Eight-Partitioning Points in 3D, and Efficiently Too

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Abstract

An *eight-partition* of a finite set of points (respectively, of a continuous mass distribution) in $\mathbb{R}^3$ consists of three planes that divide the space into 8 octants, such that each open octant contains at most 1/8 of the points (respectively, of the mass). In 1966, Hadwiger showed that any mass distribution in $\mathbb{R}^3$ admits an eight-partition; moreover, one can prescribe the normal direction of one of the three planes. The analogous result for finite point sets follows by a standard limit argument.

We prove the following variant of this result: Any mass distribution (or point set) in $\mathbb{R}^3$ admits an eight-partition for which the intersection of two of the planes is a line with a prescribed direction.

Moreover, we present an efficient algorithm for calculating an eight-partition of a set of n points in $\mathbb{R}^3$ (with prescribed normal direction of one of the planes) in time $O^*(n^{5/2})$.

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1 Introduction

Geometric methods for partitioning space, point sets, or other geometric objects are a central topic in discrete and computational geometry. Partitioning results are often proved using topological methods and also play an important role in topological combinatorics [7, 15, 17, 20]. A classical example is the famous *Ham-Sandwich Theorem*, which goes back to the work of

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Steinhaus, Banach, Stone, and Tukey (see [17, Sec. 1] for more background and references). A "discrete" version of this theorem asserts that, given any d finite point sets $P_1, \ldots, P_d$ in $\mathbb{R}^d$, there is an (affine) hyperplane H that *simultaneously bisects* all P_i, i.e., each of the two open half-spaces determined by H contains at most $|P_i|/2$ points, $1 \leq i \leq d$. This follows (by a standard limit argument, see [15, Sec. 3.1]) from the following "continuous" version: Let $\mu_1, \ldots, \mu_d$ be *mass distributions* in $\mathbb{R}^d$, i.e., finite measures such that every open set is measurable and every hyperplane has measure zero. Then there exists a hyperplane H such that $\mu_i(H^+) = \mu_i(H^-) = \frac{1}{2}\mu_i(\mathbb{R}^d)$ for $1 \leq i \leq d$, where H^+ and H^- are the two open half-spaces bounded by H.

In this paper, we are interested in another classical equipartitioning problem, first posed by Grünbaum [9] in 1960: Given a mass distribution (respectively, a finite point set) in $\mathbb{R}^d$, can one find d hyperplanes that subdivide $\mathbb{R}^d$ into 2^d open orthants, each of which contains exactly $1/2^d$ of the mass (respectively, at most $1/2^d$ of the points)? We call such a d-tuple of hyperplanes a 2^d*-partition* of the mass distribution (respectively, of the point set).

For $d = 2$, it is an easy consequence of the planar Ham-Sandwich theorem that any mass distribution (or point set) in $\mathbb{R}^2$ admits a four-partition; moreover, the four-partition can be chosen such that one of the lines has a prescribed direction (indeed, start by choosing a first line in the prescribed direction that bisects the given mass distribution; by the Ham-Sandwich Theorem, there exists a second line that simultaneously bisects the two parts of the mass on either side of the first line). Alternatively, one can also show that there is always a four-partition such that the two lines are orthogonal. Intuitively, the reason that we can impose such additional conditions is that the four-partitioning problem in the plane is *underconstrained*: A line in the plane can be described by two independent parameters, so a pair of lines have four degrees of freedom, while the condition that the four quadrants have the same mass can be expressed by three equations, leaving one degree of freedom; either one of the additional constraints uses this extra degree of freedom.

In 1966, Hadwiger [10] gave an affirmative answer to Grünbaum's question for $d = 3$ and showed that any mass distribution in $\mathbb{R}^3$ admits an eight-partition; moreover, the normal vector of one of the planes can be prescribed arbitrarily. This result was later re-discovered by Yao, Dobkin, Edelsbrunner, and Paterson [21].

▶ **Theorem 1.1** ([10, 21]). *Let μ be a mass distribution on $\mathbb{R}^3$, and let $v \in \mathbb{S}^2$. Then there exists a triple of planes (H_1, H_2, H_3) that form an eight-partition for μ and such that the normal vector of H_1 is v.*

More recently, Blagojević and Karasev [5] gave a different proof for the existence of eight-partitions and showed the following variant:

▶ **Theorem 1.2** ([5]). *Let μ be a mass distribution on $\mathbb{R}^3$. Then there exists an eight-partition (H_1, H_2, H_3) of μ such that the plane H_1 is perpendicular to both H_2 and H_3.*

Our first result is the following alternative version of eight-partitioning, which to the best of our knowledge is new:

▶ **Theorem 1.3.** *Given a mass distribution μ in $\mathbb{R}^3$ and a vector $v \in \mathbb{S}^2$, there exists an eight-partition (H_1, H_2, H_3) of μ such that the intersection of the two planes H_1 and H_2 is a line in direction v.*

As in the case of the Ham-Sandwich Theorem, each of the three theorems above also implies the existence of the corresponding type of eight-partition for finite point sets, again by a standard limit argument (see the full version [2, Lemma A.1] for details).

We remark that, in general, d hyperplanes in $\mathbb{R}^d$ are described by d^2 independent parameters, while the condition that 2^d orthants have equal mass can be expressed by $2^d - 1$ equations. For $d = 3$, this leaves $9 - 7 = 2$ degrees of freedom, which allows for any one of the additional conditions imposed in Theorems 1.1, 1.2, and 1.3, respectively. On the other hand, for $d \geq 5$, we have $d^2 < 2^d - 1$, so intuitively Grünbaum's problem is overconstrained. Avis [3] made this precise and constructed explicit counterexamples using the well-known *moment curve* $\gamma = \{(t, t^2, \ldots, t^d) \colon t \in \mathbb{R}\}$ in $\mathbb{R}^d$. The crucial fact is that any hyperplane intersects the moment curve γ in at most d points ([15, Lemma 1.6.4]). Thus, for $d \geq 5$, a mass distribution supported on γ admits no 2^d-partition because any d hyperplanes intersect γ in at most d^2 points, which subdivide γ into at most $d^2 + 1$ intervals, hence there are always at least $2^d - d^2 - 1 > 0$ orthants that do not intersect γ and hence contain no mass. The last remaining case $d = 4$ of Grünbaum's problem, i.e., the question whether any mass distribution in $\mathbb{R}^4$ admits a 16-partition by four hyperplanes, remains stubbornly open (see [4], [7, Conjecture 7.2], [15, pp. 50–51], and [17, Problem 2.1.4] for more background and related open problems).

We now turn to the algorithmic question of computing eight-partitions in $\mathbb{R}^3$.

▶ Problem 1. Given a set P of n points in $\mathbb{R}^3$, in sufficiently general position, compute three planes H_1, H_2, H_3 that form an eight-partition of the points.

