# Subexponential Algorithms in Geometric Graphs via the Subquadratic Grid Minor Property: The Role of Local Radius

**Gaétan Berthe**
LIRMM, Université de Montpellier, CNRS, Montpellier, France

**Marin Bougeret**
LIRMM, Université de Montpellier, CNRS, Montpellier, France

**Daniel Gonçalves**
LIRMM, Université de Montpellier, CNRS, Montpellier, France

**Jean-Florent Raymond**
Univ Lyon, EnsL, CNRS, LIP, F-69342, Lyon Cedex 07, France

## Abstract

We investigate the existence in geometric graph classes of subexponential parameterized algorithms for cycle-hitting problems like Triangle Hitting (TH), Feedback Vertex Set (FVS) or Odd Cycle Transversal (OCT). These problems respectively ask for the existence in a graph $G$ of a set $X$ of at most $k$ vertices such that $G - X$ is triangle-free, acyclic, or bipartite. It is know that subexponential FPT algorithms of the form $2^{o(k)}n^{\mathcal{O}(1)}$ exist in planar and even $H$-minor free graphs from bidimensionality theory [Demaine et al. 2005], and there is a recent line of work lifting these results to geometric graph classes consisting of intersection of similarly sized "fat" objects ([Fomin et al. 2012], [Grigoriev et al. 2014], or disk graphs [Lokshtanov et al. 2022], [An et al. 2023]).

In this paper we first identify sufficient conditions, for any graph class $\mathcal{C}$ included in string graphs, to admit subexponential FPT algorithms for any problem in $\mathcal{P}$, a family of bidimensional problems where one has to find a set of size at most $k$ hitting a fixed family of graphs, containing in particular FVS. Informally, these conditions boil down to the fact that for any $G \in \mathcal{C}$, the *local radius of* $G$ (a new parameter introduced in [Lokshtanov et al. 2023]) is polynomial in the clique number of $G$ and in the maximum matching in the neighborhood of a vertex. To demonstrate the applicability of this generic result, we bound the local radius for two special classes: intersection graphs of axis-parallel squares and of contact graphs of segments in the plane. This implies that any problem $\Pi \in \mathcal{P}$ (in particular, FVS) can be solved in:

- $2^{\mathcal{O}(k^{3/4}\log k)}n^{\mathcal{O}(1)}$-time in contact segment graphs,
- $2^{\mathcal{O}(k^{9/10}\log k)}n^{\mathcal{O}(1)}$ in intersection graphs of axis-parallel squares

On the positive side, we also provide positive results for TH by solving it in:

- $2^{\mathcal{O}(k^{3/4}\log k)}n^{\mathcal{O}(1)}$-time in contact segment graphs,
- $2^{\mathcal{O}(\sqrt{d}t^2(\log t)k^{2/3}\log k)}n^{\mathcal{O}(1)}$-time in $K_{t,t}$-free $d$-DIR graphs (intersection of segments with $d$ slopes)

On the negative side, assuming the ETH we rule out the existence of algorithms solving:

- TH and OCT in time $2^{o(n)}$ in 2-DIR graphs and more generally in time $2^{o(\sqrt{\Delta n})}$ in 2-DIR graphs with maximum degree $\Delta$, and
- TH, FVS, and OCT in time $2^{o(\sqrt{n})}$ in $K_{2,2}$-free contact-2-DIR graphs of maximum degree 6.

Observe that together, these results show that the absence of large $K_{t,t}$ is a necessary and sufficient condition for the existence of subexponential FPT algorithms for TH in 2-DIR.

**2012 ACM Subject Classification** Mathematics of computing → Graph algorithms; Theory of computation → Fixed parameter tractability; Theory of computation → Computational geometry

**Keywords and phrases** geometric intersection graphs, subexponential FPT algorithms, cycle-hitting problems, bidimensionality

**Digital Object Identifier** 10.4230/LIPIcs.SWAT.2024.11

**Related Version** *Full Version*: https://arxiv.org/abs/2306.17710

**Funding** *Jean-Florent Raymond*: Supported by the project GRALMECO (ANR-21-CE48-0004).


19th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2024).
Editor: Hans L. Bodlaender; Article No. 11; pp. 11:1–11:18
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

# 1 Introduction

In this paper we consider fundamental NP-hard cycle-hitting problems like Triangle Hitting (TH), Feedback Vertex Set (FVS), and Odd Cycle Transversal (OCT) where, given a graph $G$ and an integer $k$, the goal is to decide whether $G$ has a set of at most $k$ vertices hitting all its triangles (resp. cycles for FVS, and odd cycles for OCT). We consider these problems from the perspective of parameterized complexity, where the objective is to answer in time $f(k)n^{\mathcal{O}(1)}$ for some computable function $f$, and with $n$ denoting the order of $G$. It is known (see for instance [12]) that these three problems can be solved on general graphs in time $c^{\mathcal{O}(k)}n^{\mathcal{O}(1)}$ (for some constant $c$) and that, under the Exponential Time Hypothesis (ETH), the contribution of $k$ cannot be improved to a subexponential function (i.e., there are no algorithms with running times of the form $c^{o(k)}n^{\mathcal{O}(1)}$ for these problems). However, it was discovered that some problems admit subexponential time algorithms in certain classes of graphs, and there is now a well established set of techniques to design such algorithms. Let us now review these techniques and explain why they do not apply on the problems we consider here.

**Subexponential FPT algorithms in sparse graphs.** Let us start with the bidimensionality theory, which gives an explanation on the so-called *square root phenomenon* arising for planar and $H$-minor free graphs [14] for bidimensional[1] problems, where a lot of graph problems admit ETH-tight $2^{\mathcal{O}(\sqrt{k})}n^{\mathcal{O}(1)}$ algorithms. What we call a *graph parameter* here is a function $p$ mapping any (simple) graph to a natural number and that is invariant under isomorphism. The classical win-win strategy to decide if $p(G) \leq k$ for a minor-bidimensional[2] parameter (like $p = \mathsf{fvs}$, the size of a minimum feedback vertex set of $G$) is to first reduce to the case where $\boxplus(G) = \mathcal{O}(\sqrt{k})$ (where $\boxplus(G)$ denotes the maximum $k$ such that the $(k,k)$-grid is contained as a minor in $G$), and then use an inequality of the form $\mathsf{tw}(G) \leq f(\boxplus(G))$ to bound the treewidth obtained through the following property.

▶ **Definition 1** ([4]). *Given $c < 2$, a graph class $\mathcal{G}$ has the* subquadratic grid minor property *for $c$ (*SQGM *for short), denoted $\mathcal{G} \in SQGM(c)$, if $\mathsf{tw}(G) = \mathcal{O}(\boxplus(G)^c)$ for all $G \in \mathcal{G}$. We write $\mathcal{G} \in SQGM$ if there exists $c < 2$ such that $\mathcal{G} \in SQGM(c)$.*

While in general every graph $G$ satisfies the inequality $\mathsf{tw}(G) \leq \boxplus(G)^c$ for some $c < 10$ [11], the SQGM property additionally require that $c < 2$. Thus, for any $\mathcal{G} \in SQGM(c)$ and $G \in \mathcal{G}$ such that $\boxplus(G) = \mathcal{O}(\sqrt{k})$, we get $\mathsf{tw}(G) \leq \boxplus(G)^c = \mathcal{O}\left(k^{c/2}\right) = o(k)$. For instance planar graphs and more generally $H$-minor free graph [15] are known to have a treewidth linearly bounded from above by the size of their largest grid minor. In other words, these classes belong to $SQGM(1)$. The conclusion is that the SQGM property allows subexponential parameterized algorithms for minor-bidimensional problems (if the considered problem has a $2^{\mathcal{O}(\mathsf{tw}(G))}n^{\mathcal{O}(1)}$-time algorithm) on sparse graph classes. Notice that these techniques have been extended to contraction-bidimensional problems [4].

**Extension to geometric graphs.** Consider now a geometric graph class $\mathcal{G}$, meaning that any $G \in \mathcal{G}$ represents the interactions of some specified geometric objects. We consider here (Unit) Disk Graphs which correspond to intersection of (unit) disks in the plane, $d$-DIR graphs (where the vertices correspond to segments with $d$ possible slopes in $\mathbb{R}^2$), and contact-segment

[1] Informally: yes-instances are minor-closed and a solution on the $(r,r)$-grid has size $\Omega(r^2)$.
[2] See definition in [20].

graphs (where each vertex corresponds to a segment in $\mathbb{R}^2$, and any intersection point between two segments must be an endpoint of one of them). We refer to Subsection 2.2 for formal definitions. Classes of geometric graphs represented in the plane form an appealing source of candidates to obtain subexponential parameterized algorithms as there is an underlying planarity in the representation. However these graphs are no longer sparse as they may contain large cliques, and thus cannot have the SQGM property. Indeed, if $G$ is a clique of size $a$, then $\mathsf{tw}(G) = a - 1$ but $\boxplus(G) \leq \sqrt{|G|} = \sqrt{a}$. To overcome this, let us introduce the following notion where the bound on treewidth is allowed to depend on an additional parameter besides $\boxplus(G)$.

▶ **Definition 2.** *Given a graph parameter $p$ and a real $c < 2$, a graph class $\mathcal{G}$ has the* almost subquadratic grid minor property *(*ASQGM *for short) for $p$ and $c$ if there exists a function $f$ such that $\mathsf{tw}(G) = \mathcal{O}(f(p(G))\boxplus(G)^c)$. The class $\mathcal{G}$ has ASQGM$(p)$ if there exists $c < 2$ such that $\mathcal{G}$ has the ASQGM property for $p$ and $c$. The notion is naturally extended to more than one parameter.*

This notion was used implicitly in earlier work (e.g., [20]) but we chose to define it explicitly in order to highlight the contribution $f$ of the parameter $p$ to the treewidth, which is particularly relevant when it can be shown to be small (typically, polynomial). Let us now explain how ASQGM can be used to obtain subexponential parameterized algorithms on geometric graphs.

