Local Spanners Revisited

Stav Ashur
Department of Computer Science, University of Illinois, Urbana, IL, USA

Sariel Har-Peled
Department of Computer Science, University of Illinois, Urbana, IL, USA

Abstract

For a set $P \subseteq \mathbb{R}^2$ of points and a family $\mathcal{F}$ of regions, a *local t-spanner* of P is a sparse graph G over P, such that for any region $\mathscr{r} \in \mathcal{F}$ the subgraph restricted to $\mathscr{r}$, denoted by $G \cap \mathscr{r}$, is a t-spanner for all the points of $\mathscr{r} \cap P$.

We present algorithms for the construction of local spanners with respect to several families of regions such as homothets of a convex region. Unfortunately, the number of edges in the resulting graph depends logarithmically on the spread of the input point set. We prove that this dependency cannot be removed, thus settling an open problem raised by Abam and Borouny. We also show improved constructions (with no dependency on the spread) of local spanners for fat triangles, and regular k-gons. In particular, this improves over the known construction for axis-parallel squares.

We also study notions of weaker local spanners where one is allowed to shrink the region a "bit". Surprisingly, we show a near linear-size construction of a weak spanner for axis-parallel rectangles, where the shrinkage is *multiplicative*. Any spanner is a weak local spanner if the shrinking is proportional to the diameter of the region.

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1 Introduction

For a set P of points in $\mathbb{R}^d$, the *Euclidean graph* $\mathcal{K}_P = \left(P, \binom{P}{2}\right)$ of P is an undirected graph. Here, an edge $pq \in \binom{P}{2}$ is associated with the segment pq, and its weight is the (Euclidean) length of the segment. Let $G = (P, E)$ and $H = (P, E')$ be two graphs over the same set of vertices (usually H is a subgraph of G). Consider two vertices $p, q \in P$, and parameter $t \geq 1$. A path π between p and q in H is a *t-path* if the length of π in H is at most $t \cdot \mathsf{d}_G(p, q)$, where $\mathsf{d}_G(p, q)$ is the length of the shortest path between p and q in G. The graph H is a *t-spanner* of G if there is a t-path in H for every $p, q \in P$. Thus, for a set $P \subseteq \mathbb{R}^d$ of points, a graph G over P is a *t-spanner* if it is a t-spanner of the Euclidean graph $\mathcal{K}_P$. There is a lot of work on building geometric spanners, see [10] and references there in.

Fault-tolerant spanners

An *$\mathcal{F}$-fault-tolerant spanner* for $P \subseteq \mathbb{R}^d$ is a graph $G = (P, E)$ such that for any region $\mathscr{r} \in \mathcal{F}$ (i.e., the "attack"), the graph $G - \mathscr{r}$ is a t-spanner of $\mathcal{K}_P - \mathscr{r}$, where $G - \mathscr{r}$ denotes the graph after one deletes from G all the vertices in $P \cap \mathscr{r}$, and all the edges in G whose corresponding segments intersect $\mathscr{r}$ (See Definition 1 for a formal definition of this notation).

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Surprisingly, as shown by Abam et al. [3], such fault-tolerant spanners can be constructed where the attack region is any convex set. Furthermore, these spanners have a near linear number of edges.

Fault-tolerant spanners were first studied with vertex and edge faults, meaning that some arbitrary set of at most k of vertices and edges has failed. Levcopoulos et al. [8] showed the existence of k-vertex/edge fault tolerant spanners for a set P of points in some metric space. Their spanner had $\mathcal{O}(kn \log n)$ edges, and weight, i.e. sum of edge weights, bounded by $f(k) \cdot wt(MST(P))$, where $wt(MST(P))$ is the weight of $MST(P)$, for some function f. Lukovszki [9] later achieved a similar construction, improving the number of edges to $\mathcal{O}(kn)$, and was able to prove that the result is asymptotically tight.

Local spanners

Recently, Abam and Borouny [2] introduced the notion of local spanners, which can be interpreted as having the complement property to being fault-tolerant. For a family $\mathcal{F}$ of regions, a graph $G = (P, E)$ is an *$\mathcal{F}$-local t-spanner* for P if for any $\mathscr{r} \in \mathcal{F}$, the subgraph of G induced on $P \cap \mathscr{r}$ is a t-spanner. Specifically, this induced subgraph $G \cap \mathscr{r}$ contains a t-path between any $p, q \in P \cap \mathscr{r}$ (note that we keep an edge in the subgraph only if both its endpoints are in $\mathscr{r}$, see Definition 1).

Abam and Borouny [2] showed how to construct such spanners for axis-parallel squares and vertical slabs. In this work, we further extend their results. They also showed how to construct such spanners for disks if one is allowed to add Steiner points, but left the question of how to construct local spanners for disks (without Steiner points) as an open problem.

To appreciate the difficulty in constructing local spanners, observe that unlike regular spanners, the construction has to take into account many different scenarios as far as which points are available to be used in the spanner. As a concrete example, a local spanner for axis-parallel rectangles requires a quadratic number of edges, see Figure 1.1.

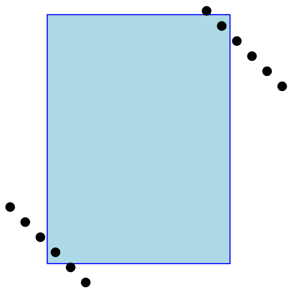

Figure 1.1 For any point in the top diagonal and bottom diagonal, there is a fat axis-parallel rectangle that contains only these two points. Thus, a local spanner requires a quadratic number of edges in this case.

Namely, regular spanners can rely on using midpoints in their path under the assurance that they are always there. For local spanners this is significantly harder as natural midpoints might “disappear”. Intuitively, a local spanner construction needs to use midpoints that are guaranteed to be present judging only from the source and destination points of the path.

A good jump is hard to find

Most constructions for spanners can be viewed as searching for a way to build a path from the source to the destination by finding a “good” jump, either by finding a way to move locally from the source to a nearby point in the right direction, as done in the θ-graph

Table 1.1 Known and new results. The notation $\mathcal{O}_\varepsilon$ hides polynomial dependency on ε which is not specified in the original work.

