the cubes $\phi(p_1') = (5,3,0)$ and $\phi(q_1') = (5,3,8)$. Next we attach the segment connecting the centers of the cubes $(5,3,8)$ and $(5,3,9)$. Next we attach (shifted) γ_{774} inside the white box. Because the white box is $[0,7]^2 \times [8,12]$ instead of $[0,7]^2 \times [0,4]$, this means that this γ_{774} connects the centers of cubes $(5,3,9)$ and $(2,6,12)$ (instead of $(5,3,1)$ and $(2,6,4)$). We denote the endpoints of γ_{774} as $p_2 = (4.5,2.5,8.5)$ and $q_2 = (1.5,5.5,11.5)$. Finally, we attach the curve to a black cube on the boundary of $[-1,8]^2 \times [-1,13]$ by adding the segment q_2q_2' where $q_2' = (1.5,5.5,12)$. This finishes the construction of the second red curve. We recapitulate that it connects the points $(4.5,2.5,-.5)$ and q_2'.

Next we obtain a *black dual complex* as follows (see Figure 7 for a simplified example): First, we take the dual graph of the collection of black cubes. Next, whenever four black distinct cubes share an edge, we add a square to the complex which is the convex hull of the centers of the four cubes. (Seemingly, we should also add a cube to the black dual complex corresponding to eight black cubes sharing a vertex but such eight cubes do not appear in our construction.)

Now we perform the second expansion. We observe that the cubes of $\phi(B_2')$ fill the box $B_3 = [-2,15]^2 \times [-2,25]$. Now we color the cubes in B_3, black, red or white as follows. (The coloring should be understood relative to the chosen box. For example, it is completely plausible if an (a,b,c) cube is red in B_2' and black in B_3. In other words, it is convenient to use the integer coordinates for both B_2' and B_3 but otherwise they should be considered as independent.) The cube $Q = (a,b,c)$ in B_3 is *black* if the corresponding point $\phi^{-1}(Q)$ in B_2' is in the black dual complex, it is *red*, if the corresponding point is on the second red curve, and it is *white* otherwise.

Bends and straight segments after the second expansion. Before we continue with construction, we describe in more detail how the bends of the first red curve, or the bends of γ_{774} as a part of the second red curve affect the colors of cubes in B_3. This will be useful later on in the proof of Theorem 1.

Consider a grid cube Q_1 in B_1. The first red curve may either pass through Q_1 as a straight segment connecting the centers of opposite squares of Q_1, or it may form a bend connecting the centers of non-opposite squares through the center of Q_1; see Figure 8, left.

Then it follows from the construction that the way how the first red curve passes through Q_1 uniquely determines the colors of 5^3 cubes in $\psi(Q_1)$ where $\psi(Q_1)$ is obtained by applying ϕ to the interior of (the polyhedron of) $\phi(Q_1)$; see Figure 8, right. We emphasize that this

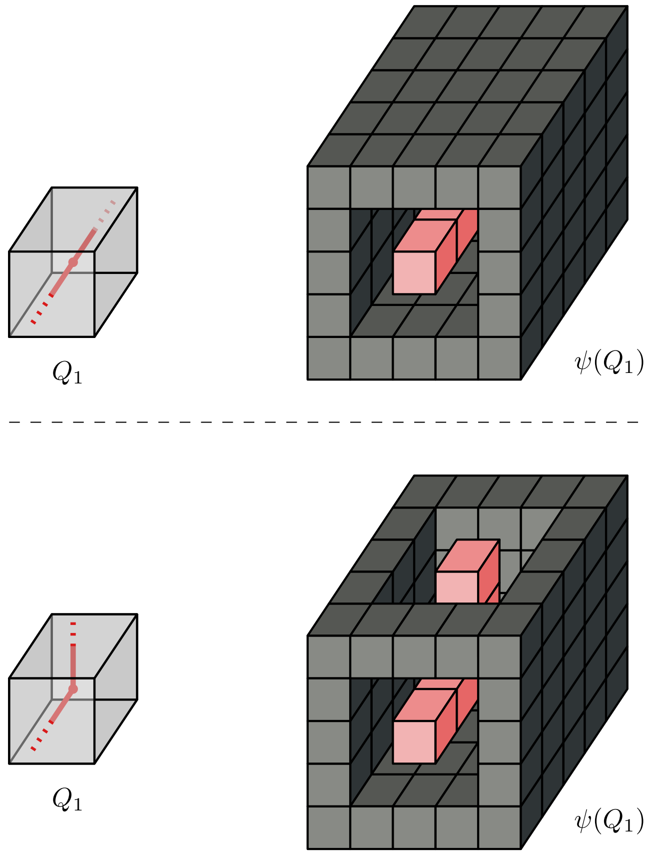

Figure 8 The two expansions of grid cubes in B_1. The white cubes are not depicted.

