Optimizing Visibility-Based Search in Polygonal Domains

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Abstract

Given a geometric domain P, visibility-based search problems seek routes for one or more mobile agents ("watchmen") to move within P in order to be able to see a portion (or all) of P, while optimizing objectives, such as the length(s) of the route(s), the size (e.g., area or volume) of the portion seen, the probability of detecting a target distributed within P according to a prior distribution, etc. The classic watchman route problem seeks a shortest route for an observer, with omnidirectional vision, to see all of P. In this paper we study bicriteria optimization problems for a single mobile agent within a polygonal domain P in the plane, with the criteria of route length and area seen. Specifically, we address the problem of computing a minimum length route that sees at least a specified area of P (minimum length, for a given area quota). We also study the problem of computing a length-constrained route that sees as much area as possible. We provide hardness results and approximation algorithms. In particular, for a simple polygon P we provide the first fully polynomial-time approximation scheme for the problem of computing a shortest route seeing an area quota, as well as a (slightly more efficient) polynomial dual approximation. We also consider polygonal domains P (with holes) and the special case of a planar domain consisting of a union of lines. Our results yield the first approximation algorithms for computing a time-optimal search route in P to guarantee some specified probability of detection of a static target within P, randomly distributed in P according to a given prior distribution.

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1 Introduction

We investigate the Quota Watchman Route problem (QWRP) and the Budgeted Watchman Route problem (BWRP) for a single mobile agent (a "watchman") within a polygonal domain P in the plane. These problems naturally arise in various applications, including motion planning, search-and-rescue, surveillance, and exploration of a polygonal

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domain, where complete coverage is not feasible due to shortage of fuel, time, etc. The QWRP seeks a route/tour that sees at least some specified area of the domain P with a shortest length, while the BWRP seeks a route/tour that sees the maximum area subject to a length constraint. Both can be seen as extensions of the well-known Watchman Route Problem (WRP) with different objectives and constraints.

The challenge in addressing the trade-off between area seen and tour length is that one is not able to exploit the optimality structure that is implied by having to see *all* of a polygon P. It is this structure, yielding an ordered sequence of "essential cuts", that allows the WRP to be solved efficiently, e.g., as an instance of the "touring polygons problem" [13].

Results. We address the challenge by establishing new structural results that enable a careful discretization and analysis, along with carefully crafted dynamic programs. We provide several new results on optimal visibility search in a polygon:

(1) We prove that the QWRP and the BWRP are (weakly) NP-hard, even in a simple polygon; this is to be contrasted with the WRP, for which exact polynomial-time algorithms are known in simple polygons.

(2) For the QWRP in a simple polygon P, we give the first fully polynomial-time approximation scheme (FPTAS), as well as a dual-approximation (with slightly more efficient running time than the FPTAS) that computes a tour having length at most $(1+\varepsilon_1)$ times the length of an optimal tour that sees area at least A (where A is the area quota), while seeing area at least $(1-\varepsilon_2)A$ for any $\varepsilon_1, \varepsilon_2 > 0$.

(3) For the BWRP in simple P, we compute, in polynomial time, a tour of length at most $(1+\varepsilon)B$ that sees a region within P of area at least that seen by an optimal tour of length at most B.

(4) In a multiply connected domain, in a polygon P with holes, we provide hardness of approximations and a $(1+\varepsilon, O(\log n))$-dual approximation ($n$ is the number of vertices of P) for the BWRP. In the special case of an arrangement of lines, we obtain polynomial-time exact algorithms for both problems.

(5) We solve two visibility-based stochastic search problems that seek to locate a static target given a prior probability distribution of its location within P: (a) compute the minimum time to achieve a specified detection probability; (b) compute a search route maximizing the probability of detection by time T for a mobile searcher.

Related Work

Chin and Ntafos introduced the classic Watchman Route Problem (WRP) [10]: compute a shortest closed route (tour) within a polygon P from which every point of P can be seen; they gave an $O(n)$-time algorithm for computing an optimal tour in a simple orthogonal polygon, and later results established polynomial-time exact algorithms for the WRP in a simple polygon P, both with and without an anchor point (depot) [5, 10, 11, 13, 26, 29, 30]. In a polygon P with holes, the WRP is NP-hard [10, 17] and is, in fact, NP-hard to approximate better than a logarithmic factor [25]; however, an $O(\log^2 n)$-approximation algorithm is known [25]. The BWRP and the QWRP are natural variants of the WRP.

Another related problem is that of maximum visibility coverage with point guards: Given an integer k, place k point guards within P to maximize the area of P seen by the guards. When k is arbitrary, the problem is NP-hard [26], since an exact solution to this problem would yield a method to compute the minimum number of guards needed to see a polygon. Viewed as a maximum coverage problem, one can greedily compute an approximation, with factor $\left(1-\frac{1}{e}\right)$, by iteratively placing a guard that sees the most unseen area [9, 26].

The BWRP is related to the ORIENTEERING problem. Given a budget constraint and an edge-weighted graph where each vertex is associated with a prize, the objective of ORIENTEERING is to find a path/tour within the length budget maximizing the total reward of the vertices visited. On the other hand, the QWRP is related to the QUOTA TRAVELING SALESPERSON problem, which aims to minimize the distance travelled to achieve a given quota of reward. The Euclidean versions of ORIENTEERING and Quota TSP have polynomial-time approximation schemes [8, 20, 24]. Both the QWRP and the BWRP can be considered a reward (the area of P seen by the watchman) collecting process; however, the main difference lies in the continuous nature of visibility, since we see portions of the domain as we travel to checkpoints, we must take into account the area that has been seen previously.

Optimal search theory has been extensively studied in discrete, graph theoretic settings; see, e.g., [18, 19, 31]. In geometric contexts, searching and target tracking have been studied in the form of VISIBILITY-BASED PURSUIT-EVASION games. In [22], the visibility-based version of the pursuit-evasion game was introduced and formulated as a geometric problem, in which an evader moves unpredictably, arbitrarily fast within a polygonal domain, and the goal is to strategically coordinate one or multiple pursuers to guarantee a finite time of detection. See the survey [12] on visibility-based pursuit-evasion games.

2 The QWRP in a Simple Polygon

2.1 Preliminaries and Hardness Results

A *simple polygon* P is a simply connected subset of $\mathbb{R}^2$ whose boundary, ∂P, is a polygonal cycle consisting of a finite set of line segments, whose endpoints are the *vertices*, $v_1, v_2, \ldots, v_n$, of P. A vertex is *reflex* (resp. *convex*) if its internal angle is at least (resp. at most) 180 degrees. We consider polygons to be closed sets, including the interior and the boundary. We use the notation $|\cdot|$ to denote the measure of several types of objects. In the case of a segment or a route γ, $|\gamma|$ denotes its length, while for a polygon P, $|P|$ denotes the area of P. For a finite set S, $|S|$ is the cardinality of S. Based on the object within the notation, the interpretation should be apparent.

Point $x \in P$ sees point $y \in P$ if the line segment connecting them lies entirely within P. The *visibility polygon* of x, denoted $V(x)$, is the closed region of P that x sees; necessarily, $V(x)$ is a simple polygon within P. For a subset $X \subset P$, let $V(X)$ be the set of points that are seen by at least one point in X; formally, $V(X) = \bigcup_{x \in X} V(x)$. The visibility polygon of a point or a segment can be computed in time $O(n)$ for a simple polygon, or in time $O(n + h \log h)$ for a polygonal domain with n vertices and h holes [27]. Given a domain P (a simple polygon or a polygon with holes) and an area quota $0 \leq A \leq |P|$, in the QWRP, the objective is to find a *tour* (a polygonal cycle) $\gamma \subset P$ of minimum length $|\gamma|$ such that $|V(\gamma)| \geq A$; see Figure 1. Note that when $A = |P|$, the QWRP is the classic WATCHMAN ROUTE PROBLEM.