Region	# edges	Paper	New # edges	Location in paper
Local $(1+\varepsilon)$-spanners				
Halfplanes	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[3]		
Axis-parallel squares	$\mathcal{O}_\varepsilon(n \log^6 n)$	[2]	$\mathcal{O}\bigl(\varepsilon^{-3} n \log n\bigr)$	Remark 27
Vertical slabs	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[2]		
Disks+Steiner points	$\mathcal{O}_\varepsilon(n \log^2 n)$	[2]		
Disks			$\mathcal{O}\bigl(\varepsilon^{-2} n \log \Phi\bigr)$	Theorem 19
			$\Omega\bigl(n \log(1 + \frac{\Phi}{n})\bigr)$	Lemma 20
Homothets of a convex body			$\mathcal{O}\bigl(\varepsilon^{-2} n \log \Phi\bigr)$	Theorem 19
Homothets of α-fat triangles			$\mathcal{O}\bigl((\alpha\varepsilon)^{-1} n\bigr)$	Theorem 23
Homothets of triangles			$\Omega\bigl(n \log(1 + \frac{\Phi}{n})\bigr)$	Lemma 21
δ-weak local $(1+\varepsilon)$-spanners				
Convex body			$\mathcal{O}\bigl((\varepsilon^{-1} + \delta^{-2})n\bigr)$	Lemma 12
$(1-\delta)$-local $(1+\varepsilon)$-spanners				
Axis-parallel rectangles			$\mathcal{O}\bigl((\varepsilon^{-2} + \delta^{-2}) n \log^2 n\bigr)$	Theorem 31

construction, or alternatively, by finding an edge in the spanner from the neighborhood of the source to the neighborhood of the destination, as done in the spanner constructions using a well-separated pair decomposition (WSPD). Usually, one argues inductively that the spanner must have (sufficiently short) paths from the source to the start of the jump, and from the end of the jump to the destination, and then, combining these implies that the resulting new path is short. These ideas guide our constructions as well. However, the availability of specific edges depends on the query region, making the search for a good jump significantly more challenging. Intuitively, the constructions have to guarantee that there are many edges available, and that at least one of them is useful as a jump regardless of the chosen region (since slight perturbation in the region might make many of these edges unavailable).

Our results

Almost local spanners

We start by showing that regular geometric spanners are local spanners if one is required to provide the spanner guarantee only to shrunken regions. Namely, if G is a $(1+\varepsilon)$-spanner of P, then for any convex region $\mathcal{C}$, the graph $G \cap \mathcal{C}$ is a spanner for $\mathcal{C}' \cap P$, where $\mathcal{C}'$ is the set of all points in $\mathcal{C}$ that are in distance at least $\delta \cdot \operatorname{diam}(\mathcal{C})$ from its boundary, for $\delta = \Omega(\sqrt{\varepsilon})$ – see Lemma 12.

Homothets

A *homothet* of a convex region $\mathcal{C}$ is a translated and scaled copy of $\mathcal{C}$. In Section 3 we present a construction of spanners which surprisingly is not only fault-tolerant for all smooth convex regions, but is also a local spanner for homothets of a prespecified convex region. This in particular works for disks, and resolves the aforementioned open problem of Abam and Borouny [2]. Our construction is somewhat similar to the original construction of Abam et al. [3]. For a parameter $\varepsilon > 0$ the construction of a local $(1+\varepsilon)$-spanner for homothets takes $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$ time, and the resulting spanner is of size $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$, where Φ is the spread of the input point set P, and $n = |P|$.

The dependency on the spread Φ in the above construction is somewhat disappointing. However, the lower bound constructions, provided in Section 3.3, show that this is unavoidable for disks or homothets of triangles.

Thus, the natural question is what are the cases where one can avoid the "curse of the spread" – that is, cases where one can construct local spanners of near-linear-size independent of the spread of the input point set.

The basic building block: $\mathcal{C}$-Delaunay triangulation

A key ingredient in the above construction is the concept of Delaunay triangulations induced by homothets of a convex body. Intuitively, one replaces the unit disk (of the standard L_2-norm) by the provided convex region. It is well known [6] that such diagrams exist, have linear complexity in the plane, and can be computed quickly. In Section 3.1 we review these results, and restate the well-known property that the $\mathcal{C}$-Delaunay triangulation is connected when restricted to a homothet of $\mathcal{C}$. By computing these triangulations for carefully chosen subsets of the input point set, we get the results stated above.

Specifically, we use well-separated and semi-separated decompositions to compute these subsets.

Fat triangles

In Section 3.4 we give a construction of local spanners for the family $\mathcal{F}$ of homothets of a given triangle $\triangle$, and get a spanner of size $\mathcal{O}((\alpha\varepsilon)^{-1} n)$ in $\mathcal{O}((\alpha\varepsilon)^{-1} n \log n)$ time, where α is the smallest angle in $\triangle$. This construction is a careful adaptation of the θ-graph spanner construction to the given triangle, and it is significantly more technically challenging than the original construction.

k-regular polygons

It seems natural that if one can handle fat triangles, then homothets of k-regular polygons should readily follow by a simple decomposition of the polygon into fat triangles. Maybe surprisingly, this is not the case – a critical configuration might involve two points that are on the interior of two non-adjacent edges of a homothet of the input polygon. We overcome this by first showing that sufficiently narrow trapezoids provide us with a good jump somewhere inside the trapezoid, assuming one computes the Delaunay triangulation induced by the trapezoid, and that the source and destination lie on the two legs of the trapezoid. Next, we show that such a polygon can be covered by a small number of narrow trapezoids and fat triangles. By building appropriate graphs for each trapezoid/triangle in the collection, we get a spanner for homothets of the given k-regular polygon, with size that has no dependency on the spread. Of course, the size does depend polynomially on k. See Section 3.5 for details, and Theorem 26 for the precise result.

Multiplicative weak local spanner for rectangles

In the final result we use a less known type of pair-decomposition to construct a weak local spanner for axis-parallel rectangles. Here, the graph G, constructed over P, has the property that for any axis-parallel rectangle R, the graph $G \cap R$ is a $(1+\varepsilon)$-spanner for all the points of $\big((1-\delta)R\big) \cap P$, where $(1-\delta)R$ is the scaling of the rectangle by a factor of $1-\delta$ around its center. Intuitively, δ is a parameterization of the weakness of the spanner, which guarantees $(1+\varepsilon)$-paths for smaller regions as δ approaches 1. Importantly, this works for narrow rectangles where this form of multiplicative shrinking is still meaningful (unlike the diameter based shrinking mentioned above). Contrast this with the lower bound (illustrated in Figure 1.1) of $\Omega(n^2)$ on the size of local spanner if one does not shrink the rectangles. See Section 4 for details of the precise result.

See Table 1.1 for a summary of known results and comparisons to the results of this paper.

2 Preliminaries

Residual graphs

▶ **Definition 1.** *Let $\mathcal{F}$ be a family of regions in the plane. For a region $\mathcal{r} \in \mathcal{F}$ and a geometric graph G on a point set P, let $G - \mathcal{r}$ be the residual graph after removing from G all the points of P in $\mathcal{r}$ and all the edges whose corresponding segments intersect $\mathcal{r}$. Similarly, let $G \cap \mathcal{r}$ denote the graph restricted to $\mathcal{r}$. Formally, let*

$$G - \mathcal{r} = \big(P \setminus \mathcal{r}, \{uv \in E \mid uv \cap \mathrm{int}(\mathcal{r}) = \emptyset\}\big) \qquad \textit{and} \qquad G \cap \mathcal{r} = \big(P \cap \mathcal{r}, \{uv \in E \mid uv \subseteq \mathcal{r}\}\big).$$

where $\mathrm{int}(\mathcal{r})$ denotes the interior of $\mathcal{r}$,

2.1 On various pair decompositions

For sets X, Y, let $X \otimes Y = \{\{x, y\} \mid x \in X,\, y \in Y, x \neq y\}$ be the set of all the (unordered) pairs of points formed by the sets X and Y.

