On the Independence Number of 1-Planar Graphs

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Abstract

An independent set in a graph is a set of vertices where no two vertices are adjacent to each other. A maximum independent set is the largest possible independent set that can be formed within a given graph G. The cardinality of this set is referred to as the independence number of G. This paper investigates the independence number of 1-planar graphs, a subclass of graphs defined by drawings in the Euclidean plane where each edge can have at most one crossing point. Borodin establishes a tight upper bound of six for the chromatic number of every 1-planar graph G, leading to a corresponding lower bound of $n/6$ for the independence number, where n is the number of vertices of G. In contrast, the upper bound for the independence number in 1-planar graphs is less studied. This paper addresses this gap by presenting upper bounds based on the minimum degree δ. A comprehensive table summarizes these upper bounds for various δ values, providing insights into achievable independence numbers under different conditions.

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1 Introduction

An *independent set* in a graph contains vertices that are not adjacent to each other. A *maximum independent set* is an independent set of largest possible size for a given graph, and the number of vertices in this set is known as the *independence number* of G and denoted by $\alpha(G)$. The size $\alpha(G)$ serves as a crucial parameter in graph theory and holds significance in algorithmic contexts. For instance, Kirkpatrick [24] and Dobkin and Kirkpatrick [19] employed the repeated removal of independent sets from triangulations to devise data structures for efficient planar point location and distance computation between convex polytopes, respectively. Biedl and Wilkinson [6] explored the size of independent sets in bounded degree triangulations. In addition, Bose, Dujmović and Wood [11] obtained graphs of bounded degree with large independent sets.

The celebrated 4-color theorem [3, 29] immediately implies that every planar graph contains an independent set of size at least $n/4$, where n is the number of vertices in the graph. Interestingly, this bound represents the maximum attainable, as there exist planar graphs without larger independent sets; for instance, consider disjoint copies of complete graphs with 4 vertices. Some weaker lower bounds are also established [1, 17] that circumvent the complexity of the 4-color theorem (as suggested by Erdős [5]) via charging and discharging arguments. Also, Caro and Roditty in [13] gives the following upper bound.

▶ **Theorem 1** ([13]). *Let G be a planar graph with minimum degree δ. Then $\alpha(G) \le \frac{2n-4}{\delta}$.*

In addition, they construct an infinite family of planar graphs with $\alpha(G) = \frac{2n-4}{\delta}$, where δ takes values of 3, 4, and 5. From an algorithmic standpoint, determining the maximum independent set in planar graphs is NP-hard, even when restricted to planar graphs of maximum degree 3 [21, 28] or planar triangle-free graphs [26]. Consequently, efforts have shifted towards approximating large independent sets through methods like approximation algorithms [2, 4, 12, 15, 20, 26], parallel algorithms [16, 18, 22], or within certain minor-free planar graphs [20, 25, 27].

There are various generalizations of planar graphs, for example a 1-*planar* graph is a graph that can be drawn in the Euclidean plane with at most one crossing per edge. In this paper, we study the independence number of 1-planar graphs. Borodin [10] establishes that every 1-planar graph G has a 6-coloring, therefore $\alpha(G) \geq n/6$. This is tight, for example a graph consisting of disjoint copies of K_6 is 1-planar and has chromatic number 6. Unlike planar graphs, there were no prior results on the upper bound for the independence number of 1-planar graphs under degree conditions.

Our Results. This paper aims to explore the upper bounds on the independence number of 1-planar graphs. We provide such upper bounds, relative to the minimum degree δ. Furthermore, we construct 1-planar graphs that (for most values of δ) match the bound, i.e., they have this minimum degree and have an independent set of that size. Our results are summarized in Table 1.

Table 1 Bounds on the independence number of 1-planar graphs of minimum degree δ and optimal 1-planar graphs. The upper bound means that no graph can have a bigger independent set, while the lower bound means that some 1-planar graph has an independent set of this size. Note that the bounds match (up to small additive constants) except for $\delta = 7$.

	$\delta = 3$	$\delta = 4$	$\delta = 5$	$\delta = 6$	$\delta = 7$	Optimal 1-planar
Upper bound	$\frac{6}{7}(n-2)$	$\frac{2}{3}(n-2)$	$\frac{4}{7}(n-2)$	$\frac{1}{2}(n-3)$	$\frac{8}{20}(n-2)$	$\frac{1}{3}(n-2)$
Lower bound	$\frac{6}{7}(n-2)$	$\frac{2}{3}(n-2)$	$\frac{4}{7}(n-2)$	$\frac{1}{2}(n-4)$	$\frac{8}{21}(n-13.5)$	$\frac{1}{3}(n-2)$

We also study optimal 1-planar graphs, which are 1-planar graphs with the maximum possible number of edges, and for these, we give an upper bound $\frac{1}{3}(n-2)$ on the independence number. In addition, we show that this upper bound is tight by providing a family of optimal 1-planar graphs that achieve this bound.