We also distinguish between the *anchored* version (in which γ must pass through a given depot point s) and the *floating* version (in which no depot is given). We provide the following NP-hardness results (proved in the full version) for both the anchored and floating cases:

▶ **Theorem 1.** *The QWRP in a simple polygon is weakly NP-hard, with or without an anchor.*

Throughout the paper, we assume a real RAM model of computation [28].

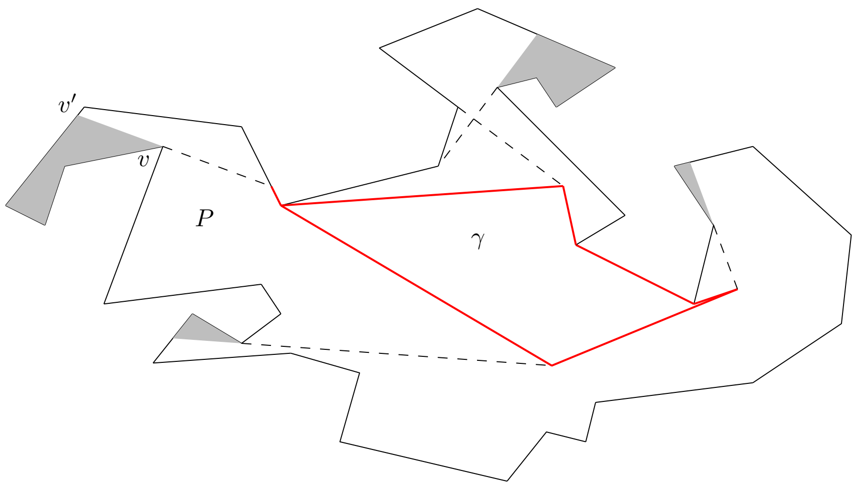

Figure 1 A route γ (red) that sees the white portion of P (the gray regions are unseen).

2.2 Structural Lemma

Let $\pi_P(x, y)$ denote the *geodesic shortest path* (shortest path constrained to stay within P) between $x \in P$ and $y \in P$; $\pi_P(x, y)$ is unique in a simple P, and is the segment xy if x sees y. For a subset $S \subseteq P$, the *relative/geodesic convex hull* of S is the minimal set that contains S and is closed under taking shortest paths. Equivalently, the relative convex hull of S is the minimum-perimeter connected subset of P that contains S. A set is *relatively convex* if it is equal to its relative convex hull, and a closed curve is relatively convex if it is the boundary of a relatively convex set. Let P_γ denote the connected region bounded by some closed polygonal chain γ. If P_γ is a (sub)polygon of P and P_γ is relatively convex, then P_γ is the relative convex hull of its convex vertices, and all reflex vertices of P_γ are necessarily reflex vertices of P. We similarly define relative convexity of an open polygonal chain γ: if γ is a connected subset of the boundary of the relative convex hull of γ, then we say that γ is relatively convex. Geodesic shortest paths and relative convex hulls have been studied extensively and can be computed efficiently [24].

An optimal solution to the QWRP in a simple polygon P satisfies a structural lemma:

▶ **Lemma 2.** *For a simple polygon P with n vertices, and no depot, an optimal QWRP tour is a relatively convex simple polygonal cycle of at most $2n$ vertices.*

Proof. Let γ be an optimal QWRP tour and let $P' = V(\gamma)$ be the visibility polygon of γ. Since γ is connected, P' is a simple subpolygon of P; some edges of P' coincide with edges of P and some are shadow chords (chords separating $V(\gamma)$ from the rest of P) supported by reflex vertices of P. Then γ is a shortest watchman route in the simple polygon P'. Thus, γ is relatively convex in P', and thus in P, and γ has at most $2n$ vertices, since P' is easily seen to have at most n vertices. (See [11, 25].)

Specifically, the polygon $P' = V(\gamma)$ is obtained from P by removing certain subpolygons ("shadow pockets") of P that are each defined by a chord, vv', extending from a reflex vertex, v, of P, along the line through v and a convex vertex of γ, to the first point v' on the boundary of P. This process introduces a (convex) vertex v' (on an edge of P, in general on its interior), and removes at least one vertex of P, on the boundary of the pocket that is cut off by the chord. Refer to Figure 1. Thus, P' has at most n vertices. For a simple polygon with n vertices, any shortest watchman route has at most $2n$ vertices [5, 10, 11, 13, 26, 29, 30]. Moreover, all reflex vertices of P' must be reflex vertices of P, hence all reflex vertices of γ must also be reflex vertices of P. ◀

If there is a specified depot $s \in \partial P$, a statement similar to Lemma 2 holds. If s is interior to P, an optimal tour $\gamma = (s, w_1, w_2, \ldots, w_k, s)$ through s need not be relatively convex; however, it is "nearly" relatively convex in that the tour obtained by replacing the two edges sw_1 and $w_k s$ with the geodesic path $\pi_P(w_1, w_k)$ is relatively convex.

2.3 Dual approximation algorithm for anchored QWRP

An optimal tour for the QWRP will, in general, have (convex) vertices that are interior to P, at locations within the continuum that are not known to come from a discrete set. This poses a challenge to algorithms that are to compute solutions for the QWRP exactly or approximately. We address this challenge by discretizing an appropriate portion of the domain P using a (Steiner) triangulation whose faces are small enough that we can afford to round an optimal tour to vertices of the triangles, while increasing the length of the tour only slightly, and assuring that the rounded tour continues to see at least as much of P as the optimal tour did. We focus here on the anchored case, with a specified depot s, which we assume to be on ∂P for now.

First, we triangulate P (in $O(n)$ time [6]), including s as a vertex of the triangulation. We then overlay, centered on s, a regular square grid of pixels of side lengths δ within an axis-aligned square of size L-by-L for a length L that is at least the optimal tour length; we specify how to determine δ and L below. The overlay of the grid with the triangulation yields a partition of P into convex cells of constant complexity, each of which we triangulate, resulting in an overall Steiner triangulation of P, such that every triangle within distance $L/2$ of s has diameter at most $\sqrt{2}\delta$ and perimeter at most 4δ; we let $S_{\delta,L}$ denote the set of vertices of these triangles. We refer to $S_{\delta,L}$ as the set of *candidate turn points* for a route.

▶ **Lemma 3.** *For an optimal tour γ for the QWRP with area quota A, there exists a polygonal tour γ' whose vertices are in the set $S_{\delta,L}$ of candidates, such that γ' is relatively convex, $|\gamma'| \le |\gamma| + (8 + 4\sqrt{2})\delta n$ and $V(\gamma) \subseteq V(\gamma')$.*

Proof. Let $c_1, c_2, \ldots$ be convex vertices of the optimal tour γ (γ is the relative convex hull of such vertices) and let $\sigma_1, \sigma_2, \ldots$ be (closed) cells of the decomposition that contain the vertices. Let γ' be the boundary of the relative convex hull of the cells. By construction, γ' is a relatively convex tour enclosing γ, implying that any point seen by γ is also seen by γ'. Furthermore, since $s \in \partial P$, it follows that s cannot be in the interior of $P_{\gamma'}$, a subpolygon of P, thus $s \in \gamma'$.

We claim that $|\gamma'|$ is at most $|\gamma| + (8+4\sqrt{2})\delta n$. For each edge e' of γ' going from σ_i to σ_j, we can bound its length by the sum of the length of the edge e of γ going from σ_i to σ_j (γ' visits the cells containing the vertices of γ in the same order) and at most two connections from endpoints of e to vertices of σ_i, σ_j, which is no more than $2\sqrt{2}\delta$, see Figure 2, right. Additionally, the part of γ' along the perimeters of $\sigma_1, \sigma_2, \ldots$ is no longer than $8\delta n$. Hence, $|\gamma'| \le |\gamma| + (8 + 4\sqrt{2})\delta n$. ◀

From Lemma 3, if $\delta = O\left(\frac{\varepsilon|\gamma|}{n}\right)$, then for approximation purposes within factor $(1+\varepsilon)$, it suffices to search for a tour whose vertices come from $S_{\delta,L}$. In fact, our algorithm returns a tour no longer than $(1+\varepsilon_1)|\gamma|$ for any $\varepsilon_1 > 0$; however, due to discretization of the area quota, we only guarantee the tour sees at least $(1-\varepsilon_2)A$ for any $\varepsilon_2 > 0$.

