The Ultimate Frontier: An Optimality Construction for Homotopy Inference

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Abstract

In our companion paper "Tight bounds for the learning of homotopy à la Niyogi, Smale, and Weinberger for subsets of Euclidean spaces and of Riemannian manifolds" we gave optimal bounds (in terms of the two one-sided Hausdorff distances) on a sample P of an input shape $\mathcal{S}$ (either manifold or general set with positive reach) such that one can infer the homotopy of $\mathcal{S}$ from the union of balls with some radius centred at P, both in Euclidean space and in a Riemannian manifold of bounded curvature. The construction showing the optimality of the bounds is not straightforward. The purpose of this video is to visualize and thus elucidate said construction in the Euclidean setting.

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Related Version *Full description of all topological transitions*: `https://hal.science/hal-04501285`

Supplementary Material *Audiovisual*: `https://youtu.be/N8wCCAcLgi8?si=wtdEDIKPNqFP6ne1`

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1 Introduction

The medial axis of a set consists of those points in ambient space with no unique closest point on the set, and the reach of a set is the distance between the set and its medial axis. In [6], Niyogi, Smale, and Weinberger showed that, given a C^2 manifold of positive reach and a sufficiently dense point sample (P) on (or near) the manifold, the union of balls of certain radii centred on the point sample captures the homotopy type of the manifold. Niyogi, Smale, and Weinberger's homotopy reconstruction result has led to generalizations including [3, 4, 5, 7].

In this video and our companion paper [2] we consider both submanifolds ($\mathcal{M}$) and general closed subsets of positive reach ($\mathcal{S}$) in the Euclidean space. While for the media contribution we concentrate on the Euclidean setting due to the ease of visualization, in [2] we also generalize the ambient space to Riemannian manifolds of bounded curvature.

We recall that the one-sided Hausdorff distance from X to Y, denoted by $d_H^o(X;Y)$, is the smallest ρ such that the union of balls of radius ρ centred at X covers Y. We denote the bound on the one-sided Hausdorff distance from P to $\mathcal{M}$ (resp. $\mathcal{S}$) by ε, and the one-sided Hausdorff distance from $\mathcal{M}$ (resp. $\mathcal{S}$) to P by δ. In [2], we achieved the following conditions on ε and δ which, if satisfied, guarantee the existence of a radius $r > 0$ such that the union of balls $\bigcup_{p\in P} B(p,r) = P \oplus B(r)$, with $\oplus$ the Minkowski sum, deformation-retracts onto $\mathcal{M}$ (resp. $\mathcal{S}$): If $\mathcal{S}$ has positive reach ($\mathcal{R}$), the condition is

$$\varepsilon + \sqrt{2}\,\delta \le (\sqrt{2}-1)\mathcal{R}. \qquad (\mathbf{CS})$$

If, moreover, $\mathcal{S} = \mathcal{M}$ is a manifold and $\delta \le \varepsilon \le \mathcal{R}$, the condition is

$$(\mathcal{R}-\delta)^2 - \varepsilon^2 \ge \left(4\sqrt{2}-5\right)\mathcal{R}^2. \qquad (\mathbf{CM})$$

The set of pairs (ε, δ) that satisfy these conditions is depicted in Figure 1.