As remarked above, the corresponding problem of computing a four-partition of a planar point can be reduced to finding a Ham-Sandwich cut of two planar point sets that are separated by a line; Megiddo [16] showed that this can be done in linear time.

Computing a Ham-Sandwich cut in $\mathbb{R}^3$ can be done efficiently, in time $O^*(n^{3/2})$ [14] (where n is the total number of points and the $O^*(\cdot)$-notation suppresses polylogarithmic factors). In general, the best known algorithm for computing Ham-Sandwich cuts in fixed dimension $d \geq 3$ runs in time $O(n^{d-1-\alpha_d})$ where $\alpha_d > 0$ a constant depending only on d [14], and a decision variant of the problem becomes computationally hard when the dimension is part of the input, see, e.g., [12]. However, the problem of computing eight-partitions in $\mathbb{R}^3$ seems significantly more difficult, and there is no known way of reducing it to the computation of a Ham-Sandwich cut; in particular, given two planes H_1 and H_2 that *four-sect* a finite point set P (in the sense that every one of the four open orthants determined by H_1 and H_2 contains at most $|P|/4$ points), there generally need not exist a third plane H_3 such that H_1, H_2, H_3 form an eight-partition.

The following concept is useful for characterizing the complexity of Problem 1. A *halving plane* for an n-point set for n odd in $\mathbb{R}^3$ in general position is a plane that passes through three of the points and contains exactly $(n-3)/2$ points on each side. Let $h_3(n)$ be the maximum number of halving planes for an n-point set $\mathbb{R}^3$ as above. The best known upper and lower bounds for $h_3(n)$ are $O(n^{5/2})$ and $\Omega(n^2 e^{\sqrt{\log n}})$, due to Sharir, Smorodinsky, and Tardos [18] and Tóth [19], respectively. A brute-force algorithm that checks every triple of halving planes solves Problem 1 in time roughly proportional to $(h_3(n))^3$.

Yao et al. [21] and Edelsbrunner [8] gave a $O(n^6)$ time algorithm for Problem 1 that computes an eight-partition (with a prescribed normal direction for one of the planes, as in Theorem 1.1) by an expensive search, using the fact that only two planes need to be identified. Fixing one plane and performing a brute-force search for the remaining two would yield an algorithm with a running time comparable to $(h_3(n))^2$.

Here, we present, to our knowledge, the fastest known algorithm for Problem 1. Roughly speaking, our algorithm runs in time near-linear in $h_3(n)$ rather than quadratic in it.

▶ **Theorem 1.4** (Algorithm). *An eight-partition of n points in general position in $\mathbb{R}^3$, with a prescribed normal vector for one of the planes, can be computed in time $O^*(n^{5/2})$; the $O^*(\cdot)$-notation suppresses polylogarithmic factors.*

Our algorithm can be seen as a constructive version of Hadwiger's proof [10]. We start by bisecting the point set by a plane with a fixed normal direction, which partitions the initial point set into two subsets of "red" and "blue" points, respectively, of equal size. After that, our algorithm finds two more planes that simultaneously four-sect both the red points and the blue points.

It remains an open question whether Theorem 1.2 or our own Theorem 1.3 can also be used to obtain an efficient algorithm for Problem 1. It would also be interesting to decide whether there is an algorithm for Problem 1 with running time $o(h_3(n))$.

2 The Topological Result

2.1 Notation and Preliminaries

In what follows, it will often be convenient to assume that the mass distributions we work with have *connected support* where the support of a mass distribution μ is $\mathrm{Supp}(\mu) := \{x \in \mathbb{R}^3 : \mu(B_r(x)) > 0 \text{ for every } r > 0\}$ and $B_r(x)$ denotes the ball of radius $r > 0$ centred at x. By a standard limit argument (see the full version [2, Lemma A.3] for details), the existence of eight-partitions for mass distributions with connected support implies the existence of eight-partitions for the general case. Hereafter, unless stated otherwise, we assume, without loss of generality, that every mass distribution has connected support.

We denote the *scalar product* of two vectors $x, y \in \mathbb{R}^3$ by $x \cdot y := \sum_{i=1}^{3} x_i y_i$. A vector $v \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ and a scalar $a \in \mathbb{R}$ determine an (affine) plane

$$H = H_v(a) := \{x \in \mathbb{R}^3 : x \cdot v = a\},$$

together with an orientation of H (given by the direction of the normal vector v). We denote by $-H := H_{-v}(-a)$ the affine plane with the same equation as H but with opposite orientation. The oriented plane H determines two open half-spaces, denoted by

$$H^+ := \{x \in \mathbb{R}^3 : x \cdot v > a\} \quad \text{and} \quad H^- := \{x \in \mathbb{R}^3 : x \cdot v < a\}.$$

More generally, let $\mathcal{H} = (H_1, \ldots, H_k)$ be an ordered k-tuple of (oriented) planes in $\mathbb{R}^3$, $k \leq 3$. In what follows, it will be convenient to identify the set $\{+, -\}$ with the group $\mathbb{Z}_2$ (where the group operation is multiplication of signs). Elements of $\{+, -\}^k = \mathbb{Z}_2^k$ are strings of signs of length k, and we will denote by $\boldsymbol{+} = + \cdots +$ the identity element of $\mathbb{Z}_2^k$.

For $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_2^k = \{+, -\}^k$, we define the *open orthant* determined by $\mathcal{H}$ and α as $\mathcal{O}_\alpha^{\mathcal{H}} := H_1^{\alpha_1} \cap \cdots \cap H_k^{\alpha_k}$. Given a mass distribution μ in $\mathbb{R}^3$, we say that an ordered k-tuple $\mathcal{H} = (H_1, \ldots, H_k)$ of planes ($k \leq 3$) forms a 2^k*-partition* of μ if every orthant contains $1/2^k$ of the mass, i.e., $\mu(\mathcal{O}_\alpha^{\mathcal{H}}) = \mu(\mathbb{R}^3)/2^k$ for every $\alpha \in \{+, -\}^k$. For $k = 1, 2, 3$, this corresponds to the notions of bisecting, four-secting, and eight-partitioning μ as mentioned in the introduction. Analogously, we say that $\mathcal{H}$ forms a 2^k*-partition* of a *finite* point set P in $\mathbb{R}^3$ if $|P \cap \mathcal{O}_\alpha^{\mathcal{H}}| \leq \frac{|P|}{2^k}$ for all α.