We now establish an ordering on the point set $S_{\delta,L}$, so that a relatively convex chain of the candidate points moves in increasing order. First, we compute $\mathcal{T}$, the tree of shortest paths rooted at s to all the candidate points; this takes $O(|S_{\delta,L}|)$ time [21]. The path from s to a candidate point s' in $\mathcal{T}$ is the geodesic shortest path $\pi_P(s, s')$. Define a *geodesic angular*

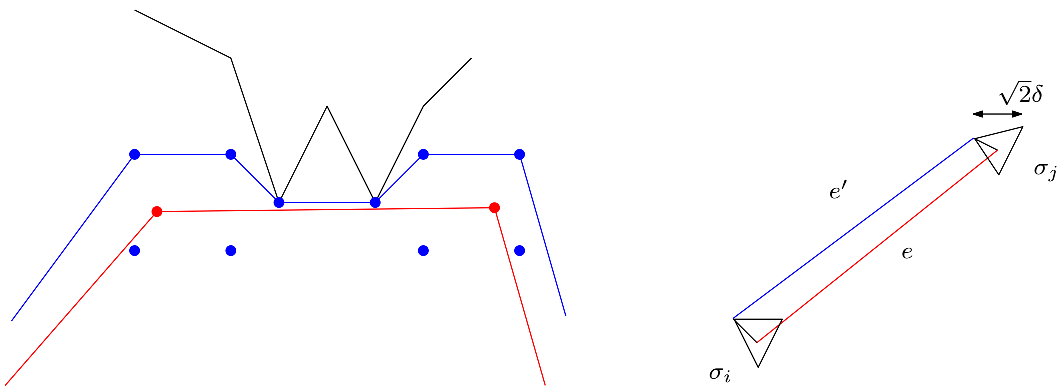

Figure 2 Left: γ' (blue) is the relative convex hull of the vertices (blue) of the cells that contain convex vertices of γ (red). Right: Each edge of γ' that traverses between two different cells σ_i, σ_j by triangle inequality, is no longer than the edge of γ between the same cells plus at most two connections to two vertices of σ_i, σ_j.

order as follows: for two candidate points s_i, s_j, if s_i is to the left of the extended geodesic shortest path between s and s_j, i.e $\pi_P(s, s_j)$ with the last segment extending up to ∂P, then s_i precedes s_j. In case of ties, we break ties by increasing distance to s. For each reflex vertex r_i of P, we add another candidate s_{r_i} to the list to account for the possibility that r_i can appear as two different vertices of a relatively convex polygonal chain; s_{r_i} obeys the aforementioned geodesic angular order but precedes every candidate point in the subtree of $\mathcal{T}$ rooted at r_i. Sort the candidates accordingly, then append $s_m := s_1$ to the end of the sorted list. Any relatively convex chain with vertices sequence oriented clockwise $(s, s_{i_1}, s_{i_2}, \ldots)$ has $1 < i_1 < i_2 < \ldots$. Without loss of generality, we consider any relatively convex polygonal chain to be oriented clockwise.

Next, we examine the optimal substructure of the problem.

▶ **Lemma 4** ([3]). *The visibility region of the geodesic shortest path $\pi_P(s, s_j)$ is the inclusion-wise minimal set among all visibility regions of all paths from s to s_j.*

Let C be a relatively convex polygonal chain from s to a candidate point s_j, and let s_i be the vertex of C immediately preceding s_j. We identify the overlap of visibility between the segment $s_i s_j$ and C_{s_i}, the subchain of C from s to s_i in Lemma 5.

▶ **Lemma 5.** $V(C_{s_i}) \cap V(s_i s_j) = V(\pi_P(s, s_i)) \cap V(s_i s_j)$.

Proof. Refer to Figure 3. Let $x \in P$ be a point seen by both $\pi_P(s, s_i)$ and $s_i s_j$. Since $x \in V(\pi_P(s, s_i))$, it follows that $x \in V(C_{s_i})$ (Lemma 4). Thus, $x \in V(C_{s_i}) \cap V(s_i s_j)$ and $V(\pi_P(s, s_i)) \cap V(s_i s_j) \subseteq V(C_{s_i}) \cap V(s_i s_j)$.

On the other hand, let $x \in P$ be seen by both C_{s_i} and $s_i s_j$. Since $x \in V(C_{s_i}) \cap V(s_i s_j)$ there exists $x_1 \in C_{s_i}$ and $x_2 \in s_i s_j$ such that $x x_1$ and $x x_2$ are contained within P. Thus, the (pseudo)triangle $x x_1 x_2$ is contained within P since P has no holes. By our ordering scheme, s_j is to the right of $\pi_P(s, s_i)$ with the last segment extended up to ∂P, while C_{s_i} is to the left of it. This implies that in the relatively convex polygon $P_{C \cup \pi_P(s, s_j)}$, x_1, x_2 are in opposite sides with respect to $\pi_P(s, s_i)$. As we pivot a line of sight around x from x_1 to x_2, it must intersect $\pi_P(s, s_i)$ at some point due to continuity as well as (relative) convexity, therefore $\pi_P(s, s_i)$ sees x. Hence, $V(C_{s_i}) \cap V(s_i s_j) \subseteq V(\pi_P(s, s_i)) \cap V(s_i s_j)$. ◀

Based on Lemma 5, the overlap of visibility between the segment $s_i s_j$, for $i < j$, and any relatively convex chain C_{s_i} from s to s_i does not depend on the vertices between s and s_i. This leads to the optimal substructure utilized by our dynamic programming algorithm.

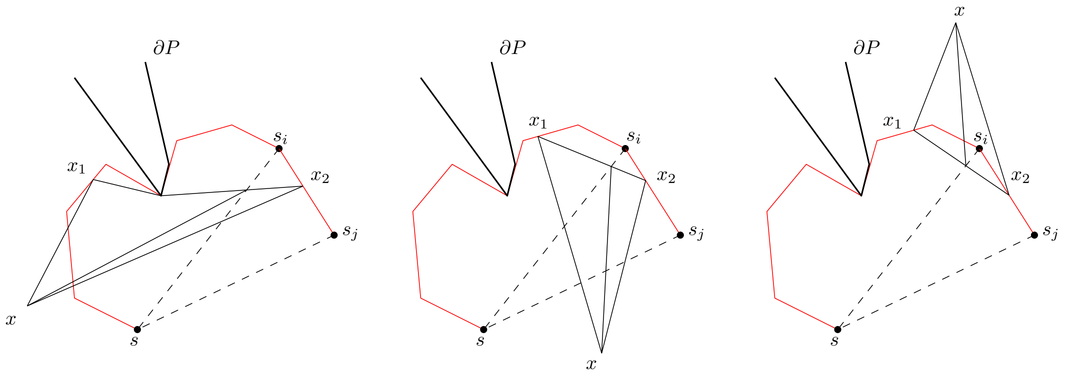

Figure 3 Proof of Lemma 5.