Our paper is organized as follows. After giving preliminaries, we first construct in Section 3 a number of 1-planar graphs as a warm-up to introduce this graph class. Specifically, we provide infinite families of 1-planar graphs with large independent sets for minimum degree $\delta = 3, 4, 5, 6, 7$. In Section 4, we then present upper bounds for the independence number of 1-planar graphs with minimum degrees $\delta = 3, 4, 5, 6, 7$. Here for $\delta = 3, 4, 5$ the upper bounds are proven with some techniques that were used to bound the size of matchings in such graphs [7]. For $\delta = 6, 7$ we prove the upper bounds by expanding and generalizing some known results. Section 5 focuses on the independence number of optimal 1-planar graphs, before we conclude in Section 6 with some further thoughts.

2 Preliminaries

Let $G = (V, E)$ be a graph on n vertices. We assume familiarity with basic terms in graph theory, such as connectivity. We refer the reader to Bondy and Murty [9] for graph theoretic notations. Throughout the paper our input is always a connected graph $G = (V, E)$ on n vertices, and $n \geq 3$. We also use the letter T to denote an *independent set*, i.e., a set of vertices without edges between them. The notation $\overline{T}$ refers to the set of vertices $V \setminus T$.

A drawing Γ of a graph G assigns vertices to points in $\mathbb{R}^2$ and edges to curves in $\mathbb{R}^2$ in such a way that edge-curves join the corresponding endpoints. In this paper we only consider *good* drawings, see [30], where the following holds:

1. no vertex-points coincide and no edge-curve intersects a vertex-point except at its two ends;
2. if two edge-curves intersect at a point p that is not a common endpoint, then they properly cross at p;
3. if three or more edge-curves intersect at a point p, then p is a common endpoint of the curves;
4. if the curves of two edges e, e' intersect twice at points $p \neq p'$, then e, e' are parallel edges and p, p' are their endpoints; and
5. if the curve of an edge e self-intersects at point p, then e is a loop and p is its endpoint.

A drawing is called *k-planar* if each edge is involved in at most k crossings; a 0-planar drawing is simply called *planar*. In this paper, all drawings are 1-planar. A graph is called *planar/1-planar* if it has a planar/1-planar drawing, respectively.

For a given drawing Γ, the *cells* are the connected regions of $\mathbb{R}^2 \setminus \Gamma$; if Γ is planar then these are also called *faces*. The unbounded cell is also called the *outer face* (even for drawings that are not planar).

3 1-planar graphs with large independent sets

In this section, we construct several families of 1-planar graphs, each corresponding to a specified minimum degree denoted by δ. These graphs are designed to possess a large independent set. We first provide a general overview of how to construct them, and in each subsection, we elaborate with full details.

3.1 1-planar graphs with large independent sets for $\delta = 3, 4, 5, 6$

All but one of our constructions use a scheme that we call a *standard construction* which we explain here in general terms (see Figure 1). Fix three integer parameters s, k, τ, where $s \geq 1$ is arbitrary (it serves to make the graph as big as we wish), while k and τ will depend on the minimum degree δ.

The standard construction (illustrated in Figure 1) starts with s *nested k-cycles*, i.e., cycles of length k that are drawn (in the 1-planar drawing that we construct) such that each next cycle is inside the previous one. We will show our drawings on the *standing flat cylinder*, i.e., a rectangle where the left and right side have been identified; the nested cycles then become horizontal lines.

The s nested cycles define $s+1$ faces; of these, $s-1$ faces (the *middle faces*) are bounded by two disjoint k-cycles while two faces (the *end faces*) are bounded by one k-cycle. Consider one middle face, say it is bounded by k-cycles P and P'. We place τ vertices $t_1, \ldots, t_\tau$ inside this middle face; these vertices (over all middle faces together, plus possibly a few more at the end-faces) become our independent set T. These vertices are drawn in white in Figure 1. We make each t_i adjacent to $\lceil \delta/2 \rceil$ vertices on one of P, P' and $\lfloor \delta/2 \rfloor$ vertices on the other. With this, the vertices in T have degree δ. The main bottleneck for τ and k is that we must be able to place these vertices so that the drawing is 1-planar and simple. Another bottleneck is that all vertices in $\overline{T} := V \setminus T$ must have degree at least δ. Both claims will be mostly proved by illustrations outlining the 1-planar embeddings.