We will parameterize oriented planes in $\mathbb{R}^3$ by $\mathbb{S}^3$, where the north pole e_4 and the south pole $-e_4$ map to the plane at infinity with opposite orientations. For this we embed $\mathbb{R}^3$ into $\mathbb{R}^4$ via the map $(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, 1)$. An oriented plane in $\mathbb{R}^3$ is mapped to an oriented affine 2-dimensional subspace of $\mathbb{R}^4$ and is extended (uniquely) to an oriented linear hyperplane. The unit normal vector on the positive side of the linear hyperplane defines a point on the sphere $\mathbb{S}^3$. Hence, there is a one-to-one correspondence between points v in $\mathbb{S}^3 \setminus \{e_4, -e_4\}$ and oriented affine planes H_v in $\mathbb{R}^3$. The positive side of the plane at infinity is $\mathbb{R}^3$ for $v = e_4$ and $\emptyset$ for $v = -e_4$. Hence $H^+_{-v} = H^-_v$ for every v. Note that planes at infinity cannot arise as solutions to the measure partitioning problem, since they produce empty orthants. Therefore we do not need to worry about the fact that the sphere includes these.

We parameterize triples of planes (called *plane configurations*) in $\mathbb{R}^3$ by $(\mathbb{S}^3)^3$, and denote by $\mathcal{H}_v$ the triple corresponding to $v \in (\mathbb{S}^3)^3$. Given a mass distribution μ on $\mathbb{R}^3$, for each $v \in (\mathbb{S}^3)^3$ and $\alpha \in \mathbb{Z}_2^3 \setminus \{+\}$, we set

$$F_\alpha(v, \mu) = \sum_{\beta \in \mathbb{Z}_2^3} (-1)^{p(\alpha,\beta)} \mu(\mathcal{O}_\beta^{\mathcal{H}_v}).$$

where $p(\alpha, \beta)$ is the number of coordinates where both α and β are $-$. As an example, with $\mathcal{H} := \mathcal{H}_v = (H_1, H_2, H_3)$ and $\alpha = --+ \in \mathbb{Z}_2^3 \setminus \{+\}$, we obtain

$$\begin{aligned} F_{--+}(\mathcal{H}, \mu) &= \sum_{\beta \in \mathbb{Z}_2^3} (-1)^{p(\alpha,\beta)} \mu(\mathcal{O}_\beta^{\mathcal{H}}) = \sum_{\beta \in \mathbb{Z}_2^3 \colon p(\alpha,\beta)=0} \mu(\mathcal{O}_\beta^{\mathcal{H}}) - \sum_{\beta \in \mathbb{Z}_2^3 \colon p(\alpha,\beta)=1} \mu(\mathcal{O}_\beta^{\mathcal{H}}) \\ &= \big(\mu(\mathcal{O}_{+++}^{\mathcal{H}}) + \mu(\mathcal{O}_{++-}^{\mathcal{H}}) + \mu(\mathcal{O}_{--+}^{\mathcal{H}}) + \mu(\mathcal{O}_{---}^{\mathcal{H}})\big) \\ &\quad - \big(\mu(\mathcal{O}_{-++}^{\mathcal{H}}) + \mu(\mathcal{O}_{-+-}^{\mathcal{H}}) + \mu(\mathcal{O}_{+-+}^{\mathcal{H}}) + \mu(\mathcal{O}_{+--}^{\mathcal{H}})\big) \\ &= \mu(H_1^+ \cap H_2^+) + \mu(H_1^- \cap H_2^-) - \mu(H_1^- \cap H_2^+) - \mu(H_1^+ \cap H_2^-). \end{aligned}$$

When μ is clear from context, we write $F_\alpha(\mathcal{H})$ instead of $F_\alpha(\mathcal{H}, \mu)$. The definitions of alternating sums for a pair of planes or a single plane are analogous.

The alternating sums have the following properties which will play an important role in the proof below; for a proof, see the full version [2, Observation 2.1].

▶ **Observation 2.1.** *Let μ be a mass distribution and fix $k = 2, 3$.*

(i) *Let $\alpha \in \mathbb{Z}_2^{k-1} \setminus \{+\}$ and let $\mathcal{H} = (H_1, \dots, H_k)$ be a k-tuple of planes, then $F_{+\alpha}(\mathcal{H}) = F_\alpha((H_2, \dots, H_k))$ (the equivalent statement holds for any other entry of a k-tuple $\alpha_1 \cdots \alpha_k$ instead of just for α_1).*

(ii) *A k-tuple $\mathcal{H}$ of planes 2^k-partitions if and only if $F_\alpha(\mathcal{H}) = 0$ for every $\alpha \in \mathbb{Z}_2^k \setminus \{+\}$.*

2.2 The Main Topological Result

Our goal is to prove the following result – a more precise statement of Theorem 1.3:

▶ **Theorem 2.2.** *Given a mass distribution μ and a direction $p \in \mathbb{S}^2$, there exists a triple $\mathcal{H} = (H_1, H_2, H_3)$ of oriented planes that eight-partition μ so that the oriented direction of the intersection $H_1 \cap H_2$ is p.*

As mentioned before, it is sufficient to prove Theorem 2.2 for mass distributions with connected support. We require a technical lemma about partitioning a mass distribution on $\mathbb{R}^2$, due to Blagojević and Karasev [5]; see the full version [2, Appendix C] for a proof.

▶ **Lemma 2.3** (Four-partitioning a mass distribution in $\mathbb{R}^2$ [5]). *Let $\mu^\#$ be a mass distribution (with connected support) on $\mathbb{R}^2$ and $v \in \mathbb{S}^1$. Then there exists a pair (ℓ_1, ℓ_2) of lines in $\mathbb{R}^2$ that four-partitions $\mu^\#$ and such that v bisects the angle between ℓ_1 and ℓ_2.*

Moreover, if we orient ℓ_1 and ℓ_2 so that ℓ_1 is in the first direction clockwise from v, and ℓ_2 is in the first direction counterclockwise, the oriented pair is unique and the lines depend continuously on v.

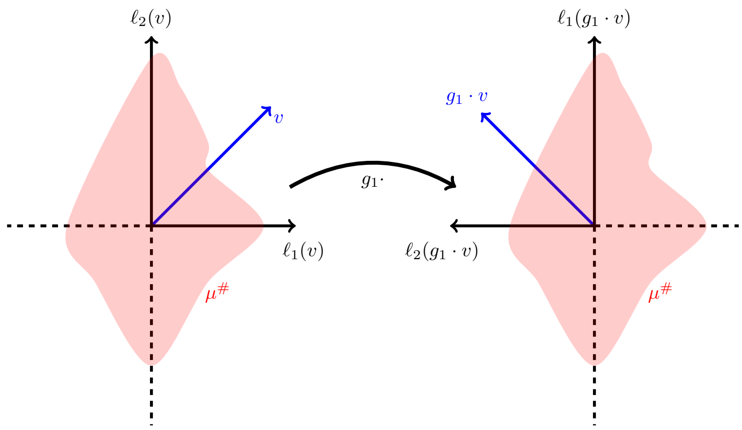

Figure 1 Example of the action of g_1.