▶ **Definition 2** (Pair decomposition). *For a point set P, a **pair decomposition** of P is a set of pairs*

$$\mathcal{W} = \big\{\{X_1, Y_1\}, \ldots, \{X_s, Y_s\}\big\},$$

such that
- **(I)** *$X_i, Y_i \subseteq P$ for every i,*
- **(II)** *$X_i \cap Y_i = \emptyset$ for every i, and*
- **(III)** *$\bigcup_{i=1}^{s} X_i \otimes Y_i = P \otimes P$.*

*Its **weight** is $\omega(\mathcal{W}) = \sum_{i=1}^{s}(|X_i| + |Y_i|)$.*

The **closest pair** distance of a set $P \subseteq \mathbb{R}^d$ of points is $\mathrm{cp}(P) = \min_{p,q \in P, p \neq q} \|pq\|$. The **diameter** of P is $\mathrm{diam}(P) = \max_{p,q \in P} \|pq\|$. The **spread** of P is $\Phi(P) = \mathrm{diam}(P)/\mathrm{cp}(P)$, which is the ratio between the diameter and closest pair distance. While in general the weight of a WSPD (defined below) can be quadratic, if the spread is bounded, the weight is near linear. For $X, Y \subseteq \mathbb{R}^d$, let $\mathsf{d}(X, Y) = \min_{p \in X, q \in Y} \|pq\|$ be the **distance** between the two sets.

▶ **Definition 3.** *Two sets $X, Y \subseteq \mathbb{R}^d$ are*

$$\begin{array}{llll} & 1/\varepsilon\textbf{-well-separated} & \textit{if} & \mathbf{max}(\mathrm{diam}(X), \mathrm{diam}(Y)) \leq \varepsilon \cdot \mathsf{d}(X, Y), \\ \textit{and} & 1/\varepsilon\textbf{-semi-separated} & \textit{if} & \mathbf{min}(\mathrm{diam}(X), \mathrm{diam}(Y)) \leq \varepsilon \cdot \mathsf{d}(X, Y). \end{array}$$

For a point set P, a **well-separated pair decomposition** *(***WSPD***) of P with parameter ε is a pair decomposition of P with a set $\mathcal{W} = \big\{\{B_1, C_1\}, \ldots, \{B_s, C_s\}\big\}$, of pairs such that for all i, the sets B_i and C_i are $(1/\varepsilon)$-well-separated. The notion of $(1/\varepsilon)$-SSPD (a.k.a.* ***semi-separated pair decomposition****) is defined analogously.*

▶ **Lemma 4** ([1])**.** *Let P be a set of n points in $\mathbb{R}^d$, with spread $\Phi = \Phi(P)$, and let $\varepsilon > 0$ be a parameter. Then, one can compute a $(1/\varepsilon)$-WSPD $\mathcal{W}$ for P of total weight $\omega(\mathcal{W}) = \mathcal{O}(n\varepsilon^{-d} \log \Phi)$. Furthermore, any point of P participates in at most $\mathcal{O}\big(\varepsilon^{-d} \log \Phi\big)$ pairs.*

▶ **Theorem 5** ([1, 7])**.** *Let P be a set of n points in $\mathbb{R}^d$, and let $\varepsilon > 0$ be a parameter. Then, one can compute a $(1/\varepsilon)$-SSPD for P of total weight $\mathcal{O}\big(n\varepsilon^{-d} \log n\big)$. The number of pairs in the SSPD is $\mathcal{O}\big(n\varepsilon^{-d}\big)$, and the computation time is $\mathcal{O}\big(n\varepsilon^{-d} \log n\big)$.*

The following claim is straightforward.

▶ **Lemma 6.** *Given an α-SSPD $\mathcal{W}$ of a set P of n points in $\mathbb{R}^d$ and a parameter $\beta \geq 2$, one can refine $\mathcal{W}$ into an $\alpha\beta$-SSPD $\mathcal{W}'$, such that $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\beta^d)$ and $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\beta^d)$.*

▶ **Definition 7.** *An* ***ε-double-wedge*** *is a region between two lines, where the angle between the two lines is at most ε.*

Two point sets X and Y that each lie in their own face of a shared ε-double-wedge are ***ε-angularly separated****.*

▶ **Theorem 8** (Proof in full version [5])**.** *Given a $(1/\varepsilon)$-SSPD $\mathcal{W}$ of n points in the plane, one can refine $\mathcal{W}$ into a $(1/\varepsilon)$-SSPD $\mathcal{W}'$, such that each pair $\Xi = \{X, Y\} \in \mathcal{W}'$ is contained in an ε-double-wedge $\times_\Xi$, such that X and Y are contained in the two different faces of the double wedge $\times_\Xi$. We have that $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\varepsilon)$ and $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\varepsilon)$. The construction time is proportional to the weight of $\mathcal{W}'$.*

▶ **Corollary 9.** *Let P be a set of n points in the plane, and let $\varepsilon > 0$ be a parameter. Then, one can compute a $(1/\varepsilon)$-SSPD for P such that every pair is ε-angularly separated. The total weight of the SSPD is $\mathcal{O}\big(n\varepsilon^{-3} \log n\big)$, the number of pairs in the SSPD is $\mathcal{O}\big(n\varepsilon^{-3}\big)$, and the computation time is $\mathcal{O}\big(n\varepsilon^{-3} \log n\big)$.*

2.2 Weak local spanners for fat convex regions

▶ **Definition 10.** *Given a convex region C, let*

$$C_{\boxminus\delta} = \big\{p \in C \;\big|\; \mathsf{d}\big(p, \mathbb{R}^2 \setminus C\big) \geq \delta \cdot \mathrm{diam}(C)\big\}.$$

In other words, $C_{\boxminus\delta}$ is the Minkowski difference of C with a disk of radius $\delta \cdot \mathrm{diam}(C)$.

▶ **Definition 11.** *Consider a (bounded) set C in the plane. Let $r_{\mathrm{in}}(C)$ be the radius of the largest disk contained inside C. Similarly, $R_{\mathrm{out}}(C)$ is the smallest radius of a disk containing C.*

The ***aspect ratio*** *of a region C in the plane is $\mathsf{ar}(C) = R_{\mathrm{out}}(C)/r_{\mathrm{in}}(C)$. Given a family $\mathcal{F}$ of regions in the plane, its aspect ratio is $\mathsf{ar}(\mathcal{F}) = \max_{C\in\mathcal{F}} \mathsf{ar}(C)$.*

Note, that if a convex region C has bounded aspect ratio, then $C_{\boxminus\delta}$ is similar to the result of scaling C by a factor of $1 - \mathcal{O}(\delta)$. On the other hand, if C is long and skinny then this region is much smaller. Specifically, if C has width smaller than $2\delta \cdot \mathrm{diam}(C)$, then $C_{\boxminus\delta}$ is empty.