It was shown in [19] that FVS can be solved in time $2^{\mathcal{O}(k^{3/4}\log k)}n^{\mathcal{O}(1)}$ in map graphs, a superclass of planar graphs where arbitrary large cliques may exist, as follows. Let $\omega(G)$ denote the order of the largest clique in a graph $G$. The first ingredient is to prove that map graphs have *ASQGM*$(\omega)$, and more precisely that $\mathsf{tw}(G) = \mathcal{O}(\omega(G)\boxplus(G))$. Then, if $\omega(G) \geq k^\epsilon$ for some $\epsilon$, the presence of such large clique allows to have subexponential branchings (as a solution of FVS must take almost all vertices of a clique). When $\omega(G) < k^\epsilon$, then the ASQGM property gives that $\mathsf{tw}(G) \leq k^\epsilon \boxplus(G) \leq k^{\frac{1}{2}+\epsilon}$ (as before we can immediately answer no if $\boxplus(G) > \mathcal{O}(\sqrt{k})$). By appropriately choosing $\epsilon$ the authors of [19] obtain the mentioned running time. The same approach also applies to unit disk graphs and has since been improved to $2^{\sqrt{k}\log k}n^{\mathcal{O}(1)}$ in [17] using a different technique, and finally improved to an optimal $2^{\sqrt{k}}(n+m)$ in [2] for similarly sized fat objets (which typically includes unit squares, but not disks, squares, nor segments).

There is also a line of work aiming at establishing ASQGM property for different classes of graphs and parameters, with for example [20] proving that (1) string graphs have ASQGM when the parameter $p$ is the number of times a string is intersected (assuming at most two strings intersect at the same point), and that (2) intersection graphs of "fat" and convex objects have ASQGM when the parameter $p(G)$ is the minimal order of a graph $H$ not subgraph of $G$ (generalizing the degree when $H$ is a star).

**When *ASQGM*$(\omega)$ does not hold.** A natural next step for FVS and TH is to consider classes that are not *ASQGM*$(\omega)$. Observe (see Figure 1) that neither disk graphs, nor contact-2-DIR graphs are in *ASQGM*$(\omega)$, and thus constitute natural candidates.

New ideas allowed the authors of [23] to obtain subexponential parameterized algorithms on disk graphs, in particular for TH and FVS. The first idea is a preliminary branching step (working on general graphs) which given an input $(G, k)$ first reduces to the case where we are given a set $M$ of size $\mathcal{O}(k^{1+\epsilon})$ such that $G - M$ is a forest and, for any $v \in M$, $N(v) \setminus M$ is an independent set (corresponding to Corollary 7, but where we consider a generic problem instead of FVS). The second idea is related to neighborhood complexity. If for a graph class $\mathcal{G}$ there is a constant $c$ such that for every $G \in \mathcal{G}$ and every $X \subseteq V(G)$,

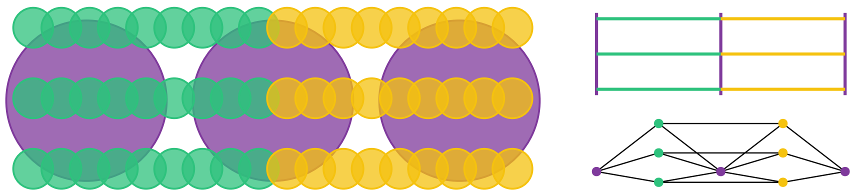

**Figure 1** Left: a representation of a disk graph. Right: a contact 2-DIR graph and the corresponding graph. In these graphs (where the left one is from [19]), $\omega(G)$ is constant, $\mathsf{tw}(G) \geq t$ (where $t = 3$ here) as it contains $K_{t,t}$ as a minor, and $\boxplus(G) = \mathcal{O}(\sqrt{t})$ as they have a feedback vertex set of size at most $t$.

$|\{N(v) \cap X : v \in V(G)\}| \leq c|X|$, then we say that $\mathcal{G}$ has *linear neighborhood complexity* with *ratio* $c$. The following theorem was originally formulated using ply (the maximum number of disks containing a fixed point) instead of clique number, but it is known [7] that these two values are linearly related in disk graphs.

▶ **Theorem 3** (Theorem 1.1 in [23]). *Disk graphs with bounded clique number have linear neighborhood complexity.*

For TH, these two ideas are sufficient to obtain a subexponential parameterized algorithm. For FVS, [23] provides the following corollary.

▶ **Corollary 4** (Corollary 1.1 in [23] restricted to FVS). *Let $G$ be a disk graph with a (non-necessarily minimal) feedback vertex set $M \subseteq V(G)$ such that for all $v \in M$, $N(v) \setminus M$ is an independent set, and such that for all $v \in V(G) \setminus M$, $N(v) \setminus M$ is non-empty. Then, the treewidth of $G$ is $\mathcal{O}(\sqrt{|M|}\omega(G)^{2.5})$.*

As they use this corollary after a branching process reducing the clique number to $k^{\epsilon}$ and as their (approximated) feedback vertex set $M$ has size $|M| = k^{1+\epsilon'}$, they obtain a sublinear treewidth and thus a subexponential parameterized algorithm for FVS (and several variants of FVS) running in time $2^{\mathcal{O}(k^{13/14}\log k)}n^{\mathcal{O}(1)}$. Recently this running time has been improved to $2^{\mathcal{O}(k^{7/8}\log k)}n^{\mathcal{O}(1)}$ when the representation is given and $2^{\mathcal{O}(k^{9/10}\log k)}n^{\mathcal{O}(1)}$ otherwise [1]. We point out that it is likely that the algorithms of [23] and [1] solving FVS in disk graphs with the respective running times $2^{\mathcal{O}(k^{13/14}\log k)}n^{\mathcal{O}(1)}$ and $2^{\mathcal{O}(k^{9/10}\log k)}n^{\mathcal{O}(1)}$, can be adapted[3] to the setting of square graphs, the later matching our bound.

**Subexponential FPT algorithms via kernels.** Another approach to obtain $2^{o(k)}n^{\mathcal{O}(1)}$ algorithms is to obtain small kernels (meaning computing in polynomial time an equivalent instance $(G', k')$ with $|G'|$ typically in $\mathcal{O}(k)$), and then use a $2^{o(n)}$ time algorithm. For FVS such a $2^{o(n)}$-time algorithm is known in string graphs from [9] or [25], and was recently generalized to induced-minor-free graph classes [22]. However, as far as we are aware, the existence of a subquadratic kernel in this graph class is currently open.

## 1.1 Our contribution

Our objective is to study the existence of subexponential parameterized algorithms for hitting problems like FVS and TH in different types of intersection graphs. Our algorithmic results are summarized in Table 1.

[3] Regarding the algorithm of [1], it would be true if their lemma to bound the number of what they call "deep vertices" can be extended to square graphs.

**Table 1** Summary of our results. All algorithms are robust, i.e., they do not need a representation.

| Upper bounds | | | | |
|---|---|---|---|---|
| Restriction | of class | Problem | Time complexity | Section |
| none | square graphs | $\Pi \in \mathcal{P}$ | $2^{\mathcal{O}(k^{9/10}\log k)}n^{\mathcal{O}(1)}$ | Section 3 |
| contact | segment graphs | $\Pi \in \mathcal{P}$ | $2^{\mathcal{O}(k^{7/8}\log k)}n^{\mathcal{O}(1)}$ | Full version |
| contact | segment graphs | TH | $2^{\mathcal{O}(k^{3/4}\log k)}n^{\mathcal{O}(1)}$ | Full version |
| $K_{t,t}$-free | $d$-DIR graphs | TH | $2^{\mathcal{O}(k^{2/3}(\log k)\sqrt{d}t^2\log t)}n^{\mathcal{O}(1)}$ | Full version |
| $K_{t,t}$-free | string graphs | TH | $2^{\mathcal{O}_t(k^{2/3}\log k)}n^{\mathcal{O}(1)}$ | Full version |

| Lower bounds (under ETH) | | | | |
|---|---|---|---|---|
| Restriction | of class | Problem | Lower bound | Section |
| none | 2-DIR | TH, OCT | $2^{o(n)}$ | Section 4 |
| Maximum degree $\Delta$, for $\Delta \geq 6$ | 2-DIR | TH, OCT | $2^{o(\sqrt{\Delta n})}$ | Full version |
| $K_{2,2}$-free contact, max degree 6 | 2-DIR | TH, FVS, OCT | $2^{o(\sqrt{n})}$ | Full version |

**Positive results via ASQGM.** In Section 3 we explain how the local radius (hereafter denoted lr), introduced recently in [24] in the context of approximation, can be used to get subexponential FPT algorithms for any problem in $\mathcal{P}$, a family of bidimensional problems where one has to find a set of size at most $k$ hitting a fixed family of graphs. This class contains in particular FVS, and Pseudo Forest Del (resp. $P_t$-Hitting) where given a graph $G$, the goal is to remove a set $S$ of at most $k$ vertices of $G$ such that each connected component of $G - S$ contains at most one cycle (resp. does not contain a path on $t$ vertices as a subgraph). We point out that these three problems are also in the list of problems mentioned in [24] that admit EPTAS in disk graphs. We first provide sufficient conditions for graph class to admit subexponential FPT algorithms for any problem in $\mathcal{P}$, after the preprocessing step of Corollary 7 (introduced for disk graphs in [23]) has been performed. These conditions mainly boil down to having $ASQGM(\omega, \mu^{N^\star})$, where $\mu^{N^\star}$ is, informally, the maximum size of matching in the neighborhood of a vertex. Then, we use the framework of [4] to show that string graphs have $ASQGM(\omega, \mathrm{lr})$. Thus, the message of Section 3 is that in order to obtain a subexponential FPT algorithm for a problem $\Pi \in \mathcal{P}$ in a given subclass of string graphs, the only challenge is to bound lr by a polynomial of $\omega$ and $\mu^{N^\star}$. Finally, we provide such bounds for square graphs (intersection of axis-parallel squares) and contact-segment graphs.