▶ **Lemma 6.** *C is a shortest relatively convex polygonal chain from s to s_j that sees at least some area $\overline{A}$ if and only if C_{s_i} is a shortest relatively convex polygonal chain from s to s_i that sees at least area $\overline{A} - |V(s_is_j) \setminus V(\pi_P(s, s_i))|$.*

Proof. We write $V(C)$ as the union of 2 disjoint sets $V(C_{s_i})$ and $V(s_is_j) \setminus V(C_{s_i})$. Notice that

$$V(s_is_j) \setminus V(C_{s_i}) = V(s_is_j) \setminus (V(C_{s_i}) \cap V(s_is_j)) = V(s_is_j) \setminus (V(\pi_P(s, s_i)) \cap V(s_is_j))$$
$$= V(s_is_j) \setminus V(\pi_P(s, s_i)),$$

therefore $|V(C_{s_i})| \geq \overline{A} - |V(s_is_j) \setminus V(\pi_P(s, s_i))|$. As a result, C_{s_i} must be the shortest chain to achieve a visibility area of $\overline{A} - |V(s_is_j) \setminus V(\pi_P(s, s_i))|$, since the existence of a shorter chain contradicts the optimality of C, and vice versa. ◀

A subproblem in the dynamic program is determined by a candidate point s_j and an area quota $\overline{A}$. Let $\pi(s_j, \overline{A})$ denote the length of a shortest relatively convex polygonal chain from s to s_j that can see area at least $\overline{A}$; and let $C(s_j, \overline{A})$ denote the associated optimal chain. Initialize $\pi(s, |V(s)|) = 0$. The Bellman recursion for each subproblem with $j = 1, 2, \ldots, m$ and all values of $\overline{A}$ would be given as follows, for all $\bar{i} < j$ such that s_j sees $s_{\bar{i}}$ and $C(s_{\bar{i}}, \overline{A} - |V(s_{\bar{i}}s_j) \setminus V(\pi_P(s, s_{\bar{i}}))|) \cup s_{\bar{i}}s_j$ is relatively convex:

$$i = \arg\min_{\bar{i}} \left\{ \pi(s_{\bar{i}}, \overline{A} - |V(s_{\bar{i}}s_j) \setminus V(\pi_P(s, s_{\bar{i}}))|) + |s_{\bar{i}}s_j| \right\},$$
$$\pi(s_j, \overline{A}) = \pi(s_i, \overline{A} - |V(s_is_j) \setminus V(\pi_P(s, s_i))|) + |s_is_j|,$$
$$C(s_j, \overline{A}) = C(s_i, \overline{A} - |V(s_is_j) \setminus V(\pi_P(s, s_i))|) \cup s_is_j.$$

Finally, return $C(s_m, A)$. Correctness of the algorithm follows from the principle of optimality.

Note that there always exists an optimal solution i to the above recursion such that $C(s_i, \overline{A} - |V(s_is_j) \setminus V(\pi_P(s, s_i))|) \cup s_is_j$ is relatively convex. Otherwise, we can shortcut $C(s_i, \overline{A} - |V(s_is_j) \setminus V(\pi_P(s, s_i))|) \cup s_is_j$ by connecting s_j to the closest reflex vertex (of P) or the point of tangency in $C(s_i, \overline{A} - |V(s_is_j) \setminus V(\pi_P(s, s_i))|)$.

Since $\overline{A}$ can take values from a continuous set, it is impractical to tabulate all such values. Instead, we bucket A into uniform intervals, and let the subproblems be defined by interval endpoints. We round down the area of any visibility polygon to the nearest interval. Since we sum up the area of at most $2(n-3) + \frac{2L}{\delta}$ visibility polygons (each of the $n-3$ diagonals in the triangulation of P and $\frac{L}{\delta}$ horizontal/vertical grid lines potentially has at most 2

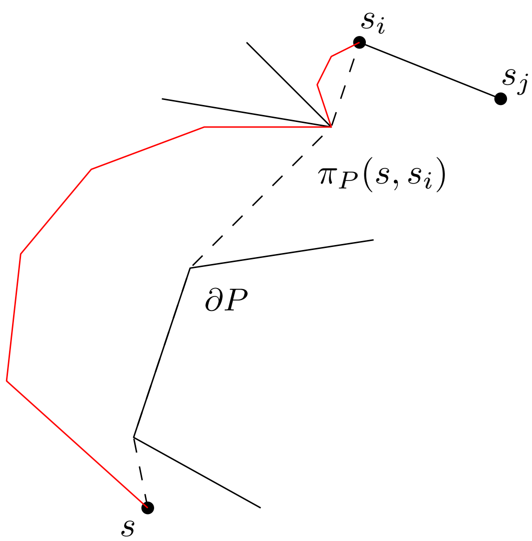

Figure 4 Solving subproblem $(s_j, \overline{A})$.

vertices of the tour returned by the dynamic programming algorithm since we enforce relative convexity), if we denote by I the length of each interval, the area lost by rounding down is at most $I\left(2(n-3)+\frac{2L}{\delta}\right)$. We run the algorithm on the "rounded down" instance with area quota $A - I\left(2(n-3)+\frac{2L}{\delta}\right)$, and since the optimal solution of the original instance is a feasible solution of the new instance, the algorithm returns a tour no longer than an optimal tour γ' (that sees at least area A).

It remains to compute an appropriate L such that an optimal tour γ is contained within the bounding box of the grid. Denote by $C_g(r)$ the geodesic disk of radius r centered at s (the locus of points whose length of the geodesic path to s is no greater than r). Let $r := r_{\min}$ where $r_{\min}$ is the smallest value of r such that $|V(C_g(r))| = A$; then, r is a lower bound on $|\gamma|$, since a tour of length r has geodesic radius at most $r/2$ and thus cannot see an area of A. We can compute r by the "visibility wave" methods in [1] by considering all $O(n^2)$ edges of the visibility graph G_v of P (nodes are vertices of P and two nodes are adjacent if they are visible to one another); we have a sequence of visibility edges hit by $C_g(r)$ for the first time in the process of increasing r, obtained by sorting the distance from every visibility edge to s in $O(n^2 \log n)$ time.

Moreover, $|\partial C_g(r)| + 2r$ is an upper bound on $|\gamma|$, since if the watchman goes from s to $\partial C_g(r)$ (s may not be on $\partial C_g(r)$), follows along $\partial C_g(r)$ then goes back to s, he sees an area of A. We argue that $|\partial C_g(r)| + 2r = O(nr)$ as follows: $\partial C_g(r)$ consists of polygonal chains that are portions of ∂P and circular arcs; the circular arcs have total length at most $2\pi r$. For each segment in the polygonal part of $\partial C_g(r)$, we can bound its length by the sum of geodesic distances from its endpoints to s (triangle inequality), which is no more than $2r$. There are at most n segments in the polygonal portions of $\partial C_g(r)$, therefore their total length is no longer than $2nr$, implying $|\gamma| \le |\partial C_g(r)| + 2r = 2nr + 2\pi r + 2r \le 6nr$.

We initialize $L := r$ and run the dynamic program with the following δ and I:

$$\delta = \frac{\varepsilon_1 L}{16 + 8\sqrt{2}n}, \qquad I = \frac{\varepsilon_1 \varepsilon_2 A}{2(n-3)\varepsilon_1 + (32 + 16\sqrt{2})n}.$$

Then, set $L := 2L$, and repeat until $L \ge 6nr$. At some point, we must have $|\gamma| \le L \le 2|\gamma|$, which means the approximate tour γ' returned by the dynamic program will be such that $V|\gamma'| \ge (1-\varepsilon_2)|V(\gamma)|$ and $|\gamma'| \le (1+\varepsilon_1)|\gamma|$. We return the shortest tour out of all tours that achieve the visibility area quota as we increase r. Since $r \leq |\gamma| = O(nr)$, the number of iterations of the doubling search is $O(\log n)$. The resulting theorem (proof details in the full version):

▶ **Theorem 7.** *Given a starting point s, a dual approximation γ' to an optimal solution γ of the QWRP, with area quota A, in a simple polygon with n vertices, satisfying $|V(\gamma')| \geq (1-\varepsilon_2)A$ and $|\gamma'| \leq (1+\varepsilon_1)|\gamma|$ for any $\varepsilon_1, \varepsilon_2 > 0$, can be computed in $O\left(\frac{n^5}{\varepsilon_1^5\varepsilon_2}\log n\right)$ time if $s \in \partial P$ or $O\left(\frac{n^9}{\varepsilon_1^9\varepsilon_2}\log n\right)$ time if $s \notin \partial P$.*

3 The BWRP in a Simple Polygon

▶ **Theorem 8** (proof in the full version)**.** *The BWRP in a simple polygon is weakly NP-hard.*

3.1 Approximation algorithm for anchored BWRP

For a given budget length $B > 0$, and any fixed $\varepsilon > 0$, we compute a route of length at most $(1+\varepsilon)B$ that sees at least as much area as an area-maximizing route γ of length B.