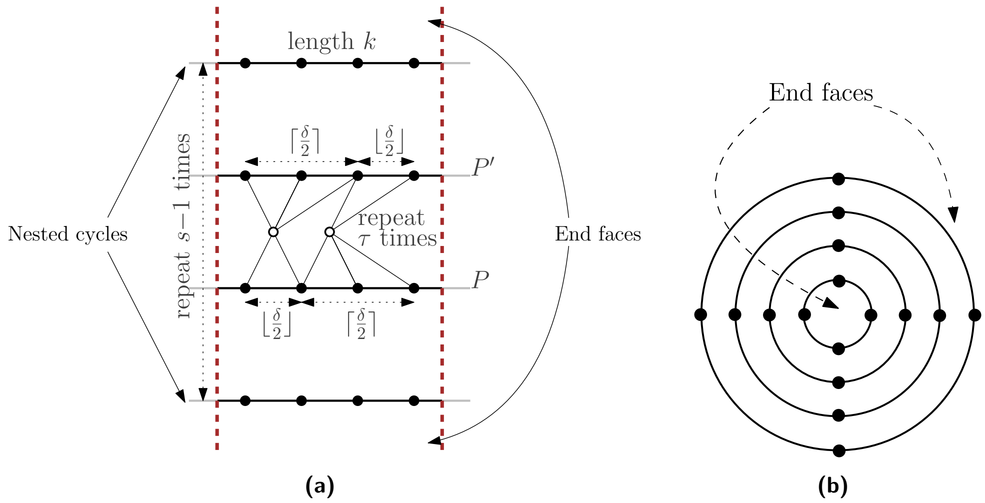

Figure 1 The standard construction and the view when the graph is drawn in the plane.

The construction inside the end-faces depends very much on δ; sometimes we add nothing at all, sometimes we add edges, sometimes we add more vertices (in T or in $\overline{T}$ or both). The bottleneck is again that the vertices on the first/last nested cycle must have degree δ or more. In total the number of vertices is $n = s \cdot k + (s-1)\tau$ (plus whatever we added at the end-faces). The size of the independent set is $|T| = (s-1)\tau$ (plus whatever we added at the end-faces).

Now we give the specific constructions. (We should mention that for $\delta = 3, 4$ these are the same as the ones given in [7] to obtain 1-planar graphs for which the maximum matching is small, though described in a different way.)

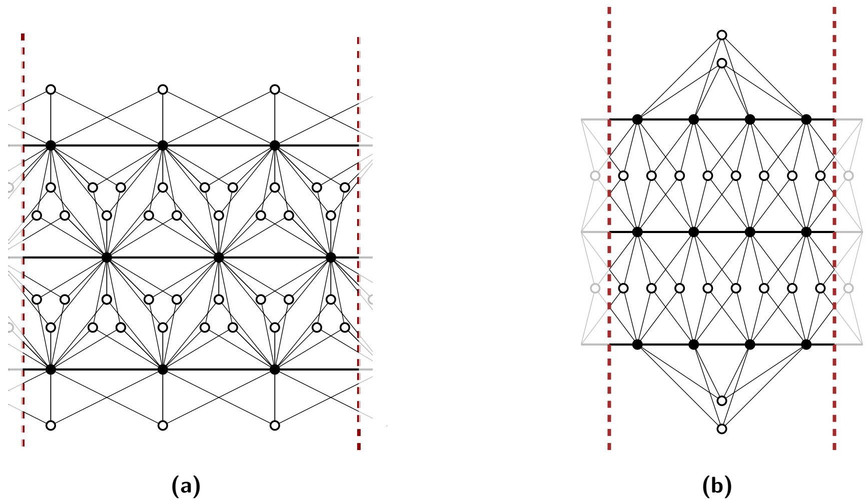

Figure 2 The graphs for $\delta = 3$ and $\delta = 4$ (for $s = 3$). Vertices in T are white, vertices in $\overline{T}$ are black.

▶ **Lemma 2.** *For any integer N and $\delta \in \{3, 4, 5, 6\}$, there exists a simple 1-planar graph with minimum degree δ and $n \geq N$ vertices with an independent set that has the size listed in Table 1 under "Lower bound".*

Proof. We follow the standard construction, choosing s big enough so that the resulting graph has at least N vertices. We choose k and τ as follows:

- For $\delta = 3$, we use $k = 3$ (so nested triangles) and $\tau = 18$. Into each end-face we add three more vertices of T that we make adjacent to all three vertices of the nested triangle that bounds the face. See Figure 2(a) to verify that this can be done such that the drawing is 1-planar and the minimum degree is 3. With this we have $n = 3s + 18(s-1) + 6 = 21s - 12$ and $|T| = 18(s-1) + 6 = 18s - 12 = \frac{6}{7}(21s - 14) = \frac{6}{7}(n-2)$.
- For $\delta = 4$, we use $k = 4$ (so nested quadrangles) and $\tau = 8$. Into each end-face we add two more vertices of T that we make adjacent to all four vertices of the nested quadrangle that bounds the face. See Figure 2(b) to verify that this can be done such that the drawing is 1-planar and the minimum degree is 4. With this we have $n = 4s + 8(s-1) + 4 = 12s - 4$ and $|T| = 8(s-1) + 4 = 8s - 4 = \frac{2}{3}(12s - 6) = \frac{2}{3}(n-2)$.
- For $\delta = 5$, we use $k = 12$ and $\tau = 16$. Into each end-face we add four more vertices of $\overline{T}$ connected as a path, and then 12 more vertices of T that we each make adjacent to two vertices of the path and three vertices of the 12-gon that bounds the face. See Figure 3, and verify that this can be done such that the drawing is 1-planar and the minimum degree is 5. With this we have $n = 12s + 16(s-1) + 32 = 28s + 16$ and $|T| = 16(s-1) + 24 = 16s + 8 = \frac{4}{7}(28s + 14) = \frac{4}{7}(n-2)$.

