Ipelets for the Convex Polygonal Geometry

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Abstract

There are many structures, both classical and modern, involving convex polygonal geometries whose deeper understanding would be facilitated through interactive visualizations. The Ipe extensible drawing editor, developed by Otfried Cheong, is a widely used software system for generating geometric figures. One of its features is the capability to extend its functionality through programs called Ipelets. In this media submission, we showcase a collection of new Ipelets that construct a variety of geometric objects based on polygonal geometries. These include Macbeath regions, metric balls in the forward and reverse Funk distance, metric balls in the Hilbert metric, polar bodies, the minimum enclosing ball of a point set, and minimum spanning trees in both the Funk and Hilbert metrics. We also include a number of utilities on convex polygons, including union, intersection, subtraction, and Minkowski sum (previously implemented as a CGAL Ipelet).

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Supplementary Material *Software (Source Code)*: `https://github.com/GeneralCoder365/umd_ipelets`, archived at `swh:1:dir:9bd3fa80632acd54ba70aaa6d413da809ef3d73d`

1 Introduction

We present several **Ipelets** for the convex polygonal geometry. These Ipelets include Macbeath regions, metric balls in the forward and reverse Funk distance, metric balls in the Hilbert metric, polar bodies, the minimum enclosing ball of a point set, and minimum spanning trees in both the Funk and Hilbert metrics. All Ipelets are programmed in Lua and freely available at `https://github.com/GeneralCoder365/umd_ipelets`. To install an Ipelet, download the file and place it in the `ipelets` subfolder of your Ipe folder.

2 Structures

In this section, we describe the geometric structures that our Ipelets compute. Many Ipelets are available that calculate geometric structures such as Poincare disks [10], free space diagrams [41], triangulations [23], circular fillets in polygons [32], graph embeddings [40], tangent lines [33], tessellations [13] and a Voronoi diagrams Ipelet [16] (a default Ipelet).

2.1 Macbeath Regions

Given a point x in a convex polygon Ω, the Macbeath region [31] around x, denoted $M_\Omega(x)$, is a useful tool in convex approximation [1, 5–7], approximate range searching [14], smooth distance approximation [2]. Additionally, it has useful relations with cap coverings [8, 9] and Hilbert balls [1]. Macbeath regions around a point are constructed by taking the original polygon and intersecting it with its reflection around the point (see Figure 1).

▶ **Definition 1** (Macbeath Region). *Given a convex polygon Ω in $\mathbb{R}^d$ and a point $x \in \mathrm{int}(\Omega)$*

$$M_\Omega(x) \;=\; x + ((\Omega - x) \cap (x - \Omega)).$$

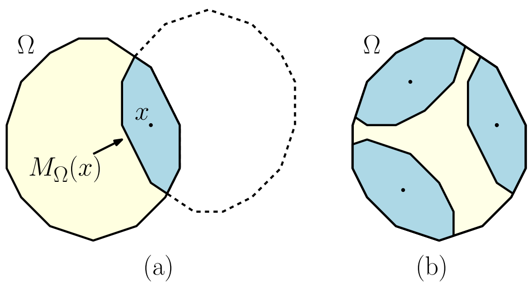

Figure 1 (a) A Macbeath region around a point x and (b) a collection of Macbeath regions.

2.2 Funk and Hilbert Balls

The Funk metric is an asymmetric metric used in the analysis of Finsler metrics. Its importance for Finsler geometry and differential geometry is the collection of the following three properties: it is non-reversible, complete, and projectively flat [39, 43, 49]. Since it is non-symmetric, to define balls, we present Ipelets for both the forward and reverse Funk metric. The Funk and reverse Funk ball around a point p in a polygonal geometry Ω with m sides is a polygon with $O(m)$ sides. Let $\|\cdot\|$ denote the Euclidean norm.