▷ Claim 9. Without loss of generality, we can assume that B is less than the length of an optimal watchman route for P; otherwise, the solution is simply an optimal WRP tour. Hence, an optimal budgeted watchman route γ is necessarily the shortest watchman route of the subpolygon $V(\gamma)$, and so γ is relatively convex (otherwise, we can shortcut γ and expand the remaining length budget to see more area, contradicting the optimality of γ).

Since it suffices to consider only relatively convex routes for the BWRP, the observation in Lemma 5 allows us to prove the structure of an optimal BWRP tour.

▶ **Lemma 10.** *C is a relatively convex polygonal chain from s to s_j of length at most $\overline{B}$ that sees the largest area possible if and only if C_{s_i} is a relatively convex polygonal chain from s to s_i of length at most $\overline{B} - |s_is_j|$ that sees the largest area possible.*

Proof. The proof is identical to that of Lemma 6 and hence omitted in this version. ◀

We decompose P by overlaying a triangulation and regular square grid of δ-sized pixels within an axis-aligned square of size B-by-B, centered on s, then sort the set of candidates $S_{\delta,B}$ according to the geodesic angular order defined for the QWRP. Similarly, there exists a route γ' of length at most $|\gamma| + (8+4\sqrt{2})\delta n$ with vertices in $S_{\delta,B}$.

Let $a(s_j, \overline{B})$ be the optimal area that a relatively convex polygonal chain from s to s_j that is no longer than $\overline{B}$ can see; and let $C(s_j, \overline{B})$ be the associated optimal chain of the subproblem $(s_j, \overline{B})$. Initialize $a(s, 0) = |V(s)|$. The Bellman recursion for each subproblem with $j = 1, 2, \ldots, m$ and all values of $\overline{B}$ would be given as follows, for all $\bar{i} < j$ such that s_j sees $s_{\bar{i}}$ and $C(s_{\bar{i}}, \overline{B} - |s_{\bar{i}}s_j|) \cup s_{\bar{i}}s_j$ is relatively convex

$$i = \arg\max_{\bar{i}} \left\{a(s_{\bar{i}}, \overline{B} - |s_{\bar{i}}s_j|) + |V(s_{\bar{i}}s_j) \setminus V(\pi_P(s, s_{\bar{i}}))|\right\}$$

$$a(s_j, \overline{B}) = a(s_i, \overline{B} - |s_is_j|) + |V(s_is_j) \setminus V(\pi_P(s, s_i))|,$$

$$C(s_j, \overline{B}) = C(s_i, \overline{B} - |s_is_j|) \cup s_is_j.$$

Then, return $\gamma' := C(s_m, B + (8+4\sqrt{2})\delta n)$.

To bound the number of subproblems, we consider a partition of an interval of length $B + (8+4\sqrt{2})\delta n$ into uniform intervals, and round up the length of any segment s_is_j to the nearest interval endpoint. Let I be the length of each interval, we run the algorithm on a new instance with budget $B + (8+4\sqrt{2})\delta n + \left(2(n-3) + \frac{2B}{\delta}\right) I$ and the subproblems defined by intervals' endpoints. The optimal solution of the original instance is a feasible solution of the new instance; thus, we find a route seeing as much as the optimal route of the original instance. The values of δ and I can be set as follows:

$$\delta = \frac{\varepsilon B}{(16+8\sqrt{2})n}, \qquad I = \frac{\varepsilon^2 B}{4(n-3)\varepsilon + (64+32\sqrt{2})n},$$

so that $B + (8+4\sqrt{2})\delta n + \left(2(n-3) + \frac{2B}{\delta}\right) I \leq (1+\varepsilon)B$. In the full paper we prove:

▶ **Theorem 11.** *Given a starting point s, a route of length at most $(1+\varepsilon)B$ seeing at least as much area as is seen by an optimal route of length B for the BWRP in a simple polygon with n vertices can be computed in $O\left(\frac{n^5}{\varepsilon^6}\right)$ time if $s \in \partial P$ or $O\left(\frac{n^9}{\varepsilon^{10}}\right)$ time if $s \notin \partial P$.*

3.2 From anchored BWRP to an FPTAS for anchored QWRP

We can adapt the algorithm for the anchored BWRP above to obtain an FPTAS for the anchored QWRP. Let γ be an optimal QWRP tour, suppose we have some L such that $|\gamma| \leq L \leq 2|\gamma|$. We divide L into $\frac{L}{\lceil \frac{2}{\varepsilon} \rceil}$ uniform intervals, each of length no greater than $\varepsilon|\gamma|$. The smallest interval endpoint L' that is no smaller than $|\gamma|$, is a $(1+\varepsilon)$-approximation to $|\gamma|$. We can iterate through interval endpoints as the budget constraint and use the approximation algorithm for the BWRP to compute a route of length at most $(1+\varepsilon)L'$ that sees as much as an area-maximizing route of length L' does, which sees more area than does γ. The result is the following theorem, which, in contrast with the earlier Theorem 7, is not a dual approximation (allowing for a relaxation of the quota constraint), but an FPTAS for optimizing the length, subject to a hard constraint on the area seen. The running time of the algorithm in the dual approximation of Theorem 7, however, is better than that of the FPTAS, so it may be preferred in some settings.

▶ **Theorem 12** (proof in the full version)**.** *Given a starting point s, an approximation γ' to an optimal solution γ of the QWRP, with area quota A, in a simple polygon with n vertices, satisfying $|V(\gamma')| \geq A$ and $|\gamma'| \leq (1+\varepsilon)|\gamma|$ for any $\varepsilon > 0$, can be computed in $O\left(\frac{n^5}{\varepsilon^6}\log n \log \frac{1}{\varepsilon}\right)$ if $s \in \partial P$ or $O\left(\frac{n^9}{\varepsilon^{10}}\log n \log \frac{1}{\varepsilon}\right)$ if $s \notin \partial P$.*

4 Floating QWRP and BWRP

When the starting point s of the tour is not specified (the so called "floating" case), the WRP tends to be trickier: known algorithms for the floating WRP are $O(n)$-factor slower than in the non-floating case [10, 5, 11, 30, 13, 26, 29]. If the optimal tour is not convex (but only relatively convex), one can iterate through all reflex vertices of P as choices for s, and thus reduce the floating version to the basic WRP; the same can be done for QWRP and BWRP. Thus, the remaining challenge is to find the shortest (strictly, not just relatively) convex tour.