▶ **Lemma 12** (Proof in full version [5]). *Given a set P of n points in the plane, and parameters $\delta, \varepsilon \in (0,1)$, one can construct a graph G over P, in $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n \log n)$ time, and with $\mathcal{O}\big((\varepsilon^{-1} + \delta^{-2})n\big)$ edges, such that for any (bounded) convex region C in the plane, we have that for any two points $p, q \in P \cap C_{\boxminus\delta}$ the graph $C \cap P$ has a $(1+\varepsilon)$-path between p and q.*

3 Local spanners of homothets of convex region

Let $\mathcal{C}$ be a bounded convex and closed region in the plane (e.g., a disk). A **homothet** of $\mathcal{C}$ is a scaled and translated copy of $\mathcal{C}$. A point set P is in **general position** with respect to $\mathcal{C}$, if no four points of P lie on the boundary of a homothet of $\mathcal{C}$, and no three points are colinear.

A graph $G = (P, E)$ is a *$\mathcal{C}$-local t-spanner* for P if for any homothet $\mathscr{r}$ of $\mathcal{C}$ we have that $G \cap \mathscr{r}$ is a t-spanner of $\mathcal{K}_P \cap \mathscr{r}$.

3.1 Delaunay triangulation for homothets

▶ **Definition 13** ([6]). *Given $\mathcal{C}$ as above, and a point set P in general position with respect to $\mathcal{C}$, the **$\mathcal{C}$-Delaunay triangulation** of P, denoted by $\mathcal{D}_{\mathcal{C}}(P)$, is the graph formed by edges between any two points $p, q \in P$ such that there is a homothet of $\mathcal{C}$ that contains only p and q and no other point of P.*

▶ **Theorem 14** ([6]). *For a set P of points, $\mathcal{D}_{\mathcal{C}}(P)$ can be computed in $\mathcal{O}(n \log n)$ time for a pre-determined convex body $\mathcal{C}$. Furthermore, the triangulation $\mathcal{D}_{\mathcal{C}}(P)$ has $\mathcal{O}(n)$ edges, vertices, and faces.*

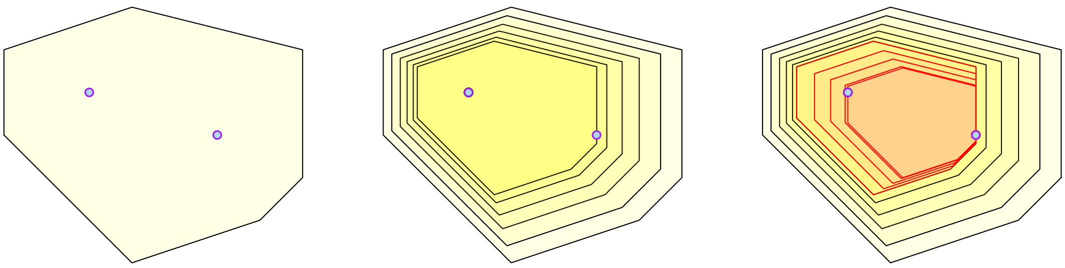

Figure 3.1 Shrinking of a homothet so that two specific points would lie on its boundary.

▶ **Lemma 15** (Proof in full version [5]). *Let $\mathcal{C}$ be a convex body, and let P be a set of points in general position with respect to $\mathcal{C}$. Then, if C is a homothet of $\mathcal{C}$ that contains two points $p, q \in P$, then there exists a homothet $C' \subseteq C$ of $\mathcal{C}$ such that $p, q \in \partial C'$.*

See Figure 3.1 for An illustration of the claim in Lemma 15.

The following standard claim, usually stated for the standard Delaunay triangulations, also holds for homothets.

▷ Claim 16 (Proof in full version [5]). Let $\mathcal{C}$ be a convex body. Given a set $P \subseteq \mathbb{R}^2$ of points in general position with respect to $\mathcal{C}$, let $\mathcal{D} = \mathcal{D}_{\mathcal{C}}(P)$ be the $\mathcal{C}$-Delaunay triangulation of P. For any homothet C of $\mathcal{C}$, we have that $\mathcal{D} \cap C$ is connected.

3.2 The generic construction

The input is a set P of n points in the plane (in general position) with spread $\Phi = \Phi(P)$, a parameter $\varepsilon \in (0,1)$, and a convex body $\mathcal{C}$ that defines the "unit" ball. The task is to construct $\mathcal{C}$-local spanner.

The algorithm computes a $(1/\vartheta)$-WSPD $\mathcal{W}$ of P using the algorithm of Lemma 4, where $\vartheta = \varepsilon/6$. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, the algorithm computes the $\mathcal{C}$-Delaunay triangulation $\mathcal{D}_\Xi = \mathcal{D}_\mathcal{C}(X \cup Y)$, and adds all the edges in $\mathcal{D}_\Xi \cap (X \otimes Y)$ to the computed graph G.

▶ Remark 17. In the above algorithm, the idea of computing a triangulation for each WSPD pair seems to be new.

3.2.1 Analysis

Size. For each pair $\Xi = \{X, Y\}$ in the WSPD, its $\mathcal{C}$-Delaunay triangulation contains at most $\mathcal{O}(|X| + |Y|)$ edges. As such, the number of edges in the resulting graph is bounded by $\sum_{\{X,Y\}\in\mathcal{W}} O\bigl(|X| + |Y|\bigr) = \mathcal{O}(\omega(\mathcal{W})) = \mathcal{O}\bigl(n\vartheta^{-2}\log\Phi\bigr)$, by Lemma 4.

Construction time. The construction time is bounded by

$$\sum_{\{X,Y\}\in\mathcal{W}} O\bigl((|X| + |Y|)\log(|X| + |Y|)\bigr) = \mathcal{O}(\omega(\mathcal{W})\log n) = \mathcal{O}\bigl(n\vartheta^{-2}\log\Phi\log n\bigr).$$

▶ **Lemma 18** (Local spanner property). *For $P, \mathcal{C}, \varepsilon$ as above, let G be the graph constructed above for the point set P. Then, for any homothet C of $\mathcal{C}$ and any two points $x, y \in P \cap C$, we have that $G \cap C$ has a $(1+\varepsilon)$-path between x and y. That is, G is a $\mathcal{C}$-local $(1+\varepsilon)$-spanner.*

Proof. Fix a homothet C of $\mathcal{C}$, and consider two points $p, q \in P \cap C$. The proof is by induction on the distance between p and q (or more precisely, the rank of their distance among the $\binom{n}{2}$ pairwise distances). Consider the pair $\Xi = \{X, Y\}$ such that $x \in X$ and $y \in Y$.

