



# Stability in Graphs with Matroid Constraints

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## Abstract

We study the following INDEPENDENT STABLE SET problem. Let  $G$  be an undirected graph and  $\mathcal{M} = (V(G), \mathcal{I})$  be a matroid whose elements are the vertices of  $G$ . For an integer  $k \geq 1$ , the task is to decide whether  $G$  contains a set  $S \subseteq V(G)$  of size at least  $k$  which is independent (stable) in  $G$  and independent in  $\mathcal{M}$ . This problem generalizes several well-studied algorithmic problems, including RAINBOW INDEPENDENT SET, RAINBOW MATCHING, and BIPARTITE MATCHING WITH SEPARATION. We show that

- When the matroid  $\mathcal{M}$  is represented by the independence oracle, then for any computable function  $f$ , no algorithm can solve INDEPENDENT STABLE SET using  $f(k) \cdot n^{o(k)}$  calls to the oracle.
- On the other hand, when the graph  $G$  is of degeneracy  $d$ , then the problem is solvable in time  $\mathcal{O}((d+1)^k \cdot n)$ , and hence is FPT parameterized by  $d+k$ . Moreover, when the degeneracy  $d$  is a constant (which is not a part of the input), the problem admits a kernel polynomial in  $k$ . More precisely, we prove that for every integer  $d \geq 0$ , the problem admits a kernelization algorithm that in time  $n^{\mathcal{O}(d)}$  outputs an equivalent framework with a graph on  $dk^{\mathcal{O}(d)}$  vertices. A lower bound complements this when  $d$  is part of the input: INDEPENDENT STABLE SET does not admit a polynomial kernel when parameterized by  $k+d$  unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$ . This lower bound holds even when  $\mathcal{M}$  is a partition matroid.
- Another set of results concerns the scenario when the graph  $G$  is chordal. In this case, our computational lower bound excludes an FPT algorithm when the input matroid is given by its independence oracle. However, we demonstrate that INDEPENDENT STABLE SET can be solved in  $2^{\mathcal{O}(k)} \cdot \|\mathcal{M}\|^{\mathcal{O}(1)}$  time when  $\mathcal{M}$  is a linear matroid given by its representation. In the same setting, INDEPENDENT STABLE SET does not have a polynomial kernel when parameterized by  $k$  unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$ .

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## 1 Introduction

We initiate the algorithmic study of computing stable (independent) sets in frameworks. The term *framework*, also known as *pregeometric graph* [28, 29], refers to a pair  $(G, \mathcal{M})$ , where  $G$  is a graph and  $\mathcal{M} = (V(G), \mathcal{I})$  is a matroid on the vertex set of  $G$ . We remind the reader that pairwise nonadjacent vertices of a graph form a *stable* or *independent* set. To avoid confusion with independence in matroids, we consistently use the term “stable set” throughout the paper. Whenever we mention independence, it is in reference to independence with respect to a matroid. We consider the following problem.

### INDEPENDENT STABLE SET

*Input:* A framework  $(G, \mathcal{M})$  and an integer  $k \geq 0$ .  
*Task:* Decide whether  $G$  has vertex set  $S \subseteq V(G)$  of size at least  $k$  that is stable in  $G$  and independent in  $\mathcal{M}$ .

The INDEPENDENT STABLE SET problem encompasses several well-studied problems related to stable sets.

When  $\mathcal{M}$  is a uniform matroid with every  $k$ -element subset of  $V(G)$  forming a basis, the INDEPENDENT STABLE SET problem seeks to determine whether a graph  $G$  contains a stable set of size at least  $k$ . This is the classic STABLE SET problem.

For a partition matroid  $\mathcal{M}$  whose elements are partitioned into  $k$  blocks and independent sets containing at most one element from each block, INDEPENDENT STABLE SET transforms into the rainbow-independence (or RAINBOW-STABLE SET) problem. To express this problem in graph terminology, consider a graph  $G$  with a vertex set  $V(G)$  colored in  $k$  colors. A set of vertices  $S$  is termed *rainbow-independent* if it is stable in  $G$  and no color occurs in  $S$  more than once [3, 25]. This concept is also known in the literature as an *independent transversal* [19, 18, 23] and an independent system of representatives [2].

Rainbow-independence generalizes the well-studied combinatorial concept of rainbow matchings [1, 13]. (Note that a matching in a graph is a stable set in its line graph.) It also has a long history of algorithmic studies. In the RAINBOW MATCHING problem, we are given a graph  $G$ , whose edges are colored in  $q$  colors, and a positive integer  $k$ . The task is to decide whether a matching of size at least  $k$  exists whose edges are colored in distinct colors. Itai, Rodeh, and Tanimoto in [24] established that RAINBOW MATCHING is NP-complete on bipartite graphs. Le and Pfender [26] strongly enhanced this result by showing that RAINBOW MATCHING is NP-complete even on paths and complete graphs. Gupta et al. [21] considered the parameterized complexity of RAINBOW MATCHING. They gave an FPT algorithm of running time  $2^k \cdot n^{\mathcal{O}(1)}$ . They also provided a kernel with  $\mathcal{O}(k^2 \Delta)$  vertices, where  $\Delta$  is the maximum degree of a graph. Later, in [22], the same set of authors obtained a kernel with  $\mathcal{O}(k^2)$  vertices for RAINBOW MATCHING on general graphs.

When  $\mathcal{M}$  is a transversal matroid, INDEPENDENT STABLE SET transforms into the BIPARTITE MATCHING WITH SEPARATION problem [30]. In this variant of the maximum matching problem, the goal is to determine whether a bipartite graph  $H$  contains a matching of size  $k$  with a separation constraint: the vertices on one side lie on a path (or a grid), and two adjacent vertices on a path (or a grid) are not allowed to be matched simultaneously. This problem corresponds to INDEPENDENT STABLE SET on a framework  $(G, \mathcal{M})$ , where  $G$  is a path (or a grid) on vertices  $U$ , and  $\mathcal{M}$  is a transversal matroid of the bipartite graph  $H = (U, W, E_H)$  whose elements are  $U$ , and the independent subsets are sets of endpoints of matchings of  $H$ . Manurangsi, Segal-Halevi, and Suksompong in [30] proved that BIPARTITE MATCHING WITH SEPARATION is NP-complete and provided approximation algorithms.

STABLE SET is a notoriously difficult computational problem. It is well-known to be NP-complete and W[1]-complete when parameterized by  $k$  [9]. On the other hand, STABLE SET is solvable in polynomial time on perfect graphs [20]. When it comes to parameterized algorithms and kernelization, STABLE SET is known to be FPT and to admit polynomial (in  $k$ ) kernel on classes of sparse graphs, like graphs of bounded degree or degeneracy [10]. The natural question is which algorithmic results about the stable set problem could be extended to INDEPENDENT STABLE SET.

- We commence with a lower bound on INDEPENDENT STABLE SET. Theorem 2 establishes that when the matroid in a framework is represented by the independence oracle, for any computable function  $f$ , no algorithm can solve INDEPENDENT STABLE SET using  $f(k) \cdot n^{o(k)}$  calls to the oracle. Moreover, we show that the lower bound holds for frameworks with bipartite, chordal, claw-free graphs, and AT-free graphs for which the classical STABLE SET problem can be solved in polynomial time. While the usual bounds in parameterized complexity are based on the assumption  $\text{FPT} \neq \text{W}[1]$ , Theorem 2 rules out the existence of an FPT algorithm for INDEPENDENT STABLE SET parameterized by  $k$  unconditionally.

- In Section 4, we delve into the parameterized complexity of INDEPENDENT STABLE SET when dealing with frameworks on  $d$ -degenerate graphs. The first result of this section, Theorem 3, demonstrates that the problem is FPT when parameterized by  $d + k$ , by providing an algorithm of running time  $\mathcal{O}((d + 1)^k \cdot n)$ . Addressing the kernelization aspect, Theorem 5 reveals that when  $d$  is a constant, INDEPENDENT STABLE SET on frameworks with graphs of degeneracy at most  $d$ , admits a kernel polynomial in  $k$ . More precisely, we prove that for every integer  $d \geq 0$ , the problem admits a kernelization algorithm that in time  $n^{\mathcal{O}(d)}$  outputs an equivalent framework with a graph on  $dk^{\mathcal{O}(d)}$  vertices. This is complemented by Theorem 6, establishing that INDEPENDENT STABLE SET on frameworks with  $d$ -degenerate graphs and partition matroids lacks a polynomial kernel when parameterized by  $k + d$  unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$ .

Shifting the focus to the stronger maximum vertex degree  $\Delta$  parameterization, Theorem 4 establishes improved kernelization bounds. Specifically, INDEPENDENT STABLE SET admits a polynomial kernel on frameworks that outputs an equivalent framework with a graph on at most  $k^2\Delta$  vertices.

- When it comes to perfect graphs, there is no hope of polynomial or even parameterized algorithms with parameter  $k$ : RAINBOW-STABLE SET is already known to be NP-complete and W[1]-complete when parameterized by  $k$  on bipartite graphs by the straightforward reduction from the dual MULTICOLORED BICLIQUE problem [9]. Also, the unconditional lower bound from Theorem 2 holds for bipartite and chordal graphs if the input matroids are given by the independence oracles.