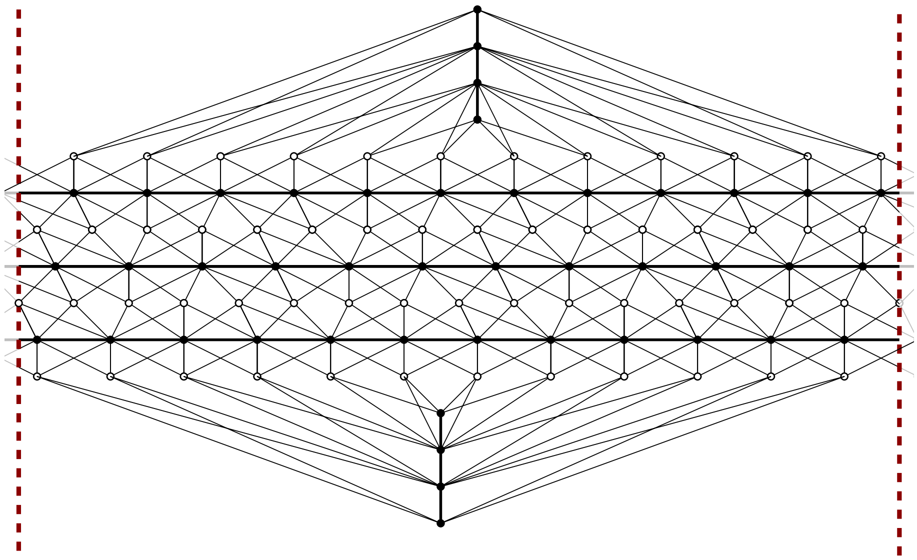

Figure 3 The graph for $\delta = 5$ (for $s = 3$).

- For $\delta = 6$, we use $k = 4$ (so nested quadrangles) and $\tau = 4$. Into each end-face we add a pair of crossing edges between the four vertices of the nested quadrangle that bounds the face. See Figure 4 to verify that this can be done such that the drawing is 1-planar and the minimum degree is 6. With this we have $n = 4s + 4(s-1) = 8s - 4$ and $|T| = 4(s-1) = 4s - 4 = \frac{1}{2}(8s - 8) = \frac{1}{2}(n-4)$. ◀

3.2 1-planar graphs with large independent sets for $\delta = 7$

In this subsection, we show how to construct a 1-planar graph with minimum degree 7 and a large independent set. This does not use the construction from the previous section since it seems impossible to use equal-length nested cycles. Instead, we prove first a construction with desirable properties by induction, and then combine two such constructions into a graph with minimum degree 7.

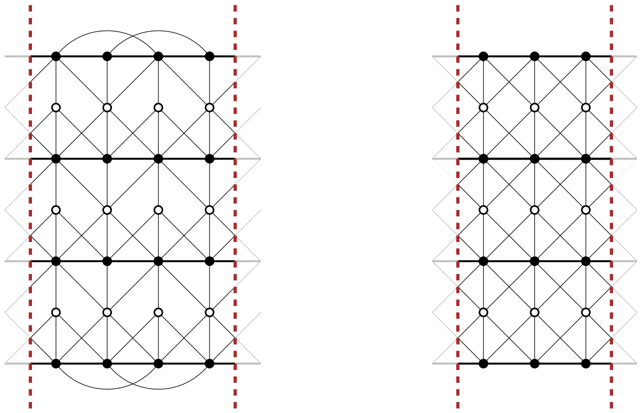

Figure 4 The graph for $\delta = 6$ (for $s = 4$). We also show a graph where all but 6 vertices have degree 6 that has an independent set of size $(n-3)/2$.

▷ Claim 3. For all $k \geq 0$, there exists a 1-planar graph G_k with $27 \cdot 2^k - 9$ vertices and an independent set T with $9 \cdot 2^k - 6$ vertices such that (in some 1-planar drawing)

- there are exactly $9 \cdot 2^k$ vertices on the outer-face, they form a cycle and each of them has degree at least 4,
- all other vertices have degree at least 7,
- no vertex of T is on the outer-face.

Proof. For the base case ($k = 0$), we need a graph with 18 vertices of which three form an independent set; see Figure 5(a) to verify all conditions.