Any convex polygon can be outer-approximated by a convex polygon with a constant number of vertices: Dudley's approximation [2, 14, 23] implies that for any length-L tour γ, there is a length-$(1+\varepsilon)L$ tour γ' with $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ vertices that sees at least as much area as γ does. To find γ' (either for QWRP or BWRP), we use the techniques from [26]: For each of the $O(n^4)$ cells of the visibility decomposition D (the visibility graph G_v of P as defined Section 2.3, with maximally extended edges), points within the cell have visibility polygons that are combinatorially equivalent, implying that the area seen by any point in

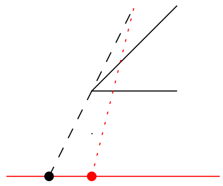

Figure 5 What is seen from the interior of a segment does not change as the segment is moved locally: whatever was hidden, but becomes seen from an interior point (red), was seen by a neighboring point (black) before the move.

the cell is given by the same formula. Moreover, if each vertex of a tour sits within a fixed cell of D and each edge of the tour passes through the same set of cells, the total area seen from the tour is given by the same formula: the interiors of the edges do not add to the area seen, so the total seen area is a function, $f(v_1, \ldots, v_k)$, of only the positions v_i of the tour's $k = O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ vertices (Fig. 5). We further decompose the cells of D by lines through all of D's vertices. If the vertices of the tour are in the same cells of this refined $O(n^8)$-complex decomposition D', then the edges of the tour pass through the same cells of D. We iterate through all $O(n^{8k})$ placements of vertices of the tour into cells of D'. For each placement, finding $\vec{v}$ maximizing the seen area, $f(\vec{v})$, amounts to solving an $O(k)$-sized system of polynomial equations having $O(k)$ algebraic degree (the Lagrangian of the problem will contain the constraint that the tour's length is L, consisting of k terms and will have to be squared $O(k)$ times before becoming a polynomial). The solution can be found in $k^{O(k)}$ time [4, Section 3.4]. We summarize in the following theorems:

▶ **Theorem 13.** *Let γ be an optimal QWRP (no starting point) tour. In $n^{O\left(\frac{1}{\sqrt{\varepsilon}}\right)}$ time, a tour of length at most $(1+\varepsilon)|\gamma|$ can be found that sees at least as much area as does γ.*

▶ **Theorem 14.** *In $n^{O\left(\frac{1}{\sqrt{\varepsilon}}\right)}$ time, a BWRP (no starting point) tour of length $(1+\varepsilon)L$ can be found that sees at least as much area as does any tour of length L.*

5 Domains that are a Union of Lines

We consider the QWRP and the BWRP in a domain P that is a connected union (arrangement) of lines; it suffices to truncate the lines within a bounding box that encloses all vertices of the line arrangement, so that P is bounded. Such domains, which are a special case of polygons with holes, have been studied in the context of the Watchman Route Problem [15, 16]. In this setting, the QWRP seeks to minimize the length of a route contained within P that visits at least a specified number of (truncated) lines, and the BWRP seeks to maximize the number of lines visited by a route within P of length at most some budget. In this section we do not assume a depot s is specified.

For a set of lines $\mathcal{L}$, let $\mathcal{A}(\mathcal{L})$ denote the arrangement formed by $\mathcal{L}$. We assume that not all of the lines are parallel so that the union is connected. All of a line can be seen from any point incident on it. Let $G(\mathcal{L})$ denote the weighted planar graph with vertex $V(\mathcal{A}(\mathcal{L}))$ of intersections between lines. Two vertices in the graph are connected by an edge with Euclidean weight if they share the same edge in $\mathcal{L}$. Since the watchman is constrained to travel within $\mathcal{A}(\mathcal{L})$ and the only time the route can have turning points not in $V(\mathcal{A}(\mathcal{L}))$ is when it traces out the same edge of $G(\mathcal{L})$ consecutively in opposite directions, turning somewhere interior to that edge. However, then we can shortcut that portion of the route altogether while maintaining visibility coverage, it is easy to see that any optimal route for BWRP or QWRP is polygonal and its vertices is a subset of $V(\mathcal{A}(\mathcal{L}))$.

We first explain our results for the QWRP, then we use them to solve the BWRP. The following observation is essential to our algorithm: a line intersects a tour γ if and only if it intersects the convex hull of γ. We define the *quota intersecting convex hull* problem as follows: compute a cyclic sequence $(v_1, v_2, \ldots, v_h)$ of vertices $v_i \in V(\mathcal{A}(\mathcal{L}))$ in convex position such that the number of lines intersecting the convex polygon $(v_1, v_2, \ldots, v_h)$ is at least some specified $Q > 0$ and $\sum_i^h |\pi(v_i, v_{i+1})|$ is minimized ($v_{h+1} = v_1$), where $\pi(s,t) = \pi_G(s,t)$ is the shortest path connecting s and t in $G(\mathcal{L})$. We show the relationship between the quota intersecting convex hull problem and the QWRP in an arrangement of lines.

▶ **Lemma 15.** *An optimal solution to the quota intersecting convex hull problem yields an optimal solution to the QWRP in an arrangement of lines.*

Proof. Suppose $(v_1, v_2, \ldots, v_h)$ is an optimal solution to the quota intersecting convex hull problem of length L intersecting Q lines. We concatenate $\pi(v_1, v_2), \ldots, \pi(v_{h-1}, v_h)$ and $\pi(v_h, v_1)$ to form γ. Every line intersecting the convex polygon $(v_1, v_2, \ldots, v_h)$ must intersect γ as well. Thus, γ is a route of length $\sum_i^h |\pi(v_i, v_{i+1})| = L$ seeing Q lines.

We claim that there is no solution γ' to the QWRP intersecting Q lines that is strictly shorter than γ. Suppose to the contrary, take the convex hull of γ', which has vertices in $V(\mathcal{A}(\mathcal{L}))$ since vertices of γ' are in $V(\mathcal{A}(\mathcal{L}))$. The vertices of the convex hull of γ' form a cyclic sequence that is feasible for the quota intersecting convex hull problem, and the length is exactly $|\gamma'|$ (or γ' could be shortened while still intersecting Q lines), which is strictly smaller than L. Thus, γ' yields a feasible cyclic sequence intersecting Q lines while the length is shorter than γ, violating the assumption that $(v_1, v_2, \ldots, v_h)$ is optimal. ◀

Note that there can be many optimal solutions to the quota intersecting convex hull problem yielding the same tour (Figure 6).

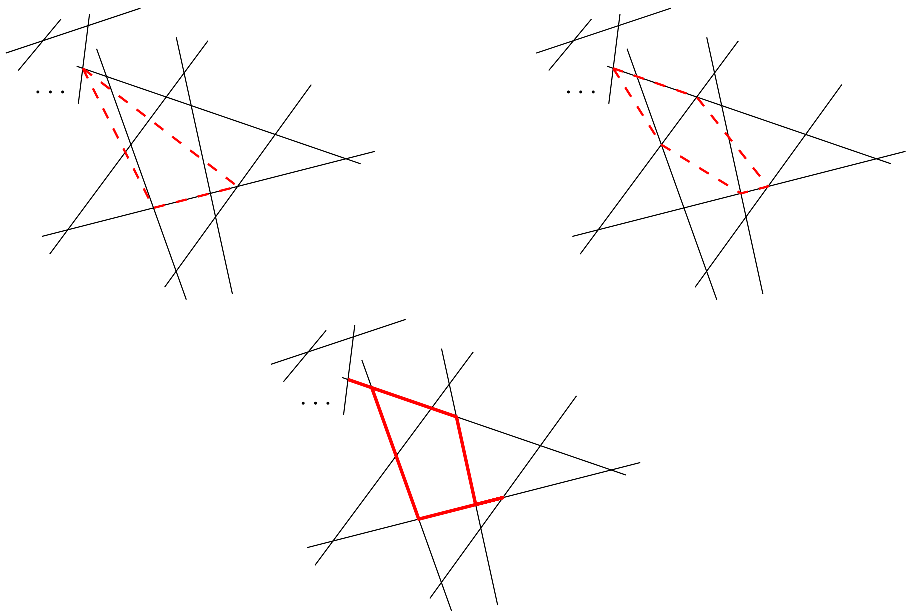

Figure 6 Multiple optimal solutions to the quota intersecting convex hull problem corresponding to the same optimal solution of the QWRP.

We give a dynamic programming algorithm to solve the quota intersecting convex hull problem. Fix one vertex to be the lowest vertex, let that vertex be v_1. We will examine all possible choices of v_1, and find the optimal cyclic sequence with each choice. Let $\{v_2, v_3, \ldots, v_{m-1}\}$ be the list of vertices above v_1 sorted by increasing angle with the left horizontal ray passing through v_1, breaking ties by increasing distance to v_1. Then, set a new element $v_m := v_1$ and append it to the list. Thus, an optimal cyclic sequence $(v_1, v_{i_2}, v_{i_3}, \ldots, v_m)$ has $1 < i_2 < i_3 < \ldots < m$. We can restrict ourselves to ordered pairs (v_i, v_j) of consecutive vertices where $1 \le i < j \le m$ and either $i \ne 1$ or $j \ne m$ in the sequence.