Interestingly, it is still possible to design FPT algorithms for frameworks with chordal graphs when the input matroids are given by their representations. In Theorem 8, we show that INDEPENDENT STABLE SET can be solved in  $2^{\mathcal{O}(k)} \cdot \|A\|^{\mathcal{O}(1)}$  time by a one-sided error Monte Carlo algorithm with false negatives on frameworks with chordal graphs and linear matroids given by their representations  $A$ . When it concerns kernelization, Theorem 9 shows that INDEPENDENT STABLE SET on frameworks with chordal graphs and partition matroids does not admit a polynomial kernel when parameterized by  $k$  unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$ .

## 2 Preliminaries

In this section, we introduce the basic notation used throughout the paper and provide some auxiliary results.

**Graphs.** We use standard graph-theoretic terminology and refer to the textbook of Diestel [11] for missing notions. We consider only finite undirected graphs. For a graph  $G$ ,  $V(G)$  and  $E(G)$  are used to denote its vertex and edge sets, respectively. Throughout the paper, we use  $n$  to denote the number of vertices if it does not create confusion. For a graph  $G$  and a subset  $X \subseteq V(G)$  of vertices, we write  $G[X]$  to denote the subgraph of  $G$  induced by  $X$ . We denote by  $G - X$  the graph obtained from  $G$  by the deletion of every vertex of  $X$  (together with incident edges). For  $v \in V(G)$ , we use  $N_G(v)$  to denote the (*open*) *neighborhood* of  $v$ , that is, the set of vertices of  $G$  that are adjacent to  $v$ ;  $N_G[v] = N_G(v) \cup \{v\}$  is the *closed neighborhood* of  $v$ . For a set of vertices  $X$ ,  $N_G(X) = (\bigcup_{v \in X} N_G(v)) \setminus X$  and  $N_G[X] = \bigcup_{v \in X} N_G[v]$ . We use  $d_G(v) = |N_G(v)|$  to denote the *degree* of  $v$ ;  $\delta(G)$  and  $\Delta(G)$  denote the minimum and maximum degree of a vertex in  $G$ , respectively. For a nonnegative integer  $d$ ,  $G$  is *d-degenerate* if for every subgraph  $H$  of  $G$ ,  $\delta(H) \leq d$ . Equivalently, a graph  $G$  is *d-degenerate* if there is an ordering  $v_1, \dots, v_n$  of the vertices of  $G$ , called *elimination ordering*, such that  $d_{G_i}(v_i) \leq d$  for every  $i \in \{1, \dots, n\}$  where  $G_i = G[\{v_i, \dots, v_n\}]$ . Given a *d-degenerate* graph  $G$ , the elimination ordering can be computed in linear time [32]. The *degeneracy* of  $G$  is the minimum  $d$  such that  $G$  is *d-degenerate*. We remind that a graph  $G$  is *bipartite* if its vertex set can be partitioned into two sets  $V_1$  and  $V_2$  in such a way that each edge has one endpoint in  $V_1$  and one endpoint in  $V_2$ . A graph  $G$  is *chordal* if it has no induced cycles on at least four vertices. A graph  $G$  is said to be *claw-free* if it does not contain the *claw* graph  $K_{1,3}$  as an induced subgraph. An independent set of three vertices such that each pair can be joined by a path avoiding the neighborhood of the third is called an *asteroidal triple* (AT). A graph is *AT-free* if it does not contain asteroidal triples.

**Matroids.** We refer to the textbook of Oxley [34] for the introduction to Matroid Theory. Here we only briefly introduce the most important notions.

► **Definition 1.** A pair  $\mathcal{M} = (V, \mathcal{I})$ , where  $V$  is a ground set and  $\mathcal{I}$  is a family of subsets, called independent sets of  $\mathcal{M}$ , is a matroid if it satisfies the following conditions, called independence axioms:

(11)  $\emptyset \in \mathcal{I}$ ,

(12) if  $X \subseteq Y$  and  $Y \in \mathcal{I}$  then  $X \in \mathcal{I}$ ,

(13) if  $X, Y \in \mathcal{I}$  and  $|X| < |Y|$ , then there is  $v \in Y \setminus X$  such that  $X \cup \{v\} \in \mathcal{I}$ .

We use  $V(\mathcal{M})$  and  $\mathcal{I}(\mathcal{M})$  to denote the ground set and the family of independent sets of  $\mathcal{M}$ , respectively, unless  $\mathcal{M}$  is clear from the context. An inclusion-maximal set of  $\mathcal{I}$  is called a *base*; it is well-known that all bases of  $\mathcal{M}$  have the same cardinality. A function  $r: 2^V \rightarrow \mathbb{Z}_{\geq 0}$  such that for every  $X \subseteq V$ ,

$$r(X) = \max\{|Y|: Y \subseteq X \text{ and } Y \in \mathcal{I}\}$$

is called the *rank function* of  $\mathcal{M}$ . The *rank* of  $\mathcal{M}$ , denoted  $r(\mathcal{M})$ , is  $r(V)$ ; equivalently, the rank of  $\mathcal{M}$  is the size of any base of  $\mathcal{M}$ . Let us remind that a set  $X \subseteq V$  is independent if and only if  $r(X) = |X|$ . The *closure* of a set  $X$  is the set  $\text{cl}(X) = \{v \in V: r(X \cup \{v\}) = r(X)\}$ . The matroid  $\mathcal{M}' = (V \setminus X, \mathcal{I}')$ , where  $\mathcal{I}' = \{Y \in \mathcal{I}: Y \subseteq V \setminus X\}$ , is said to be obtained from  $\mathcal{M}$  by the *deletion* of  $X$ . The *restriction* of  $\mathcal{M}$  to  $X \subseteq V$  is the matroid obtained from  $\mathcal{M}$  by the deletion of  $V \setminus X$ . If  $X$  is an independent set then the matroid  $\mathcal{M}'' = (V \setminus X, \mathcal{I}'')$ , where  $\mathcal{I}'' = \{Y \subseteq V \setminus X: Y \cup X \in \mathcal{I}\}$ , is the *contraction* of  $\mathcal{M}$  by  $X$ . For a positive integer  $k$ , the *k-truncation* of  $\mathcal{M} = (V, \mathcal{I})$  is the matroid  $\mathcal{M}'$  with the same ground set  $V$  such that  $X \subseteq V$  is independent in  $\mathcal{M}'$  if and only if  $X \in \mathcal{I}$  and  $|X| \leq k$ . Because in INDEPENDENT STABLE SET, we are interested only in independent sets of size at most  $k$ , we assume throughout our paper that the rank of the input matroids is upper bounded by  $k$ . Otherwise, we replace  $\mathcal{M}$  by its *k-truncation*.

In our paper, we assume in the majority of our algorithmic results that the input matroids in instances of INDEPENDENT STABLE SET are given by independence oracles. An *independence oracle* for  $\mathcal{M}$  takes as its input a set  $X \subseteq V$  and correctly returns either **yes** or **no** in unit time depending on whether  $X$  is independent or not. We assume that the memory used to store oracles does not contribute to the input size; this is important for our kernelization results. Notice that given an independence oracle, we can greedily construct an inclusion-maximal independent subset of  $X$  and this can be done in  $\mathcal{O}(|X|)$  time. Note also that the oracle for  $\mathcal{M}$  can be trivially transformed to an oracle for the  $k$ -truncation of  $\mathcal{M}$ .

Our computational lower bounds, except the unconditional bound in Theorem 2, are established for partition matroids. The *partition* matroid for a given partition  $\{V_1, \dots, V_\ell\}$  of  $V$  is the matroid with the ground set  $V$  such that a set  $X \subseteq V$  is independent if and only if  $|X \cap V_i| \leq 1$  for each  $i \in \{1, \dots, \ell\}$  (in a more general setting, it is required that  $|V \cap X_i| \leq d_i$  where  $d_1, \dots, d_\ell$  are some constant but we only consider the case  $d_1 = \dots = d_\ell = 1$ ).

Matroids also could be given by their representations. Let  $\mathcal{M} = (V, \mathcal{I})$  be a matroid and let  $\mathbb{F}$  be a field. An  $r \times n$ -matrix  $A$  is a *representation of  $\mathcal{M}$  over  $\mathbb{F}$*  if there is a bijective correspondence  $f$  between  $V$  and the set of columns of  $A$  such that for every  $X \subseteq V$ ,  $X \in \mathcal{I}$  if and only if the set of columns  $f(X)$  consists of linearly independent vectors of  $\mathbb{F}^r$ . Equivalently,  $A$  is a representation of  $\mathcal{M}$  if  $\mathcal{M}$  is isomorphic to the *column* matroid of  $A$ , that is, the matroid whose ground set is the set of columns of the matrix and the independence of a set of columns is defined as the linear independence. If  $\mathcal{M}$  has a such a representation, then  $\mathcal{M}$  is *representable* over  $\mathbb{F}$  and it is also said  $\mathcal{M}$  is a *linear* (or  $\mathbb{F}$ -*linear*) matroid.

