Sparse Cuts in Hypergraphs from Random Walks on Simplicial Complexes

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Abstract

There are a lot of recent works on generalizing the spectral theory of graphs and graph partitioning to k-uniform hypergraphs. There have been two broad directions toward this goal. One generalizes the notion of graph conductance to hypergraph conductance [Louis, Makarychev – TOC'16; Chan, Louis, Tang, Zhang – JACM'18]. In the second approach, one can view a hypergraph as a simplicial complex and study its various topological properties [Linial, Meshulam – Combinatorica'06; Meshulam, Wallach – RSA'09; Dotterrer, Kaufman, Wagner – SoCG'16; Parzanchevski, Rosenthal – RSA'17] and spectral properties [Kaufman, Mass – ITCS'17; Dinur, Kaufman – FOCS'17; Kaufman, Openheim – STOC'18; Oppenheim – DCG'18; Kaufman, Openheim – Combinatorica'20].

In this work, we attempt to bridge these two directions of study by relating the spectrum of *up-down walks* and *swap walks* on the simplicial complex, a downward closed set system, to hypergraph expansion. More precisely, we study the simplicial complex obtained by downward closing the given hypergraph and random walks between its levels $X(l)$, i.e., the sets of cardinality l. In surprising contrast to random walks on graphs, we show that the spectral gap of swap walks and up-down walks between level m and l with $1 < m \leqslant l$ cannot be used to infer any bounds on hypergraph conductance. Moreover, we show that the spectral gap of swap walks between $X(1)$ and $X(k-1)$ cannot be used to infer any bounds on hypergraph conductance. In contrast, we give a Cheeger-like inequality relating the spectra of walks between level 1 and l for any $l \leqslant k$ to hypergraph expansion. This is a surprising difference between swaps walks and up-down walks!

Finally, we also give a construction to show that the well-studied notion of *link expansion* in simplicial complexes cannot be used to bound hypergraph expansion in a Cheeger-like manner.

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1 Introduction

In recent years, there have been two broad directions of generalizations of graph partitioning and the spectral theory of graphs to hypergraphs. One direction attempts to generalize the notion of conductance in graphs to hypergraphs [23, 8]. The graph expansion (also referred to as graph conductance) is defined as

$$\phi_G \stackrel{\text{def}}{=} \min_{\substack{S\subseteq V \\ \mathsf{vol}_G(S)\leqslant \frac{\mathsf{vol}_G(V)}{2}}} \phi_G(S), \text{ where } \phi(S) \stackrel{\text{def}}{=} \frac{w(\partial_G(S))}{\mathsf{vol}_G(S)}$$

with $\mathsf{vol}_G(S)$ being the sum of degrees of the vertices in S and $\partial_G(S)$ being the edges crossing the boundary of the set S, hence $w(\partial_G(S))$ is the sum of weights of the edges on the boundary. Analogously, the hypergraph expansion/conductance is defined as

$$\phi_H \stackrel{\text{def}}{=} \min_{\substack{S\subseteq V \\ \mathsf{vol}_H(S)\leqslant \frac{\mathsf{vol}_H(V)}{2}}} \phi_H(S), \text{ where } \phi_H(S) \stackrel{\text{def}}{=} \frac{\Pi\left(\partial_H(S)\right)}{\mathsf{vol}_H(S)}$$

with $\mathsf{vol}_H(S)$ being the sum of degrees of the vertices in S, and $\partial_H(S)$ being the edges crossing the boundary of the set S, and $\Pi(\partial_H(S))$ is the sum of the weight of edges on the boundary.

Another direction views a hypergraph as a *simplicial complex*, a downward closed set system, and studies its various topological properties [22, 24, 12, 26] and spectral properties [19, 11, 20, 21, 25]. The work [11] introduced a generalization of random walks on graphs to random walks over the faces[1] of the simplicial complex; this random walk has found numerous applications in a myriad of other problems [11, 9, 4, 3, 1], etc., to state a few.

There has been a lot of work on understanding the relationship between random walks on graphs (including the spectra of the random walk operator) and graph partitioning. The celebrated Cheeger's inequality gives one such relation between the graph expansion and the second eigenvalue of the random walk matrix λ_2 as,

$$\frac{1-\lambda_2}{2} \leqslant \phi_G \leqslant \sqrt{2(1-\lambda_2)}.$$

In this work, we aim to bridge the gap between these two directions by studying the relationship between hypergraph expansion and random walks on the corresponding simplicial complex.

In a seminal work, [5] showed that if a graph has a "small" threshold rank[2], then they can compute a near-optimal assignment to unique games in time exponential in the threshold rank. The works [7, 15] gave an SoS hierarchy-based algorithm generalizing this result to any 2-CSP. The work [2] introduces the notion of *swap walks* and uses that to define a notion of threshold rank for simplicial complexes. Using their notion of threshold rank, they generalized the results of [7, 15] to k-CSPs. Further, [5] showed that large threshold rank graphs must have a small non-expanding set (they also gave a polynomial time algorithm to compute such a set). A natural open question from the work of [2, 17] is whether hypergraphs with large threshold rank (the hypergraph analogue is called non-splittability) have a small, non-expanding set. Our first result answers this question negatively.

[1] The faces (the hyperedges) here may belong to different levels. A level $X(l)$ denotes the set of hyperedges of cardinality l.
[2] the number of "large" eigenvalues of the adjacency matrix, see Definition 30 for formal definition.