Now assume that we have graph G_k with $9 \cdot 2^k$ vertices on the outer-face F_k. Insert $9 \cdot 2^k$ new vertices in F_k (let T_{k+1} be the set of added vertices) and make each of them adjacent to three vertices of F_k; Figure 5(b) shows that this can be done while retaining 1-planarity and keeping T_{k+1} on the outer-face. With this, all vertices in F_k receive three more neighbours and hence now have degree 7 or more. Insert $18 \cdot 2^k$ new vertices into the outer-face of the resulting graph, and connect them in a cycle that will form the outer-face F_{k+1} of the new graph G_{k+1}. Make each vertex of T_{k+1} adjacent to four vertices of F_{k+1}; the figure shows that can be done while remaining 1-planar. Also, with this all vertices on F_{k+1} receive two neighbours in T_{k+1}; this plus the cycle among them ensures that they have degree 4 while everyone else has degree at least 7. As desired T_{k+1} forms an independent set and has no edges to vertices of the independent set T_k of G_k since those are not on F_k by inductive hypothesis.

It remains to verify the claim on the size. Independent set $T_k \cup T_{k+1}$ has size $9 \cdot 2^k - 6 + 9 \cdot 2^k = 9 \cdot 2^{k+1} - 6$. The outer-face F_{k+1} of G_{k+1} has $18 \cdot 2^k = 9 \cdot 2^{k+1}$ vertices, and finally $|V(G_{k+1})| = |V(G_k)| + |T_{k+1}| + |F_{k+1}| = 27 \cdot 2^k - 9 + 9 \cdot 2^k + 18 \cdot 2^k = 27 \cdot 2^{k+1} - 9$. ◁

▶ **Lemma 4.** *For any integer N, there exists a simple 1-planar graph with minimum degree 7 and $n \geq N$ vertices with an independent set of size $\frac{8}{21}(n - 13.5)$.*

Proof. Let $k = \lceil \log_2((N+18)/63) \rceil$ and start with two copies of G_k, placed such that the two outer-faces F_k, F'_k of the two copies together bound one face. Into this face, insert $9 \cdot 2^k$ vertices that we call U_{k+1}, grouped into $3 \cdot 2^k$ paths of three vertices each. Each vertex of U_{k+1} is adjacent to three vertices each of F_k and F'_k; Figure 6 shows that this can be done while remaining 1-planar.

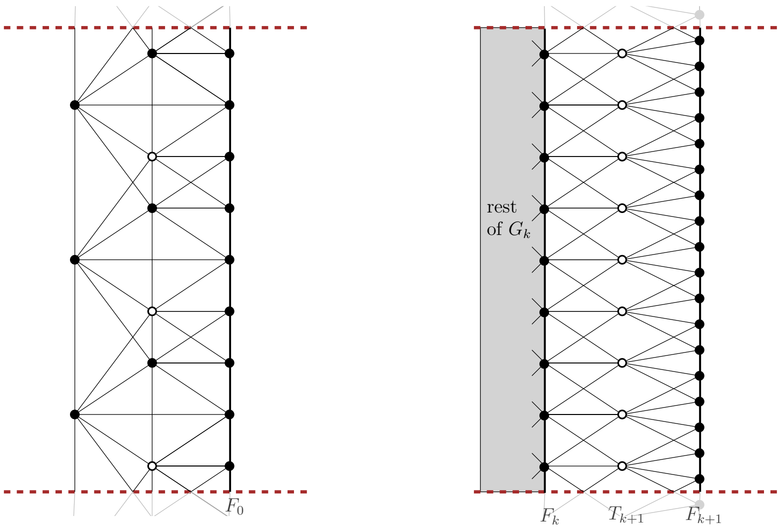

Figure 5 The base case and the induction step for building the graph G_k. For ease of reading we now show the construction on the rolling cylinder, rather than the standing one.

Since each vertex of U_{k+1} also has at least one neighbour in U_{k+1}, and each vertex of F_k and F'_k receives three more neighbours, the resulting graph G has minimum degree 7. Define T to consist of the two independent sets of the two copies of G_k as well as the $6 \cdot 2^k$ end-vertices of the paths in U_{k+1}; this is an independent set. See Figure 5.

It remains to analyze the size of G and T. Since G contains two copies of G_k, plus U_{k+1}, it has

$$n = 2 \cdot 27 \cdot 2^k - 2 \cdot 9 + 9 \cdot 2^k = 63 \cdot 2^k - 18 \geq N$$

vertices. Likewise T contains two copies of the independent set of G_k, plus the ends of the $3 \cdot 2^k$ paths, hence

$$|T| = 2 \cdot 9 \cdot 2^k - 2 \cdot 6 + 6 \cdot 2^k = 24 \cdot 2^k - 12.$$

Since $\frac{8}{21}(63 \cdot 2^k - 18 - 13.5) = 24 \cdot 2^k - \frac{144}{21} - \frac{108}{21} = 24 \cdot 2^k - 12$, the bound holds. ◀

With this we have proved all lower-bound entries in Table 1.