**Parameterized Complexity.** We refer to the books of Cygan et al. [9] and Fomin et al. [16] for an introduction to the area. Here we only briefly mention the notions that are most important to state our results. A *parameterized problem* is a language  $L \subseteq \Sigma^* \times \mathbb{N}$  where  $\Sigma^*$  is a set of strings over a finite alphabet  $\Sigma$ . An input of a parameterized problem is a pair  $(x, k)$  where  $x$  is a string over  $\Sigma$  and  $k \in \mathbb{N}$  is a *parameter*. A parameterized problem is *fixed-parameter tractable* (or FPT) if it can be solved in time  $f(k) \cdot |x|^{\mathcal{O}(1)}$  for some computable function  $f$ . The complexity class FPT contains all fixed-parameter tractable parameterized problems. A *kernelization algorithm* or *kernel* for a parameterized problem  $L$  is a polynomial-time algorithm that takes as its input an instance  $(x, k)$  of  $L$  and returns an instance  $(x', k')$  of the same problem such that (i)  $(x, k) \in L$  if and only if  $(x', k') \in L$  and (ii)  $|x'| + k' \leq f(k)$  for some computable function  $f: \mathbb{N} \rightarrow \mathbb{N}$ . The function  $f$  is the *size* of the kernel; a kernel is *polynomial* if  $f$  is a polynomial. While every decidable parameterized problem is FPT if and only if the problem admits a kernel, it is unlikely that all FPT problems have polynomial kernels. In particular, the *cross-composition* technique proposed by Bodlaender, Jansen, and Kratsch [5] could be used to prove that a certain parameterized problem does not admit a polynomial kernel unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$ .

We conclude the section by defining RAINBOW-STABLE SET.

#### RAINBOW-STABLE SET

*Input:* A graph  $G$  and a partition  $\{V_1, \dots, V_k\}$  of  $V(G)$  into  $k$  sets, called color classes.

*Task:* Decide whether  $G$  has a stable set  $S$  of size  $k$  such that  $|S \cap V_i| = 1$  for each  $i \in \{1, \dots, k\}$ .

As mentioned, RAINBOW-STABLE SET is a special case of INDEPENDENT STABLE SET for partition matroids where  $k$  is the number of subsets in the partition defining the input matroid.

### 3 Unconditional computational lower bound

Because INDEPENDENT STABLE SET generalizes the classical STABLE SET problem, INDEPENDENT STABLE SET is NP-complete [17] and W[1]-complete [12]. However, when the input matroids are given by their independence oracles, we obtain an unconditional computational lower bound. Moreover, we show that the lower bound holds for several graph classes for which the classical STABLE SET problem can be solved in polynomial time. For this, we remind that STABLE SET is polynomial on claw-free and AT-free graphs by the results of Minty [33] and Broersma et al. [8], respectively.

► **Theorem 2.** *There is no algorithm solving INDEPENDENT STABLE SET for frameworks with matroids represented by the independence oracles using  $f(k) \cdot n^{o(k)}$  oracle calls for any computable function  $f$ . Furthermore, the bound holds for bipartite, chordal, claw-free, and AT-free graphs.*

**Proof.** First, we show the bound for claw-free and AT-free and then explain how to modify the proof for other graph classes.

Let  $p$  and  $q$  be positive integers. We define the graph  $G_{p,q}$  as the disjoint union of  $G_i$  constructed as follows for each  $i \in \{1, \dots, p\}$ .

- For each  $j \in \{1, \dots, q\}$ , construct two vertices  $a_{i,j}$  and  $b_{i,j}$ ; set  $A_i = \{a_{i,1}, \dots, a_{i,q}\}$  and  $B_i = \{b_{i,1}, \dots, b_{i,q}\}$ .
- Make  $A_i$  and  $B_i$  cliques.
- For each  $j \in \{1, \dots, q\}$  and for all distinct  $h, j \in \{1, \dots, q\}$ , make  $a_{i,h}$  and  $b_{i,j}$  adjacent. Equivalently, each  $G_i$  is obtained by deleting a perfect matching from the complete graph  $K_{2q}$ . By the construction,  $G_{p,q}$  is both claw-free and AT-free and has  $2pq$  vertices. Consider a family of indices  $j_1, \dots, j_p \in \{1, \dots, q\}$  and set  $W = \bigcup_{i=1}^p \{a_{i,j_i}, b_{i,j_i}\}$ . We define the matroid  $\mathcal{M}_W$  with the ground set  $V(G_{p,q})$  as follows for  $k = 2p$ :
- Each set  $X \subseteq V(G_{p,q})$  of size at most  $k - 1$  is independent and any set of size at least  $k + 1$  is not independent.
- A set  $X \subseteq V(G_{p,q})$  of size  $k$  is independent if and only if either  $X = W$  or there is  $i \in \{1, \dots, p\}$  such that  $|A_i \cap X| \geq 2$  or  $|B_i \cap X| \geq 2$  or there are distinct  $h, j \in \{1, \dots, q\}$  such that  $a_{i,h}, b_{i,j} \in X$ .

Denote by  $\mathcal{I}_W$  the constructed family of independent sets. We will now show that  $\mathcal{M}_W$  is indeed a matroid.

► **Claim 2.1.**  $\mathcal{M}_W = (V(G_{p,q}), \mathcal{I}_W)$  is a matroid.

Proof of Claim 2.1. We have to verify that  $\mathcal{I}_W$  satisfies the independence axioms (I1)–(I3). The axioms (I1) and (I2) for  $\mathcal{I}_W$  follow directly from the definition of  $\mathcal{I}_W$ . To establish (I3), consider arbitrary  $X, Y \in \mathcal{I}_W$  such that  $|X| < |Y|$ . If  $|X| < k - 1$  then for any  $v \in Y \setminus X$ ,  $Z = X \cup \{v\} \in \mathcal{I}_W$  because  $|Z| \leq k - 1$ .

Suppose  $|X| = k - 1$  and  $|Y| = k$ . If there is  $i \in \{1, \dots, p\}$  such that  $|A_i \cap X| \geq 2$  or  $|B_i \cap X| \geq 2$  or there are distinct  $h, j \in \{1, \dots, q\}$  such that  $a_{i,h}, b_{i,j} \in X$  then for any  $v \in Y \setminus X$ , the set  $Z = X \cup \{v\}$  has the same property and, therefore,  $Z \in \mathcal{I}_W$ . Assume that this is not the case. By the construction of  $G_{p,q}$ , we have that for each  $i \in \{1, \dots, p\}$ ,  $|X \cap A_i| \leq 1$  and  $|X \cap B_i| \leq 1$ , and, furthermore, there is  $j \in \{1, \dots, q\}$  such that  $X \cap (A_i \cup B_i) \subseteq \{a_{i,j}, b_{i,j}\}$ . Because  $|X| = k - 1$ , we can assume without loss of generality that there are indices  $h_1, \dots, h_p \in \{1, \dots, q\}$  such that  $X \cap (A_i \cup B_i) = \{a_{i,h_i}, b_{i,h_i}\}$  for  $i \in \{1, \dots, p - 1\}$  and  $X \cap (A_p \cup B_p) = \{a_{p,h_p}\}$ . Recall that  $W = \bigcup_{i=1}^p \{a_{i,j_i}, b_{i,j_i}\}$  for  $j_1, \dots, j_p \in \{1, \dots, q\}$ . If there is  $v \in Y \setminus X$  such that  $v \neq b_{p,j_p}$  then consider  $Z = X \cup \{v\}$ .



We have that there is  $i \in \{1, \dots, p\}$  such that  $|A_i \cap Z| \geq 2$  or  $|B_i \cap Z| \geq 2$  or there are distinct  $h, j \in \{1, \dots, q\}$  such that  $a_{i,h}, b_{i,j} \in Z$ , that is,  $Z \in \mathcal{I}_W$ . Now we assume that  $Y \setminus X = \{b_{p,j_p}\}$ . Then  $Y = W$  and we can take  $v = b_{p,j_p}$ . We obtain that  $X \cup \{v\} = Y \in \mathcal{I}_W$ . This concludes the proof.  $\triangleleft$

We show the following lower bound for the number of oracle queries for frameworks  $(G_{p,q}, \mathcal{M}_W)$ .

► **Claim 2.2.** *Solving INDEPENDENT STABLE SET for instances  $(G_{p,q}, \mathcal{M}_W, k)$  with the matroids  $\mathcal{M}_W$  defined by the independence oracle for an (unknown) stable set  $W$  of  $G_{p,q}$  of size  $k$  demands at least  $q^p - 1$  oracle queries.*

Proof of Claim 2.2. Notice that every stable set of  $X$  of size  $k$  contains exactly two vertices of each  $G_i$  and, moreover, there is  $j \in \{1, \dots, q\}$  such that  $X \cap V(G_i) = \{a_{i,j}, b_{i,j}\}$ . Because the only stable set of this structure that is independent with respect to  $\mathcal{M}_W$  is  $W$ , the task of INDEPENDENT STABLE SET boils down to finding an unknown stable set  $W$  of  $G_{p,q}$  of size  $k$  using oracle queries. Querying the oracle for sets  $X$  of size at most  $k - 1$  or at least  $k + 1$  does not provide any information about  $W$ . Also, querying the oracle for  $X$  of size  $k$  with the property that there is  $i \in \{1, \dots, p\}$  such that  $|A_i \cap X| \geq 2$  or  $|B_i \cap X| \geq 2$  or there are distinct  $h, j \in \{1, \dots, q\}$  such that  $a_{i,h}, b_{i,j} \in X$  also does not give any information because all these are independent. Hence, we can assume that the oracle is queried only for sets  $X$  of size  $k$  with the property that for each  $i \in \{1, \dots, p\}$ , there is  $j \in \{1, \dots, q\}$  such that  $X \cap V(G_i) = \{a_{i,j}, b_{i,j}\}$ , that is, the oracle is queried for stable sets of size  $k$ . The graph  $G_{p,q}$  has  $q^p$  such sets. Suppose that the oracle is queried for at most  $q^p - 2$  stable sets of size  $k$  with the answer no. Then there are two distinct stable sets  $W$  and  $W'$  of size  $k$  such that the oracle was queried neither for  $W$  nor  $W'$ . The previous queries do not help to distinguish between  $W$  and  $W'$ . Hence, at least one more query is needed. This proves the claim.  $\triangleleft$

Now, we are ready to prove the claim of the theorem. Suppose that there is an algorithm  $\mathcal{A}$  solving INDEPENDENT STABLE SET with at most  $f(k) \cdot n^{g(k)}$  oracle calls for computable functions  $f$  and  $g$  such that  $g(k) = o(k)$ . Without loss of generality, we assume that  $f$  and  $g$  are monotone non-decreasing functions. Because  $g(k) = o(k)$ , there is a positive integer  $K$  such that  $g(k) < k/2$  for all  $k \geq K$ . Then for each  $k \geq K$ , there is a positive integer  $N_k$  such that for every  $n \geq N_k$ ,  $(f(k) \cdot n^{g(k)} + 1)k^{k/2} < n^{k/2}$ .