For $1 \le i < j \le m$ and either $i \ne 1$ or $j \ne m$, denote by $\mathcal{L}_{i,j}$ the lines that intersect the segment $v_i v_j$ (including at the endpoints v_i, v_j). Each vertex v_j and a quota value $\overline{Q}$ define a subproblem. Let $\pi(v_j, \overline{Q})$ be the shortest length of a sequence of vertices in convex position from $(v_1, \ldots, v_j)$ intersecting at least $\overline{Q}$ lines. Starting from v_1, we initialize $\pi(v_1, |\mathcal{L}_{1,1}|) = 0$ with the associated sequence (v_1). For $j = 2, \ldots, m$ and $\overline{Q} = 1, 2, \ldots, n$, we solve the subproblems $(v_j, \overline{Q})$ by the following Bellman recursion, for all $i < j$ such that the sequence associated with $(v_i, \overline{Q} - |\mathcal{L}_{i,j} \setminus \mathcal{L}_{1,i}|)$ and v_j are in convex position (Figure 7)

$$\pi(v_j, \overline{Q}) = \min_i \left\{ \pi(v_i, \overline{Q} - |\mathcal{L}_{i,j} \setminus \mathcal{L}_{1,i}|) + |\pi(v_i, v_j)| \right\}.$$

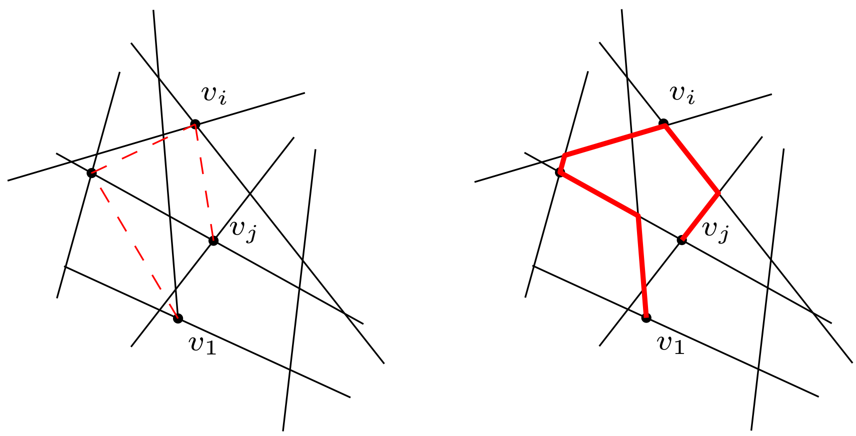

Figure 7 Solving subproblem $(v_j, \overline{Q})$. The sequence of vertices in convex position is drawn on the left with a dashed red chain, and the corresponding part of γ is drawn with solid red segments on the right.

Correctness of the algorithm follows from these two claims:

- Given a sequence of vertices in convex position $(v_1, \ldots, v_i, v_j)$, any line intersecting both $\pi(v_i, v_j)$ and the subsequence from v_1 to v_i, $(v_1, \ldots, v_i)$ must intersect the segment $v_1 v_i$ and vice versa due to continuity and convexity. If a sequence $(v_1, \ldots, v_i, v_j)$ is the shortest among all sequences from v_1 to v_j intersecting at least $\overline{Q}$ lines, then the subsequence from v_1 to v_i is the shortest sequence from v_1 to v_i intersecting $\overline{Q} - |\mathcal{L}_{i,j} \setminus \mathcal{L}_{1,i}|$.
- The sequence $(v_1, \ldots, v_i)$ associated with optimal solution i to the Bellman recursion is such that $(v_1, \ldots, v_i) \cup (v_j)$ are in convex position.

▶ **Theorem 16** (proof in the full version). *The QWRP and the BWRP in an arrangement of lines can be solved in $O(n^7)$ time.*

▶ Remark 17. When $Q = n$, the algorithm solves the classic WRP in an arrangement of lines, thus our result improves upon the $O(n^8)$ solution in [16].

6 The QWRP and BWRP in a Polygon With Holes

6.1 Hardness of approximation

▶ **Theorem 18** (proof in the full version). *The QWRP in a polygon with holes cannot be approximated, in polynomial time, within a factor of* $c \log n$ *for some constant* $c > 0$*, unless* $P = NP$*.*

▶ **Theorem 19** (proof in the full version). *The BWRP in a polygon with holes cannot be approximated, in polynomial time, within a factor of* $(1 - \varepsilon)$ *for arbitrary* $\varepsilon > 0$*, unless* $P = NP$*.*

6.2 Approximation algorithm for the BWRP in a polygon with holes

We decompose P into small convex cells and obtain the set of candidates $S_{\delta,B}$ as in the case with the BWRP in a simple polygon.

▶ **Theorem 20.** *There exists a route* γ' *whose vertices are a subset of* $S_{\delta,B}$ *such that* $V(\gamma) \subseteq V(\gamma')$ *and* $|\gamma'| \leq (1 + \varepsilon)B$*.*

Proof. Consider an edge e of γ whose endpoints lie in cells σ_i and σ_j. We append $\partial\sigma_i$ and $\partial\sigma_j$ to γ. If the endpoints of e are not vertices of σ_i and σ_j, we replace e with a set of edges whose endpoints are candidate points. The procedure can be described with a physical analog: we slide an elastic string between the two intersection points of e with σ_i and σ_j towards the exterior of γ until the two endpoints coincide with vertices of σ_i and σ_j, the string is pulled taut and never passes through a hole; see Figure 8. The result is a geodesic path e' between two vertices of σ_i and σ_j that is no longer than $|e| + 2\sqrt{2}\delta$.

Repeating the process for every edge of γ, we obtain γ'. If a point x seen by γ is inside of $P_{\gamma'}$, the extended line of vision between x and γ must intersect with γ' since there is no hole between γ and γ'. If x is outside of $P_{\gamma'}$, then the line of vision between x and γ must intersect γ' due to the Jordan Curve Theorem. Thus, γ' sees everything that γ sees, moreover γ' passes through s, a vertex in the decomposition. Since γ has $O(n^2)$ vertices [25], for an appropriate choice of $\delta = O\left(\frac{\varepsilon B}{n^2}\right)$ we have $|\gamma'| \leq (1 + \varepsilon)B$. ◀

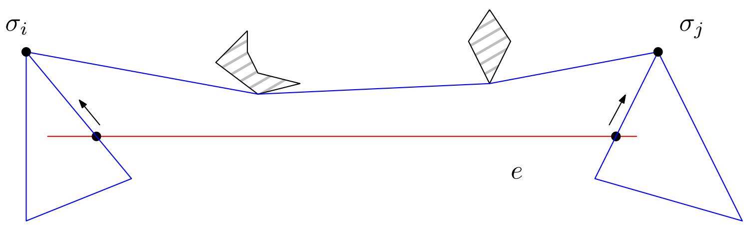

Figure 8 Replacing each edge (red) with the perimeters of the two cells containing its endpoints and a geodesic path of the same homotopy type (blue).

We apply a known result for the Submodular Orienteering problem [7]: Given a weighted directed graph G, two nodes s and t (which need not be distinct), a budget $B > 0$, and a monotone submodular reward function defined on the nodes, find an s-t walk that maximizes the reward, under the constraint that the length of the walk is no greater than B.