Proof of Theorem 2.2. Without loss of generality, let $v = (0, 0, 1)$. Our proof proceeds in two steps. In the first step, we construct a map $\Phi\colon \mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{R}^4$ whose zeros codify equipartitions of μ; then we prove that Φ is equivariant with respect to a suitable choice of actions of $G := \mathbb{Z}_4 \times \mathbb{Z}_2$ on the two spaces. In the second step we show that any continuous G-equivariant map $\Psi\colon \mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{R}^4$ has to have a zero.

Step 1. The key step in constructing the map Φ is to show that we can parameterize pairs of planes that have intersection direction p and four-sect μ, by a vector in $\mathbb{S}^1$.

We project μ to the xy-plane to obtain a mass distribution $\mu^{\#}$ on $\mathbb{R}^2$. Lemma 2.3 guarantees that, once we fix a direction $v \in \mathbb{S}^1 \hookrightarrow \mathbb{S}^2$ (inclusion as the horizontal equator in $\mathbb{S}^2$) there are two lines in the xy-plane $\ell_1 = \mathbb{R}\vec{\ell}_1(v) + a_0(v)$ and $\ell_2 = \mathbb{R}\vec{\ell}_2(v) + a_2(v)$ that four-sect the projected measure $\mu^{\#}$. Define $H_i(v) := (h_i(v), a_i(v))$ to be the (oriented) span of $\ell_i(v)$ and v; the two planes now four-sect μ and have the desired intersection direction.

Now let g_1 be a generator of $\mathbb{Z}_4 \times \{+\} \subseteq G$ and define its action on $\mathbb{S}^1$ by a counterclockwise rotation by $\frac{\pi}{2}$. We use $g_1 \cdot v$ to denote the action of g_1 on v. Then, by the uniqueness in Lemma 2.3, we have that

$$\vec{\ell}_1(g_1 \cdot v) = \vec{\ell}_2(v) \quad \text{and} \quad \vec{\ell}_2(g_1 \cdot v) = -\vec{\ell}_1(v). \tag{1}$$

Intuitively, if we consider the planar problem with the bisecting vector v rotated by $\frac{\pi}{2}$, by uniqueness the affine lines that split the measure are the same. However, while the direction chosen as $\vec{\ell}_1$ in the rotated problem is the direction $\vec{\ell}_2$ in the previous configuration, the “rotated” $\vec{\ell}_2$ is $-\vec{\ell}_1$ in the original problem (see Figure 1).

Using this construction, we can define a function $\mathbb{S}^1 \to \mathbb{S}^3 \times \mathbb{S}^3$ by $v \mapsto (H_1(v), H_2(v))$. It follows from eq. (1) that $g_1 \cdot v$ is mapped to $(H_2(v), -H_1(v))$, therefore, if we fix the corresponding action[1] of $\mathbb{Z}_4$ on $\mathbb{S}^3 \times \mathbb{S}^3$, the map is $\mathbb{Z}_4$-equivariant.

The group $\{e\} \times \mathbb{Z}_2$ acts by antipodality on $\mathbb{S}^3$; therefore, if G acts on $(\mathbb{S}^3 \times \mathbb{S}^3) \times \mathbb{S}^3$ component-wise, the map $\Phi\colon \mathbb{S}^1 \times \mathbb{S}^3 \to (\mathbb{S}^3 \times \mathbb{S}^3) \times \mathbb{S}^3$ defined as $\Phi(v, w) := (H_1(v), H_2(v), w)$ is G-equivariant.

[1] Formally, for any $(x, y) \in \mathbb{S}^3 \times \mathbb{S}^3$ the generator g_1 of $\mathbb{Z}_4 \times \{+\} \subseteq G$ acts by $g_1 \cdot (x, y) = (y, -x)$.

By construction, the first two planes are always a four-partition of the mass distribution, therefore by Observation 2.1, a configuration $\Phi(v,w)$ is an eight-partition if and only if the four alternating sums with $\alpha_3 = -$ (i.e., $\alpha = ++-$, $-+-$, $+--$ and $---$) are 0.

To compute the action of G on the alternating sums, it is enough to specify what happens on g_1 (a generator of $\mathbb{Z}_4 \times \{e\}$) and g_2 (a generator of $\{e\} \times \mathbb{Z}_2$). If we act with g_1, we obtain

$$\begin{aligned}
&F_{++-}(g_1 \cdot \Phi(v,w)) = F_{++-}(\Phi(v,w)),\\
&F_{+--}(g_1 \cdot \Phi(v,w)) = -F_{-+-}(\Phi(v,w)),\\
&F_{-+-}(g_1 \cdot \Phi(v,w)) = F_{+--}(\Phi(v,w)), \text{ and}\\
&F_{---}(g_1 \cdot \Phi(v,w)) = -F_{---}(\Phi(v,w)),
\end{aligned}$$

while acting with g_2 produces

$$\begin{aligned}
&F_{++-}(g_2 \cdot \Phi(v,w)) = -F_{++-}(\Phi(v,w)),\\
&F_{+--}(g_2 \cdot \Phi(v,w)) = -F_{-+-}(\Phi(v,w)),\\
&F_{-+-}(g_2 \cdot \Phi(v,w)) = -F_{+--}(\Phi(v,w)), \text{ and}\\
&F_{---}(g_2 \cdot \Phi(v,w)) = -F_{---}(\Phi(v,w)),
\end{aligned}$$

for every $(v,w) \in \mathbb{S}^1 \times \mathbb{S}^3$.

Finally, we can choose a linear G-action on $\mathbb{R}^4$ that is consistent with the previous equations. In particular, if we define

$$g_1 \cdot (x,y,z,u) = (x,-z,y,-u) \quad \text{and} \quad g_2 \cdot (x,y,z,u) = (-x,-y,-z,-u),$$

then the map $\Psi\colon \mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{R}^4$, given by

$$(v,w) \mapsto (F_{++-}(v,w), F_{+--}(v,w), F_{-+-}(v,w), F_{---}(v,w))$$

is G-equivariant. By Observation 2.1, the zeros of Ψ are exactly the configurations of planes that eight-partition the measure and have the desired intersection property.

Step 2. Suppose now, for a contradiction, that Ψ does not have a zero. This means that it is possible to define a G-equivariant map $\overline{\Psi}\colon \mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{S}^3$ by $\overline{\Psi}(v,w) := \frac{\Psi(v,w)}{\|\Psi(v,w)\|}$.

Denote by Ψ_a, for $a \in \mathbb{S}^1$, the map $\Psi_a\colon \mathbb{S}^3 \to \mathbb{S}^3$, $\Psi_a(p) = \overline{\Psi}(a,p)$; this function has two key properties:

(i) for any $a \in \mathbb{S}^1$, Ψ_a is antipodal;

(ii) for any $a, b \in \mathbb{S}^1$, Ψ_a and Ψ_b are homotopic.