can be used even for cubes $(3, 2, 1)$ and $(3, 2, 4)$ where γ_{444} starts and ends because the red curve is extended in these cubes. (And it does not make a bend due to the properties of γ_{444}.) We also remark that the boxes $\psi(Q_1)$ in general overlap. For example, if Q_1 and Q_1' share a square, then $\psi(Q_1)$ and $\psi(Q_1')$ share 25 cubes.

A very analogous description of the expansion can be provided for the cubes in the white box W of B_2': This time we consider a cube Q_2 in the white box W. There are again two ways how the second red curve may pass through Q_2; see Figure 9, left. This determines the colors (red or white) of 3^3 cubes of $\phi(Q_2)$; see Figure 9, right. This applies also to the cubes where γ_{774} starts or ends because the second red curve is extended beyond these endpoints.

Doubling. Finally, we extend B_3 to the box $B_3' = [-2, 15]^2 \times [-30, 25]$ – this is our final bounding box. Again we color cubes in B_3' black, red or white. For cubes that belong to both B_3 as well as B_3' we keep the colors as they are in B_3. Inside $[-2, 15]^2 \times [-30, -3]$ we take a mirror copy of B_3 so that the colors are symmetric along the *mirror plane* μ given by the equation $z = -2.5$. It remains to color the cubes $(a, b, -2)$ for $a, b \in \{-1, \dots, 15\}$ (these cubes have their centers on the aforementioned plane). Among these cubes, we color the cube $(9, 5, -2)$ red, all other cubes are white. (Note that the red cube $(9, 5, -2)$ shares a square with exactly two other red cubes $(9, 5, -1)$ and $(9, 5, -3)$.

This finishes the construction. The desired animal A from Theorem 1 is the grid complex consisting of the black and the red cubes in B_3'. It remains to check that A is indeed an animal and that the result of removing any cube or adding any cube within the bounding box is not an animal. This is done in the next section.

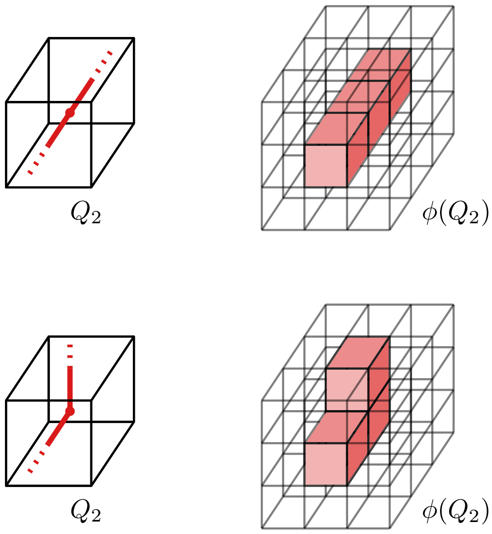

Figure 9 The second expansions of grid cubes in the white box of B_3. The white cubes are drawn as transparent.

4 Correctness of the construction

In this section, we finish the proof of Theorem 1.

4.1 A is an animal

In this subsection we work in PL (piecewise-linear) category [15]. In particular, a ball means a PL ball and a manifold means a PL manifold.[5] We will repeatedly need a special case of Lemma 3.25 from [15] which can be phrased in dimension 3 as follows:

▶ **Lemma 2.** *Let M_1 and N be 3-manifolds meeting in a disk (a. k. a. 2-ball) on the boundary of both M_1 and N. Assume that N is a 3-ball. Then $M := M_1 \cup N$ is a 3-ball if and only if M_1 is a 3-ball.*