Consider instances  $(G_{p,q}, \mathcal{M}_W, k)$  for even  $k \geq K$  where  $p = k/2$  and  $q \geq N_k/k$ . We have that  $k = 2p$  and  $n = 2pq$ . Then  $\mathcal{A}$  applied to such instances would use at most  $f(k) \cdot n^{g(k)} < \left(\frac{n}{k}\right)^{k/2} - 1 = q^p - 1$  oracle queries contradicting Claim 2.2. This completes the proof for claw and AT-free graphs.

Now we sketch the proof of Theorem 2 for bipartite graphs. For positive integers  $p$  and  $q$ , we define  $H_{p,q}$  as the disjoint union of the graphs  $H_i$  for  $i \in \{1, \dots, p\}$  constructed as follows.

- For each  $j \in \{1, \dots, q\}$ , construct three vertices  $a_{i,j}$ ,  $b_{i,j}$ , and  $c_{i,j}$ ; set  $A_i = \{a_{i,1}, \dots, a_{i,q}\}$ ,  $B_i = \{b_{i,1}, \dots, b_{i,q}\}$ , and  $C_i = \{c_{i,1}, \dots, c_{i,q}\}$ .
- For each  $j \in \{1, \dots, q\}$ , make  $a_{i,j}$  and  $b_{i,j}$  adjacent to every  $c_{i,h}$  for  $h \in \{1, \dots, q\}$  such that  $h \neq j$ .

Notice that  $H_{p,q}$  is a bipartite graph with  $3pq$  vertices. We define  $R = \bigcup_{i=1}^p (A_i \cup B_i)$ . Consider a family of indices  $j_1, \dots, j_p \in \{1, \dots, q\}$  and set  $W = \bigcup_{i=1}^p \{a_{i,j_i}, b_{i,j_i}\}$ . Note that  $W$  is a stable set of  $H_{p,q}$  of size  $2p$ . We define the matroid  $\mathcal{M}_W$  with the ground set  $V(H_{p,q})$  by setting a set  $X \subseteq V(H_{p,q})$  to be independent if and only if

- for each  $i \in \{1, \dots, p\}$ ,  $|C_i \cap X| \leq 1$  and
- it holds that

- either  $X \cap R = W$ ,
- or  $|X \cap R| < 2p$ ,
- or  $|X \cap R| = 2p$  and there is  $i \in \{1, \dots, p\}$  such that  $|A_i \cap X| \geq 2$  or  $|B_i \cap X| \geq 2$  or there are distinct  $h, j \in \{1, \dots, q\}$  such that  $a_{i,h}, b_{i,j} \in X$ .

We denote by  $\mathcal{I}_W$  the constructed family of independent sets and prove that  $\mathcal{M}_W$  is a matroid.

► **Claim 2.3.**  $\mathcal{M}_W = (V(H_{p,q}), \mathcal{I}_W)$  is a matroid.

Proof of Claim 2.3. Let  $S = \bigcup_{i=1}^p C_i$  and consider  $\mathcal{M}_1 = (S, \mathcal{I}_1)$  where  $\mathcal{I}_1$  is the set of all  $X \subseteq S$  such that  $|X \cap C_i| \leq 1$  for  $i \in \{1, \dots, p\}$ . Clearly,  $\mathcal{M}_1$  is a partition matroid. Now consider  $\mathcal{M}_2 = (R, \mathcal{I}_2)$  where  $\mathcal{I}_2$  consists of sets  $X \subseteq D$  such that either  $X = W$ , or  $|X| < 2p$ , or  $|X| = 2p$  and there is  $i \in \{1, \dots, p\}$  such that  $|A_i \cap X| \geq 2$  or  $|B_i \cap X| \geq 2$  or there are distinct  $h, j \in \{1, \dots, q\}$  such that  $a_{i,h}, b_{i,j} \in X$ . We observe that  $\mathcal{M}_2$  is a matroid and the proof of this claim is identical to the proof of Claim 2.1. To complete the proof, it remains to note that  $\mathcal{M}_W = \mathcal{M}_1 \cup \mathcal{M}_2$ , that is, a set  $X \in \mathcal{I}_W$  if and only if  $X = Y \cup Z$  for  $Y \in \mathcal{I}_1$  and  $Z \in \mathcal{I}_2$ . This implies that  $\mathcal{M}_W$  is a matroid [34]. ◀

We consider instances  $(H_{p,q}, \mathcal{M}_W, k)$  of INDEPENDENT STABLE SET with the matroid  $\mathcal{M}_W$  defined by the independence oracle for an (unknown)  $W$  and  $k = 3p$ . By the definition of  $\mathcal{M}_W$ , any stable set  $X$  of  $H_{p,q}$  of size  $k$  that is independent with respect to  $\mathcal{M}_W$  has the property that  $|X \cap C_i| = 1$  for every  $i \in \{1, \dots, p\}$ . The construction of  $H_{p,q}$  implies that if  $c_{i,j} \in X \cap C_i$  then  $X \cap (A_i \cup B_i) \subseteq \{a_{i,j}, b_{i,j}\}$ . Because  $|X| = k = 3p$ , we obtain that  $X \cap (A_i \cup B_i) = \{a_{i,j}, b_{i,j}\}$ . Then by the construction of  $\mathcal{M}_W$ , we obtain that  $X = \bigcup_{i=1}^p \{a_{i,j_i}, b_{i,j_i}, c_{i,j_i}\}$  where  $W = \bigcup_{i=1}^p \{a_{i,j_i}, b_{i,j_i}\}$ , that is,  $X$  is uniquely defined by  $W$ . In the same way as in Claim 2.2 we obtain that solving INDEPENDENT STABLE SET for instances  $(H_{p,q}, \mathcal{M}_W, k)$  with the matroids  $\mathcal{M}_W$  defined by the independence oracle for an (unknown)  $W$  demands at least  $q^p - 1$  oracle queries. Similarly to the case of claw and AT-free graphs, we conclude that the existence of an algorithm for INDEPENDENT STABLE SET using  $f(k) \cdot n^{o(k)}$  oracle calls would lead to a contradiction. This finishes the proof for bipartite graphs.

For chordal graphs, we modify the construction of  $H_{p,q}$  by making each  $C_i$  a clique. Then  $H_{p,q}$  becomes chordal but we can apply the same arguments to show that solving INDEPENDENT STABLE SET for instances  $(H_{p,q}, \mathcal{M}_W, k)$  with the matroids  $\mathcal{M}_W$  defined by the independence oracle for an (unknown)  $W$  demands at least  $q^p - 1$  oracle queries. This completes the proof. ◀

## 4 Independent Stable Set on sparse frameworks

In this section, we consider INDEPENDENT STABLE SET for graphs of bounded maximum degree and graphs of bounded degeneracy. First, we observe that the problem is FPT when parameterized by the solution size and the degeneracy by giving a recursive branching algorithm.

► **Theorem 3.** *INDEPENDENT STABLE SET can be solved in  $\mathcal{O}((d+1)^k \cdot n)$  time on frameworks with  $d$ -degenerate input graphs.*

**Proof.** The algorithm is based on the following observation. Let  $(G, \mathcal{M})$  be a framework such that for every  $v \in V(G)$ ,  $\{v\} \in \mathcal{I}$ . Then there is a stable set  $X$  of  $G$  that is independent with respect to  $\mathcal{M}$  whose size is maximum such that  $X \cap N_G[v] \neq \emptyset$ . To see this, let  $X$  be a stable set that is also independent in  $\mathcal{M}$  and such that  $X \cap N_G[v] = \emptyset$ . Because  $\{v\}$  and



$X$  are independent, there is  $Y \subseteq X$  of size  $|X| - 1$  such that  $Z = Y \cup \{v\}$  is independent. Because  $N_G(v) \cap Z = \emptyset$  and  $Y$  is a stable set,  $Z$  is a stable set. Thus, set  $Z$  of size  $|X|$  is stable in  $G$  and is independent in  $\mathcal{M}$ . This proves the observation.

Consider an instance  $(G, \mathcal{M}, k)$  of INDEPENDENT STABLE SET. Because  $G$  is a  $d$ -degenerate graph, there is an elimination ordering  $v_1, \dots, v_n$  of the vertices of  $G$ , that is,  $d_{G_i}(v_i) \leq d$  for every  $i \in \{1, \dots, n\}$  where  $G_i = G[\{v_i, \dots, v_n\}]$ . Recall that such an ordering can be computed in linear time [32].