If $xy \in \mathcal{D}_\Xi$ then the claim holds, so assume this is not the case. By the connectivity of $\mathcal{D}_\Xi \cap C$, see Claim 16, there must be points $x' \in X \cap C$, $y' \in Y \cap C$, such that $x'y' \in E(\mathcal{D}_\Xi)$. Indeed, let $x \in X \cap C$, $y \in Y \cap C$, and let π be some (x, y)-path guaranteed to exist by connectivity. π must contain an edge with one endpoint in $X \cap C$ and the other in $Y \cap C$. As such, by construction, we have that $x'y' \in E(G)$. Furthermore, by the separation property, we have that

$$\max(\operatorname{diam}(X), \operatorname{diam}(Y)) \le \vartheta\, \mathsf{d}(X, Y) \le \vartheta\ell,$$

where $\ell = \|xy\|$. In particular, $\|x'x\| \le \vartheta\ell$ and $\|y'y\| \le \vartheta\ell$. As such, by induction, we have $\mathsf{d}_G(x, x') \le (1+\varepsilon)\|xx'\| \le (1+\varepsilon)\vartheta\ell$ and $\mathsf{d}_G(y, y') \le (1+\varepsilon)\|yy'\| \le (1+\varepsilon)\vartheta\ell$. Furthermore, $\|x'y'\| \le (1+2\vartheta)\ell$. As $x'y' \in E(G)$, we have

$$\begin{aligned}\mathsf{d}_G(x, y) &\le \mathsf{d}_G(x, x') + \|x'y'\| + \mathsf{d}_G(y', y) \le (1+\varepsilon)\vartheta\ell + (1+2\vartheta)\ell + (1+\varepsilon)\vartheta\ell\\ &\le (2\vartheta + 1 + 2\vartheta + 2\vartheta)\ell\\ &= (1+6\vartheta)\ell \le (1+\varepsilon)\|xy\|,\end{aligned}$$

if $\vartheta \le \varepsilon/6$. ◀

The result. We thus get the following.

▶ **Theorem 19.** *Let $\mathcal{C}$ be a convex body in the plane, let P be a given set of n points in the plane (in general position with respect to $\mathcal{C}$), and let $\varepsilon \in (0, 1/2)$ be a parameter. The above algorithm constructs a $\mathcal{C}$-local $(1+\varepsilon)$-spanner G. The spanner has $\mathcal{O}\bigl(\varepsilon^{-2}n\log\Phi\bigr)$ edges, and the construction time is $\mathcal{O}\bigl(\varepsilon^{-2}n\log\Phi\log n\bigr)$. Formally, for any homothet C of $\mathcal{C}$, and any two points $p, q \in P \cap C$, we have a $(1+\varepsilon)$-path in $G \cap C$.*

3.3 Lower bounds

3.3.1 A lower bound for local spanner for disks

The result of Theorem 19 is somewhat disappointing as it depends on the spread of the point set (logarithmically, but still). Next, we show a lower bound proving that this dependency is unavoidable, even in the case of disks.

Some intuition. A natural way to attempt a spread-independent construction is to try and emulate the construction of Abam et al. [3] and use an SSPD instead of a WSPD, as the total weight of the SSPD is near linear (with no dependency on the spread). Furthermore, after some post-processing, one can assume every pair $\Xi = \{X, Y\}$ is angularly ε-separated – that is, there is a double wedge with angle $\leq \varepsilon$, such that X and Y are on different sides of the double wedge. The problem is that for a disk $\bigcirc$, it might be that the bridge edge between X and Y that is in $\mathcal{D}_\Xi \cap \bigcirc$ is much longer than the distance between the two points of interest. This somewhat counter-intuitive situation is illustrated in Figure 3.2.

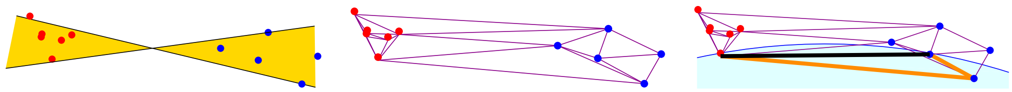

Figure 3.2 A bridge too far – the only surviving bridge between the red and blue points is too far to be useful if the sets of points are not well separated.

▶ **Lemma 20.** *For $\varepsilon = 1/4$, and parameters n and Φ, there is a point set P of $n + \lceil \log_2 \Phi \rceil$ points in the plane, with spread $\mathcal{O}(n\Phi)$, such that any local $(1+\varepsilon)$-spanner of P for disks must have $\Omega\big(n(1 + \log \frac{\Phi}{n})\big)$ edges, as long as $\sqrt{n} \leq \Phi \leq n2^n$.*

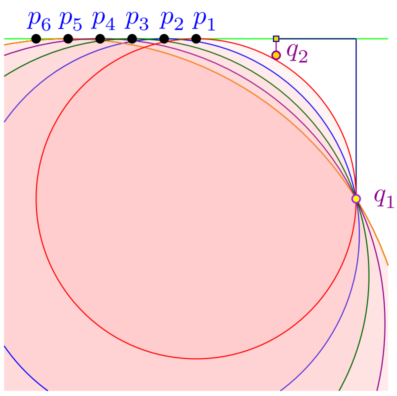

Figure 3.3 The set of disks D_1, and the construction of q_2.

Proof. Let $p_i = (-i, 0)$, for $i = 1, \ldots, n$. Let $M = 1 + \lceil \log_2 \Phi \rceil$, $x_1 = n2^M$ and $q_1 = (x_1, -1)$. For a point p on the x-axis, and a point q below the x-axis and to the right of p, let $\bigcirc_{\downarrow}^{p}(q)$ be the disk whose boundary passes through p and q, and its center has the same x-coordinate as p. In the jth iteration, for $j = 2, \ldots, M-1$, Let $x_j = n2^{M-j+1} = x_{j-1}/2$, and let $y_j < 0$ be the maximum y-coordinate of a point that lies on the intersection of the vertical line $x = x_j$ and the union of disks $D_1 \cup \cdots \cup D_j$ where

$$D_j = \left\{ \bigcirc_{\downarrow}^{p_i}(q_{j-1}) \;\middle|\; i = 1, \ldots, n \right\},$$

see Figure 3.3 for an illustration of D_1.

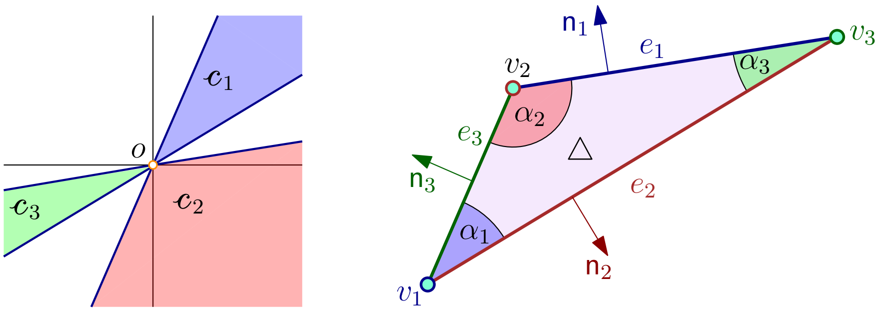

Figure 3.4 For the triangle $\triangle$ with angles α_1, α_2, and α_3 we create the cones c_1, c_2, and c_3.