We point out that in our companion paper [5] we prove that FVS admits an algorithm running in time $2^{\mathcal{O}(k^{10/11}\log k)}n^{\mathcal{O}(1)}$ for pseudo-disk graphs. As square and segment graphs are in particular pseudo-disk graphs, this generalizes the graph class where subexponential parameterized algorithms exist, but to the price of a worst running time. Moreover, our result in [5] is obtained via kernelization techniques which require a representation of the input graph (i.e., this algorithm is not robust), and the reduction rules behind the kernel are tailored for FVS and not applicable for any problem $\Pi \in \mathcal{P}$.

**Negative results.** An interesting difference between disk graphs and $d$-DIR graphs is that Theorem 3 (about the linear neighborhood complexity) no longer holds for $d$-DIR graphs, because of the presence of large bicliques. Thus, it seems that $K_{t,t}$ is an important subgraph differentiating the two settings and this fact is confirmed by the two following results. First we show (see sketch in Section 4 and full proof in the full version of the paper) in that assuming the ETH, there is no algorithm solving TH and OCT in time $2^{o(n)}$ on $n$-vertex 2-DIR graphs and more generally in time $2^{o(\sqrt{\Delta n})}$ in 2-DIR graphs with maximum degree $\Delta$. We note that the result for OCT was already proved in [26] as a consequence of algorithmic lower bounds for homomorphisms problems in string graphs. In our second negative result, we prove that assuming the ETH, the problems TH, OCT, and FVS cannot be solved in time $2^{o(\sqrt{n})}$ on $n$-vertex $K_{2,2}$-free contact-2-DIR graphs. Notice that that our $2^{o(\sqrt{n})}$ lower-bounds match those known for the same problems in planar graphs [10].

**Positive results for TH.** In the full version of the paper we observe that, for any hereditary graph class with sublinear separators, the preliminary branching step in Corollary 7 of [23] directly leads to a subexponential parameterized algorithm for TH. This implies the $2^{c_t k^{2/3} \log k} n^{\mathcal{O}(1)}$ algorithm for $K_{t,t}$-free string graphs. Recall that according to our negative result in the full version of the paper, the $K_{t,t}$-free assumption is necessary. To improve the constant $c_t$ in special cases, we provide in the full version of the paper bounds on the neighborhood complexity of two subclasses that may be of independent interest: $K_{t,t}$-free $d$-DIR graphs have linear neighborhood complexity with ratio $\mathcal{O}(dt^3 \log t)$, and contact-segment graphs have linear neighborhood complexity. These bounds lead to improved running times for TH in the corresponding graph classes (see Table 1).

Due to space constraints, the proofs of the statements marked with the ✂ symbol have been deferred to the full version [6].

# 2 Preliminaries

## 2.1 Basics

In this paper logarithms are binary and all graphs are simple, loopless and undirected. Unless otherwise specified we use standard graph theory terminology, as in [16] for instance. Given a graph $G$, we denote by $\omega(G)$ the maximum order of a clique in $G$. We denote by $d_G(v)$ the degree of $v \in V(G)$, or simply $d(v)$ when $G$ is clear from the context. The *distance* between two vertices of a graph is the minimum length (in number of edges) of a path linking them, and the *diameter* of a graph is the maximum distance between two of its vertices. The *radius* of a graph is the smallest integer $r \geq 0$ such that there exists a vertex $v$ such that every vertex in the graph is at a distance at most $r$ from $v$. A *$t$-bundle* [24] is a matching of size $t$ plus a vertex connected to the $2t$ vertices of the matching. We say that $B$ is a $t$-bundle of a graph $G$ if $G[B]$ is a $t$-bundle plus possibly some extra edges. A set $S \subseteq V(G)$ is a $t$-bundle hitting set of $G$ if $S \cap B \neq \emptyset$ for any $t$-bundle $B$ of $G$. We denote by $\boxplus(G)$ the maximum $k$ such that the $(k,k)$-grid is contained as a minor in $G$. We denote by $\mathsf{tw}(G)$ the treewidth of $G$, and $\mu(G)$ the size of a maximum matching of $G$.

In Section 3 we provide subexponential parameterized algorithms for a class of problems $\mathcal{P}$ that we will now define. We restrict our attention to *hitting problems*, where for a fixed graph family $\mathcal{F}$, the input is a graph $G$ and an integer $k$, and the goal is to decide if there exists $S \subseteq V(G)$ with $|S| \leq k$ such that $G - S \in \mathcal{F}$. A general setting where our results hold is described by the class $\mathcal{P}$ defined below and inspired by the problems tackled in [24] .

▶ **Definition 5.** *We denote by $\mathcal{P}$ the class of all hitting problems $\Pi$ such that:*

1. *$\Pi$ is bidimensional ;*
2. *there is an integer $c_\Pi > 0$ such that for any solution $S$ in a graph $G$, and any $c_\Pi$-bundle $B$ of $G$, $S \cap B \neq \emptyset$; and*
3. *$\Pi$ can be solved on a graph $G$ in time $\mathsf{tw}(G)^{\mathcal{O}(\mathsf{tw}(G))}$.*

▷ Claim 6. FVS, Pseudo Forest Del and $P_t$-Hitting for $t \leq 5$ belong to $\mathcal{P}$.

Proof. It is well known that these three problems are bidimensional. For the second condition, one can check that $c_\Pi$ is equal to 1 for FVS (as a 1-bundle is a triangle) and equal to 2 for Pseudo Forest Del and $P_t$-Hitting when $t \leq 5$. For the last condition, as FVS corresponds to hit all $K_3$ as minor and Pseudo Forest Del correspond to hit all $\{H_0, H_1, H_2\}$ as a minor (with $H_i$ is formed by two triangles sharing $i$ vertices), these two problems can be solved in $\mathsf{tw}(G)^{\mathcal{O}(\mathsf{tw}(G))}$ by [3]. For $P_t$-Hitting the result holds by [13]. ◁

## 2.2 Graph classes

A summary of graph classes considered in this article is presented in Figure 2.

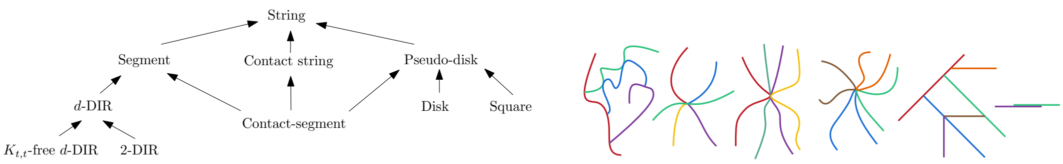


**Figure 2** Left: inclusion between graph classes. Right: from left to right, four representations of contact string graphs, then a representation of 3-DIR contact-segment graph, and finally on the right an example of an intersection between segments not allowed in a representation of a contact-segment graph.

In this article, we are mainly concerned with geometric graphs described by the intersection or contact of objects in the Euclidean plane. The most general class we consider are *string graphs*, which are intersection graphs of strings (a.k.a. Jordan arcs). Intersection graphs of segments in $\mathbb{R}^2$ are called *segment graphs*. If a segment graph can be represented with at most $d$ different slopes, we call it a *$d$-DIR graph*.[4] These classes of intersection graphs admit *contact* subclasses, where the representations should not contain *crossings*. That is, two strings either intersect tangentially, or they intersect at an endpoint of one of them. In a segment contact representation, any point belonging to two segments must be an endpoint of at least one of these segments. If a point belongs to several strings or segments, the above property must hold for any pair of them. This defines *contact string graphs*, *contact-segment graphs* and *contact $d$-DIR graphs*.

## 2.3 Preliminary branching steps

Our algorithms make use of the following preprocessing branching which was formulated in [23] for FVS for disk graphs. Here we restate it for any problem in $\mathcal{P}$ and for any graph class where the maximum clique can be approximated in polynomial time. A proof of this statement (included in the full version of the paper for completeness) can be obtained by closely following that in [23].

▶ **Corollary 7** (✂). *Let $\Pi \in \mathcal{P}$. Let $\mathcal{G}$ be a hereditary graph class where the maximum clique can be $\alpha$-approximated for some constant factor $\alpha \geq 1$ in polynomial time. There exists a $2^{\mathcal{O}\left(\frac{k}{p}\log k\right)} n^{\mathcal{O}(1)}$-time algorithm that, given an instance $(G, k)$ of $\Pi$ and an integer $p \in [6\alpha c_\Pi, k]$, where $G \in \mathcal{G}$, returns a collection $\mathcal{C}$ of size $2^{\mathcal{O}\left(\frac{k}{p}\log k\right)}$ of tuples $(G', M, k')$ such that:*

[4] In general two $d$-DIR graphs may require different sets of slopes in their representation but in the case $d = 2$ it is known that the segments can be assumed to be axis-parallel, which we will do.

1. *For any $(G', M, k') \in \mathcal{C}$, $(G', k')$ is an instance of $\Pi$ where $G'$ is an induced subgraph of $G$, $\omega(G') \leq p$, and $k' \leq k$;*
2. *$M$ is a $c_\Pi$-bundle hitting set of $G'$ with $|M| = \mathcal{O}(pk)$, and for any $v \in M$, $\mu(G'[N(v) \setminus M]) < c_\Pi$; and*
3. *$(G, k)$ is a yes-instance of $\Pi$ if and only if there exists $(G', M, k') \in \mathcal{C}$ such that $(G', k')$ is a yes-instance of $\Pi$.*

## 3 Positive results via ASQGM

### 3.1 From $ASQGM(\omega, \mu^{N^\star})$ to subexponential algorithms

In this section we provide subexponential paramterized algorithms for problems of $\mathcal{P}$ in any class that has the $ASQGM(\omega, \mu^{N^\star})$ property.