Let G_1 be the visibility graph on the candidates set with Euclidean edge weights. Let G_2 be the line graph of G_1: nodes of G_2 correspond to edges of G_1, and two nodes in G_2 are adjacent if their respective edges in G_1 are incident. The weight of an edge of G_2 is the sum of the weights of the two edges in G_1 corresponding to its endpoints, divided by two, thus a closed walk of length B in G_1 corresponds to a closed walk of length B in G_2 and vice versa. We apply the approximation algorithm from [7] on G_2 to compute a closed walk from any node in G_2 corresponding to an edge incident with s, with the area of visibility as the reward function and budget $(1+\varepsilon)B$. The reason for using the line graph G_2 is that, in the Submodular Orienteering problem, rewards are associated with nodes, while in the context of the BWRP, rewards are accumulated when traversing edges of the visibility graph G_1. We obtain the following:

▶ **Theorem 21** (proof in the full version)**.** *Given a polygon P with holes with n vertices, let $\beta \geq 2$ be any constant of choice and OPT be the maximum area that a route of length B can see. The BWRP has a dual approximation algorithm that computes a tour of length at most $(1+\varepsilon)B$ that sees an area of at least $\Omega\left(\frac{OPT \log \beta}{\log n}\right)$, with running time $\left(\frac{n}{\varepsilon} \log B\right)^{O\left(\beta \log \frac{n}{\varepsilon} / \log \beta\right)}$.*

7 Optimal Visibility-based Search for a Randomly Distributed Target

Our results can be applied to solve two problems of searching a randomly distributed static target in a simple polygon P: Given a prior distribution of the target's location in P, (1) compute a route that achieves a given detection probability within the minimum amount of time, where the target is detected if the watchman can see it; and (2) (dual to (1)) for a given time budget T, compute a search route maximizing the probability of detecting the target by time T. Denote by $\mu(.)$ the probability measure on all subsets of P; $\mu(P_1)$ is the probability measure of $P_1 \subseteq P$, i.e the probability that the target is in P_1, then

- $0 \leq \mu(.) \leq 1, \mu(\varnothing) = 0, \mu(P) = 1$,
- $\mu(P_1 \cup P_2) = \mu(P_1) + \mu(P_2)$ if $P_1 \cap P_2 = \varnothing$.

We assume that we have access to $\mu(.)$ via an oracle: Given a triangular region in P, the oracle returns its probability measure in $O(1)$ time. Thus, for a point or a segment, the probability measure of its visibility region can be computed in $O(n)$ time. Furthermore, if the watchman has constant speed, a time constraint/objective is equivalent to that of length. An optimal search route for each problem can be computed using the algorithms given with probability measure instead of area.

References

1. Esther M. Arkin, Alon Efrat, Christian Knauer, Joseph S. B. Mitchell, Valentin Polishchuk, Günter Rote, Lena Schlipf, and Topi Talvitie. Shortest path to a segment and quickest visibility queries. *Journal of Computational Geometry*, 7(2):77–100, 2016.
2. Efim M. Bronshteyn and L. D. Ivanov. The approximation of convex sets by polyhedra. *Siberian Mathematical Journal*, 16(5):852–853, 1975.
3. Kevin Buchin, Valentin Polishchuk, Leonid Sedov, and Roman Voronov. Geometric secluded paths and planar satisfiability. In *36th International Symposium on Computational Geometry (SoCG 2020)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.
4. John Canny. *The Complexity of Robot Motion Planning*. MIT press, 1988.
5. Svante Carlsson, Håkan Jonsson, and Bengt J. Nilsson. Finding the shortest watchman route in a simple polygon. *Discrete & Computational Geometry*, 22:377–402, 1999.
6. Bernard Chazelle. Triangulating a simple polygon in linear time. *Discrete & Computational Geometry*, 6(3):485–524, 1991.
7. Chandra Chekuri and Martin Pal. A recursive greedy algorithm for walks in directed graphs. In *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS'05)*, pages 245–253. IEEE, 2005.

8 Ke Chen and Sariel Har-Peled. The Euclidean orienteering problem revisited. *SIAM Journal on Computing*, 38(1):385–397, 2008.

9 Otfried Cheong, Alon Efrat, and Sariel Har-Peled. Finding a guard that sees most and a shop that sells most. *Discrete & Computational Geometry*, 37:545–563, 2007.

10 Wei-Pang Chin and Simeon Ntafos. Optimum watchman routes. In *Proceedings of the 2nd Annual Symposium on Computational Geometry*, pages 24–33, 1986.

11 Wei-Pang Chin and Simeon Ntafos. Shortest watchman routes in simple polygons. *Discrete & Computational Geometry*, 6(1):9–31, 1991.

12 Timothy H. Chung, Geoffrey A. Hollinger, and Volkan Isler. Search and pursuit-evasion in mobile robotics: A survey. *Autonomous Robots*, 31:299–316, 2011.

13 Moshe Dror, Alon Efrat, Anna Lubiw, and Joseph S. B. Mitchell. Touring a sequence of polygons. In *Proceedings of the 35th Annual ACM Symposium on Theory of Computing*, pages 473–482, 2003.

14 Richard M. Dudley. Metric entropy of some classes of sets with differentiable boundaries. *Journal of Approximation Theory*, 10(3):227–236, 1974.

15 Adrian Dumitrescu, Joseph S. B. Mitchell, and Paweł Żyliński. The minimum guarding tree problem. *Discrete Mathematics, Algorithms and Applications*, 6(01):1450011, 2014.

16 Adrian Dumitrescu, Joseph S. B. Mitchell, and Paweł Żyliński. Watchman routes for lines and line segments. *Computational Geometry*, 47(4):527–538, 2014.

17 Adrian Dumitrescu and Csaba D. Tóth. Watchman tours for polygons with holes. *Computational Geometry*, 45(7):326–333, 2012.

18 James N. Eagle. The optimal search for a moving target when the search path is constrained. *Operations Research*, 32(5):1107–1115, 1984.

19 James N. Eagle and James R. Yee. An optimal branch-and-bound procedure for the constrained path, moving target search problem. *Operations Research*, 38(1):110–114, 1990.

20 Lee-Ad Gottlieb, Robert Krauthgamer, and Havana Rika. Faster algorithms for orienteering and k-tsp. *Theoretical Computer Science*, 914:73–83, 2022.

21 Leonidas Guibas, John Hershberger, Daniel Leven, Micha Sharir, and Robert Tarjan. Linear time algorithms for visibility and shortest path problems inside simple polygons. In *Proceedings of the 2nd Annual Symposium on Computational Geometry*, pages 1–13, 1986.

22 Leonidas J. Guibas, Jean-Claude Latombe, Steven M. LaValle, David Lin, and Rajeev Motwani. Visibility-based pursuit-evasion in a polygonal environment. In *Algorithms and Data Structures: 5th International Workshop, WADS'97 Halifax, Nova Scotia, Canada August 6–8, 1997 Proceedings 5*, pages 17–30. Springer, 1997.

23 Sariel Har-Peled and Mitchell Jones. Proof of Dudley's convex approximation. *arXiv preprint*, 2019. `arXiv:1912.01977`.

24 Joseph S. B. Mitchell. Geometric shortest paths and network optimization. *Handbook of Computational Geometry*, 334:633–702, 2000.

25 Joseph S. B. Mitchell. Approximating watchman routes. In *Proceedings of the 24th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 844–855, 2013.

26 Simeon Ntafos and Markos Tsoukalas. Optimum placement of guards. *Information Sciences*, 76(1-2):141–150, 1994.

27 Joseph O'Rourke. Visibility. In *Handbook of Discrete and Computational Geometry*, pages 875–896. Chapman and Hall/CRC, 2017.

28 Michael Ian Shamos. *Computational Geometry.* Yale University, 1978.

29 Xuehou Tan. Fast computation of shortest watchman routes in simple polygons. *Information Processing Letters*, 77(1):27–33, 2001.

30 Xuehou Tan, Tomio Hirata, and Yasuyoshi Inagaki. Corrigendum to "an incremental algorithm for constructing shortest watchman routes". *International Journal of Computational Geometry & Applications*, 9(03):319–323, 1999.

31 K. E. Trummel and J. R. Weisinger. The complexity of the optimal searcher path problem. *Operations Research*, 34(2):324–327, 1986.