Let R be the union of red cubes in A, let K_+ be the union of black cubes in A above the mirror plane μ and K_- be the union of black cubes in A below μ. Note that K_+ and K_- are disjoint (because they are separated by μ which does not intersect any black cube) while both of them meet R in a square. Indeed, K_+ meets R in the square shared by the cubes $(3, 11, 24)$ and $(3, 11, 25)$. (Recall that $q'_2 = (1.5, 5.5, 12)$ is one of the endpoints of the second red path, thus the center of the cube $\phi(q'_2) = (3, 11, 24)$ is the endpoint of the path which is the dual graph of R.) It is also straightforward to check that there are no other places where R and K_+ would intersect by following the second red curve inside the white box and the (first) red curve inside B_1 and checking the result of expansion(s) with an aid of Figures 8 and 9. (It may be also useful to compare this with the 2-dimensional analogue; see Figure 4.) The argument for K_- and R is the same, mirror symmetric.

Thus, in order to show that A is an animal, it is sufficient to show that R, K_+ and K_- are 3-balls (using Lemma 2 twice, first for R and K^+, then for $R \cup K^+$ and K^-).

[5] We work in dimension 3. It is well known that the topological category and PL category coincide in dimension 3. However, we still need a lemma stated in the PL world. (And we will not be really using that those two categories coincide.)

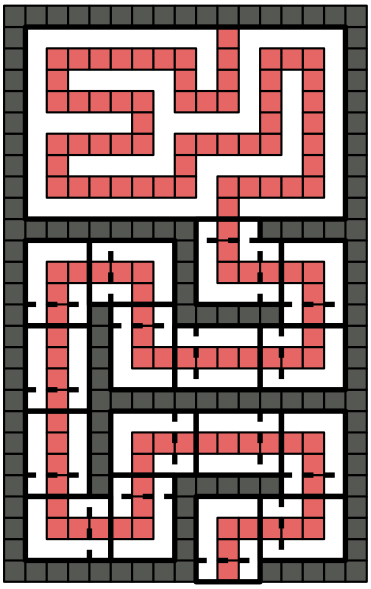

Figure 10 Checking that A is an animal. The 3×4 boxes correspond to the $3 \times 3 \times 4$ boxes in the 3-dimensional setting. The 3×3 squares inside them correspond to the $3 \times 3 \times 3$ boxes in dimension 3. The final bend in the 2-dimensional picture does not appear in the dimension 3.

It is easy to check that R is a 3-ball because the dual graph of the cubes in R is a path. Thus, R can be obtained from a single cube by adding cubes one by one meeting the previous collection in a square. It follows from a repeated application of Lemma 2 that R is a 3-ball.

Now we check that K^+ is a ball (the case of K^- is symmetric). The idea is that we obtain K^+ from the box B_3 (which is a 3-ball) by drilling a hole into it. It is hard to provide a 3-dimensional figure for this approach but it is easy to display an analogy in 2-dimensional setting; see Figure 10. In detail, we follow the first red path cube by cube in B_1 and in each case when we meet a cube Q_1 of B_1, we consider the box $\psi(Q_1)$; see Figure 8. From this box, we remove a "subbox" of dimensions 3, 3 and 4. This subbox consists of 27 (red or white) cubes not touching the boundary of $\psi(Q_1)$ and 9 cubes (1 red, eight white) corresponding to the side (square) from which we have entered Q_1. Due to Lemma 2, each such step preserves the fact that we have a 3-ball. At the very last cube of B_1, we are in the "straight" case (due to the properties of γ_{444}) and we remove all $3 \cdot 3 \cdot 5$ red or white cubes inside the affected 5^3 cubes. At this moment, the red and white cubes are inside the box $\phi(W) = [-1, 14]^2 \times [16, 24]$. Now, we remove $\phi(W)$ in a single step obtaining K^+. We again know that we got a 3-ball by Lemma 2. This finishes the proof that A is an animal.

4.2 Additions and removals

Now we prove that no cube within the bounding box of A can be added to A or removed from A while keeping that we have an animal. We will need the following lemma characterizing removable or addable cubes, which must be a folklore.