▶ **Definition 8.** *Given a graph $G$, a* subneighborhood function *of $G$ is any function $N^\star : V(G) \to 2^{V(G)}$ such that for any $v \in V(G)$, $N^\star(v) \subseteq N(v)$. Moreover, if for any $u \in V(G)$, $|\{v \in V(G),\ u \in N^\star(v)\}| \leq c$ for some $c \in \mathbb{N}$ then we say that $N^\star$ has* $c$-bounded occurrences*.*

*Given a subneighborhood function $N^\star$, we define $\mu^{N^\star}(v)$ as the maximum number of edges of a matching in $G[N^\star(v)]$. We denote by $\mu^{N^\star}(G)$ the maximum of $\mu^{N^\star}$ over $V(G)$.*
For example in square graphs, we will fix a representation $\mathcal{S}$, and define $N^\star(v)$ as the set of neighbors of $v$ whose square is smaller than the one of $v$.

The main theorem from this subsection is the following. Recall that $\mathcal{P}$ encompasses fundamental algorithmic problems such as FVS, Pseudo Forest Del and $P_t$-Hitting for $t \leq 5$ (Claim 6).

▶ **Theorem 9.** *Let $\Pi$ be a problem of $\mathcal{P}$ and $\mathcal{C}$ be a hereditary graph class such that:*
- *maximum clique can be $\mathcal{O}(1)$-approximated in polynomial time in $\mathcal{C}$;*
- *for any $G \in \mathcal{C}$, there exists a subneighborhood function $N^\star$ that has $\mathcal{O}(\omega(G)^{c_1})$-bounded occurrences for some $c_1 \in \mathbb{N}$; and*
- *$\mathcal{C}$ has the $ASQGM(\omega, \mu^{N^\star})$ property, i.e., there exists a multivariate polynomial $P$ such that for all $G \in \mathcal{C}$, we have $\mathsf{tw}(G) = \mathcal{O}(P(\omega(G), \mu^{N^\star}(G)) \cdot \boxplus(G))$.*

*Then, $\Pi$ admits a parameterized subexponential algorithm on $\mathcal{C}$. More precisely, for $\epsilon > 0$ such that $P(k^\epsilon, k^{(c_1+2)\epsilon}) = \mathcal{O}(k^{\frac{1}{2}-\epsilon})$, $\Pi$ admits a parameterized subexponential algorithm on $\mathcal{C}$ running in time $2^{\mathcal{O}(k^{1-\epsilon}\log(k))}$. This algorithm does not need a representation except if one is required for finding the $\mathcal{O}(1)$-approximation of a maximum clique.*

▶ **Lemma 10.** *Let $\Pi$ be a problem of $\mathcal{P}$. Consider a graph $G$ and $N^\star$ a $c$-bounded occurrences subneighborhood function of $G$. Let $M \subseteq V(G)$ be a $c_\Pi$-bundle hitting set of $G$ such that for any vertex $v \in M$, $\mu(G[N(v)] - M) < c_\Pi$. Then for every positive integer $\tau \geq c_\Pi$, there exists a set $B \subseteq V(G)$ of size $|B| = \frac{c|M|}{\tau - c_\Pi + 1}$ such that $\mu^{N^\star}(G - B) \leq \tau$.*

**Proof.** Let $\tau$ a positive integer with $\tau \geq c_\Pi$, and let us define $B = \{v \in V(G) : \mu^{N^\star}(v) \geq \tau\}$ the set of vertices with "big" $\mu^{N^\star}$ in $G$. Let us first prove that for any $v \in B$, $|N^\star(v) \cap M| \geq \mu^{N^\star}(v) - c_\Pi + 1$. Let $E' \subseteq E(G)$ be a maximum matching in $G[N^\star(v)]$ with $|E'| = \mu^{N^\star}(v)$. Observe that we cannot have $c_\Pi$ edges $e \in E'$ such that $V(e) \cap M = \emptyset$ as if $v \notin M$, then vertices of $E'$ together with $v$ would form a $c_\Pi$-bundle not hit by $M$, a contradiction, and if $v \in M$, this would contradict the hypothesis $\mu(G[N(v)] - M) < c_\Pi$. Thus, there is at least $|E'| - c_\Pi + 1$ edges of $E'$ intersecting $M$, leading to the desired inequality. Thus, we get

$$|B|\tau \leq \sum_{v \in B} \mu^{N^\star}(v) \leq \sum_{v \in B} (|N^\star(v) \cap M| + c_\Pi - 1).$$

Moreover, as for any $v \in V(G)$ there are at most $c$ vertices $u$ such that $v \in \mathrm{N}^\star(u)$, we get $\sum_{v \in B} |\mathrm{N}^\star(v) \cap M| \le c|M|$ by the pigeonhole principle (if the inequality was false, then there would exists $v \in M$ with $|\{u : v \in \mathrm{N}^\star(u)\}| > c$). This leads to $|B| = \frac{c|M|}{\tau - c_\Pi + 1}$. ◀

We are now ready to describe the general algorithm to solve $\Pi$.

**Proof of Theorem 9.** Given an instance $(G, k)$ of $\Pi$, we first use Corollary 7 with $p = k^\epsilon$ to obtain in time $2^{\mathcal{O}(k^{1-\epsilon}\log(k))}$ the set of $2^{\mathcal{O}(k^{1-\epsilon}\log(k))}$ triples $(G_2, M, k_2)$ with $k_2 \le k$, $|M| = \mathcal{O}(k^{1+\epsilon})$, and $\omega(G_2) \le k^\epsilon$.

In order to solve $\Pi$ on $(G, k)$, it is now enough to solve it on these instances $(G_2, k_2)$. Observe that applying the Lemma 10 to such $(G_2, k_2, M)$ triple with $\tau \ge c_\Pi$ gives a set $B$ of size at most $\frac{c|M|}{\tau - c_\Pi + 1} = \mathcal{O}(\frac{\omega(G_2)^{c_1} k^{1+\epsilon}}{\tau - c_\Pi + 1}) = \mathcal{O}(\frac{k^{1+\epsilon+\epsilon c_1}}{\tau - c_\Pi + 1})$ such that $G_3 = G_2 \setminus B$ verifies $\mu^{\mathrm{N}^\star}(G_3) \le \tau$.

By assumption on the *ASQGM* property we then have $\mathsf{tw}(G_3) = \mathcal{O}(P(k^\epsilon, \tau)\boxplus(G))$. Moreover $\mathsf{tw}(G_2) \le \mathsf{tw}(G_3) + |B| = \mathcal{O}(P(k^\epsilon, \tau)\boxplus(G)) + \mathcal{O}\left(\frac{k^{1+\epsilon+\epsilon c_1}}{\tau - c_\Pi + 1}\right)$ as removing a vertex decreases the treewidth by at most 1. We set $\tau = k^{(c_1+2)\epsilon}$. By assumption we have $P(k^\epsilon, k^{(c_1+2)\epsilon}) = \mathcal{O}(k^{\frac{1}{2}-\epsilon})$. As $\Pi$ is bidimensionnal, there exists $c_1$ such that if $\boxplus(G) > c_1\sqrt{k}$, then $(G, k)$ is a no-instance.

Thus, as $\mathsf{tw}(G_2) = \mathcal{O}\left(k^{\frac{1}{2}-\epsilon}\boxplus(G)\right) + \mathcal{O}\left(\frac{k^{1+\epsilon+\epsilon c_1}}{\tau}\right) = \mathcal{O}\left(k^{\frac{1}{2}-\epsilon}\boxplus(G)\right) + \mathcal{O}(k^{1-\epsilon})$, observe that if $\boxplus(G) \le c_1\sqrt{k}$, then there exists a constant $c$ such that $\mathsf{tw}(G_2) \le ck^{1-\epsilon}$. Thus, we use the treewidth approximation of [21] on $G_2$ with $\ell = ck^{1-\epsilon}$ to obtain in $2^{\mathcal{O}(\ell)}n^{\mathcal{O}(1)}$ either a $2\ell + 1$ treewidth decomposition, or conclude that $\mathsf{tw}(G_2) > \ell$. In the later case, this implies that $\boxplus(G) > c_1\sqrt{k}$, and thus we can conclude that $(G, k)$ is a no instance. Otherwise, by definition of problems in $\mathcal{P}$ we can solve $\Pi$ in time $\mathsf{tw}^{\mathcal{O}(\mathsf{tw}(G_2))}$, which gives the claimed overall time complexity of $2^{\mathcal{O}(k^{1-\epsilon}\log(k))} \times \mathsf{tw}(G_2)^{\mathcal{O}(\mathsf{tw}(G_2))} = 2^{\mathcal{O}(k^{1-\epsilon}\log(k))}$. ◀

## 3.2 From $ASQGM(\omega, \mathrm{lr})$ to $ASQGM(\omega, \mu^{\mathrm{N}^\star})$

To be able to use Theorem 9, we need to deal with graph classes that have the $ASQGM(\omega, \mu^{\mathrm{N}^\star})$ property. This section provides a general framework for obtaining this property via *local radius*. The local radius was originally introduced by Lokshtanov et al. [24] for disks graphs in the context of approximation algorithms. Here we first extend this definition to string graphs. To that end, we will see string graphs as graphs admitting a *thick representation*. In such a representation every vertex $v$ of the considered graph $G$ corresponds to a subset $\mathcal{D}_v$ of the plane that is homeomorphic to a disk, two intersecting such regions have an intersection with non-empty interior, and the number of maximal connected regions $\mathbb{R}^2 \setminus \bigcup_{v \in V(G)} \partial\mathcal{D}_v$ is finite.