Let $q_j = (x_j, 0.99 y_j)$.

Clearly, the point q_j lies outside all the disks of $D_1 \cup \ldots \cup D_j$. The construction now continues to the next value of j. Let $P = \{p_1, \ldots, p_n, q_2, \ldots, q_M\}$. We have that $|P| = n + M - 1$.

The minimum distance between any points in the construction is 1 (i.e., $\|p_1 p_2\|$). Indeed $x_{M-1} = 4n$ and thus $\|q_{M-1} p_1\| \geq 2n$. The diameter of P is $\|p_1 q_1\| = \sqrt{(n + n2^M)^2 + 1} \leq 2n2^M$. As such, the spread of P is bounded by $\leq n2^{M+1} = \mathcal{O}(n\Phi)$.

For any i and j, consider the disk $\bigcirc_{\downarrow}^{p_i}(q_j)$. This disk does not contain any point of $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n$ since its interior lies below the x-axis. By construction it does not contain any point $q_{j+1}, \ldots, q_{M-1}$. This disk potentially contains the points $q_{j-1}, \ldots, q_1$, but observe that for any index $k \in [\![j-1]\!] = \{1, \ldots, j-1\}$, we have that

$$\|p_i q_k\| = \sqrt{\left(i + n2^{M-k+1}\right)^2 + \left(y(q_j)\right)^2},$$

which implies that $n2^{M-k+1} \leq \|p_i q_k\| < n(2^{M-k+1} + 2)$. We thus have that

$$\frac{\|p_i q_k\|}{\|p_i q_j\|} \geq \frac{n2^{M-k+1}}{n(2^{M-j+1} + 2)} = \frac{2^{M-j} \cdot 2^{j-k}}{2^{M-j} + 1} = \frac{2^{j-k}}{1 + 1/2^{M-j}} \geq \frac{2}{1 + 1/2} = \frac{4}{3} > 1 + \varepsilon,$$

since $j \in [\![M-1]\!]$. Namely, the shortest path in G between p_i and q_j, cannot use any of the points $q_1, \ldots q_{j-1}$. As such, the graph G must contain the edge $p_i q_j$. This implies that $|E(G)| \geq n(M-1)$, which implies the claim. ◀

3.3.2 A lower bound for triangles

▶ **Lemma 21** (Proof in full version [5]). *For any $n > 0$, and $\Phi = \Omega(n)$, one can compute a set P of $n + \mathcal{O}(\log \Phi)$ points, with spread $\mathcal{O}(\Phi n)$, and a triangle $\triangle$, such that any $\triangle$-local $(3/2)$-spanner of P requires $\Omega\big(n \log(1 + \frac{\Phi}{n})\big)$ edges.*

3.4 Local spanners for fat triangles

While local spanners for homothets of an arbitrary convex body are costly, if we are given a triangle $\triangle$ with the single constraint that $\triangle$ is not too "thin", then one can construct a $\triangle$-local t-spanner with a number of edges that does not depend on the spread of the points.

▶ **Definition 22.** *A triangle $\triangle$ is α-**fat** if the smallest angle in $\triangle$ is at least α.*

3.4.1 Construction

The input is a set P of n points in the plane, an α-fat triangle $\triangle$, and an approximation parameter $\varepsilon \in (0, 1)$. Let v_i denote the ith vertex of $\triangle$, α_i be the adjacent angle, and let e_i denote the opposing edge, for $i \in [\![3]\!]$. Let $\mathcal{c}_i = \{(p - v_i)t \mid p \in e_i \text{ and } t \geq 0\}$ denote the cone with an apex at the origin induced by the ith vertex of $\triangle$. Let n_i be the outer normal of $\triangle$ orthogonal to e_i. See Figure 3.4 for an illustration. Let $\mathcal{C}_i$ be a minimum size partition of $\mathcal{c}_i$ into cones each with angle in the range $[\beta/2, \beta]$, where $\beta = \varepsilon\alpha/\gamma$, and $\gamma > 1$ is some constant discussed shortly. For each point $p \in P$, and a cone $\mathcal{c} \in \mathcal{C}_i$, let $\mathsf{nn}_i(p, \mathcal{c})$ be the first point in $(P - p) \cap (p + \mathcal{c})$ ordered by the direction n_i (it is the "nearest-neighbor" to p in $p + \mathcal{c}$ with respect to the direction n_i).

The result

Let G be the graph over P formed by connecting every point $p \in P$ to $\mathsf{nn}_i(p, \mathcal{c})$, for all $i \in [\![3]\!]$ and $\mathcal{c} \in \mathcal{C}_i$. We get the following result (see full version [5] for details).

▶ **Theorem 23.** *Let P be a set of n points in the plane, and let $\varepsilon \in (0, 1)$ be an approximation parameter. The above algorithm computes a $\triangle$-local $(1 + \varepsilon)$-spanner G for an α-fat triangle $\triangle$. The construction time is $\mathcal{O}\big((\alpha\varepsilon)^{-1} n \log n\big)$, and the spanner G has $\mathcal{O}\big((\alpha\varepsilon)^{-1} n\big)$ edges.*

3.5 A local spanner for nice polygons

3.5.1 A good jump for narrow trapezoids

As a reminder, a trapezoid is a quadrilateral with two parallel edges, known as its *bases*. The other two edges are its *legs*. For $\varepsilon \in (0, 1/4)$, a trapezoid T is **ε-narrow** if the length of each of its legs is at most $\varepsilon \cdot \mathrm{diam}(T)$.

▶ **Lemma 24** (Proof in full version [5]). *Let $\varepsilon \in (0, 1)$ be some parameter, and $\vartheta = \varepsilon/16$. Let X, Y be two point sets that are $(1/\vartheta)$-semi separated and ϑ-angularly separated (see Definition 7), and let T be a ϑ-narrow trapezoid, with two points $p \in X$ and $q \in Y$ lying on the two legs of T. Then, one can compute a homothet $T' \subseteq T$ of T such that*

1. *there are two points $p' \in X$ and $q' \in Y$, such that $p'q'$ is an edge of the T-Delaunay triangulation of $X \cup Y$, and*
2. *we have that $(1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon) \|pq\|$.*

3.5.2 Breaking a nice polygon into narrow trapezoids

For a convex polygon $\mathcal{C}$, its **sensitivity**, denoted by $\mathrm{sen}(\mathcal{C})$, is the minimum distance between any two non-adjacent edges (this quantity is no bigger than the length of the shortest edge in the polygon). A convex polygon $\mathcal{C}$ is **t-nice**, if the outer angle at any vertex of the polygon is at least $2\pi/t$, and the length of the longest edge of $\mathcal{C}$ is $\mathcal{O}(\mathrm{sen}(\mathcal{C}))$. As an example, a k-regular polygon is k-nice.