▶ **Lemma 3.** *Let A be an animal and Q be a grid cube. Let A' be the grid complex obtained from A by removing Q if Q is in A, or by adding Q if Q is not in A; and let as assume that A' is an animal as well. Let $N(Q)$ be the grid complex formed by the 26 grid cubes different from Q which intersect Q. Finally, let $A^+ := A \cap N(Q)$ and A^- be the subcomplex of $N(Q)$ formed by those cubes which do not belong to A. Then Q meets both A^+ and A^- in a disk.*

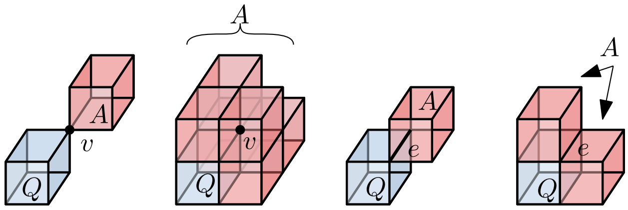

Figure 11 Cases when the singular points appear. Only the cubes that contain v or e are displayed.

Proof. First of all, in the proof we can assume that A does not contain Q and thus A' is obtained by adding Q to A. If this is not the case, we can just swap A and A' in the proof (the definitions of A^+ and A^- are resistant to this swap).

Now, we set $Q^+ := Q \cap A^+ = Q \cap A$ and $Q^- := Q \cap A^-$. Next we observe that both Q^+ and Q^- are pure 2-dimensional; that is, every vertex and edge is contained in a square. See Figure 11 when following the proof. For vertices, it is sufficient to show that there are no isolated vertices because a vertex in an edge belongs to a square if the edge belongs to a square. If Q^+ contains an isolated vertex v (that is, not contained in any edge of Q^+), then v is a singular point of A' contradicting that A' is an animal. If Q^- contains an isolated vertex v, then v is a singular point of A. If Q^+ contains an edge e which is not contained in any square of Q^+, then any interior point of e is a singular point of A' and if Q^- contains an edge e which is not contained in any square of Q^-, then any interior point of e is a singular point of A.

At this moment we know that Q^+ and Q^- are formed by collections of squares and the edges and vertices in these squares. Note that a square of Q belongs to Q^+ if and only if it does not belong to Q^-. Thus it is sufficient to rule out the following cases: One of Q^+ or Q^- contains no square or one of Q^+ or Q^- contains exactly two opposite squares. (All other cases yield disks.) We immediately rule out the case $Q^+ = \emptyset$ as A' would be disconnected in this case. In all other cases, we utilize the Mayer-Vietoris exact sequence for reduced homology [9, Chap. 2.2] using that $A' = A \cup Q$ and $Q^+ = A \cap Q$:

$$\cdots \to \tilde{H}_{n+1}(A') \to \tilde{H}_n(Q^+) \to \tilde{H}_n(A) \oplus \tilde{H}_n(Q) \to \tilde{H}_n(A') \to \cdots$$

From the fact that A and A' are animals and Q is a cube, we get that $\tilde{H}_{n+1}(A'), H_n(A), H_n(Q)$ and $H_n(A')$ are all trivial. From exactness it follows that $H_n(Q^+)$ is trivial as well. But this rules out all previous cases. (If Q^+ consists of two opposite squares, then $\tilde{H}_0(Q^+) \cong \mathbb{Z}$; if Q^+ consists of four squares missing two opposite squares, then $\tilde{H}_1(Q^+) \cong \mathbb{Z}$; and if Q^+ consists of all six squares, then $\tilde{H}_2(Q^+) \cong \mathbb{Z}$.) ◀

We finish the proof of Theorem 1 by distinguishing whether we remove a red or a black cube or whether we add a with cube within the bounding box. For clarity of the structure of the proof, it is useful to state this in three separate lemmas.

▶ **Lemma 4.** *Removing a red cube Q from A yields a grid complex A' which is not an animal.*

▶ **Lemma 5.** *Adding a white cube Q in the bounding box B_3' to A yields a grid complex A' which is not an animal.*

▶ **Lemma 6.** *Removing a black cube Q from A yields a grid complex A' which is not an animal.*

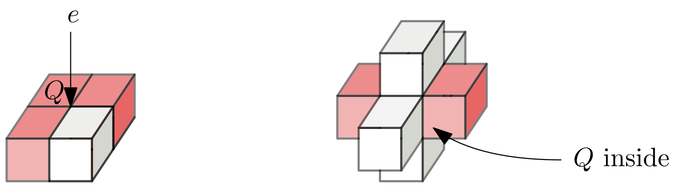

Figure 12 A neighborhood of a red cube Q. (Only some of the cubes for which we can determine the color are displayed.)