To turn a string representation into a thick one, it simply suffices to thicken each string by a small enough amount so that no new intersections occur. On the other hand, note that any thick representation can be turned into a string representation by replacing each connected subset of the plane $\mathcal{D}_u$ by a string that almost completely fills its interior. Note that a thick representation is not necessarily a pseudo-disk representation as here, the intersection of two regions, $\mathcal{D}_u \cap \mathcal{D}_v$, may not be connected, or it may also be that $\mathcal{D}_u \setminus \mathcal{D}_v$ is not connected. Thick representations allow us to extend the definition of *local radius* to all string graphs. The next definition is illustrated Figure 4.

▶ **Definition 11.** *Let $G$ be a string graph and $\mathcal{S}$ be a thick representation of it. Let $\mathcal{X}$ be the set of all maximal connected region $\mathcal{R}$ of $\mathbb{R}^2 \setminus \bigcup_{D \in \mathcal{S}} \partial D$, contained in at least one object of $\mathcal{S}$. We define the arrangement graph of $\mathcal{S}$, denoted $A_\mathcal{S}$, by:*

- *adding one vertex of each region of* $\mathcal{X}$
- *adding an edge between two vertices if the boundaries of their regions share a common arc.*

*Moreover, for each* $v \in G$*, we denote* $\mathcal{R}_\mathcal{S}(v) \subseteq \mathcal{X}$ *the set of regions included in* $\mathcal{D}_v$ *(recall that* $\mathcal{D}_v$ *is the region associated to* $v$*), and* $V_\mathcal{S}(v) \subseteq V(A_\mathcal{S})$ *the set of vertices associated to the regions of* $\mathcal{R}_\mathcal{S}(v)$ *(implying* $|V_\mathcal{S}(v)| = |\mathcal{R}_\mathcal{S}(v)|$*). Finally, we denote* $A_\mathcal{S}(v) = A_\mathcal{S}[V_\mathcal{S}(v)]$.

▶ **Definition 12** (from [24], extended here to string graphs)**.** *Let* $G$ *be a string graph.*
- *Given a thick representation* $\mathcal{S}$ *of* $G$*,*
  - *for any* $v \in V(G)$*, we define* $\mathrm{lr}_\mathcal{S}(v)$ *as the radius of the graph* $A_\mathcal{S}(v)$
  - *we define* $\mathrm{lr}_\mathcal{S}(G) = \min_{v \in V(G)} \mathrm{lr}_\mathcal{S}(v)$
- *the local radius* $\mathrm{lr}(G)$ *of* $G$ *is the minimum over all thick representation* $\mathcal{S}$ *of* $G$ *of* $\mathrm{lr}_\mathcal{S}(G)$*.*

In order to show ASQGM we use the framework of Baste and Thilikos [4] (originally designed for the classic SQGM property), that we recall now.

▶ **Definition 13** (Contractions [4])**.** *Given a non-negative integer* $c$*, two graphs* $H$ *and* $G$*, and a surjection* $\sigma : V(G) \to V(H)$ *we write* $H \leq_\sigma^c G$ *if*
- *for every* $x \in V(H)$*, the graph* $G[\sigma^{-1}(x)]$ *has diameter at most* $c$ *and*
- *for every* $x, y \in V(H)$*,* $xy \in E(H) \iff G[\sigma^{-1}(x) \cup \sigma^{-1}(y)]$ *is connected.*

*We say that* $H$ *is a* $c$*-diameter contraction of* $G$ *if there is a surjection* $\sigma$ *such that* $H \leq_\sigma^c G$ *and we write this* $H \leq^c G$*. Moreover, if* $\sigma$ *is such that for every* $x \in V(H)$*,* $|\sigma^{-1}(x)| \leq c'$*, then we say that* $H$ *is a* $c'$*-size contraction of* $G$*, and we write* $H \leq^{(c')} G$*. If there exists an integer* $c$ *such that* $H \leq^c G$*, then we say that* $H$ *is a contraction of* $G$*.*

▶ **Definition 14** ($(c_1, c_2)$-extension [4])**.** *Given a class of graph* $\mathcal{G}$ *and two non-negative integers* $c_1$ *and* $c_2$*, we define the* $(c_1, c_2)$*-extension of* $\mathcal{G}$*, denoted by* $\mathcal{G}^{(c_1,c_2)}$*, as the class containing every graph* $H$ *such that there exist a graph* $G \in \mathcal{G}$ *and a graph* $J$ *that satisfy* $G \leq^{(c_1)} J$ *and* $H \leq^{c_2} J$ *(see Figure 3).*

$$G \in \mathcal{G} \xleftarrow{\quad c_1\text{-size contraction}\quad} J \xrightarrow{\quad c_2\text{-diameter contraction}\quad} H \in \mathcal{G}^{(c_1,c_2)}$$

**Figure 3** A graphical representation of the definition of $\mathcal{G}^{(c_1,c_2)}$.

▶ **Lemma 15** (implicit in the proof of [4, Theorem 15])**.** *For every integers* $c_1, c_2$ *and* $G \in \mathcal{P}^{(c_1,c_2)}$*, with* $\mathcal{P}$ *the class of planar graphs, we have* $\mathsf{tw}(G) = \mathcal{O}(c_1 c_2 \boxplus(G))$*.*

The main result of this section is the following.

▶ **Theorem 16.** *String graphs have the ASQGM*$(\omega, \mathrm{lr})$ *property, more precisely for a string graph* $G$ *we have* $\mathsf{tw}(G) = \mathcal{O}(\omega(G)\,\mathrm{lr}(G) \boxplus(G))$*.*

**Proof.** Let $G$ be a string graph, and $\mathcal{S}$ a thick representation such that $\mathrm{lr}_\mathcal{S}(G) = \mathrm{lr}(G)$. Let us define a graph $J$ as follows, Figure 4 is a representation of the construction. For any maximal connected region $\mathcal{R}$ of $\mathbb{R}^2 \setminus \bigcup_{D \in \mathcal{S}} \partial D$, we add to $J$ a clique $K_\mathcal{R}$ of size $\mathrm{ply}(\mathcal{R})$. Then, for any pair of regions $\{\mathcal{R}_1, \mathcal{R}_2\}$ that share a common arc, we add all edges between $K_{\mathcal{R}_1}$ and $K_{\mathcal{R}_2}$. For any $v \in V(G)$, we associate a set $X(v) \subseteq V(J)$ such that for any $\mathcal{R} \in \mathcal{R}_\mathcal{S}(v)$, $|X(v) \cap K_\mathcal{R}| = 1$, and such that $X(v) \cap X(u) = \emptyset$ for any $u \neq v$. Notice that the condition $X(v) \cap X(u) = \emptyset$ is possible as $|K_\mathcal{R}| = \mathrm{ply}(\mathcal{R})$, and thus any vertex $v$ can take its "private" vertex in $X(v) \cap \mathcal{R}$ for any $\mathcal{R} \in \mathcal{R}_\mathcal{S}(v)$.

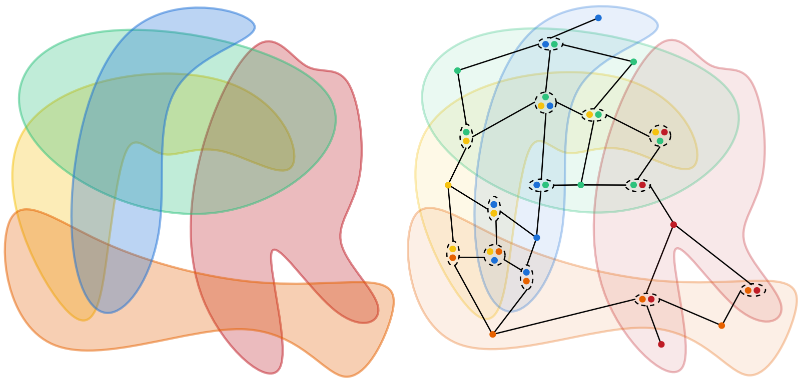

**Figure 4** Left: thick representation of a string graph $G$. Right: Illustrates both $A_{\mathcal{S}}$ and the graph $J$ used in the proof of Theorem 16. To visualise $A_{\mathcal{S}}$, consider that each black dotted ellipse is a single vertex (we have $|V(A_{\mathcal{S}})| = 23$). Moreover, if $v$ is the vertex represented in red, we have $|V_{\mathcal{S}}(v)| = 6$ and $\mathrm{lr}_{\mathcal{S}}(v) = 2$. To visualise $J$: for each maximal connected region $\mathcal{R}$ of $\mathbb{R}^2 \setminus \bigcup_{D\in\mathcal{S}} \partial D$, the clique $K_{\mathcal{R}}$ with more than one vertex is represented by a black dotted ellipse around the clique. For readability only one edge is represented between two cliques instead of the complete bipartite graph.

Let us prove that $G$ is a $\mathrm{lr}(G)$-diameter contraction of $J$ by defining a surjection $\sigma : V(J) \to V(G)$ as follows. For any $v \in V(G)$, we define $\sigma^{-1}(v) = X(v)$ (informally we contract all vertices in $X(v)$). As for any $v \in V(G)$, $J[X(v)]$ is isomorphic to $A_{\mathcal{S}}(v)$, we immediately have $\mathrm{diam}(J[\sigma^{-1}(v)]) = \mathrm{lr}(G)$. Moreover, it is straightforward to check that for every $x, y \in V(G)$, $xy \in E(G) \iff J[\sigma^{-1}(x) \cup \sigma^{-1}(y)]$ is connected. Now, observe that $A_{\mathcal{S}}$ (which is planar) is a $\mathrm{ply}(\mathcal{S})$-size contraction of $J$ using $\sigma' : V(J) \to V(A_{\mathcal{S}})$ such that for any $v \in V(A_{\mathcal{S}})$, $v$ corresponding to a region $\mathcal{R}$ of the plane delimited by the boundaries of the objects of $\mathcal{S}$, $\sigma'^{-1}(v) = K_R$. As $\mathrm{ply}(\mathcal{S}) \le \omega(G)$, we get the desired result. ◀

The following corollary is immediate from Theorem 9 and Theorem 16.