If there is  $v \in V(G)$  such that  $\{v\} \notin \mathcal{I}$ , then we delete  $v$  from the framework as such vertices are trivially irrelevant. From now on, we assume that  $\{v\} \in \mathcal{I}$  for any  $v \in V(G)$ . If  $k = 0$ , then  $\emptyset$  is a solution, and we return yes and stop. If  $k \geq 1$  but  $V(G) = \emptyset$ , then we conclude that the answer is no and stop. We can assume that  $V(G) \neq \emptyset$  and  $k \geq 1$ .

Let  $u$  be the first vertex in the elimination ordering. Clearly,  $d_G(u) \leq d$ . We branch on at most  $d + 1$  instances  $(G - v, \mathcal{M}/v, k - 1)$  for  $v \in N_G[u]$ , where  $\mathcal{M}/v$  is the contraction of  $\mathcal{M}$  by  $\{v\}$ . By our observation,  $(G, \mathcal{M}, k)$  is a yes-instance of INDEPENDENT STABLE SET if and only if at least one of the instances  $(G - v, \mathcal{M}/v, k - 1)$  is a yes-instance.

In each step, we have at most  $d + 1$  branches and the depth of the search tree is at most  $k$ . Note that we do not need to recompute the elimination ordering when a vertex is deleted; instead, we just delete the vertex from the already constructed ordering. This means we can use the ordering constructed for the original input instance. Thus, the total running time is  $\mathcal{O}((d + 1)^k \cdot n)$ . This concludes the proof.  $\blacktriangleleft$

For bounded degree graphs, we prove that INDEPENDENT STABLE SET has a polynomial kernel when parameterized by  $k$  and the maximum degree.

**► Theorem 4.** *INDEPENDENT STABLE SET admits a polynomial kernel on frameworks with graphs of maximum degree at most  $\Delta$  such that the output instance contains a graph with at most  $k^2 \Delta$  vertices.*

**Proof.** Let  $(G, \mathcal{M}, k)$  be an instance of INDEPENDENT STABLE SET with  $\Delta(G) \leq \Delta$ . Recall that by our assumption,  $r(\mathcal{M}) \leq k$ . If  $r(\mathcal{M}) < k$  then  $(G, \mathcal{M}, k)$  is a no-instance. In this case, our kernelization algorithm returns a trivial no-instance of constant size and stops. Now we can assume that  $r(\mathcal{M}) = k$ . If  $k = 0$  then we return a trivial yes-instance as  $\emptyset$  is a solution. If  $\Delta = 0$ , then any base of  $\mathcal{M}$  is a solution, and we return a trivial yes-instance. Now we can assume that  $k \geq 1$  and  $\Delta \geq 1$ .

We set  $W_0 = \emptyset$ . Then for  $i = 1, \dots, \ell$  where  $\ell = k\Delta$ , we greedily select a maximum-size independent set  $W_i \subseteq V(G) \setminus (\bigcup_{j=0}^{i-1} W_j)$ . Our kernelization algorithm returns the instance  $(G', \mathcal{M}', k)$  where  $G' = G[\bigcup_{i=1}^{\ell} W_i]$  and  $\mathcal{M}'$  is the restriction of  $\mathcal{M}$  to  $V(G')$ . It is straightforward to see that  $|V(G')| \leq k^2 \Delta$  as  $|W_i| \leq r(\mathcal{M}) = k$  and the new instance can be constructed in polynomial time. We claim that  $(G, \mathcal{M}, k)$  is a yes-instance of INDEPENDENT STABLE SET if and only if  $(G', \mathcal{M}', k)$  is a yes-instance.

Because  $G'$  is an induced subgraph of  $G$ , any stable set of  $G'$  is a stable set of  $G$ . This immediately implies that if  $(G', \mathcal{M}', k)$  is a yes-instance then any solution to  $(G', \mathcal{M}', k)$  is a solution to  $(G, \mathcal{M}, k)$  and, thus,  $(G, \mathcal{M}, k)$  is a yes-instance. Suppose that  $(G, \mathcal{M}, k)$  is a yes-instance. It means that  $G$  contains a stable set of size  $k$  independent in  $\mathcal{M}$ . We show that there is a stable set  $X \subseteq V(G')$  of  $G$  of size  $k$  that is independent with respect to  $\mathcal{M}$ .

To show this, let  $X$  be a stable set of size  $k$  that is independent in  $\mathcal{M}$  with the maximum number of vertices in  $V(G')$ . For the sake of contradiction, assume that there is  $u \in X \setminus V(G')$ . We define  $Y = X \setminus \{u\}$ . Consider the set  $W_i$  for some  $i \in \{1, \dots, \ell\}$ . By the construction of the set, we have that  $u \in \text{cl}(W_i)$ . Then it holds that  $r(Y \cup W_i) \geq r(X)$ . This implies that there is  $w_i \in W_i$  such that  $r(Y \cup \{w_i\}) = r(X) = k$ . Because this property holds for

## 22:10 Stability in Graphs with Matroid Constraints

arbitrary  $i \in \{1, \dots, \ell\}$ , we obtain that there are  $\ell = k\Delta$  vertices  $w_1, \dots, w_\ell \in V(G')$  such that for any  $i \in \{1, \dots, \ell\}$ ,  $r(Y \cup \{w_i\}) = k$ . Notice that  $w_i \notin Y$  for  $i \in \{1, \dots, \ell\}$  and  $|N_G(Y)| \leq (k-1)\Delta$ . Therefore, there is  $i \in \{1, \dots, \ell\}$  such that  $w_i$  is not adjacent to any vertex of  $Y$ . Then  $Z = Y \cup \{w_i\}$  is a stable set of  $G$ . However,  $|Z \cap V(G')| > |X \cup V(G')|$  contradicting the choice of  $X$ . This proves that there is a stable set  $X \subseteq V(G')$  of  $G$  of size  $k$  that is independent in  $\mathcal{M}$ . Then  $X$  is a solution to  $(G', \mathcal{M}', k)$ , that is,  $(G', \mathcal{M}', k)$  is a yes-instance. This concludes the proof.  $\blacktriangleleft$

Theorem 4 is handy for kernelization with parameter  $k$  when the degeneracy of the graph in a framework is a constant.

► **Theorem 5.** *For every integer  $d \geq 0$ , INDEPENDENT STABLE SET admits a polynomial kernel with running time  $n^{\mathcal{O}(d)}$  on frameworks with graphs of degeneracy at most  $d$  such that the output instance contains a graph with  $dk^{\mathcal{O}(d)}$  vertices.*

**Proof.** Let  $(G, \mathcal{M}, k)$  be an instance of INDEPENDENT STABLE SET where the degeneracy of  $G$  is at most  $d$ . We assume without loss of generality that  $r(\mathcal{M}) = k$ . Otherwise, if  $r(\mathcal{M}) < k$ , then  $(G, \mathcal{M}, k)$  is a no-instance, and we can return a trivial no-instance of constant size and stop. If  $d = 0$ , then  $G$  is an edgeless graph, and any set of vertices forming a base of  $\mathcal{M}$  is a stable set of size  $k$  that is independent with respect to  $\mathcal{M}$ , that is,  $(G, \mathcal{M}, k)$  is a yes-instance. Then we return a trivial yes-instance and stop. From now on, we assume that  $d \geq 1$ . Also, we assume that  $k \geq 2$ . Otherwise, if  $k = 0$ , the empty set is a trivial solution. If  $k = 1$  then because  $r(\mathcal{M}) = k \geq 1$ , there is a vertex  $v$  such that  $\{v\} \in \mathcal{I}(\mathcal{M})$  and  $\{v\}$  is an independent set of size  $k$ . In both cases, we return a trivial yes-instance and stop.

Since  $G$  is a  $d$ -degenerate graph, it admits an elimination ordering  $v_1, \dots, v_n$  of the vertices of  $G$ , that is,  $d_{G_i}(v_i) \leq d$  for every  $i \in \{1, \dots, n\}$  where  $G_i = G[\{v_i, \dots, v_n\}]$ . Recall that such an ordering can be computed in linear time [32]. For a set of vertices  $X \subseteq V(G)$ , we use  $F(X)$  to denote the set of common neighbors of the vertices of  $X$  that occur before the vertices of  $X$  in the elimination ordering. Note that because  $G$  is a  $d$ -degenerate graph,  $F(X) = \emptyset$  if  $|X| > d$ . For an integer  $i \geq 1$ ,  $f_i(G) = \max\{|F(X)| : X \subseteq V(G) \text{ and } |X| = i\}$ . Clearly,  $f_i(G) = 0$  if  $i > d$ .

For each  $h = d, \dots, 1$ , we apply the following reduction rule starting with  $h = d$ . Whenever the rule deletes some vertices, we do not recompute the elimination ordering; instead, we use the induced ordering obtained from the original one by vertex deletions.

► **Reduction Rule 5.1.** *Set  $d_h = d + f_{h+1}(G)$ . For each  $X \subseteq V(G)$  such that  $|X| = h$ , do the following:*

- (i) set  $W_0 = \emptyset$ ,
- (ii) for  $i = 1, \dots, \ell$  where  $\ell = kd_h$ , greedily select a maximum-size independent set  $W_i \subseteq F(X) \setminus (\bigcup_{j=0}^{i-1} W_j)$ ,
- (iii) delete the vertices of  $D = F(X) \setminus (\bigcup_{i=1}^{\ell} W_i)$  and restrict  $\mathcal{M}$  to  $V(G) \setminus D$ .