However, the map induced by g_1 on the sphere is antipodal, and, hence, has degree -1. Thus we have $[\Psi_a] = [\Psi_{g_1 \cdot a}] = [g_1 \cdot \Psi_a] = -[\Psi_a]$. It follows that Ψ_a is null-homotopic, contradicting the Borsuk-Ulam theorem. For technical details, see the full version [2, Propositions B.1 and B.2]. ◀

Theorem 2.2, along with a standard limit argument, implies the following.

▶ **Theorem 2.4.** *Let $P \subseteq \mathbb{R}^3$ be a finite set of points and $p \in \mathbb{S}^2$ a fixed direction. Then there exists a triple $\mathcal{H} = (H_1, H_2, H_3)$ of oriented planes that eight-partitions P, so that the oriented direction of the intersection $H_1 \cap H_2$ is p.*

3 The Algorithm

We can deduce the existence of eight-partitions of a finite point set $P \subset \mathbb{R}^3$ of a certain advantageous form from Theorem 1.1.

▶ **Observation 3.1.** *Let $k > 0$ be an integer and $P \subseteq \mathbb{R}^3$ be a set of $n = 8k + 7$ points in general position*[2]*. Then, there exists a triple of planes (H_1, H_2, H_3) that eight-partitions P with the following properties:*

(i) *H_1 is horizontal (i.e., parallel to the xy-plane) and passes through the z-median point of P. From here on, we refer to the $4k + 3$ points that lie below (resp., above) H_1 as* red *(resp.,* blue*) points and denote the sets R (resp. B).*

(ii) *H_2 and H_3 each contain exactly three points, and each open octant contains exactly k points.*

(iii) *H_2, H_3 each bisect R and B, and the pair (H_2, H_3) four-partitions both R and B. Furthermore, H_2 and H_3 contain at least one point of each color.*

Proof. Since the set $X := \{(H_1, H_2, H_3) \mid H_1 \text{ is horizontal}\} \subset (\mathbb{S}^3)^3$ is compact, by Theorem 1.1 and standard limiting arguments, there exists a configuration $\mathcal{H}_\infty = (H_1, H_2, H_3)$ that eight-partitions the point set with H_1 horizontal; moreover, any plane H_i bisects and any pair (H_i, H_j) four-partitions the set P. For details, see the full version [2, Corollary A.2].

Furthermore, note that any eight-partition has at most k points of P in each of the eight open octants, one point in H_1, and at most three points in each of H_2 and H_3, by general position, for a total of at most $8k + 1 + 2 \cdot 3 = 8k + 7 = n$ points. So, in fact, all the inequalities are equalities: there *must* be *exactly* k points in each open quadrant and exactly three points of $R \cup B$ in each of H_2 and H_3.

The assertions are straightforward to verify from the above discussion. ◀

▶ **Theorem 3.2** (Computation of an eight-partition)**.** *Let $P \subseteq \mathbb{R}^3$ be a set of $n > 0$ points in general position and $v \in \mathbb{S}^2$. An eight-partition (H_1, H_2, H_3) of P with v being the normal vector of H_1 can be computed in time $O^*(n + m)$, where m is the maximum complexity of the intersection of the median levels of two sets of $n/2$ planes.*

▶ Remark. Since $m = O(h_3(n)) = O(n^{5/2})$ (see the full version [2, Section 3]), we can compute an eight-partition in time $O^*(n^{5/2})$.

The rest of this section is devoted to the proof of Theorem 3.2. We assume, without loss of generality, that $v = e_3 = (0, 0, 1)$ is the vertical vector, so H_1 is required to be horizontal. If $n \leq 7$, the statement holds trivially – set H_1 to be the horizontal plane containing any point of P, and H_2, H_3 to contain at most three distinct points each, so that the octants do not contain any points. From here on, we will assume that $n = 8k + 7$, for an integer $k > 0$. Otherwise, we add dummy points to P (in general position) until the number of points is of the required form and run the algorithm. Once the algorithm terminates, we discard the dummy points, resulting in an eight-partition with at most k points in each octant.

We now describe the algorithm to construct an eight-partition of P satisfying the properties in Observation 3.1. Let H_1 be the horizontal plane containing the z-median point of P, and, without loss of generality, identify H_1 with the xy-plane. Consider the sets R and B of $4k + 3$ points each lying below and above, respectively, H_1. We assume, without loss of generality, that B is contained in the half-space $x < 0$ and R is contained in the half-space

[2] No four points in a plane, no three points in a vertical plane, and no two points in a horizontal plane.

$x > 0$. Otherwise, since no point in $R \cup B$ has $z = 0$, there exists a plane H containing the y-axis and with sufficiently small negative slope in the x direction such that all red points are below H and all blue points are above H. Applying a generic sheer transformation (so as not to violate the general position assumption) that fixes the xy-plane and maps H to the plane $x = 0$, we obtain point sets with the required properties.

It will be convenient to work in the dual space, where a point $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ is mapped to the non-vertical plane $p^* : z = p_1 x + p_2 y - p_3$ in $\mathbb{R}^3$, and vice versa (see [11, Chapter 25.2] for standard properties of the duality transform). Let $R^* = \{p^* : p \in R\}$ be the set of red planes dual to points in R and set $\mathcal{A}(R) := \mathcal{A}(R^*)$ to be the arrangement formed by the set R^*. The set of blue planes B^* and the blue arrangement $\mathcal{A}(B)$ are defined analogously. We will write $\mathcal{A} := \mathcal{A}(R \cup B)$ for the arrangement formed by the planes in $R^* \cup B^*$. For a (dual) point $p \in \mathbb{R}^3$, we set $R_p^+, R_p^- \subseteq R^*$ to be the set of red planes lying strictly above and below p, respectively. For a pair p, q of (dual) points, put

$$X(p,q) := |R_p^+ \cap R_q^+| - |R_p^+ \cap R_q^-| - |R_p^- \cap R_q^+| + |R_p^- \cap R_q^-|.$$

The sets $B_p^+, B_p^- \subseteq B^*$ and the function $Y(p,q)$ are defined analogously for B^*.

Let L be the intersection of the median levels[3] of $\mathcal{A}(B)$ and $\mathcal{A}(R)$. It is not hard to check that L is a connected y-monotone polygonal curve and, moreover, can be computed in time $O*(n+m)$ using standard tools [1, 6]; see the full version [2, Lemmas 4.3 and 4.4] for details.

We now return to the computation of the eight-partition (H_1, H_2, H_3). By the general position assumption, H_2 and H_3 cannot be vertical, so H_2 and H_3 correspond to vertices in $\mathcal{A}$, by Observation 3.1. With the above setup, we can reformulate the problem of computing H_2 and H_3 as follows.

▷ Claim 3.3 (The dual alternating sign functions). Computing H_2 and H_3 is equivalent to identifying a pair of vertices $p, q \in L$ such that $Y(p,q) = X(p,q) = 0$.