▶ **Definition 2** ((Forward) Funk Metric). *Given an open convex body Ω in $\mathbb{R}^d$ and two distinct points $p, q \in \mathrm{int}(\Omega)$, let q' be the intersection of the ray p through q with $\partial\Omega$ (Figure 2(a)):*

$$F_\Omega(p,q) \;=\; \ln \frac{\|p-q'\|}{\|q-q'\|}, \quad \textit{and} \quad F_\Omega(p,p) = 0.$$

▶ **Definition 3** ((Reverse) Funk Metric). *Given an open convex body Ω in $\mathbb{R}^d$ and two distinct points $p, q \in \mathrm{int}(\Omega)$, let p' be the intersection of the ray q through p with $\partial\Omega$ (see Figure 2(b)):*

$${}^rF_\Omega(p,q) \;=\; \ln \frac{\|p'-q\|}{\|p'-p\|}, \quad \textit{and} \quad {}^rF_{\Omega(p,p)} = 0.$$

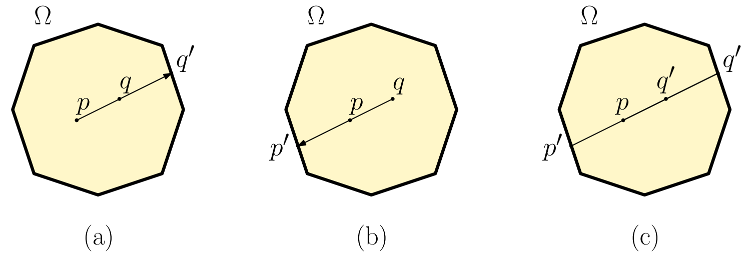

Figure 2 (a) The Funk metric, (b) the reverse Funk metric, and (c) the Hilbert metric. Seen in these are the rays and segments used in the definitions of the respective metrics.

The Hilbert polygonal geometry is a generalization of the Cayley-Klein model of hyperbolic geometry to arbitrary convex bodies [17] and the average of the forward and reverse Funk metrics [38]. It has applications in convex approximation [1,3,46], clustering [36], and graph embeddings [37]. It is also related to flags [45,46]. Several classical algorithms have been developed for the polygonal metric space including Voronoi diagrams [15,27] and Delaunay triangulations [26]. The Hilbert ball around a point p in a polygonal geometry Ω with m sides will be a polygon with $O(m)$ sides [35]. Both Funk and Hilbert balls are constructed with the use of spokes representing the intersection of lines through the center of the ball and the vertices of Ω with $\partial\Omega$ (see Figure 3).

▶ **Definition 4** (Hilbert metric)**.** *Given an open convex body Ω in $\mathbb{R}^d$ and two distinct points $p, q \in \Omega$, let p' and q' be the intersection of line pq with $\partial\Omega$ such that the points lie in order $\langle p', p, q, q'\rangle$ (see Figure 2(c)) then:*

$$H_\Omega(p,q) \;=\; \frac{1}{2}\ln\left(p',p;q,q'\right) = \frac{1}{2}(F_\Omega(p,q) + {}^rF_\Omega(p,q)), \quad \textit{and} \quad H_\Omega(p,p) = 0.$$

Where $(p',p;q,q')$ is the cross ratio of the four points.

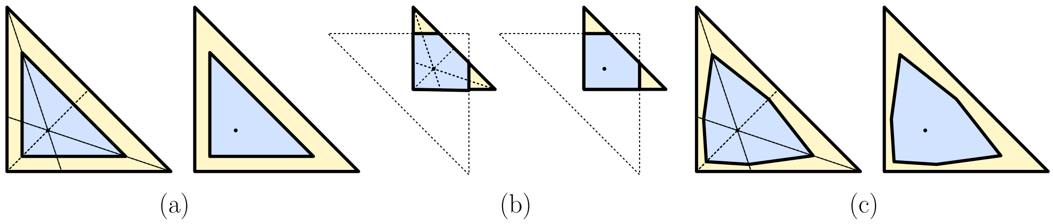

Figure 3 (a) Funk, (b) reverse Funk, and (c) Hilbert unit balls (with and without spokes).

2.3 Polar Bodies

The polar body of a convex polygon is a classical duality with modern applications in Minkowski geometry [28], quantum mechanics and information theory [19–22], and convex approximation [4,7,34]. The key quality of the polar body of a convex polygon is that pointed corners in the primal space become flatter in the dual (see Figure 4).