Joint preliminaries for the proofs of all three lemmas. In all three proofs, we use the notation Q^+ and Q^- from the proof of Lemma 3. That is, $Q^+ := Q \cap A^+$ and $Q^- := Q \cap A^-$ where A^+ and A^- are as in the statement of Lemma 3. (In other words Q^+ is formed by the intersection of Q with the black and red cubes while Q^- is formed by the intersection of Q with the white cubes. (We consider the cubes outside the bounding box also as white.)

In all three proofs, **we assume for contradiction** that A' is an animal. Using Lemma 3 we deduce that both Q^+ and Q^- are disks. We will consider different cases of how Q is positioned in A. Each such case will yield a contradiction with this disk property.

We also emphasize that in all three cases, it is sufficient to consider the cubes which are either in B_3 or meet the mirror plane μ; other cases are mirror symmetric.

Proof of Lemma 4. The dual graph of the collection of the red cubes R is a path. If Q corresponds to a vertex of degree two of this path, then either there is a bend at Q in which case, Q^- contains an edge not in square, thus Q^- is not a disk; or there is no bend and Q^+ consists of two opposite squares – see Figure 12 covering both cases. Both cases show that A' is not an animal. The only remaining case of Q red is that Q is the cube $(3, 11, 24)$. In this case, Q^+ again consists of two opposite squares as $(3, 11, 25)$ is black, $(3, 11, 23)$ is red while all eight cubes $(a, b, 24)$ intersecting Q but different from Q are white. ◀

Proof of Lemma 5. We first resolve the easy cases: If Q intersects the mirror plane μ, then Q^+ meets both K^+ and K^-, thus it immediately follows that Q^+ has at least two components. Thus it is not a disk. Similarly, if Q belongs to the box $[-2, 15]^3$ (which is the union of $\psi(Q_1)$ for cubes Q_1 in B_1), then it follows from the construction that Q meets both R and K^+ (see also Figure 8) but it does not meet $R \cap K^+$. It again follows that Q^+ has at least two components.

It remains to consider the case that Q belongs to the box $\psi(W) = [-1, 14]^2 \times [16, 24]$. (Note that the cubes in $[-2, 15]^2 \times [16, 25]$ but not in $\psi(W)$ are all black.)

We observe that the cubes (a, b, c) in $\psi(W)$ for which a, b and c is odd are red. These cubes are central cubes in boxes $\psi(Q_2)$ where Q_2 is in (the white box) W; see Figure 9. We will call these cubes *central*.

Now we separately consider the case when Q is on the boundary of $\psi(W)$. In this case Q has to meet both a red cube and a black cube. If Q does not meet the intersection of R and K^+, then Q^+ has two components which is the required contradiction. If Q meets the intersection then, Q is one of the eight white cubes meeting the red cube $\phi(q_2') = (3, 11, 24)$ and in the same layer as this red cube. In this the neighborhood of any such Q could be fully reconstructed from Figure 6; however we provide an argument independent of the exact way how γ_{774} passes through the white box (except that we know that there is no bend at q_2); see Figure 13. If Q shares an edge with $\phi(q_2')$, then this edge is not in any square of Q^+, thus Q^+ is not a disk. If Q shares a square with $\phi(q_2')$, then Q^+ shares an edge with one of the four red (central) cubes depicted in layer 23 of Figure 13 different from $\phi(q_2)$. This edge is not in any square of Q^+, thus Q^+ is not a disk.

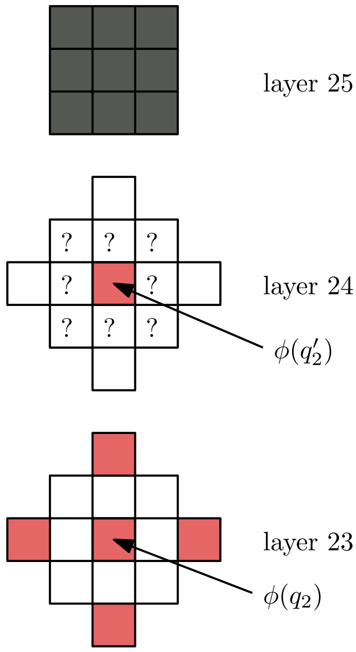

Figure 13 A neighborhood of a white cube Q which meets both R and K^+; Q is one of the eight cubes marked with "?". (Only some of the cubes are displayed.)