Proof. By Observation 3.1(ii), the eight-partition (H_1, H_2, H_3) has exactly k points in each of the eight open octants. Setting $p := H_2^*$ and $q := H_3^*$, we obtain that $|R_p^\pm \cap R_q^\pm| = |B_p^\pm \cap B_q^\pm| = k$ for all combinations of signs. Therefore $Y(p,q) = X(p,q) = 0$, as claimed.

We now argue the other direction. Let $p, q \in L$ be vertices such that $X(p,q) = Y(p,q) = 0$. Since p and q lie on L, $H_2 := p^*$ and $H_3 := q^*$ bisect both R and B and contain exactly three points each, at least one of each color. Hence, it suffices to show that (H_2, H_3) is a four-partition of both R and B, i.e., $|R_p^\pm \cap R_q^\pm|, |B_p^\pm \cap B_q^\pm| \le k$ for all combinations of signs. Indeed, this implies that each octant formed by (H_1, H_2, H_3), contains exactly k points completing the proof.

Let $a_r := |R_p^+ \cap R_q^+|$, $b_r := |R_p^+ \cap R_q^-|$, $c_r := |R_p^- \cap R_q^+|$, and $d_r := |R_p^- \cap R_q^-|$. Define a_b, b_b, c_b, d_b analogously for the blue planes. Without loss of generality, for a contradiction, suppose $a_r > k$.

We first consider the case $a_r \ge k+2$. Since p lies on the median level of $\mathcal{A}(R)$, we have $a_r + b_r \le |R_p^+| = 2k+1$, implying $b_r \le k-1$. Similarly, since q lies on the median level of $\mathcal{A}(R)$, we have $c_r \le k-1$. Recall that, by assumption, $X(p,q) = a_r + d_r - b_r - c_r = 0$, implying $d_r = b_r + c_r - a_r \le k-4$. Hence, $a_r + b_r + c_r + d_r \le 4k-4$, so p and q together are contained in $4k+3-(a_r+b_r+c_r+d_r) \ge 7$ red planes, contradicting the general position assumption.

[3] The *median level* in an arrangement of $2k+1$ non-vertical planes in $\mathbb{R}^3$ is defined as the closure of the set of all points which lie on a unique plane of the arrangement and have exactly k planes below it.

We may now assume $a_r = k+1$. Following the same reasoning we obtain $b_r \le k$, $c_r \le k$, and $d_r = b_r + c_r - a_r \le k-1$. This implies $a_r + b_r + c_r + d_r \le 4k$, and, in particular, that p and q together are contained in at least 3 red planes. Now consider the blue planes and note that $a_b + b_b + c_b + d_b \le 4k$ – this is clear if each of sets $B_p^{\pm} \cap B_q^{\pm}$ contains at most k blue planes, otherwise it follows by the same argument as above. Hence, p and q together are contained in $4k + 3 - (a_b + b_b + c_b + d_b) \ge 3$ blue planes.

By Observation 3.1(ii), p and q are contained in at most 6 planes of $R^* \cup B^*$. Combined with the argument above, this implies p and q together are contained in exactly 3 planes of each color. It follows that $a_r + b_r + c_r + d_r = a_b + b_b + c_b + d_b = 4k$, which, by the assumption $a_r = k+1$, implies $b_r = c_r = k$ and $d_r = k-1$. Since $|R_p^-| = 2k+1$ and $b_r + d_r = 2k-1$, there are exactly 2 red planes containing q below p. Similarly, since $|R_q^-| = 2k+1$ and $b_r + d_r = 2k-1$, there are exactly 2 red planes containing p below q. But then p and q are contained in a total of 4 red planes, a contradiction.

This exhausts all possibilities and, hence, $|R_p^{\pm} \cap R_q^{\pm}|, |B_p^{\pm} \cap B_q^{\pm}| \le k$ for all combinations of signs, completing the proof. $\lhd$

To summarize, once we construct L in time $O^*(n+m)$, to compute an eight-partition, it is sufficient, by Claim 3.3, to find two vertices $p, q \in L$ satisfying $X(p,q) = Y(p,q) = 0$. In particular, it is possible to construct an eight-partition by enumerating all the $\Theta(m^2)$ pairs of vertices in L; the exact running time depends on how efficiently one can check candidate pairs. Below, we describe how to reduce the amount of remaining work to $O((m+n)\log m)$.

Speed up. Recall that L is connected and y-monotone. We orient L in the y-direction and view it as an alternating sequence of edges and vertices denoted by $x_1, x_2, \ldots, x_m$, with x_1, x_m being edges.

We extend the definition of X, Y as follows. If x_i, x_j are both edges, we pick arbitrary points p and q in the open edges x_i and x_j, respectively, and set $X(x_i, x_j) \coloneqq X(p,q)$ and $Y(x_i, x_j) \coloneqq Y(p,q)$; the cases where x_i is an edge or x_j is an edge, but not both, are handled analogously. Note that specifying the (open) edges containing p and q is sufficient to determine X and Y, hence the definition is unambiguous. Define $\pi\colon [m]^2 \to \mathbb{Z}^2$ by $\pi(i,j) \coloneqq (X(x_i,x_j), Y(x_i,x_j))$. With this setup, our goal is to identify a point $(i,j) \in [m]^2$ (corresponding to a pair of vertices on L) such that $\pi(i,j) = \mathbf{0}$.

We define a *grid curve* C to be a sequence of points $(i_1, j_1), \ldots, (i_t, j_t)$ in $\mathbb{Z}^2$ such that $(i_{\ell+1}, j_{\ell+1}) \in \{(i_\ell, j_\ell), (i_{\ell\pm1}, j_\ell)(i_\ell, j_{\ell\pm1})\}$ for each $\ell \in [t-1]$. In words, a grid curve is a walk in $\mathbb{Z}^2$ which, at each step, does not move at all or moves by exactly one unit up/down/left/right. A curve is *closed* if $(i_1, j_1) = (i_t, j_t)$. A grid curve is *simple* if non-consecutive points are distinct (we think of the start and end points as consecutive) – so the curve does not revisit a point after it moves away from the point.

To each grid curve C, we associate a piecewise linear curve $\overline{C}$ in $\mathbb{R}^2$, consisting of line segments connecting consecutive points $(i_\ell, j_\ell), (i_{\ell+1}, j_{\ell+1})$ of C for each $\ell \in [t-1]$. For a curve $\overline{C}$ not passing through the origin $\mathbf{0}$, the winding number $w(\overline{C})$ about $\mathbf{0}$ is defined in the standard way. In particular, provided $\overline{C}$ misses the origin,

$$w(C) = w(\overline{C}) = \begin{cases} 0 & \text{if } \overline{C} \text{ does not wind around } \mathbf{0}, \\ n > 0 & \text{if } \overline{C} \text{ winds around } \mathbf{0} \; n \text{ times counterclockwise}, \\ n < 0 & \text{if } \overline{C} \text{ winds around } \mathbf{0} \; -n \text{ times clockwise}. \end{cases}$$

For a rigorous definition of the winding number, see [13, Chapter 4.4.4]. Slightly abusing notation, we set $w(C) \coloneqq w(\overline{C})$.