▶ **Corollary 17.** *Given an hereditary graph class $\mathcal{C}$ which is a subclass of string graphs such that*

- *maximum clique can be $\mathcal{O}(1)$-approximated in polynomial time,*
- *for any $G \in \mathcal{C}$, there exists a subneighborhood function $\mathrm{N}^\star$ that has $\mathcal{O}(\omega(G)^{c_1})$-bounded occurrences for some $c_1 \in \mathbb{N}$, and*
- *there exists a multivariate polynomial such that for any $G \in \mathcal{C}$, $\mathrm{lr}(G) = P(\omega(G), \mu^{\mathrm{N}^\star}(G))$*

*Then, any problem $\Pi \in \mathcal{P}$ admits a parameterized subexponential algorithm on $\mathcal{C}$. More precisely, let $P'(\omega(G), \mu^{\mathrm{N}^\star}(G)) = \omega(G)P(\omega(G), \mu^{\mathrm{N}^\star}(G))$. For any $\epsilon > 0$ such that $P'(k^\epsilon, k^{(c_1+2)\epsilon}) = \mathcal{O}(k^{\frac{1}{2}-\epsilon})$, FVS can be solved in time $\mathcal{O}^*(k^{\mathcal{O}(k^{1-\epsilon})})$. This algorithm does not need a representation except if one is required for finding the $\mathcal{O}(1)$-approximation of a maximum clique.*

## 3.3 Upper bounding the local radius for square graphs

Again we provided in the previous section a generic result (Corollary 17) but so far it might not be clear to the reader which graph classes may satisfy its requirements. To demonstrate the applicability of this result, we show here that square graphs do. This requires to define an appropriate $\mathrm{N}^\star$ and prove that $\mathrm{lr}(G) = \omega(G)^{\mathcal{O}(1)} \cdot \mu^{\mathrm{N}^\star}(G)^{\mathcal{O}(1)}$. A second application is for contact-segment graphs, but due to space constraints we had to move the proof to the full version [6].

We say that a graph $G$ is a *square graph* if it is the intersection graph of some collection of (closed) axis-parallel squares in the plane. In the following by *square* we always mean closed and axis-parallel square. By slightly altering the sizes and positions of the squares in a collection we can obtain a collection where exactly the same pairs of squares intersect and, in addition, all the side lengths of the squares are different from each other and no two sides squares are aligned. Furthermore this can easily be performed in polynomial time. From now on we will assume that all the representations we consider satisfy this property.

The first requirement of Corollary 17 is provided by following lemma from [8], which describes an EPTAS for the clique problem in the more general case of the intersection graph of a fixed convex geometric shape with a central symmetry, while allowing rescaling.

▶ **Theorem 18** ([8]). *There is a polynomial-time 2-approximation of maximum clique in intersection graphs of squares, even when no representation is provided.*

▶ **Definition 19.** *Given a square representation $\mathcal{S} = \{\mathcal{D}_v\}_{v\in V(G)}$ of a graph $G$, we denote $\ell_{\mathcal{S}}(\mathcal{D}_v)$ the length of a side of the square $\mathcal{D}_v$, $N^-_{\mathcal{S}}(v)$ (resp. $N^+_{\mathcal{S}}(v)$) the set of vertices $u$ such that $u \in N_G(v)$ and $\ell_{\mathcal{S}}(\mathcal{D}_u) < \ell_{\mathcal{S}}(\mathcal{D}_v)$ (resp. $>$). When $\mathcal{S}$ is clear from the context, we will instead write $\ell$, $N^-$ and $N^+$.*

As the lengths of all sides differ, $\{N^+(v), N^-(v)\}$ is a partition of $N(v)$ for every vertex $v$.

▶ **Lemma 20.** *Given a square representation $\mathcal{S}$ of a graph $G$, $N^-$ is a $\mathcal{O}(\omega(G))$-occurrences bounded subneighborhood function.*

**Proof.** $N^-$ is clearly a subneighborhood function. For $v \in V(G)$, observe that a square larger than $\mathcal{D}_v$ has to contain one of the four corners of $\mathcal{D}_v$ if the two squares intersect. But a corner of $\mathcal{D}_v$ cannot be contained in more than $\omega(G)$ squares. Hence there are at most $4\omega(G)$ vertices $u \in V(G)$ such that $v \in N^-(u)$, and so $N^-$ is $4\omega(G)$-occurrences bounded. ◀

We will prove that choosing $N^* = N^-$ allows us to bound the local radius.

▶ **Definition 21.** *Given a square graph $G$ with representation $\mathcal{S}$, for any $v \in G$, we define $H(v)$ as a minimum vertex cover of $G[N^-(v)]$, $I(v) = N^-(v) \setminus H(v)$, and $X(v) = H(v) \cup N^+(v)$.*

▷ Claim 22. For every vertex $v$ of a square graph $G$ with representation $\mathcal{S}$, the following properties hold:

1. $I(v)$ is an independent set of $G$;
2. $|H(v)| \leq 2\,\mu^{\mathrm{N}^\star}(G)$;
3. $|N^+(v)| = \mathcal{O}(\omega(G))$ (as in the proof of Lemma 20);
4. $|X(v)| = \mathcal{O}(\mu^{\mathrm{N}^\star}(G) + \omega(G))$; and
5. $\{X(v), I(v)\}$ is a partition of $N(v)$.

▶ **Definition 23.** *For a curve $\mathcal{C} : [0,1] \to \mathbb{R}^2$ such that for $t \in [0,1]$, $\mathcal{C}(t) = (x(t), y(t))$, we say that $\mathcal{C}$ is* monotonic *if the functions $x$ and $y$ are monotonic. For $k \geq 2$ we say that $\mathcal{C}$ is $k$-monotonic if it is the composition[5] of $k$ monotonic curves.*

Recall in the next Lemma that $\mathcal{D}_{I(v)}$ denotes the union of all squares in $I(v)$.

[5] A curve $\mathcal{C}(t) = (x(t), y(t))$ is the *composition* of $k$ curves $(\mathcal{C}_i(t) = (x_i(t), y_i(t)))_{i\in\{1,\dots,k\}}$ if $(x(0), y(0)) = (x_1(0), y_1(0))$, $(x(1), y(1)) = (x_k(1), y_k(1))$, $(x_i(1), y_i(1)) = (x_{i+1}(0), y_{i+1}(0))$ for every $i \in \{1, \dots, k-1\}$ and the set of points $\{(x(t), y(t)\}, t \in [0,1]\}$ is the union of the $\{(x_i(t), y_i(t)\}, t \in [0,1]\}$ for $i \in \{1, \dots, k\}$.

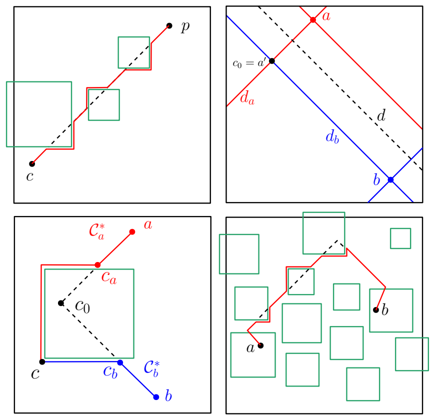

**Figure 5** Illustrations of the construction used in the proof of the Lemma 24. Squares of $I(v)$ are represented in green. Top left: construction used for the Claim 25. Top right: construction used for the Claim 26. Bottom left: construction used for Claim 27. Observe that in this situation $c_a$ and $c_b$ are next to opposite sides of the square containing $c_0$, that $\mathcal{C}_a^*$ can be extended in an counterclockwise direction, and $\mathcal{C}_b^*$ in a clockwise direction, which ensure the existence of a common point $c$ of their monotonic extensions. Bottom right: an example of a 4-monotonic curve between $a$ and $b$ obtained by the construction of Lemma 24. Observe that only two squares of $I(v)$ are crossed.

▶ **Lemma 24.** *Let $G$ be a square graph and $\mathcal{S}$ a representation. Let $v \in V(G)$ and $a, b$ two points contained in $\mathcal{D}_v$. There exists a 4-monotonic curve $\mathcal{C}$ contained in $\mathcal{D}_v$ joining the point $a$ to the point $b$, and crossing at most twice a boundary of the squares of $I(v)$.*

**Proof.** In what follows, what we call a *diagonal line (resp. half line)* any line (resp. half line) having an angle $+45°$ or $-45°$ with the horizontal axis, and a *diagonal of a point $p$ in the plan* a diagonal half line whose endpoint is $p$.

The first step for the creation of the curve is to reduce to the case where the point $a$ and $b$ are outside $\mathcal{D}_{I(v)}$. If this is not the case, for example if $a$ in contained in a square $s = \mathcal{D}_u$ with $u \in I(v)$, we create a rectilinear curve from $a$ toward the outside of $s$, in a direction such that the intersection of the curve with the boundary of $s$ is still in $\mathcal{D}_v$ (see the construction in Figure 5 for an example of such reduction). As such curve is monotonic and crosses the boundary of a square of $I(v)$ exactly once, after the reduction we are in the case where we want to construct a 2-monotonic curve between two points of $\mathcal{D}_v \setminus \mathcal{D}_{I(v)}$ such that no square of $I(v)$ is crossed. In what follow we suppose we have reduced to this case and we still denote $a$ and $b$ the two points of $\mathcal{D}_v \setminus \mathcal{D}_{I(v)}$ we want to join by a curve.

▷ Claim 25. Given two points $c, p \in \mathcal{D}_v \setminus \mathcal{D}_{I(v)}$ on the same diagonal line, there is a monotonic curve included in $\mathcal{D}_v \setminus \mathcal{D}_{I(v)}$ between $c$ and $p$.