It is easy to see that the rule can be applied in  $n^{\mathcal{O}(d)}$  time. We show that the rule is *safe*, that is, it returns an equivalent instance of the problem.

► **Claim 5.1.** *Reduction Rule 5.1 is safe.*

**Proof of Claim 5.1.** Let  $X \subseteq V(G)$  be of size  $h$ . Denote by  $G'$  the graph obtained from  $G$  by applying steps (i)–(iii) for  $X$  and let  $\mathcal{M}$  be the restriction of  $\mathcal{M}$  to  $V(G) \setminus D$ . We prove that  $(G, \mathcal{M}, k)$  is a yes-instance of INDEPENDENT STABLE SET if and only if  $(G', \mathcal{M}', k)$  is a yes-instance. Clearly, this is sufficient for the proof of the claim. Since  $G'$  is an induced

subgraph of  $G$ , any solution to  $(G', \mathcal{M}', k)$  is a solution to  $(G, \mathcal{M}, k)$ . Thus, if  $(G', \mathcal{M}', k)$  is a yes-instance then the same holds for  $(G, \mathcal{M}, k)$ . Hence, it remains to show that if  $(G, \mathcal{M}, k)$  is a yes-instance then  $(G', \mathcal{M}', k)$  is a yes-instance as well.

We use the following axillary observation: for every  $v \in V(G) \setminus X$ ,  $|N_G(v) \cap F(X)| \leq d_h$ . To see this, consider  $v \in V(G) \setminus X$ , and denote by  $L$  and  $R$  the sets of vertices of  $F(X)$  that are prior and after  $v$ , respectively, in the elimination ordering. By the definition of an elimination ordering,  $|N_G(v) \cap R| \leq d$ . For  $N_G(v) \cap L$ , we have that  $N_G(v) \cap L \subseteq F(X \cup \{v\})$ . Then  $|N_G(v) \cap L| \leq |F(X \cup \{v\})| \leq f_{h+1}$ . We conclude that  $|N_G(v) \cap F(X)| = |N_G(v) \cap L| + |N_G(v) \cap R| \leq d + f_{h+1} = d_h$ . This proves the observation.

Suppose that  $G$  has a stable set  $Y$  of size  $k$  that is independent with respect to  $\mathcal{M}$ . Among all these sets, we select  $Y$  such that  $Y \cap D$  has the minimum size. We claim that  $Y \cap D = \emptyset$ . The proof is by contradiction and is similar to the proof of Theorem 4. Assume that there is  $u \in Y \cap D$  and let  $Z = Y \setminus \{u\}$ . For each  $i \in \{1, \dots, \ell\}$ ,  $u \in \text{cl}(W_i)$  by the construction of  $W_i$ . Thus,  $r(Z \cup W_i) \geq r(Y)$  and for each  $i \in \{1, \dots, \ell\}$ , there is  $w_i \in W_i$  such that  $r(Z \cup \{w_i\}) = r(Y) = k$ . Therefore, there are  $\ell$  vertices  $w_1, \dots, w_\ell \in F(X) \setminus D$  such that for any  $i \in \{1, \dots, \ell\}$ ,  $r(Z \cup \{w_i\}) = k$ . Notice that  $w_i \notin Z$  for all  $i \in \{1, \dots, \ell\}$  and  $Y \cap X = \emptyset$  because  $u$  is adjacent to every vertex of  $X$ . By the above observation, we have that  $|N_G(Z) \cap F(X)| \leq (k-1)d_h$ . Since  $\ell = kd_h > (k-1)d_h$ , there is  $i \in \{1, \dots, \ell\}$  such that  $w_i \notin N_G(Z)$ . Then  $Y' = Z \cup \{w_i\}$  is a stable set of  $G$ . Because  $Y'$  is independent with respect to  $\mathcal{M}$  and  $u \notin D$ , this leads to a contradiction with the choice of  $Y$ . We conclude that there is a stable set  $Y$  of  $G$  of size  $k$  that is independent with respect to  $\mathcal{M}$  such that  $Y \cap D = \emptyset$ . This means that  $Y$  is a solution to  $(G', \mathcal{M}', k)$ , that is,  $(G', \mathcal{M}', k)$  is a yes-instance of INDEPENDENT STABLE SET. This concludes the proof.  $\triangleleft$

Denote by  $(G', \mathcal{M}', k)$  the instance of INDEPENDENT STABLE SET obtained after applying Reduction Rule 5.1. We prove that the maximum degree of  $G'$  is bounded.

► **Claim 5.2.**  $\Delta(G') \leq dk^{2d+1}$ .

Proof of Claim 5.2. For  $i \in \{1, \dots, d\}$ , denote by  $G_i$  the graph obtained from  $G$  by applying Reduction Rule 5.1 for  $h = d, \dots, i$ . Note that  $G' = G_1$ . Because  $r(\mathcal{M}) = k$ , for each set  $W_j$  selected in step (ii) of Reduction Rule 5.1,  $|W_j| \leq k$ . Therefore,  $|\bigcup_{j=1}^\ell W_j| \leq k\ell = k^2d_h$ . Notice that for  $h = d$ ,  $f_{h+1}(G) = 0$  and, therefore,  $d_h = d$ . This implies that  $f_d(G_d) \leq k^2d$ . For  $i < d$ , we have that  $f_i(G_i) \leq k^2d_i = k^2(d + f_{i+1}(G_{i+1}))$ . Therefore,  $f_i(G_i) \leq d \sum_{j=i}^d k^{2(j-i+1)}$  and, as  $k \geq 2$ ,

$$f_1(G') \leq f_1(G_1) \leq d \sum_{j=1}^d k^{2j} = d \sum_{j=0}^d k^{2j} - d = d \frac{k^{2(d+1)} - 1}{k^2 - 1} - d \leq dk^{2d+1} - d.$$

Therefore, each vertex  $v$  of  $G'$  has at most  $dk^{2d+1} - d$  neighbors in  $G'$  that are prior  $v$  in the elimination ordering. Because  $v$  has at most  $d$  neighbors that are after  $v$  in the ordering,  $d_{G'}(v) \leq dk^{2d+1}$ . This concludes the proof.  $\triangleleft$

Because the maximum degree of  $G'$  is bounded, we can apply Theorem 4. Applying the kernelization algorithm from this theorem to  $(G', \mathcal{M}', k)$ , we obtain a kernel with at most  $dk^{2d+3}$  vertices. This concludes the proof of the theorem.  $\blacktriangleleft$

In Theorem 5, we proved that INDEPENDENT STABLE SET admits a polynomial kernel on  $d$ -degenerate graphs when  $d$  is a fixed constant. We complement this result by showing that it is unlikely that the problem has a polynomial kernel when parameterized by both  $k$  and  $d$ .

► **Theorem 6.** *INDEPENDENT STABLE SET on frameworks with  $d$ -degenerate graphs and partition matroids does not admit a polynomial kernel when parameterized by  $k + d$  unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$ .*

**Proof.** We use the fact that RAINBOW-STABLE SET is a special case of INDEPENDENT STABLE SET and show that RAINBOW-STABLE SET does not admit a polynomial kernel when parameterized by  $k + d$  unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$  where  $k$  is the number of color classes.

We use cross-composition from RAINBOW-STABLE SET. We say that two instances  $(G, \{V_1, \dots, V_k\})$  and  $(G', \{V'_1, \dots, V'_{k'}\})$  are *equivalent* if  $|V(G)| = |V(G')|$  and  $k = k'$ . Consider  $t$  equivalent instances  $(G_i, \{V_1^i, \dots, V_k^i\})$  of RAINBOW-STABLE SET for  $i \in \{1, \dots, t\}$  where each graph has  $n$  vertices. We assume that  $t = 2^p$  for some  $p \geq 1$ . Otherwise, we add  $2^{\lceil \log t \rceil} - t$  copies of  $(G_1, \{V_1^1, \dots, V_k^1\})$  to achieve the property for  $p = \lceil \log t \rceil$ ; note that by this operation, we may add at most  $t$  instances. Then we construct the instance  $(G, \{V_1, \dots, V_{k+p}\})$  of RAINBOW-STABLE SET as follows.

- Construct the disjoint union of copies of  $G_1, \dots, G_t$ .
- For each  $i \in \{1, \dots, p\}$ ,
  - construct two adjacent vertices  $u_i$  and  $v_i$ ,
  - for each  $j \in \{1, \dots, t\}$ , consider the binary encoding of  $j - 1$  as a string  $s$  with  $p$  symbols and make all the vertices of  $G_j$  adjacent to  $u_i$  if  $s[i] = 0$  and make them adjacent to  $v_i$ , otherwise, for  $i \in \{1, \dots, p\}$ .
- Define  $k + p$  color classes  $V_i = \bigcup_{j=1}^t V_i^j$  for  $i \in \{1, \dots, k\}$  and  $V_{k+i} = \{u_i, v_i\}$  for  $i \in \{1, \dots, p\}$ .