Our algorithm proceeds as follows:

Step 1 Set $C \coloneqq T$ (see Definition 3.4). If $\pi(C)$ meets $\mathbf{0}$, then stop – we have found a point that maps to $\mathbf{0}$ (see Lemma 3.6). Otherwise $w(\pi(C))$ is odd, by Lemma 3.7.

Step 2 Construct two simple closed curves C_1, C_2 so that (a) $\overline{C} = \overline{C}_1 + \overline{C}_2$, (b) at least one of $\pi(C_1), \pi(C_2)$ has odd winding number (unless they meet $\mathbf{0}$), (c) the regions enclosed by $\overline{C}_1$ and $\overline{C}_2$ partition the region enclosed by $\overline{C}$, and (d) the area enclosed by each of $\overline{C}_1, \overline{C}_2$ is a fraction of that enclosed by $\overline{C}$ (see Lemma 3.9).

Step 3 If $\pi(C_1)$ or $\pi(C_2)$ meets $\mathbf{0}$, then stop – we found a point that maps to $\mathbf{0}$, by Lemma 3.6.

Step 4 Compute $w(\pi(C_1))$ and $w(\pi(C_2))$, and replace C with the one with the odd winding number. Goto Step 2.

We now proceed to fill in the details, starting with the definition of the initial curve T.

▶ **Definition 3.4** (The triangular grid curve T). *The simple closed grid curve T traverses a* triangular *path defined as follows:*

- *Starting with the bottom horizontal side of the grid $[m]^2$, T traverses the points*

$$(x_1, x_1), (x_2, x_1), \ldots, (x_m, x_1),$$

- *continuing along the right vertical side of the grid $[m]^2$ along the points*

$$(x_m, x_1), (x_m, x_2), \ldots, (x_m, x_m),$$

- *finally, traversing back* diagonally *along*

$$(x_m, x_m), (x_{m-1}, x_m), (x_{m-1}, x_{m-1}), (x_{m-2}, x_{m-1}), \ldots, (x_1, x_2), (x_1, x_1).$$

Along the diagonal side of T, we are really only interested in points of the form (x_ℓ, x_ℓ) with $\ell \in [m]$. However, since this doesn't give a grid curve, we "patch" it up by introducing intermediate points. Fortunately, this does not change the desired properties of T.

▶ **Lemma 3.5.** *If C is a grid curve, then $\pi(C)$ is a grid curve. Moreover, if L has already been computed, $\pi(C)$ can be computed in time $O(n + |C|)$.*

Proof. Consider a step in C from (x_i, x_j) to (x_{i+1}, x_j), where x_i is an edge of L and x_{i+1} is a vertex. Then x_{i+1} is contained in the planes that contain x_i and one additional plane H. Suppose, without loss of generality, that H is red. This means that the cardinality of one of the sets $R_p^{\pm}$ changes by one. Hence, the cardinality of $R_p^{\pm} \cap R_q^{\pm}$, for each combination of signs, changes by at most one – if H contains q, nothing changes. It follows that the function X changes by at most one, and the function Y remains unchanged.

Note that, up to symmetry, only one such transition or its reverse occurs in a single step of C. We've shown that each step causes either X or Y (but not both) to change by at most one, and, hence, $\pi(C)$ is a grid curve.

The computation can be carried out in constant time per incident edge-vertex pair of C, since L has been already computed, after a $O(n)$-time initialization that computes X, Y at an arbitrary starting point of C by brute force. ◀

Lemma 3.5 immediately implies the following.

▶ **Lemma 3.6.** *If $\overline{\pi(C)}$ meets $\mathbf{0}$, then some point of C is mapped to $\mathbf{0}$.*

A key property of the triangular grid curve T is the following.

▶ **Lemma 3.7.** *If $\mathbf{0} \notin \pi(T)$, then $w(T)$ is odd.*

Proof. Let $N := 4k+2$, and let H, V, D be the images (under π) of the horizontal, vertical, diagonal sides of T, respectively. Note that $\pi(T)$ is the concatenation of H, V, and D in that order.

Observe that if $p = q = x_i$ with $i \in [m]$, then $|R_p^+ \cap R_q^-| = |R_p^- \cap R_q^+| = 0$. Hence, $X(x_i, x_i) \in \{4k+1, 4k+2\}$ depending on whether x_i is contained in one or two red planes. Similarly, $Y(x_i, x_i) \in \{4k+1, 4k+2\}$. Hence $\pi(x_i, x_i) \in \{(N, N), (N-1, N-1)\}$ and, in particular, $\pi(x_i, x_i) = (N, N)$ if x_i is an edge. Along with Lemma 3.5, this implies that the grid curve D is a closed walk on the points in $\{N-2, N-1, N, N+1\}^2 \setminus \{\mathbf{0}\}$ starting and ending at the point (N, N).

Noting that x_1 and x_m are half-lines contained in the same red plane, and that every red plane that lies above x_1 lies below x_m and vice versa, we obtain $\pi(x_m, x_1) = (-N, -N)$. Hence, H is a grid curve from the point (N, N) to $(-N, -N)$ and V is a grid curve from the point $(-N, -N)$ to (N, N).

The discussion above implies that $w(T)$ is equal to the winding number of the concatenation of V and H. We argue below that V is the image of H under a rotation by 180° around the origin, i.e., the map $(x, y) \mapsto (-x, -y)$. Since, by assumption, neither H nor V contain $\mathbf{0}$, the concatenation of H and V has odd winding number as claimed.

Specifically, we show that $\pi(x_i, x_1) = -\pi(x_m, x_i)$. Since π is symmetric in the two arguments, this follows from $\pi(x_1, x_i) = -\pi(x_m, x_i)$. As mentioned before, every plane that lies above x_1 lies below x_m and vice versa. That is, $R_{x_1}^+ = R_{x_m}^-$ and $R_{x_1}^- = R_{x_m}^+$, and similarly $B_{x_1}^+ = B_{x_m}^-$ and $B_{x_1}^- = B_{x_m}^+$. The assertion is now straightforward to verify. ◀

▶ **Lemma 3.8.** *If $w(\pi(C))$ is odd, then there is a point $(i, j) \in \mathbb{Z}^2$ enclosed by $\overline{C}$ with $\pi(i, j) = \mathbf{0}$.*

Proof. A *grid square* S is a simple closed grid curve of the form

$$(i, j), (i+1, j), (i+1, j+1), (i, j+1), (i, j)$$

with $(i, j) \in \mathbb{Z}^2$. A *square* is $\overline{S}$ for some grid square S. If there is a grid square S enclosed by $\overline{C}$ such that $\pi(S)$ meets $\mathbf{0}$, then we are done by Lemma 3.6. Otherwise, note that $\overline{\pi(C)}$ is the sum of the images of the corresponding squares. Hence, there is a grid square S with $w(\pi(S))$ odd. By Lemma 3.5, $\pi(S)$ is a grid curve. By enumerating all possibilities (see Fig. 2), we conclude that $w(\pi(S))$ cannot be odd. ◀

Next, we show how to decompose a curve C. We restrict our attention to "trapezoidal" curves: Such a curve is the boundary of the intersection of the region bounded by the initial triangle T with a grid-aligned rectangle. This property is maintained inductively.