Proof. The construction is represented in Figure 5. The curve is created by starting from the point $c$, then by following the diagonal line toward $p$. When encountering a square $s = \mathcal{D}_u$ of a vertex $u \in I(v)$, it is always possible of getting around $s$ in order to join back the diagonal on the other side, and doing so in a direction such that the curve is still monotonic and contained in $\mathcal{D}_v$. ◁

▷ Claim 26. There are diagonals $d_a$ of $a$ and $d_b$ of $b$ intersecting on a point $c_0 \in \mathcal{D}_v$.

Proof. Consider the line $d$ parallel to the top left to bottom right diagonal of $\mathcal{D}_v$ (see Figure 5), at equal distances of the points $a$ and $b$. By symmetry of the square and of the variables $a$ and $b$, we can suppose that $d$ goes from top left to bottom right, is above the diagonal of $\mathcal{D}_v$, and that $a$ is above $d$. The symmetric $a'$ of the point $a$ relatively to $d$ is inside $\mathcal{D}_v$ and is contained in a diagonal of both $a$ and $b$. ◁

Now, if $c_0 \in \mathcal{D}_v \setminus \mathcal{D}_{I(v)}$, composing the two curves toward $c_0$ given by the previous claim gives the wanted result.

It remains to deal with the case where $c_0$ lies in some square $s = \mathcal{D}_u$ for $u \in I(v)$. Let $c_a$ be a point of $d_a$ between $a$ and the square $s$, at an infinitely small distance outside of $s$. Claim 25 gives a monotonic curve $\mathcal{C}_a^*$ from $a$ to $c_a$. In the same way we define $c_b$ and $\mathcal{C}_b^*$.

▷ Claim 27. There exists a point $c \in \mathcal{D}_v \setminus \mathcal{D}_{I(v)}$ such that $\mathcal{C}_a^*$ and $\mathcal{C}_b^*$ can be extended to $c$ while still being monotonic and contained in $\mathcal{D}_v \setminus \mathcal{D}_{I(v)}$.

Proof. We can assume that $d_a$ and $d_b$ are perpendicular as otherwise the points $a$ and $b$ are on the same diagonal and so Claim 25 gives the wanted result by taking $c = b$. Observe that if $c_a$ and $c_b$ are arbitrarily close to the same side of $s$, then prolonging $\mathcal{C}_a^*$ toward $c_b$ would keep the curve monotonic, as $\mathcal{C}_a^*$ was already going toward $d_b$ as $d_a$ and $d_b$ intersect in $s$. So taking $c = c_b$ would give the wanted result.

Otherwise if $c_a$ and $c_b$ are at arbitrarily small distance from two different sides, observe that the curve $\mathcal{C}_a^*$ can be extended running alongside the boundary of $s$ until crossing 2 corners. The same is true for $\mathcal{C}_b^*$ so the only situation where those extensions do not cross each other would be if $c_a$ and $c_b$ are next to opposite side of $s$, and that the orientations of $d_a$ and $d_b$ force the extensions of $\mathcal{C}_a*$ and $\mathcal{C}_b^*$ to go in the same direction around $s$. However, this is impossible: as $d_a$ and $d_b$ cross each other inside of $s$, one extension will go clockwise around $s$ and the other counterclockwise (see Figure 5). This ensures that $\mathcal{C}_a^*$ and $\mathcal{C}_b^*$ can be extended around $s$ while still being monotonic in order for them to join on a point $c$ while staying outside of $\mathcal{D}_{I(v)}$. ◁

Composing the two curves obtained by the above claim gives a path as wanted. ◀

We are now ready to prove the main combinatorial statement of this section.

▶ **Lemma 28.** *Let $G$ be a square graph. There exists a subneighborhood function* $\mathrm{N}^\star$ *which is $\omega(G)$-occurrences bounded and such that* $\mathrm{lr}(G) = \mathcal{O}(\mu^{\mathrm{N}^\star}(G) + \omega(G))$.

**Proof.** Let $\mathcal{S}$ be a square representation of $G$, and let $\mathrm{N}^\star$ as defined in Definition 19, which is $\omega(G)$-occurrences bounded according to Lemma 20. Let us now prove that $\mathrm{lr}_{\mathcal{S}}(G) = \mathcal{O}(|X(v)|)$. This will imply the required result as $\mathrm{lr}(G) \leq \mathrm{lr}_{\mathcal{S}}(G)$ and $|X(v)| = \mathcal{O}(\mu^{\mathrm{N}^\star}(G) + \omega(G))$ by Claim 22. To that end, let us bound the diameter of $A_{\mathcal{S}}[V_{\mathcal{S}}(v)]$. Let $u, v$ be two vertices of $A_{\mathcal{S}}[V_{\mathcal{S}}(v)]$, and let us bound the distance between these two vertices. Remember that any vertex in $A_{\mathcal{S}}[V_{\mathcal{S}}(v)]$ corresponds to an inclusion-wise maximal rectangular region of the plane included in $\mathcal{D}_v$, and delimited by edges of squares of $\mathcal{S}$. Let $a$ and $b$ be points in the regions of $u$ and $v$ respectively. Notice that to any curve inside $\mathcal{D}_v$ we can associate a path in $A_{\mathcal{S}}[V_{\mathcal{S}}(v)]$ by considering the sequence of regions visited by $\mathcal{C}$, and associate to each of the region its corresponding vertex in $A_{\mathcal{S}}[V_{\mathcal{S}}(v)]$ (see Figure 6). Thus, we will upper bound the distance from $u$ to $v$ in $A_{\mathcal{S}}[V_{\mathcal{S}}(v)]$ by constructing a curve $\mathcal{C}$ from $a$ to $b$, and by counting the length of the sequence of regions visited by $\mathcal{C}$.

We use for $\mathcal{C}$ the 4-monotonic curve between $a$ and $b$ defined in Lemma 24. Observe the following property $\pi_0$: any monotonic curve inside $\mathcal{D}_v$ crosses at most $4|X(v)|$ sides of squares in $X(v)$. Indeed, as each square in $X(v)$ has at most 4 sides intersecting $\mathcal{D}_v$, and any

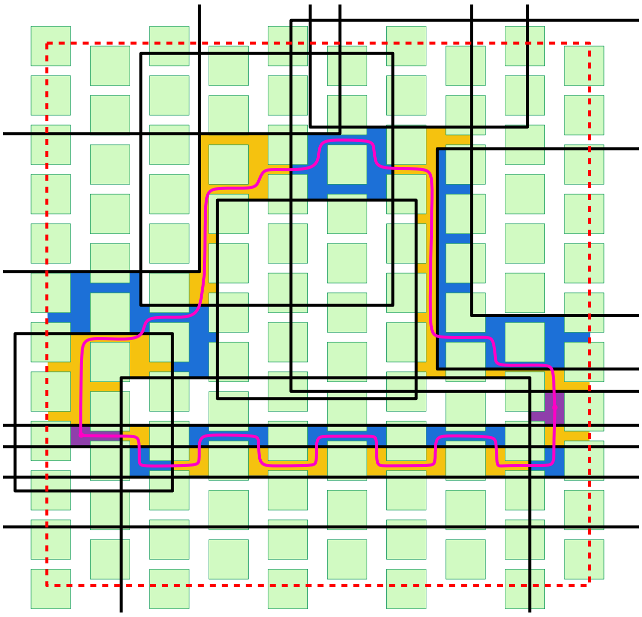

**Figure 6** Examples of paths in the configuration graph, with $\mathcal{D}_v$ represented with a dashed red square, $I(v)$ by green squares and the sides of the squares of $X(v)$ in black. Here we can see two curves between the two purple regions, $\mathcal{C}_1$ (that goes up and then down) and $\mathcal{C}_2$, and the path in $A_{\mathcal{S}}(v)$ associated to each curve as in the proof of Lemma 28, where the regions traversed by the paths are alternatively colored blue and yellow. Notice that $\mathcal{C}_1$ is 2-monotone, whereas $\mathcal{C}_2$ is $c$-monotone, where $c$ could be made arbitrary large by creating more and smaller squares in $I(v)$. As $c$ is large, there is a side of a square in $X(v)$ crossed many times (eight) by $\mathcal{C}_2$, and thus we do not use curve like $\mathcal{C}_2$ in the proof.

side, as a vertical or horizontal segment intersecting in $\mathcal{D}_v$, can be crossed at most one time by a monotonic curve. Observe also that, each time $\mathcal{C}$ leaves its current region, $\mathcal{C}$ must cross a side of a square in $N(v)$. However, the total number of crossings between $\mathcal{C}$ and a side of a square in $N(v)$ is at most $16|X(v)| + 4$, as each of the four monotonic part of $\mathcal{C}$ crosses at most $4|X(v)|$ sides of squares in $X(v)$ (by $\pi_0$), and $\mathcal{C}$ crosses at most 4 sides of squares in $I(v)$ (the worst case being when $a \neq a'$, and $\mathcal{C}_{a \to a'}$ crosses the corner of the square in $I(v)$ containing $a$, and same for $b, b'$). Thus, the curve $\mathcal{C}$ goes from a region to the next one at most $16|X(v)| + 4$ times, implying that the diameter of $A_{\mathcal{S}}[V_{\mathcal{S}}(v)]$, and so the local radius $\mathrm{lr}_{\mathcal{S}}(G)$, are in $\mathcal{O}(|X(v)|)$. ◀

As announced in the introduction of the section, we are now able to apply Corollary 17.

▶ **Theorem 29.** *Any problem $\Pi \in \mathcal{P}$ can be solved in time $2^{\mathcal{O}(k^{9/10}\log(k))}n^{\mathcal{O}(1)}$ in square graphs, even when no representation is given.*

**Proof.** Let $\Pi \in \mathcal{P}$. According to Theorem 18, Lemma 28, we can apply Corollary 17 with $c_1 = 1$, and $P(x, y) = x + y$. This implies that for any $\epsilon$ such that $k^\epsilon(k^\epsilon + k^{3\epsilon}) = \mathcal{O}(k^{\frac{1}{2}-\epsilon})$, $\Pi$ can be solved in $\mathcal{O}^*(k^{\mathcal{O}(k^{1-\epsilon})})$ in square graphs. Taking $\epsilon = \frac{1}{10}$ leads to the claimed complexity. ◀

## 4 ETH based hardness results

Let us here sketch the lower bounds. Full proofs are provided in the full version.