Figure 14 The cube Q meeting the central cubes in edges and the U-turn.

It remains to consider the case when Q is $\phi(W)$ but not on the boundary of it. In this case, Q meets no black cube. In this case, Q may meet central cubes in vertices (if all coordinates of Q are even), in edges (if one of the coordinates of Q is odd), or in squares (if two of the coordinates of Q are odd). In addition, a we remark that a cube with at most one coordinate odd is necessarily white (this easily follows from an inspection of Figure 9).

If Q meets central cubes in vertices, then all six cubes neighboring Q (that is sharing a square) must be white, thus Q^- is not a disk. If Q meets the central cubes in squares, then two opposite cubes neighboring Q are red while the remaining four cubes are white. This means that Q^- is not a disk. Finally assume that Q meets the central cubes in edges; see Figure 14. Consider the four cubes neighboring Q which meet central cubes in squares. These cubes may be white or red, the remaining two neighbors of Q are white. All four cubes cannot be red. If three of the four cubes are red, then this configuration comes from a U-turn on γ_{774} which is a contradiction that γ_{774} does not form U-turns. Finally, if two or less of the cubes are red, then either there is an edge in Q^+ shared with one of the central cubes which is not in any square of Q^+, or exactly two opposite neighbors of Q are red. In each of these cases Q^+ cannot be a disk. ◀

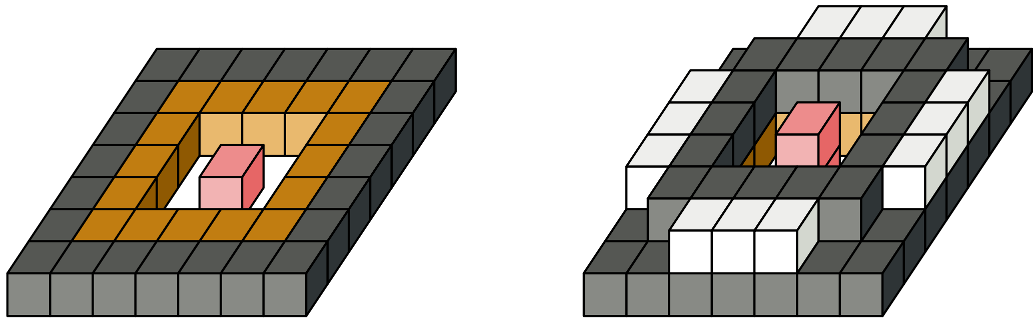

Figure 15 The cubes on the boundary of B_3 which intersect a white cube on the boundary of B_3.

Proof of Lemma 6. We start our considerations with the case that Q is on the boundary of B_3, then it intersects a white cube outside B_3 as well as it intersects a white cube inside B_3. If it does not intersect a white cube on the boundary of B_3, then we get that Q^- has at least two components. If it intersects a white cube on the boundary of B_3, then Q is one of the 16 cubes emphasized in brown in Figure 15, left. (The left picture shows a part of the lowest level of B_3.) Each such cube intersects one of the emphasized white cubes in Figure 15, right in a vertex or edge that cannot belong to a square of Q^-. Thus Q^- cannot be a disk. (Note that we crucially use that the first red curve does not have a bend in cube $(3, 2, 1)$, considered in B_1.)

Similarly, if Q is in layer 15, which is on the boundary of $[-2, 15]^3$, then intersects a white cube in $[-2, 15]^3$ as well as a white cube outside $[-2, 15]^3$. If Q does not intersect a white cube in layer 15, then we get that Q^- has at least two components. If Q intersects a white in layer 15, then we are essentially in the same case as before, only the Figure 15 should be mirrored through a plane orthogonal to the z-axis. (Here we use that the first red curve does not have a bend in cube $(3, 2, 4)$, considered in B_1.) We resolve this case in the same way as the analogous case before.

Finally, there are no black cubes in $\phi(W)$ thus we may assume that Q is in $[-2, 15]^3$ but not on the boundary. That is, Q is in the box $[-1, 14]^3$. Here the consideration is essentially the same as for the last cases of the previous lemma (only the picture is somewhat rescaled). But we of course provide full detail.