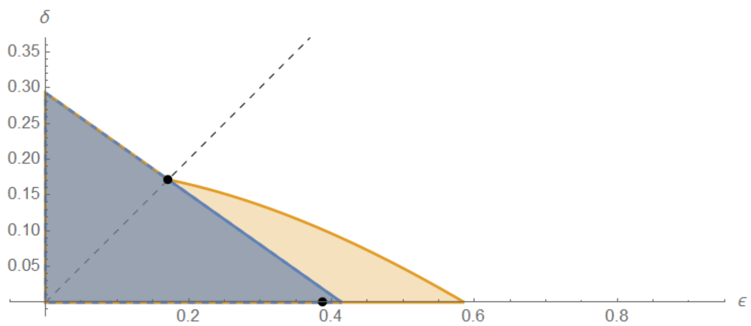


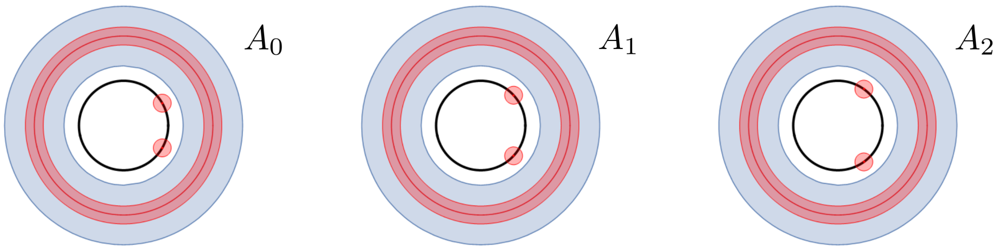
Figure 1 In blue we depict the region in (ε, δ)-space for which there exists a radius r such that the union of balls $P \oplus B(r)$ captures the homotopy type of a set of positive reach $\mathcal{R} = 1$. We do the same in yellow for a manifold. We stress that for $\delta \ge \varepsilon$, the fact that a set of positive reach is also a manifold does not lead to better bounds. The black points indicate the bounds that were known to Niyogi, Smale, and Weinberger.

In [2] we proved that the bounds **CS** and **CM** are optimal for sets of dimension at least 2 in the following sense: If the conditions are not satisfied, we can construct a set of positive reach $\mathcal{S}$ (resp. manifold $\mathcal{M}$) and a sample P, such that the homology of the thickening $P \oplus B(r)$ always differs from the homology of $\mathcal{S}$ (resp. $\mathcal{M}$).

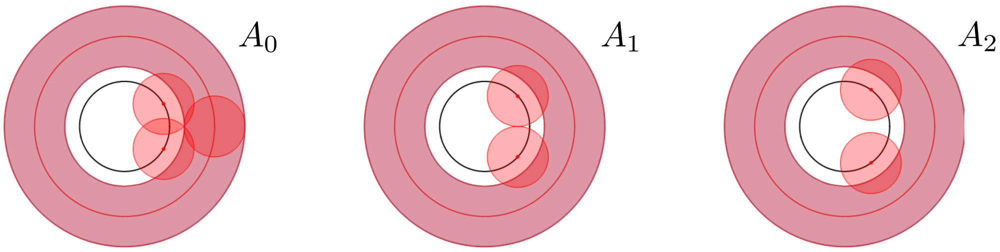
In the following we construct the set $\mathcal{S}$ (resp. the manifold $\mathcal{M}$) and the sample P that show the optimality of our bounds. The main goal of the video is to visualize the construction.

2 The Construction

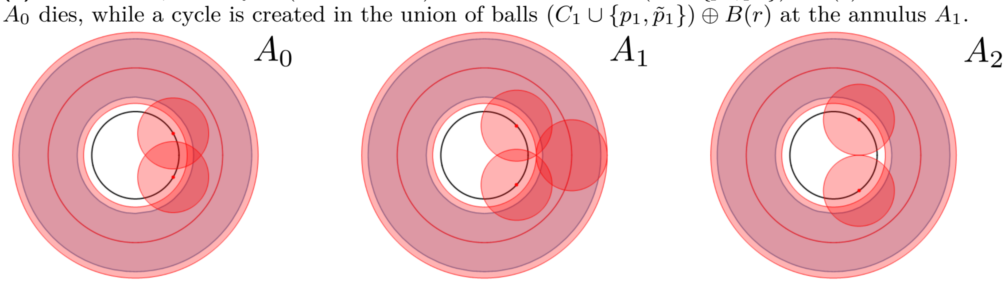
Note that due to rescaling it suffices to construct sets of reach $\mathrm{rch}(\mathcal{S}) = \mathcal{R} = 1$.



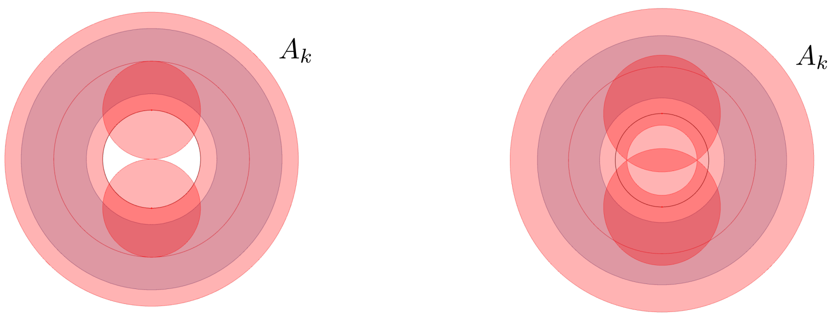
(a) For all $r < r_0$, the union of balls $(C_i \cup \{p_i, \tilde{p}_i\}) \oplus B(r)$ has three connected components.



(b) At radius r_1, the 1-cycle (created at r_0) in the union of balls $(C_0 \cup \{p_0, \tilde{p}_0\}) \oplus B(r)$ at the annulus A_0 dies, while a cycle is created in the union of balls $(C_1 \cup \{p_1, \tilde{p}_1\}) \oplus B(r)$ at the annulus A_1.