It is straightforward to see that the instance  $(G, \{V_1, \dots, V_{k+p}\})$  of RAINBOW-STABLE SET can be constructed in polynomial time. We claim that  $(G, \{V_1, \dots, V_{k+p}\})$  is a yes-instance of RAINBOW-STABLE SET if and only if there is  $j \in \{1, \dots, t\}$  such that  $(G_j, \{V_1^j, \dots, V_k^j\})$  is a yes-instance of RAINBOW-STABLE SET.

Suppose that  $(G_j, \{V_1^j, \dots, V_k^j\})$  is a yes-instance for some  $j \in \{1, \dots, t\}$ . Then there is a stable set  $X \subseteq V(G_j)$  of size  $k$  such that  $|X \cap V_i^j| = 1$  for  $i \in \{1, \dots, k\}$ . Let  $s$  be the string with  $p$  symbols that is the binary encoding of  $j - 1$ . Consider the set  $Y \subseteq \bigcup_{i=1}^p \{u_i, v_i\}$  such that for each  $i \in \{1, \dots, p\}$ ,  $Y$  contains either  $u_i$  or  $v_i$ , and  $u_i$  is in  $Y$  whenever  $s[i] = 1$ . Observe that  $Z = X \cup Y$  is a stable set of  $G$  and it holds that  $|Z \cap V_h| = 1$  for each  $h \in \{1, \dots, p+k\}$ . This means that  $(G, \{V_1, \dots, V_{k+p}\})$  is a yes-instance of RAINBOW-STABLE SET.

For the opposite direction, assume that  $(G, \{V_1, \dots, V_{k+p}\})$  is a yes-instance of RAINBOW-STABLE SET. Then there is a stable set  $Z$  of  $G$  of size  $k' = k + p$  such that  $|Z \cap V_h| = 1$  for each  $h \in \{1, \dots, p+k\}$ . Let  $Y = Z \cap (\bigcup_{i=1}^p \{u_i, v_i\})$  and  $X = Z \setminus Y$ . By the construction of color classes and because  $Y$  is a stable set, for each  $i \in \{1, \dots, p\}$ ,  $Y$  contains either  $u_i$  or  $v_i$ . Also, we have that  $X \subseteq \bigcup_{j=1}^t V(G_j)$ . Consider the binary string  $s$  of length  $p$  such that  $s[i] = 1$  if  $u_i \in Y$  and  $s[i] = 0$ , otherwise, for all  $i \in \{1, \dots, p\}$ . Notice that the vertices of  $G_j$  such that  $s$  is the binary encoding of  $j - 1$  are not adjacent to the vertices of  $Y$  and for every  $j' \in \{1, \dots, t\}$  distinct from  $j$ , all the vertices of  $G_{j'}$  are adjacent to at least one vertex of  $Y$ . This implies that  $X \subseteq V(G_j)$ . Therefore,  $X$  is a stable set of  $G_j$  of size  $k$  and  $|X \cap V_i^j| = 1$  for  $i \in \{1, \dots, k\}$ , that is,  $(G_j, \{V_1^j, \dots, V_k^j\})$  is a yes-instance of RAINBOW-STABLE SET.

Notice that each vertex  $v \in V(G_j)$  for  $j \in \{1, \dots, t\}$  is adjacent in  $G$  to at most  $n - 1$  vertices of  $G_j$  and  $p$  vertices of  $\bigcup_{i=1}^p \{u_i, v_i\}$ . Therefore, the degeneracy of  $G$  is at most  $n + \log t$ . Also, we have the number of color classes  $k' = k + p \leq n + \log t$ . Then because RAINBOW-STABLE SET is NP-complete and  $(G, \{V_1, \dots, V_{k+p}\})$  is a yes-instance of RAINBOW-STABLE SET if and only if there is  $j \in \{1, \dots, t\}$  such that  $(G_j, \{V_1^j, \dots, V_k^j\})$  is a yes-instance of

RAINBOW-STABLE SET, the result of Bodlaender, Jansen, and Kratsch [5] implies that RAINBOW-STABLE SET does not admit a polynomial kernel unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$  when parameterized by the number of color classes  $k$  and the degeneracy of the input graph. This concludes the proof.  $\blacktriangleleft$

## 5 Independent Stable Set on chordal graphs

For chordal graphs, we show that INDEPENDENT STABLE SET is FPT in the case of linear matroids when parameterized by  $k$  by demonstrating a dynamic programming algorithm over tree decompositions exploiting representative sets [28, 31, 15, 27].

Let  $\mathcal{M} = (V, \mathcal{I})$  be a matroid and let  $\mathcal{S}$  be a family of subsets of  $V$ . For a positive integer  $q$ , a subfamily  $\widehat{\mathcal{S}}$  is  $q$ -representative for  $\mathcal{S}$  if the following holds: for every set  $Y \subseteq V$  of size at most  $q$ , if there is a set  $X \in \mathcal{S}$  disjoint from  $Y$  with  $X \cup Y \in \mathcal{I}$  then there is  $\widehat{X} \in \widehat{\mathcal{S}}$  disjoint from  $Y$  with  $\widehat{X} \cup Y \in \mathcal{I}$ . We write  $\widehat{\mathcal{S}} \subseteq_{rep}^q \mathcal{S}$  to denote that  $\widehat{\mathcal{S}} \subseteq \mathcal{S}$  is  $q$ -representative for  $\mathcal{S}$ . We use the results of Fomin et al. [15] to compute representative families for linear matroids. A family of sets  $\mathcal{S}$  is said to be a  $p$ -family for an integer  $p \geq 0$  if  $|S| = p$  for every  $S \in \mathcal{S}$ , and we use  $\|A\|$  to denote the bit-length of the encoding of a matrix  $A$ .

► **Proposition 7** ([15, Theorem 3.8]). *Let  $M = (V, \mathcal{I})$  be a linear matroid and let  $\mathcal{S} = \{S_1, \dots, S_t\}$  be a  $p$ -family of independent sets. Then there exists  $\widehat{\mathcal{S}} \subseteq_{rep}^q \mathcal{S}$  of size at most  $\binom{p+q}{p}$ . Furthermore, given a representation  $A$  of  $M$  over a field  $\mathbb{F}$ , there is a randomized Monte Carlo algorithm computing  $\widehat{\mathcal{S}} \subseteq_{rep}^q \mathcal{S}$  of size at most  $\binom{p+q}{p}$  in  $\mathcal{O}(\binom{p+q}{p} t p^\omega + t \binom{p+q}{q}^{\omega-1}) + \|A\|^{\mathcal{O}(1)}$  operations over  $\mathbb{F}$ , where  $\omega$  is the exponent of matrix multiplication.<sup>1</sup>*

The following theorem is proved by the bottom-up dynamic programming over a nice tree decomposition where representative sets are used to store partial solutions. Due to space constraints, the proof is omitted in this extended abstract and can be found in the full version [14].

► **Theorem 8.** *INDEPENDENT STABLE SET can be solved in  $2^{\mathcal{O}(k)} \cdot \|A\|^{\mathcal{O}(1)}$  time by a one-sided error Monte Carlo algorithm with false negatives on frameworks with chordal graphs and linear matroids given by their representations  $A$ .*

The algorithm in Theorem 8 is randomized because it uses the algorithm from Proposition 7 to compute representative sets. For some linear matroids, the algorithm can be derandomized using the deterministic construction of representative sets given by Lokshtanov et al. [27]. In particular, this can be done for linear matroids over any finite field and the field of rational numbers.

We complement Theorem 8 by proving that it is unlikely that INDEPENDENT STABLE SET admits a polynomial kernel when parameterized by  $k$  in the case of chordal graphs.

► **Theorem 9.** *INDEPENDENT STABLE SET on frameworks with chordal graphs and partition matroids does not admit a polynomial kernel when parameterized by  $k$  unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$ .*

**Proof.** In the same way as in the proof of Theorem 6, we prove that RAINBOW-STABLE SET does not admit a polynomial kernel when parameterized by  $k$  on chordal graphs unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$  where  $k$  is the number of color classes.

<sup>1</sup> The currently best value is  $\omega \approx 2.3728596$  [4].

We construct a cross-composition from RAINBOW-STABLE SET. Again, we say that two instances  $(G, \{V_1, \dots, V_k\})$  and  $(G', \{V'_1, \dots, V'_{k'}\})$  are *equivalent* if  $|V(G)| = |V(G')|$  and  $k = k'$ . Consider  $t$  equivalent instances  $(G_i, \{V_1^i, \dots, V_k^i\})$  of RAINBOW-STABLE SET for  $i \in \{1, \dots, t\}$  where each graph is chordal and has  $n$  vertices. Then we construct the instance  $(G, \{V_0, V_1, \dots, V_k\})$  of RAINBOW-STABLE SET as follows.

- Construct the disjoint union of copies of  $G_1, \dots, G_t$ .
- Construct a clique  $K$  with  $t$  vertices  $v_1, \dots, v_t$ .
- For each  $j \in \{1, \dots, t\}$ , make  $v_j$  adjacent to all the vertices of every  $G_i$  for  $i \in \{1, \dots, t\}$  that is distinct from  $j$ .
- Define  $k + 1$  color classes  $V_0 = K$  and  $V_i = \bigcup_{j=1}^t V_i^j$  for  $i \in \{1, \dots, k\}$ .