4 Upper bounds on the independence number

All our approaches to prove the upper bounds rely on bounding the maximum size of a bipartite 1-planar graph where one side of the bipartition is the bounded degree independent set. A previously known result here gives tight upper bounds for 1-planar graphs with minimum degree 3,4 and 5. For $\delta = 6$, by counting differently, we improve the bound. For $\delta = 7$ we improve the existing result by using a charging/discharging argument.

4.1 Upper bounds on the independence number for $\delta = 3, 4, 5$

To obtain our upper bounds in this section, we use the following lemma from [7] on independent sets in 1-planar graphs.

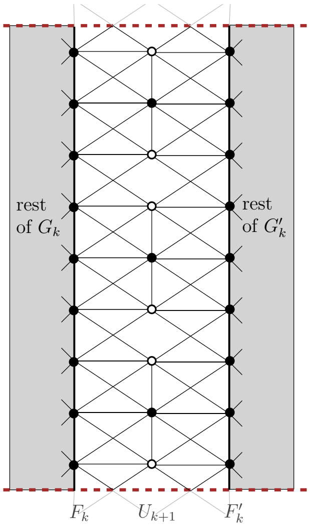

Figure 6 Combining two copies of G_k.

▶ **Lemma 5** ([7]). *Let G be a simple 1-planar graph. Let T be a non-empty independent set in G where $\deg(t) \geq 3$ for all $t \in T$. Let T_d be the vertices in T that have degree d. Then*

$$2|T_3| + \sum_{d\geq 4}(3d-6)|T_d| \leq 12|\overline{T}| - 24. \tag{1}$$

The notation of "minimum degree" is normally only defined for an entire graph, but we now use it also for a subset T of vertices, so T has minimum degree δ if all vertices in T have degree at least δ (but vertices in $\overline{T}$ may have smaller degrees).
Given the upper bound established in Lemma 5, we are able to use a counting argument to obtain the following upper bounds.

▶ **Corollary 6.** *Let G be a simple 1-planar graph and T be an independent set with minimum degree $\delta = 3$. Then $|T| \leq \frac{6}{7}(n-2)$.*

Proof. We have $2|T| = 2\sum_{d\geq 3}|T_d| \leq 2|T_3| + \sum_{d\geq 4}(3d-6)|T_d| \leq 12(n-|T|) - 24$ and therefore $14|T| \leq 12n - 24$. ◀

▶ **Corollary 7.** *Let G be a simple 1-planar graph and T be an independent set with minimum degree $\delta = 4$. Then $|T| \leq \frac{2}{3}(n-2)$.*

Proof. Since T_3 is empty, we have $6|T| = \sum_{d\geq 4} 6|T_d| \leq 2|T_3| + \sum_{d\geq 4}(3d-6)|T_d| \leq 12(n-|T|) - 24$ and therefore $18|T| \leq 12n - 24$. ◀

▶ **Corollary 8.** *Let G be a simple 1-planar graph and T be an independent set with minimum degree $\delta = 5$. Then $|T| \leq \frac{4}{7}(n-2)$.*

Proof. Since T_3 and T_4 are empty, we have $9|T| = \sum_{d\geq 5} 9|T_d| \leq 2|T_3| + \sum_{d\geq 4}(3d-6)|T_d| \leq 12(n-|T|) - 24$ and therefore $21|T| \leq 12n - 24$. ◀

4.2 Upper bounds on the independence number for $\delta = 6$

Note that if we apply the above Lemma for $\delta = 6$, we get a bound of $12|T| = \sum_{d\geq 6} 12|T_d| \leq 2|T_3| + \sum_{d\geq 4}(3d-6)|T_d| \leq 12(n-|T|) - 24$ and therefore $24|T| \leq 12n - 24$ which means $\frac{1}{2}(n-2)$. However, we are able to get a slightly better bound by using an alternative argument.

▶ **Lemma 9.** *Let G be a simple 1-planar graph. Then for any independent set T with minimum degree δ we have $|T| \leq \frac{3n-8-\chi}{\delta}$ where $\chi = 1$ if n is odd and $\chi = 0$ otherwise.*

Proof. Consider the 1-planar bipartite subgraph G^- of G that consists of the edges between T and $\overline{T}$. This graph has n vertices and has (by a result by Karpov [23]) at most $3n - 8 - \chi$ edges. Every vertex of T has no neighbour in T, so all its incident edges are in G^-. Therefore $\delta|T| \leq E(G^-) \leq 3n - 8 - \chi$ which implies the result. ◀

▶ **Corollary 10.** *Let G be a simple 1-planar graph and T be an independent set with minimum degree $\delta = 6$. Then $|T| \leq \frac{1}{2}(n-3)$.*

Proof. If n is odd then $|T| \leq \frac{1}{6}(3n-9) = \frac{1}{2}(n-3)$. If n is even then by integrality $|T| \leq \lfloor \frac{1}{6}(3n-8) \rfloor = \lfloor \frac{n}{2} - \frac{4}{3} \rfloor = \frac{n}{2} - 2 = \frac{1}{2}(n-4)$. ◀

4.3 Upper bounds on the independence number for $\delta = 7$

We notice that by using the counting argument as above, the upper bounds that can be obtained for $\delta = 7$ are

$\frac{3n-8}{7}, \frac{4n-8}{9}$

which are quite weak. We are able to obtain a better upper bound by revisiting the charging/discharging argument that was used in the proof of Lemma 5 (this was hinted at in [7], and many parts of the proof below are directly taken from there). We furthermore generalize the statement to graphs with parallel edges (in [7] simplicity of the graph was used only for $\delta = 3$). Specifically we assume that the graph has no loops and a *bigon-free* 1-planar drawing Γ, i.e., there is no cell whose boundary consists of two parallel uncrossed edges.