▶ **Lemma 25** (Proof in full version [5]). *Let t be a positive integer. Given a t-nice polygon $\mathcal{C}$, and a parameter ϑ, one can cover it by a set $\mathcal{T}$ of $\mathcal{O}(t^4/\vartheta^3)$ ϑ-narrow trapezoids, such that for any two points $p, q \in \partial\mathcal{C}$ that belong to two edges of $\mathcal{C}$ that are not adjacent, there exists a narrow trapezoid $T \in \mathcal{T}$, such that p and q are located on two different short legs of T.*

3.5.3 Constructing the local spanner for nice polygons

▶ **Theorem 26** (Proof in full version [5]). *Let $\mathcal{C}$ be a k-nice convex polygon, P be a set of n points in the plane, and let $\varepsilon \in (0,1)$ be a parameter. Then, one can construct a $\mathcal{C}$-local $(1+\varepsilon)$-spanner of P. The construction time is $\mathcal{O}\big((k^4/\varepsilon^6)n\log^2 n\big)$, and the resulting graph has $\mathcal{O}\big((k^4/\varepsilon^6)n\log n\big)$ edges. In particular these bounds hold if $\mathcal{C}$ is a k-regular polygon.*

▶ Remark 27. For axis-parallel squares Theorem 26 implies a local spanner with $\mathcal{O}\big(\varepsilon^{-6}n\log n\big)$ edges. However, for this special case, the decomposition into narrow trapezoid can be skipped. In particular, in this case, the resulting spanner has $\mathcal{O}(\varepsilon^{-3}n\log n)$ edges. We do not provide the details here, as it is only a minor improvement over the above, and requires quite a bit of additional work – essentially, one has to prove a version of Lemma 24 for squares. We leave the question of whether this bound can be further improved as an open problem for further research.

4 Weak local spanners for axis-parallel rectangles

4.1 Orthant separated pair decomposition

For the purpose of building the spanners in this section, we use a variation of a pair decomposition introduced by Agarwal et al. [4]. For two points $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$ in $\mathbb{R}^d$, let $p \prec q$ denote that q **dominates** p coordinate-wise. That is $p_i < q_i$, for all i. More generally, let $p <_i q$ denote that $p_i < q_i$. For two point sets $X, Y \subseteq \mathbb{R}^d$, we use $X <_i Y$ to denote that $x <_i y \quad \forall x \in X, y \in Y$. In particular X and Y are *i-coordinate separated* if $X <_i Y$ or $Y <_i X$. A pair $\{X, Y\}$ is **orthant-separated**, if X and Y are i-coordinate separated, for all $i = 1, \ldots, d$.

A **orthant-separated pair decomposition** of a point set $P \subseteq \mathbb{R}^d$, is a pair decomposition (see Definition 2) $\mathcal{W} = \big\{\{X_1, Y_1\}, \ldots, \{X_s, Y_s\}\big\}$ of P such that $\{X_i, Y_i\}$ are orthant-separated for all i.

In the full version of the paper [5], we prove the properties regarding the computational and combinatorial complexity of OSPDs that are used in the proof of Theorem 31

4.2 Weak local spanner for axis-parallel rectangles

For a parameter $\delta \in (0,1)$, and an interval $I = [b, c]$, let $(1-\delta)I = [t - (1-\delta)r, t + (1-\delta)r]$, where $t = (b+c)/2$, and $r = (c-b)/2$, be the shrinking of I by a factor of $1-\delta$.

Let $\mathcal{R}$ be the set of all axis-parallel rectangles in the plane. For a rectangle $R \in \mathcal{R}$ with $R = I \times J$, let $(1-\delta)R = (1-\delta)I \times (1-\delta)J$ denote the rectangle resulting from shrinking R by a factor of $1-\delta$.

▶ **Definition 28.** *Given a set P of n points in the plane, and parameters $\varepsilon, \delta \in (0,1)$, a graph G is a* $(1-\delta)$-local $(1+\varepsilon)$-spanner *for rectangles, if for any axis-parallel rectangle R, we have that $G \cap R$ is a $(1+\varepsilon)$-spanner for all the points in $\big((1-\delta)R\big) \cap P$.*

Observe that rectangles in $\mathcal{R}$ might be quite "skinny", so the previous notion of shrinkage used before is not useful in this case.

4.2.1 Construction for a single orthant separated pair

Consider a pair $\Xi = \{X, Y\}$ in a OSPD of P. The set X is orthant-separated from Y, that is, there is a point c_Ξ such that X and Y are contained in two opposing orthants in the partition of the plane formed by the vertical and horizontal lines through c_Ξ.

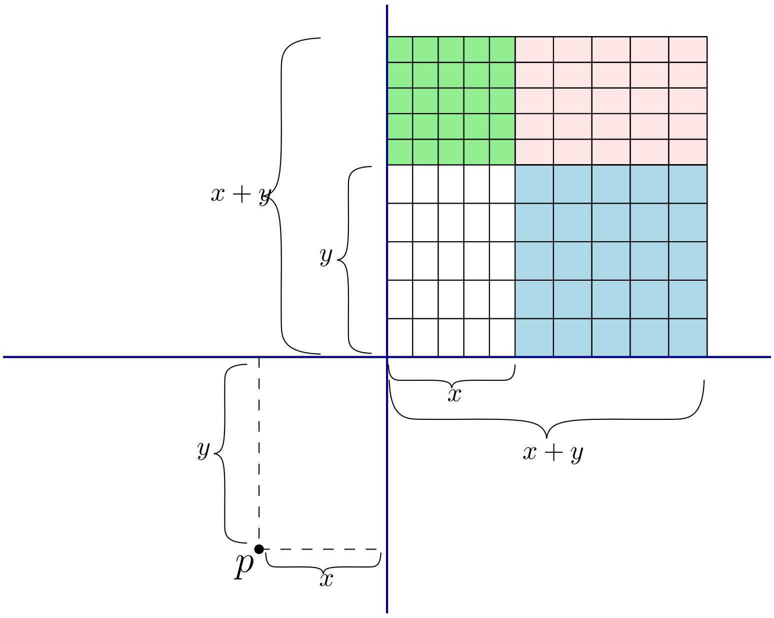

Figure 4.1 The construction of the grid $\mathsf{K}(p, \Xi)$ for a point $p = (-x, -y)$ and a pair Ξ.

For simplicity of exposition, assume that $c_\Xi = (0,0)$, and $X \prec (0,0) \prec Y$. That is, the points of X are in the negative orthant, and the points of Y are in the positive orthant.

For a point $p = (-x, -y) \in X$ we construct a non-uniform grid $\mathsf{K}(p, \Xi)$ in the square $[0, x+y]^2$. To this end, we first partition it into four subrectangles

$B_{\nwarrow} = [0, x] \times [y, x+y]$	$B_{\nearrow} = [x, x+y] \times [y, x+y]$
$B_{\swarrow} = [0, x] \times [0, y]$	$B_{\searrow} = [x, x+y] \times [0, y]$.