▶ **Lemma 3.9.** *Given a trapezoidal curve C whose image misses $\mathbf{0}$, with $w(\pi(C))$ odd, we can construct two trapezoidal curves C_1 and C_2 so that*

- **(i)** *the region R surrounded by $\overline{C}$ is partitioned into region R_1 surrounded by $\overline{C}_1$ and region R_2 surrounded by $\overline{C}_2$.*
- **(ii)** $\operatorname{area}(R_1), \operatorname{area}(R_2) \le c \cdot \operatorname{area}(R)$*, for an absolute constant $c < 1$.*
- **(iii)** *either $\mathbf{0}$ is in the image of C_1 and C_2 or $w(\pi(C)) = w(\pi(C_1)) + w(\pi(C_2))$.*

Proof. Note that the image of a grid square cannot have odd winding number, therefore R is not a grid square. As long as R is at least two units high, divide it by a horizontal grid chord into pieces with heights as equal as possible producing two regions R_1 and R_2. The curves C_1 and C_2 are the boundaries of the regions (refer to Fig. 3). If the height of R is one, perform a similar partition by a vertical chord into to near-equal-width pieces.

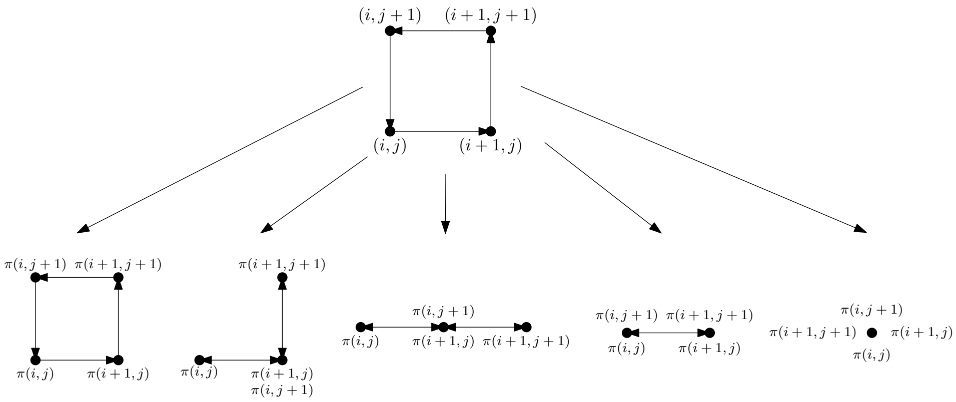

Figure 2 Up to symmetries, the different possibilities for the image under π of a grid square S, which is always always a grid curve in $\mathbb{Z}^2$, by Lemma 3.5.

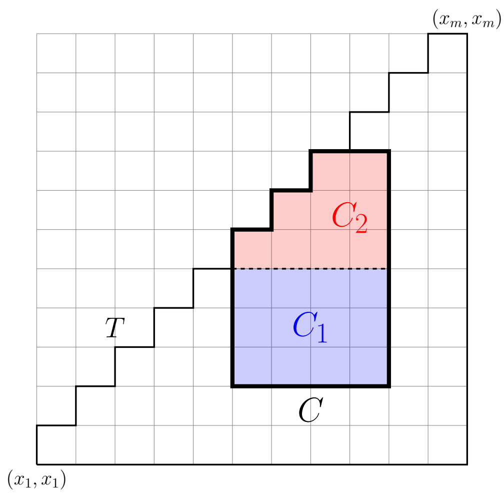

Figure 3 Curve C; the blue region is bounded by C_1, and the red by C_2, with the horizontal dividing chord drawn dashed.

Property (i) is satisfied by construction. If the image of the new chord misses $\mathbf{0}$, then both C_1 and C_2 avoid $\mathbf{0}$ and property (iii) follows from the properties of the winding number on the plane. Finally, an easy calculation shows that, if the split height is even, then each part contains at most $3/4$ of the original area; this fraction can rise to $5/6$ if R has odd height (the extreme case is achieved at height of three), which proves property (ii). ◀

▶ **Lemma 3.10.** *Given a simple closed grid curve C in $[m]^2$ we can determine whether $\pi(C)$ contains a zero. If not, we can compute $w(\pi(C))$, all in time $O(|C| + n)$.*

Proof. By Lemma 3.5, we can trace $\pi(C)$ step by step and, in particular, detect whether $\mathbf{0} \in \pi(C)$. So suppose this is not the case.

Consider the (open) ray ρ from the origin directed to the right in $\mathbb{Z}^2$. To determine the winding number of the curve $\overline{\pi(C)}$ not containing the origin, it is sufficient to count the number of times $\overline{\pi(C)}$ crosses the ray ρ. We may compute this by computing π for every vertex of C in order and counting the number of times $(X, 0)$ occurs along it, with $X > 0$.

As $\overline{\pi(C)}$ may partially overlap ρ, we need to check whether $\overline{\pi(C)}$ arrives at $(X, 0)$ with $X > 0$ from below the X-axis and (possibly after staying on the axis for a while) departs into the region above X-axis, or vice versa. That would count as a signed crossing. Arriving from below and returning below, or arriving from above and returning above, does not count as a crossing.

All of the above calculations can be done in time O(1) per step of $\pi(C)$, after proper initialization, by Lemma 3.5. ◀

Running time. We now analyze the running time of the algorithm we described. We can traverse a length-$O(m)$ closed grid curve C, compute its image $\pi(C)$, and check whether it passes through the origin in time $O(m + n)$ by Lemma 3.5. One can check whether $\pi(C)$ winds around the origin an odd number of times by Lemma 3.10, also in time $O(m + n)$.

The number of rounds of the main loop of the algorithm is $O(\log m)$, as the starting curve cannot enclose an area larger than $O(m^2)$ and areas shrink by a constant factor in every iteration, by Lemma 3.9. Combining everything together, we conclude that L can be computed in $O^*(n + m)$ time, and the algorithm can then identify the pair of vertices of L corresponding to an eight-partition in at most $O(\log m)$ rounds, each costing at most $O(m + n)$. This concludes the proof of Theorem 3.2.

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