▶ **Definition 5** (Polar Body)**.** *Given a convex body Ω in $\mathbb{R}^d$ containing the origin, let $\langle\cdot,\cdot\rangle$ denote the inner product, the polar dual of Ω, Ω° is defined to be:*

$$\Omega^\circ \;=\; \{q \in \mathbb{R}^d \,:\, \langle p,q\rangle \leq 1, \forall p \in \Omega\},$$

If Ω is in $\mathbb{R}^d$ with vertices $\{(a_i, b_i, \dots)\}_{i=1}^m$, its polar dual Ω° is the polygon defined as the intersection of the half-spaces $\{a_i x + b_i y + \cdots \leq 1\}_{i=1}^m$.

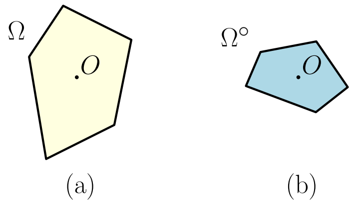

Figure 4 (a) The original body and (b) its polar body with origin O.

2.4 Funk and Hilbert Minimum Spanning Trees

The minimum spanning tree (MST) of a graph G, is the smallest tree contained in G by edge weight that touches every vertex of G. Modern applications include the analysis of brain networks [11], evolutionary algorithms [12], and clustering [25]. The Funk (the maximum of both the Funk and reverse Funk distances) and Hilbert minimum spanning trees have yet to be studied. An example of each appears below (see Figure 5).

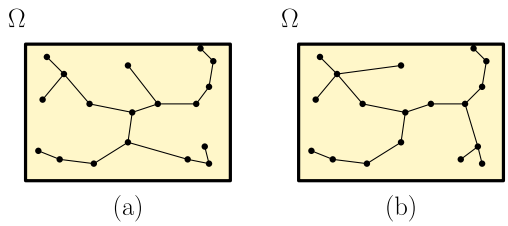

Figure 5 (a) The Hilbert MST and (b) Funk MST of a pointset in a convex body Ω.

3 Operations

In this section we describe the geometric operations that our Ipelets compute.

3.1 Boolean Operations

Boolean operations on convex polygons are fundamental to many problems in computational geometry [18]. We implemented three such operations: polygon subtraction, union, and intersection (see Figure 6).

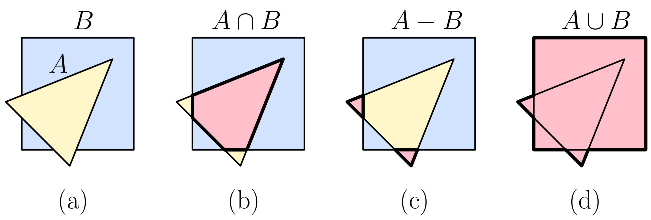

Figure 6 Boolean operations between two convex polygons, a triangle A and a square B, (a) the original pair, (b) the intersection, (c) the subtraction, and (d) the union.

3.2 Minkowski Sum

The Minkowski sum of two polygons is used extensively in many modern fields of research for collision detection [42, 47, 48], solid modeling [24, 44], and motion planning [29, 30]. It is defined as the pairwise addition of all points in both polygons (see Figure 7(a)). The Minkowski sum of two polygons can be thought of as the region traced out by the centroid of one polygon moving along the boundary of the other (see Figure 7(b)). In the Ipelet, the Minkowski sum is shifted to the centroid of the two polygons.

▶ **Definition 6** (Minkowski Sum). *Given two polygons* Ω *and* Ψ *the Minkowski sum of the two polygons is defined to be:*

$$\Omega \oplus \Psi \;=\; \{a + b \in \mathbb{R}^d \,:\, a \in \Omega, b \in \Psi\}.$$

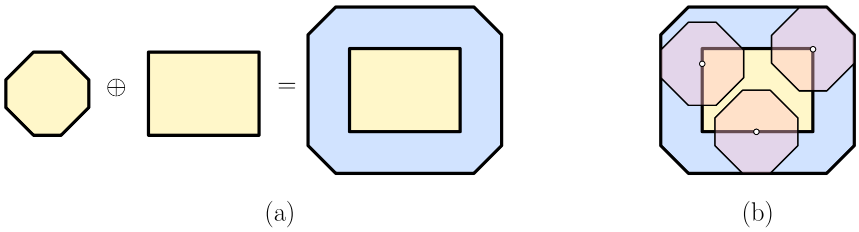

Figure 7 (a) Minkowski sum of two polygons, (b) collision detection using the two polygons.

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