▶ **Lemma 11.** *Let G be a 1-planar graph with an independent set T that has minimum degree $\delta \geq 4$. Graph G may have parallel edges, but assume that it has no loops and a bigon-free 1-planar drawing Γ. Then*

$$|T| \leq \frac{4}{\delta + \lceil \frac{\delta}{3} \rceil}(n-2).$$

Proof. We use a charging scheme, where we assign some *charges* (units of weight) to edges in G (as well as to some edges that we add to G), redistribute these charges to the vertices in T, and then count the number of charges in two ways to obtain the bound.

As a first step, delete all edges within $\overline{T}$ so that G becomes bipartite. Also add any edge to Γ that connects T to $\overline{T}$ and that can be added without a crossing. We are allowed to add parallel edges, as long as they do not form a bigon. Both operations can only increase degrees of vertices in T, so it suffices to prove the bound in the resulting drawing Γ'.

As shown in [7], for any vertex $t \in T$ there cannot be three consecutive crossed edges in the circular ordering of edges at t. For if there were three such edges (say $(t, s_1), (t, s_2), (t, s_3)$) then the edge that crosses (t, s_2) has one endpoint in $\overline{T}$; we could have added an uncrossed edge from this endpoint to t, and since it would be before or after (t, s_2) in the circular ordering at t it would not have formed a bigon. This contradicts maximality.

We assign charges as follows: Let E_- be the uncrossed edges of Γ; each of those receives 2 charges. Let $E_\times$ be the crossed edges of Γ; each of those receives 1 charges. We know (see [7]) that $\frac{1}{2}|E_\times| + |E_-| \leq 2n - 4$ (this holds even with parallel edges if the drawing is bigon-free and has no loops). Hence

$$\#\text{charges} \quad = \quad 2|E_-| + 1|E_\times| \leq 4n - 8. \tag{2}$$

For $t \in T$, let $c(t)$ be the total charges of incident edges of t and write d for the degree of t. We know that there are at least $\lceil \frac{d}{3} \rceil$ uncrossed edges at t since there are no three consecutive crossed edges. Thus t obtains $2\lceil \frac{d}{3} \rceil$ charges from three uncrossed edges, and at least $d - \lceil \frac{d}{3} \rceil$ further charges from the remaining edges. Hence $c(t) \geq d + \lceil \frac{d}{3} \rceil \geq \delta + \lceil \frac{\delta}{3} \rceil$ and

$$\#\text{charges} \quad = \quad \sum_{t \in T} c(t) \geq |T|(\delta + \lceil \tfrac{\delta}{3} \rceil). \tag{3}$$

Combining this with (2) gives $|T|(\delta + \lceil \frac{\delta}{3} \rceil) \leq 4n - 8$ as desired. ◀

▶ **Corollary 12.** *Let G be a simple 1-planar graph and T be an independent set with minimum degree $\delta = 7$. Then $|T| \leq \frac{2}{5}(n-2)$.*

Proof. The proof follows from Lemma 11 by setting $\delta = 7$ ◀

With this we have proved all upper-bound entries in Table 1.

5 Optimal 1-planar graphs

Caro and Roditty in [13] showed that if G is a planar graph with order $n \geq 4$ and minimum degree δ, the equality $\alpha(G) = \frac{2n-4}{\delta}$ holds if and only if G can be formed from a planar graph H, all of whose faces are bounded by δ-cycles, by adding a vertex of degree δ inside each region. In particular, for planar graph the upper bound on the independence number is tight for *maximal planar graph*, i.e., planar graphs that have the maximum possible number $3n-6$ of edges.

In the same spirit, one should ask what the independence number can be for 1-planar graphs that have the maximum possible number of edges. It is known that every 1-planar graph has at most $4n-8$ edges, and a simple 1-planar graph G is called *optimal* if it has exactly $4n-8$ edges. An optimal 1-planar graph can equivalently be defined as the graphs obtained by taking a planar quadrangulated graph Q (i.e., all faces are bounded by 4-cycles) and inserting a pair of crossing edges into each face. Numerous results are known for such graphs, see [8]. In particular, a simple optimal 1-planar graph has exactly $n-2$ pairs of crossing edges, all vertex-degrees are even, and the minimum degree is 6.