(c) At radius r_2, the cycle in the union of balls at the annulus A_1 dies, while a cycle is created in the union of balls at the annulus A_2.



(d) The set $(C_k \cup \{p_k, \tilde{p}_k\}) \oplus B(r)$ at radius $r_k = 1 - \delta$. The two "gaps" are identical.

(e) The two "gaps" of the set $(C_k \cup \{p_k, \tilde{p}_k\}) \oplus B(r)$ disappear simultaneously.

Figure 2 The changing homology of the thickening $P \oplus B(r)$ in the annuli A_0, A_1, A_2, and A_k. The set $\mathcal{S}$ is in blue, the sample P in red, and the thickening $P \oplus B(r)$ in pink. The black circles indicate the location of the two isolated sample points of P associated to each annulus.

2.1 Sets of Positive Reach

The construction for sets of positive reach goes as follows: We define $\mathcal{S}$ to be a union of annuli A_i in $\mathbb{R}^2$, each of which has inner radius 1 and outer radius $1+2\varepsilon$. We lay the annuli in a row at distance at least 2 away from each other. We number the annuli from $i = 0$.

The sample P consists of circles C_i of radius $1+\varepsilon$ lying in the middle of the annuli ($C_i \subseteq A_i$), and pairs of points $\{p_i, \tilde{p}_i\}$. Each pair $\{p_i, \tilde{p}_i\}$ lies in the disk inside the annulus A_i, at a distance δ from A_i, and the two points lie at a distance $2r_i$ from each other. The bisector of p_i and $\tilde{p}_i$ intersects the circle C_i in two points. We let q_i be the intersection point that is closest to p_i (and thus $\tilde{p}_i$). We denote the circumradius of $p_i\tilde{p}_i q_i$ by R_i and note that $R_i \geq r_i$.

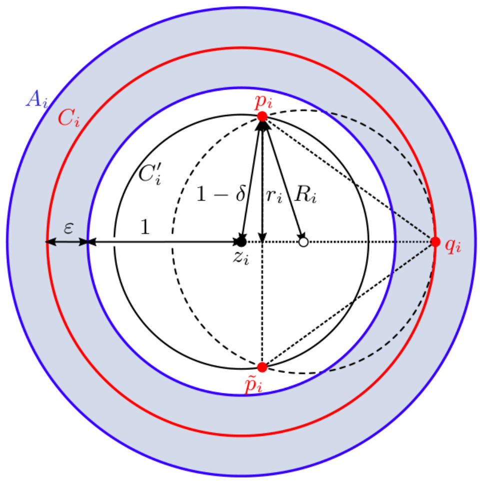


Figure 3 Each annulus A_i is sampled by a circle C_i and a pair of points $\{p_i, \tilde{p}_i\}$. The circumradius is indicated by R_i.

We set $r_0 = \frac{\delta+\varepsilon}{2}$ and, for $i \geq 0$,

$$r_{i+1} = \begin{cases} R_i, & \text{if } R_i < 1-\delta, \\ 1-\delta, & \text{otherwise.} \end{cases}$$

We stop the sequence at the first value of $i = k$ such that $r_k = 1-\delta$. Our constructed set $\mathcal{S}$ consists of the finitely many annuli $A_0 \cup A_1 \cup \ldots \cup A_k$ and our sample P is defined as $\bigcup_{0 \leq i \leq k} C_i \cup \{p_i, \tilde{p}_i\}$. The topological transitions of the thickening of the sample $(P \oplus B(r))$ are illustrated in Figure 2 and described in the supplementary material [1].

2.2 Manifolds

The construction for manifolds goes as follows: We define $\mathcal{M}$ to be a union of tori of revolution $T_i \subseteq \mathbb{R}^3$. Each of these tori is the 1-offset of a circle (in the horizontal plane) of radius 2 in $\mathbb{R}^3$. We number the tori from $i = 0$, and lay them out in a row at a distance at least 2 apart from one another. Due to this assumption, the reach of $\mathcal{M}$ equals 1.

The sample P consists of sets C_i which are tori with a part cut out, and pairs of points $\{p_i, \tilde{p}_i\}$ lying inside the hole of each torus T_i. To construct each set C_i we take the δ-offset of T_i, keep the part that lies inside the solid torus bounded by T_i, and remove an ε-neighbourhood of the circle obtained by revolving the point $(1, 0, 0)$ around the z-axis; see the red set in Figures 4 and 5.