Let $\tau \geq 4/\varepsilon + 4/\delta$ be an integer number. We partition each of these rectangles into a $\tau \times \tau$ grid, where each cell is a copy of the rectangle scaled by a factor of $1/\tau$. See Figure 4.1. This grid has $\mathcal{O}(\tau^2)$ cells. For a cell C in this grid, let $Y \cap \mathsf{C}$ be the points of Y contained in it. We connect p to the left-most and bottom-most points in $Y \cap \mathsf{C}$. This process generates two edges in the constructed graph for each grid cell (that contains at least two points), and $\mathcal{O}(\tau^2)$ edges overall.

The algorithm repeats this construction for all the points $p \in X$, and does the symmetric construction for all the points of Y.

4.2.2 The spanner construction algorithm

The algorithm computes a OSPD $\mathcal{W}$ of P. For each pair $\Xi \in \mathcal{W}$, the algorithm generates edges for Ξ using the algorithm of Section 4.2.1 and adds them to the generated spanner G.

4.2.3 Correctness

For a rectangle R, let $\overleftrightarrow{R} = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \exists x' \in \mathbb{R} \text{ such that } (x', y) \in R\right\}$ be its expansion into a horizontal slab. Restricted to a rectangle R', the resulting set is $\overleftrightarrow{R} \cap R'$, depicted in Figure 4.2. Similarly, we denote

$$\updownarrow R = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \exists y' \in \mathbb{R} \text{ such that } (x, y') \in R\right\}.$$

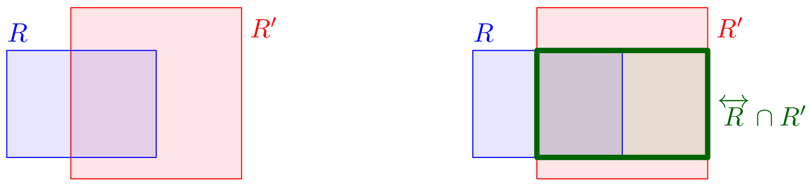

Figure 4.2 Left: The two rectangles R, R'. Right: In green $\overleftrightarrow{R} \cap R'$, the restriction of the slab $\overleftrightarrow{R}$ to the rectangle R'.

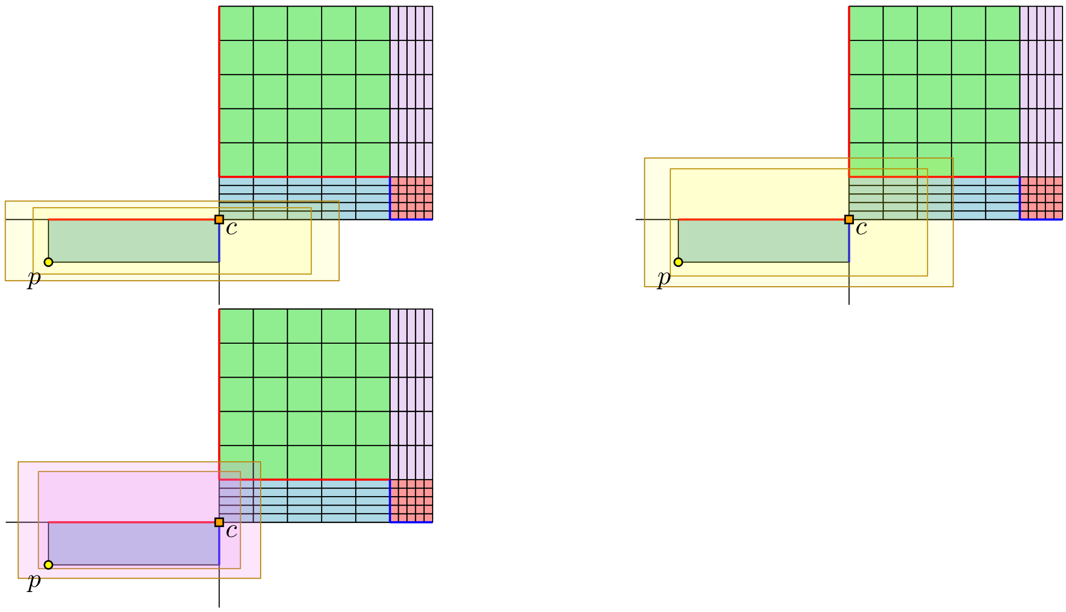

Figure 4.3 An illustration of $\mathsf{K}(p, \Xi)$ with three rectangles and their shrunken version.

▶ **Lemma 29** (Proof in full version [5]). *Assume that $\delta < 1/2$, and $\tau \geq \lceil 20/\varepsilon + 20/\delta \rceil$. Consider a pair $\Xi = \{X, Y\}$ in the above construction, and a point $p = (-x, -y) \in X$ with its associated grid $\mathsf{K} = \mathsf{K}(p, \Xi)$. Consider any axis-parallel rectangle R, such that $p \in (1-\delta)R = I \times J$, and $(1-\delta)R$ intersects a cell $\mathsf{C} \in \mathsf{K}$. We have the following.*

1. *If $\mathsf{C} \subseteq (1-\delta)R$ then $(1-\delta)^{-1}\mathsf{C} \subseteq R$.*
2. $\mathrm{diam}(\mathsf{C}) \leq (\varepsilon/4)\mathsf{d}(p, \mathsf{C})$.
3. *If $x \geq y$ and $\mathsf{C} \subseteq R_{\swarrow} \cup R_{\searrow}$ then $(1-\delta)^{-1}\mathsf{C} \subseteq R$.*
4. *If $x \leq y$ and $\mathsf{C} \subseteq R_{\swarrow} \cup R_{\nwarrow}$ then $(1-\delta)^{-1}\mathsf{C} \subseteq R$.*
5. *If $x \geq y$ and $\mathsf{C} \subseteq R_{\nwarrow}$, then $(1-\delta)^{-1}\big(\overleftrightarrow{(1-\delta)R} \cap \mathsf{C}\big) \subseteq R$.*
6. *If $x \leq y$ and $\mathsf{C} \subseteq R_{\searrow}$, then $(1-\delta)^{-1}\Big(\updownarrow\big((1-\delta)R\big) \cap \mathsf{C}\Big) \subseteq R$.*

▶ **Lemma 30** (Proof in full version [5]). *For any axis-parallel rectangle R, and any two points $p, q \in (1-\delta)R \cap P$, there exists a $(1+\varepsilon)$-path between p and q in G.*

▶ **Theorem 31** (Proof in full version [5]). *Let P be a set of n points in the plane, and let $\varepsilon, \delta \in (0,1)$ be parameters. The above algorithm constructs, in $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n \log^2 n)$ time, a graph G with $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n \log^2 n)$ edges. The graph G is a $(1-\delta)$-local $(1+\varepsilon)$-spanner for axis-parallel rectangles. Formally, for any axis-parallel rectangle R, we have that $R \cap P$ is an $(1+\varepsilon)$-spanner for all the points of $\big((1-\delta)R\big) \cap P$.*

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