By computing the coordinates in the first and the second expansion we observe that for any cube Q_1 in B_1, the central red cube of $\psi(Q_1)$ has coordinates of form $(4i+1, 4j+1, 4k+1)$ where i, j and k are integers. We say that a coordinate of a cube in $[-2, 15]^3$ is *blackish* if it gives remainder -1 modulo 4. The aforementioned observation implies (by checking Figure 8) that the cubes where at least two coordinates are blackish are black. If one coordinate is blackish, then the cube can be of any color, and if none of the coordinates is blackish, then the cube is red or white.

Now if all coordinates of Q are blackish, then the cubes neighboring with Q (that is, sharing a square with Q) are all black, thus Q^+ cannot be a disk. If exactly one coordinate of Q is blackish, then Q has exactly two white opposite white neighbors (with no coordinate blackish) and all remaining four neighbors are black. (In Figure 8 this corresponds to a setting that Q is on some side of $\psi(Q_1)$ for some Q_1 without a hole formed by the white and red cubes.)

The final case is to consider the setting when exactly two coordinates of Q are blackish. Without loss of generality we assume that the blackish coordinates are x and z. That is, there are integers i, j, k and $r \in \{0, 1, 2\}$ such that Q is a $(4i - 1, 4j + r, 4k - 1)$ cube; see

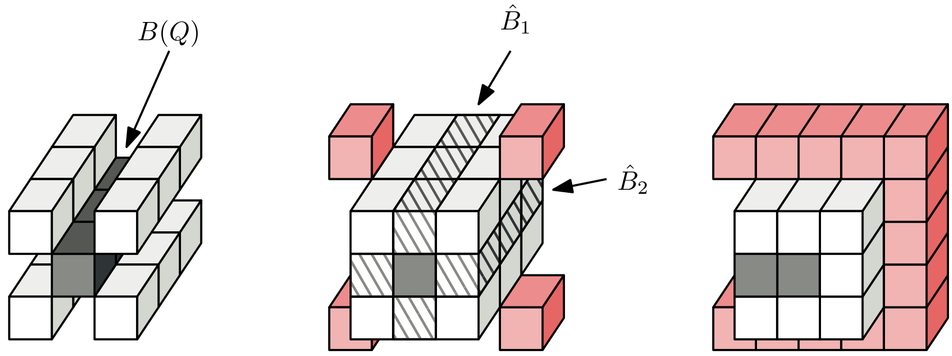

Figure 16 Neighborhood of Q in the last case.

Figure 16, left. Q is one of the three black cubes. By $B(Q)$ we denote the $1 \times 3 \times 1$ box formed by these three cubes (independently of the position of Q in this box). Then all twelve cubes $(4i - 1 \pm 1, 4j + r', 4k - 1 \pm 1)$ for $r' \in \{0, 1, 2\}$ are white; see Figure 16, left again. Next we consider the four cubes $(4i - 1 \pm 2, 4j + 1, 4k - 1 \pm 2)$, these cubes have to be red as they are central cubes of $\phi(Q_2)$ for some cubes Q_2; see Figure 16, middle. We also consider four $1 \times 3 \times 1$ boxes $\hat{B}_1, \dots \hat{B}_4$ sharing with $B(Q)$ rectangles of dimensions 1 and 3. The cubes in each of these four boxes $\hat{B}_i$ are either all white or all black. Similarly as in the proof of Lemma 5 we distinguish how many of these boxes are white.

All four boxes cannot be white (this would correspond to a loop on γ_{444}. If three of the boxes are white, this corresponds to a U-turn on γ_{444}; see Figure 16, right. If two opposite boxes among $\hat{B}_1, \dots \hat{B}_4$ are white then Q neighbors two opposite white cubes while the remaining neighbors of Q are black. Thus Q^+ is an annulus and not a disk. In all remaining cases Q contains an edge share with one of the twelve white cube in Figure 16. This edge belongs to Q^- but it does not belong to any square of Q^-. Thus Q^- is not a disk, a contradiction. We have contradicted all cases which finishes the proof of the lemma and therefore it finishes the proof of Theorem 1 as well. ◀

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