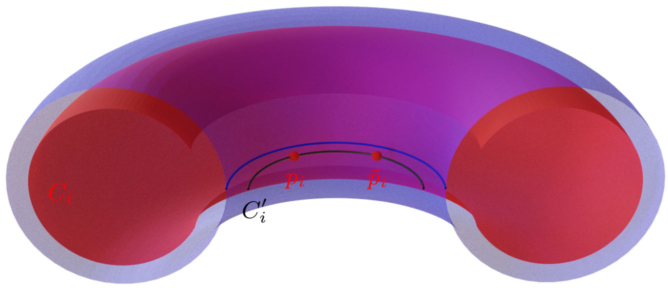


Figure 4 The (half of the) torus T_i depicted in blue; the sample – the set C_i and the points p_i and $\tilde{p}_i$ – in red. In black we indicate the circle C_i'. The closest point projection of this circle onto $\mathcal{M}$ is indicated in blue.

Let C_i' be the circle found by revolving the point $(1-\delta, 0, 0)$ around the z-axis. Each pair of points p_i and $\tilde{p}_i$ lies on C_i' at a distance $2r_i$ from each other. Let q_i and $\tilde{q}_i$ be those two points in the intersection of the bisector of p_i and $\tilde{p}_i$ and the set C_i, that lie closest to p_i and $\tilde{p}_i$. Note that q_i and $\tilde{q}_i$ lie on the boundary of C_i (where we think of C_i as a manifold with boundary), and $\{q_i, \tilde{q}_i\} = \pi_{C_i}\left(\frac{p_i+\tilde{p}_i}{2}\right)$. Denote the circumradius of the simplex $p_i\tilde{p}_i q_i \tilde{q}_i$ by R_i.

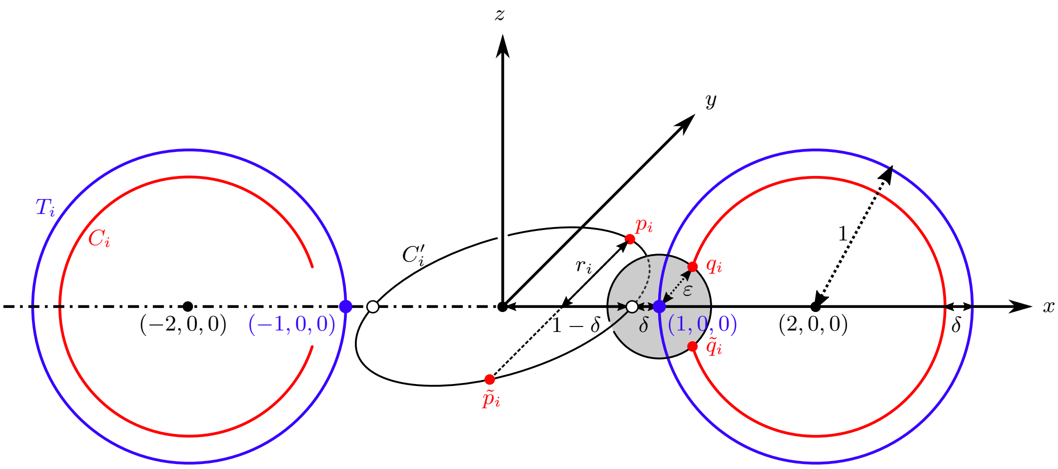


Figure 5 The sets T_i, C_i and C_i' are obtained by rotating around the z-axis, respectively, the blue circles, the red arcs and the white point.

We define the distance $2r_i$ between each pair of points p_i and $\tilde{p}_i$ inductively. We set the distance r_0 such that the balls $B(p_0, r)$ and $B(\tilde{p}_0, r)$ start to intersect at the same value of r as the balls $B(q_0, r)$ and $B(\tilde{q}_0, r)$ start to intersect:

$$r_0 = \tfrac{1}{2} d\left(q_0, \tilde{q}_0\right) = \sqrt{\epsilon^2 - \left(\frac{\epsilon^2 - \delta^2 + 2\delta}{2}\right)^2}.$$

We then define

$$r_{i+1} = \begin{cases} R_i, & \text{if } R_i < 1-\delta, \\ 1-\delta, & \text{otherwise.} \end{cases}$$

We stop the sequence at the first value of $i = k$ such that $r_i = 1 - \delta$.

Finally, the manifold $\mathcal{M}$ consists of the finitely many tori $T_0 \cup T_1 \cup \ldots \cup T_k$, and the sample P is defined as $\bigcup_{0 \le i \le k} (C_i \cup \{p_i, \tilde{p}_i\})$. The topological transitions of the thickening of the sample $(P \oplus B(r))$ are described in the supplementary material [1].

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