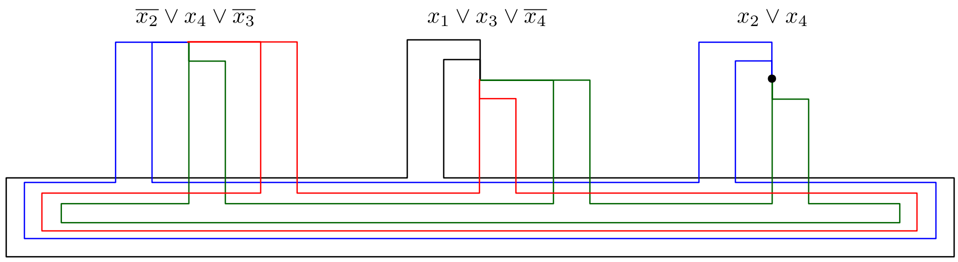


**Figure 7** The construction for the formula $(\overline{x_2} \vee x_4 \vee \overline{x_3}) \wedge (x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_2 \vee x_4)$. The zero-length segments at each corner of the $k$-polygons are not represented, while that added for the clause with two variables is depicted with a black dot.

▶ **Theorem 30.** *Under the ETH, TH and OCT cannot be solved in time $2^{o(n)}$ on $n$-vertex 2-DIR graphs.*

**Sketch of Proof.** Let $\varphi$ be a 3-SAT instance with $n$ variables $x_1, \dots, x_n$ and $m$ clauses $C_1, \dots, C_m$. In these clauses, we do not have 3 literals all positive or all negative. We can ensure this by adding only few variables and few clauses.

Let us now construct a 2-DIR graph $G$ from the formula $\varphi$. In this graph, each variable $x_i$ is represented by a polygon with $k_i$ vertical segments, $k_i$ horizontal segments, and with also $2k_i$ trivial segments (i.e. points) that are placed in each corner of the polygon, where $k_i$ is some number linear in the number of clauses containing $x_i$. See Figure 7 for an illustrative example. There, one can see that these polygons form concentric rectangles, from which small parts escape from above. These escaping parts allow interactions with other polygons, corresponding to variables from a same clause.

The idea of the reduction is that, $\varphi$ is satisfiable if and only if $G$ has a TH (resp. OCT) of size $K = \sum_{1 \le i \le n} k_i$. Furthermore, such hitting set will be of the following form. For the polygon corresponding to $x_i$, the hitting set will be either formed by the $k_i$ vertical segments, or by the $k_i$ horizontal segments. This is ensured by the triangles induced at each corner of the polygon. Furthermore, the choice of vertical or horizontal segments, depends on the interactions among polygons, and will correspond to a valuation of the variable $x_i$. ◀

In the full version [6] we also provide a refined bound of Theorem 30 depending on the maximum degree, and another negative result in $K_{2,2}$-free contact 2-DIR graphs.

## 5 Discussion

In this paper we gave subexponential FPT algorithms for cycle-hitting problems in intersection graphs. A general goal is to characterize the geometric graph classes that admit subexponential FPT algorithms for the problems we considered. In particular, an interesting open problem is whether FVS admits a subexponential parameterized algorithm in 2-DIR graphs.

## References


1 Shinwoo An, Kyungjin Cho, and Eunjin Oh. Faster algorithms for cycle hitting problems on disk graphs. In *Algorithms and Data Structures: 18th International Symposium, WADS 2023, Montreal, QC, Canada, July 31 – August 2, 2023, Proceedings*, pages 29–42, Berlin, Heidelberg, 2023. Springer-Verlag. doi:10.1007/978-3-031-38906-1_3.

**2** Shinwoo An and Eunjin Oh. Feedback vertex set on geometric intersection graphs. In *32nd International Symposium on Algorithms and Computation (ISAAC 2021)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2021.

**3** Julien Baste, Ignasi Sau, and Dimitrios M Thilikos. Hitting minors on bounded treewidth graphs. iv. an optimal algorithm. *arXiv preprint*, 2019. `arXiv:1907.04442`.

**4** Julien Baste and Dimitrios M Thilikos. Contraction bidimensionality of geometric intersection graphs. *Algorithmica*, 84(2):510–531, 2022.

**5** Gaétan Berthe, Marin Bougeret, Daniel Gonçalves, and Jean-Florent Raymond. Feedback vertex set for pseudo-disk graphs in subexponential fpt-time, 2024. To appear on Arxiv.

**6** Gaétan Berthe, Marin Bougeret, Daniel Gonçalves, and Jean-Florent Raymond. Subexponential parameterized algorithms in square graphs and intersection graphs of thin objects, 2024. `arXiv:2306.17710`.

**7** Marthe Bonamy, Edouard Bonnet, Nicolas Bousquet, Pierre Charbit, and Stéphan Thomassé. Eptas for max clique on disks and unit balls. In *2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 568–579. IEEE, 2018.

**8** Édouard Bonnet, Nicolas Grelier, and Tillmann Miltzow. Maximum clique in disk-like intersection graphs. In Nitin Saxena and Sunil Simon, editors, *40th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2020, December 14-18, 2020, BITS Pilani, K K Birla Goa Campus, Goa, India (Virtual Conference)*, volume 182 of *LIPIcs*, pages 17:1–17:18. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2020. `doi:10.4230/LIPICS.FSTTCS.2020.17`.

**9** Édouard Bonnet and Paweł Rzążewski. Optimality program in segment and string graphs. *Algorithmica*, 81:3047–3073, 2019.

**10** Liming Cai and David Juedes. On the existence of subexponential parameterized algorithms. *Journal of Computer and System Sciences*, 67(4):789–807, 2003.

**11** Julia Chuzhoy and Zihan Tan. Towards tight (er) bounds for the excluded grid theorem. *Journal of Combinatorial Theory, Series B*, 146:219–265, 2021.

**12** Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer Publishing Company, Incorporated, 1st edition, 2015.

**13** Marek Cygan, Dániel Marx, Marcin Pilipczuk, and Michał Pilipczuk. Hitting forbidden subgraphs in graphs of bounded treewidth. *Information and Computation*, 256:62–82, 2017. `doi:10.1016/j.ic.2017.04.009`.

**14** Erik D. Demaine, Fedor V. Fomin, Mohammadtaghi Hajiaghayi, and Dimitrios M. Thilikos. Subexponential parameterized algorithms on bounded-genus graphs and $H$-minor-free graphs. *Journal of the ACM (JACM)*, 52(6):866–893, 2005.

**15** Erik D Demaine and MohammadTaghi Hajiaghayi. Linearity of grid minors in treewidth with applications through bidimensionality. *Combinatorica*, 28(1):19–36, 2008.

**16** Reinhard Diestel. Graph theory 3rd ed. *Graduate texts in mathematics*, 173(33):12, 2005.

**17** Fedor V. Fomin, Daniel Lokshtanov, Fahad Panolan, Saket Saurabh, and Meirav Zehavi. Finding, hitting and packing cycles in subexponential time on unit disk graphs. *Discrete & Computational Geometry*, 62:879–911, 2019.

**18** Fedor V. Fomin, Daniel Lokshtanov, and Saket Saurabh. Bidimensionality and geometric graphs. In *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '12, pages 1563–1575, USA, 2012. Society for Industrial and Applied Mathematics.

**19** Fedor V. Fomin, Daniel Lokshtanov, and Saket Saurabh. Excluded grid minors and efficient polynomial-time approximation schemes. *Journal of the ACM (JACM)*, 65(2):1–44, 2018.

**20** Alexander Grigoriev, Athanassios Koutsonas, and Dimitrios M Thilikos. Bidimensionality of geometric intersection graphs. In *SOFSEM 2014: Theory and Practice of Computer Science: 40th International Conference on Current Trends in Theory and Practice of Computer Science, Novỳ Smokovec, Slovakia, January 26-29, 2014, Proceedings 40*, pages 293–305. Springer, 2014.

**21** Tuukka Korhonen. A single-exponential time 2-approximation algorithm for treewidth. In *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 184–192. IEEE, 2022.

**22** Tuukka Korhonen and Daniel Lokshtanov. Induced-minor-free graphs: Separator theorem, subexponential algorithms, and improved hardness of recognition. In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM, 2024.

**23** Daniel Lokshtanov, Fahad Panolan, Saket Saurabh, Jie Xue, and Meirav Zehavi. Subexponential parameterized algorithms on disk graphs (extended abstract). In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2005–2031. SIAM, 2022.

**24** Daniel Lokshtanov, Fahad Panolan, Saket Saurabh, Jie Xue, and Meirav Zehavi. A framework for approximation schemes on disk graphs. In Nikhil Bansal and Viswanath Nagarajan, editors, *Proceedings of the 2023 ACM-SIAM Symposium on Discrete Algorithms, SODA 2023, Florence, Italy, January 22-25, 2023*, pages 2228–2241. SIAM, 2023. doi:10.1137/1.9781611977554.CH84.

**25** Jana Novotná, Karolina Okrasa, Michał Pilipczuk, Paweł Rzążewski, Erik Jan van Leeuwen, and Bartosz Walczak. Subexponential-time algorithms for finding large induced sparse subgraphs. *Algorithmica*, 83:2634–2650, 2021.

**26** Karolina Okrasa and Paweł Rzążewski. Subexponential algorithms for variants of the homomorphism problem in string graphs. *Journal of Computer and System Sciences*, 109:126–144, 2020.