▶ **Lemma 13.** *Let G be a simple optimal 1-planar graph. Then for any independent set T we have $|T| \leq \frac{1}{3}(n-2)$.*

Proof. Fix an arbitrary vertex $t \in T$, say it has degree $d \geq 6$. In the 1-planar drawing of G, the cyclic order of edges around t alternates between uncrossed and crossed edges. Therefore half of the incident edges of t are crossed, and we assign all these crossings to t. This does not double-count crossings, because (in an optimal 1-planar graph) the four endpoints of a crossing induce K_4 and so at most one of them can belong to T. We assigned at least three crossings to every vertex in T, and there are exactly has $n-2$ crossings, so $|T| \leq \frac{1}{3}(n-2)$. ◀

▶ **Lemma 14.** *For any integer N, there exists a simple optimal 1-planar graph with $n \geq N$ vertices and an independent set of size $\frac{1}{3}(n-2)$.*

Proof. Let H be a $2s$-*prism*, i.e., it consists of two cycles of length $2s$ with corresponding vertices of the cycles connected by an edge. Here $s \geq \max\{4, \frac{N-2}{6}\}$. Graph H has $4s$ vertices and is bipartite; we let T be one of its colors classes and note that $|T| = 2s$. See also Figure 7.

Now obtain graph G by adding the dual graph H^* to H and connecting every dual vertex v_F of H^* to all vertices of the face F of H that v_F represents. It is well-known that this gives an optimal 1-planar graph, and since H has $2s+2$ faces, we have $n = |V(G)| = 6s+2 \geq N$. No two vertices of T were connected, so T is an independent set of G of size $2s = \frac{1}{3}(n-2)$. ◀

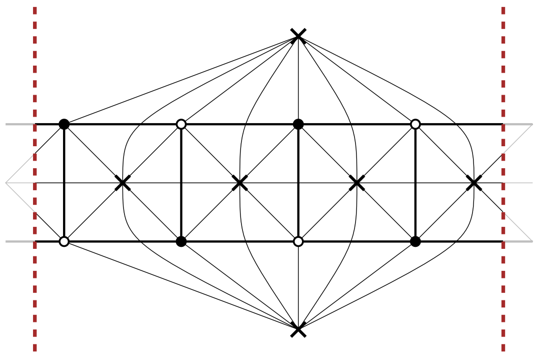

Figure 7 Graph H (bold) for $s = 2$ and the resulting optimal 1-planar graph that has an independent set (white) of size $\frac{n-2}{3}$.

Combining the two results we obtain:

▶ **Theorem 15.** *The independence number of optimal* 1*-planar graphs is exactly* $\frac{n-2}{3}$ *for the only feasible minimum degree* $\delta = 6$.

6 Further thoughts

In this paper, we studied upper bounds on the independence number of 1-planar graphs of minimum degree δ (for $\delta = 3, 4, 5, 6, 7$). This considered all interesting cases, because for $\delta = 2$ the complete bipartite graph $K_{2,n-2}$ is 1-planar (in fact, planar), so the independence number can be arbitrarily close to n, and there are no simple 1-planar graphs with minimum degree $\delta = 8$. We also provided 1-planar graphs for these minimum degrees that have large independent sets.

For $\delta = 3, 4, 5$, our lower and upper bound match exactly (see also Table 1). For $\delta = 6$, our bounds are within a very small constant of each other. We do leave a larger gap between upper and lower bounds for $\delta = 7$.

One reason for this gap is that the arguments used for upper bounds (Lemmas 5, 9, and 11) are ignoring some information: they do not use that the entire *graph* has minimum degree δ, but they only use that the independent set T has minimum degree δ. While this (surprisingly) does not seem to make a difference for $\delta = 3, 4, 5$, it makes a tiny (additive) difference for $\delta = 6$ and a noticeable (multiplicative) difference for $\delta = 7$. For $\delta = 6$, we can construct a graph with an independent set T of the (maximum possible size) $(n-3)/2$ if we allow just six vertices in $\overline{T}$ to have smaller degree; see also Figure 4. It is also not hard to construct a 1-planar graph with an independent set T of size $\frac{2}{5}n - O(1)$ that has minimum degree 7 (but some vertices of $\overline{T}$ have smaller degree); roughly speaking take two graphs G_k (from Lemma 4) and identify the vertices of the two copies of F_k (we leave the calculations to the reader). So Lemma 11 as written is tight, but can we prove a smaller upper bound in the scenario where vertices of $\overline{T}$ also must have minimum degree δ?

Last but not least, it would be interesting to explore algorithmic questions around finding independent sets of a certain size. For example, it is easy to find an independent set of size $\frac{n}{8}$ in any 1-planar graph (because they are 7-degenerate and so can be 8-coloured in linear time). With more effort, we can even 7-color the graph in linear time, so find an independent set of size at least $\frac{n}{7}$ [14]. But can we find, say, an independent set of size $\frac{n}{3} - O(1)$ in an optimal 1-planar graph efficiently?

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