It is straightforward to see that  $G$  is chordal and the instance  $(G, \{V_0, V_1, \dots, V_k\})$  of RAINBOW-STABLE SET can be constructed in polynomial time. We claim that  $(G, \{V_0, V_1, \dots, V_k\})$  is a yes-instance of RAINBOW-STABLE SET if and only if there is  $j \in \{1, \dots, t\}$  such that  $(G_j, \{V_1^j, \dots, V_k^j\})$  is a yes-instance of RAINBOW-STABLE SET.

Suppose that  $(G_j, \{V_1^j, \dots, V_k^j\})$  is a yes-instance for some  $j \in \{1, \dots, t\}$ . Then there is a stable set  $X \subseteq V(G_j)$  of size  $k$  such that  $|X \cap V_i^j| = 1$  for  $i \in \{1, \dots, k\}$ . By the construction of  $G$ , the vertex  $v_j \in K$  is not adjacent to any vertex of  $G_j$ . Thus,  $Y = X \cup \{v_j\}$  is stable set of  $G$  such that  $|Y \cap V_i| = 1$  for each  $i \in \{0, \dots, k\}$ . Therefore,  $(G, \{V_0, V_1, \dots, V_k\})$  is a yes-instance of RAINBOW-STABLE SET.

For the opposite direction, assume that  $(G, \{V_0, V_1, \dots, V_k\})$  is a yes-instance of RAINBOW-STABLE SET. Then there is a stable set  $Y$  of  $G$  of size  $k + 1$  such that  $|Y \cap V_i| = 1$  for each  $i \in \{0, \dots, k\}$ . In particular,  $|Y \cap V_0| = 1$ . Then there is  $j \in \{1, \dots, t\}$  such that  $v_j \in Y$ . By the construction of  $G$ , we have that  $X = Y \setminus \{v_j\} \subseteq V(G_j)$ . Then  $|X \cap V_i^j| = 1$  for each  $i \in \{1, \dots, k\}$ , that is,  $(G_j, \{V_1^j, \dots, V_k^j\})$  is a yes-instance of RAINBOW-STABLE SET.

Le and Pfender in [26] proved that RAINBOW MATCHING remains NP-complete on paths. This implies that RAINBOW-STABLE SET is also NP-complete on paths, and hence on chordal graphs. Because the number of color classes is  $k + 1 \leq n + 1$  and RAINBOW-STABLE SET is NP-complete on chordal graphs, we can apply the result of Bodlaender, Jansen, and Kratsch [5]. This concludes the proof. ◀

## 6 Conclusion

In this paper, we investigated the parameterized complexity of the INDEPENDENT STABLE SET problem for various classes of graphs where the classical STABLE SET problem is tractable. We derived kernelization results and FPT algorithms, complemented by complexity lower bounds. We believe exploring INDEPENDENT STABLE SET on other natural graph classes with similar properties would be interesting. For instance, STABLE SET is solvable in polynomial time on claw-free graphs [33] and AT-free graphs [8]. While our unconditional lower bound from Theorem 2 applies to these classes, it does not rule out the possibility of FPT algorithms for frameworks with *linear* matroids. A similar question arises regarding graphs with a polynomial number of minimal separators [6, 7].

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## References

- 1 Ron Aharoni, Eli Berger, Maria Chudnovsky, David M. Howard, and Paul D. Seymour. Large rainbow matchings in general graphs. *Eur. J. Comb.*, 79:222–227, 2019. doi:10.1016/J.EJC.2019.01.012.
- 2 Ron Aharoni, Eli Berger, and Ran Ziv. Independent systems of representatives in weighted graphs. *Combinatorica*, 27(3):253–267, 2007.



- 3 Ron Aharoni, Joseph Briggs, Jinha Kim, and Minki Kim. Rainbow independent sets in certain classes of graphs. *Journal of Graph Theory*, 104(3):557–584, 2023.
- 4 Josh Alman and Virginia Vassilevska Williams. A refined laser method and faster matrix multiplication. In Dániel Marx, editor, *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms, SODA 2021, Virtual Conference, January 10 - 13, 2021*, pages 522–539. SIAM, 2021. doi:10.1137/1.9781611976465.32.
- 5 Hans L. Bodlaender, Bart M. P. Jansen, and Stefan Kratsch. Kernelization lower bounds by cross-composition. *SIAM J. Discret. Math.*, 28(1):277–305, 2014. doi:10.1137/120880240.
- 6 Vincent Bouchitté and Ioan Todinca. Treewidth and minimum fill-in: Grouping the minimal separators. *SIAM J. Comput.*, 31(1):212–232, 2001. doi:10.1137/S0097539799359683.
- 7 Vincent Bouchitté and Ioan Todinca. Listing all potential maximal cliques of a graph. *Theor. Comput. Sci.*, 276(1-2):17–32, 2002. doi:10.1016/S0304-3975(01)00007-x.
- 8 Hajo Broersma, Ton Kloks, Dieter Kratsch, and Haiko Müller. Independent sets in asteroidal triple-free graphs. *SIAM J. Discret. Math.*, 12(2):276–287, 1999. doi:10.1137/S0895480197326346.
- 9 Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015. doi:10.1007/978-3-319-21275-3.
- 10 Marek Cygan, Fabrizio Grandoni, and Danny Hermelin. Tight kernel bounds for problems on graphs with small degeneracy. *ACM Trans. Algorithms*, 13(3):43:1–43:22, 2017. doi:10.1145/3108239.
- 11 Reinhard Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.
- 12 Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013. doi:10.1007/978-1-4471-5559-1.
- 13 Arthur A. Drisko. Transversals in row-latin rectangles. *Journal of Combinatorial Theory, Series A*, 84(2):181–195, 1998. doi:10.1006/jcta.1998.2894.
- 14 Fedor V. Fomin, Petr A. Golovach, Tuukka Korhonen, and Saket Saurabh. Stability in graphs with matroid constraints, 2024. arXiv:2404.03979.
- 15 Fedor V. Fomin, Daniel Lokshtanov, Fahad Panolan, and Saket Saurabh. Efficient computation of representative families with applications in parameterized and exact algorithms. *J. ACM*, 63(4):29:1–29:60, 2016. doi:10.1145/2886094.
- 16 Fedor V. Fomin, Daniel Lokshtanov, Saket Saurabh, and Meirav Zehavi. *Kernelization: theory of parameterized preprocessing*. Cambridge University Press, 2019.
- 17 M. R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.
- 18 Alessandra Graf, David G. Harris, and Penny Haxell. Algorithms for weighted independent transversals and strong colouring. *ACM Trans. Algorithms*, 18(1):1:1–1:16, 2022. doi:10.1145/3474057.
- 19 Alessandra Graf and Penny Haxell. Finding independent transversals efficiently. *Comb. Probab. Comput.*, 29(5):780–806, 2020. doi:10.1017/S0963548320000127.
- 20 Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2. Springer Science & Business Media, 2012.
- 21 Sushmita Gupta, Sanjukta Roy, Saket Saurabh, and Meirav Zehavi. Parameterized algorithms and kernels for rainbow matching. *Algorithmica*, 81(4):1684–1698, 2019. doi:10.1007/S00453-018-0497-3.
- 22 Sushmita Gupta, Sanjukta Roy, Saket Saurabh, and Meirav Zehavi. Quadratic vertex kernel for rainbow matching. *Algorithmica*, 82(4):881–897, 2020. doi:10.1007/S00453-019-00618-0.
- 23 Penny Haxell. On forming committees. *The American Mathematical Monthly*, 118(9):777–788, 2011.
- 24 Alon Itai, Michael Rodeh, and Steven L. Tanimoto. Some matching problems for bipartite graphs. *J. ACM*, 25(4):517–525, 1978. doi:10.1145/322092.322093.

## 22:16 Stability in Graphs with Matroid Constraints

- 25 Jinha Kim, Minki Kim, and O-joung Kwon. Rainbow independent sets on dense graph classes. *Discrete Applied Mathematics*, 312:45–51, 2022.
- 26 Van Bang Le and Florian Pfender. Complexity results for rainbow matchings. *Theor. Comput. Sci.*, 524:27–33, 2014. doi:10.1016/J.TCS.2013.12.013.
- 27 Daniel Lokshtanov, Pranabendu Misra, Fahad Panolan, and Saket Saurabh. Deterministic truncation of linear matroids. *ACM Trans. Algorithms*, 14(2):14:1–14:20, 2018. doi:10.1145/3170444.
- 28 L. Lovász. Flats in matroids and geometric graphs. In *Combinatorial surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977)*, pages 45–86, 1977.
- 29 László Lovász. *Graphs and geometry*, volume 65 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2019. doi:10.1090/coll/065.
- 30 Pasin Manurangsi, Erel Segal-Halevi, and Warut Suksompong. On maximum bipartite matching with separation. *Inf. Process. Lett.*, 182:106388, 2023. doi:10.1016/J.IPL.2023.106388.
- 31 Dániel Marx. A parameterized view on matroid optimization problems. *Theor. Comput. Sci.*, 410(44):4471–4479, 2009. doi:10.1016/j.tcs.2009.07.027.
- 32 David W. Matula and Leland L. Beck. Smallest-last ordering and clustering and graph coloring algorithms. *J. ACM*, 30(3):417–427, 1983. doi:10.1145/2402.322385.
- 33 George J. Minty. On maximal independent sets of vertices in claw-free graphs. *J. Comb. Theory, Ser. B*, 28(3):284–304, 1980. doi:10.1016/0095-8956(80)90074-X.
- 34 James Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011. doi:10.1093/acprof:oso/9780198566946.001.0001.