▶ **Theorem 1** (Informal Version of Theorem 34 and Corollary 35). *For any $n \geqslant 6, k \geqslant 3$, there exists a k-uniform hypergraph H with at least n vertices such that $\phi_H \geqslant \frac{1}{k}$ but for any m, l, if either $m, l \geqslant 2$ or $m = k - l$, the swap walk from $X(m)$ to $X(l)$ has threshold rank at least $\Omega_k(n)$ (for any $\tau \in [-1, 1]$ as choice of threshold). Moreover, H is not $(\tau, \Omega_k(n))$-splittable for any $\tau \in [-1, 1]$.*

For a splittable hypergraph, there is some l, such that the swap walk graph between $X(l)$ and $X(k-l)$ has low threshold rank. Then, it follows from Theorem 1 that there are non-splittable expanding hypergraphs (see Corollary 35 for the precise statement).

[2, 9] show that for a high dimensional expander (HDX)[3] the swap walks indeed have a large spectral gap[4]. However, we are interested in the case when the hypergraph instance is not an HDX. One recalls that for a non-expanding graph, Cheeger's inequality and Fiedler's algorithm allow us to compute a combinatorial sparse cut in the graph. Similarly, we ask whether one can compute a sparse cut in the input hypergraph in this setting.

Unfortunately, in the light of Theorem 1, computing a sparse cut in the hypergraph when swap walks (in the setting studied by [2, 17]; see Theorem 34 for the precise statement) have a small spectral gap is generally not possible. This is in surprising contrast to the case of graphs where the swap walk reduces to the usual random walk, and the second largest eigenvalue of the random walk matrix is related to graph expansion via Cheeger's inequality.

Next, we investigate whether the spectral gap of the up-down walk introduced by [11] can be related to hypergraph expansion. More formally, we investigate whether the spectral gap of the up-down walk between levels $X(m)$ and $X(l)$ $(l > m)$ be related to the hypergraph expansion in a Cheeger-like manner. Here, the answer depends on m and l. We first show that if $m \geqslant 2$, then no such relation is possible.

▶ **Theorem 2** (Informal Version of Theorem 36). *For any positive integers n, k with $n \geqslant 6, k \geqslant 3$, there exists a k-uniform hypergraph H on at least n vertices such that $\phi_H \geqslant \frac{1}{k}$ and for all positive integers $2 \leqslant m < l \leqslant k$ the threshold rank of the up-down walk matrix between levels $X(m)$ and $X(l)$ is at least $\Omega_k(n)$ (for any $\tau \in [-1, 1]$ as choice of threshold).*

Contrasting this, we show that if $m = 1$, then such a relationship is indeed possible.

▶ **Theorem 3** (Informal Version of Theorem 18). *Given a hypergraph, where the second largest eigenvalue of the up-down walk matrix (of simplicial complex induced by the hypergraph) between levels $X(1)$ and $X(l)$, for some $l \in [k]$ is $1 - \varepsilon$ we have $\frac{\varepsilon}{k} \leqslant \phi_H \leqslant 4\sqrt{\varepsilon}$. Furthermore, there exists a polynomial time algorithm which, when given such a hypergraph, outputs a set S such that its expansion in the hypergraph $\phi_H(S) \leqslant 4\sqrt{\varepsilon}$.*

Theorem 3 and Theorem 1 also show a surprising difference between up-down walks and swap walks whereby we can compute sparse cut on the hypergraph using up-down walk from $X(1)$ to $X(l), l \in [k]$ using a Cheeger-like inequality, whereas it is not possible (in general) to compute a sparse cut by considering the spectrum of swap walks from $X(1)$ to $X(k-1)$.

Yet another notion of spectral expansion called *link expansion* of a simplicial complex has been studied recently in many works [19, 11, 20, 21, 25] having applications in [11, 9, 4, 3, 1] (see Definition 9 for formal definition). Our final result shows that hypergraphs with large hypergraph expansion and arbitrarily small link expansion exist. Therefore, hypergraph expansion cannot be bounded by link expansion in a Cheeger-like manner.

[3] For formal definition see Definition 9.

[4] For a linear operator $\mathsf{A} : V \to W$ where $V \neq W$ the spectral gap refers to $\sigma_1(A) - \sigma_2(A)$, while for a linear operator $B : V \to V$, it refers to $\lambda_1(A) - \lambda_2(A)$.

▶ **Theorem 4** (Informal Version of Theorem 43). *Let n, k be any positive integers such that $n \geqslant 3k$ and $k \geqslant 3$, there exists a k-uniform hypergraph H on $n + k - 2$ vertices such that the link expansion of the induced simplicial complex X is at most $\mathcal{O}(\frac{1}{n^2})$ and the expansion of H is at least $\Omega_k(1)$.*

To the best of our knowledge, this is the first construction to show this.

The work [23] (see Remark 1.9) used an example similar in spirit to our constructions to show that another notion of expansion on simplicial complexes called co-boundary expansion is incomparable to the hypergraph expansion. In particular, they constructed a class of k-uniform hypergraphs, each with co-boundary expansion (at dimension k) as one but containing hypergraphs with essentially arbitrary hypergraph expansion. Still, [23] did not give an explicit example that shows a separation between hypergraph expansion and quantities like the link expansion, spectral gap, or threshold rank of the random walks on a simplicial complex (i.e., up-down walk, swap walk).

The m-dimensional co-boundary expansion may also seem related to the expansion of the up-walk from the level $m - 1$ to m as both of these consider the ratio of the number of m-dimensional faces containing a set of $m - 1$-dimensional faces to the volume of the set with the only difference being how the volume is computed. Yet, we do not know if such a relation exists. One may similarly compare the expansion of the down-walk and the boundary expansion. But still, Steenbergen, Klivian, and Mukherjee [28] and Gundert and Wagner [14] were able to show that for the m-dimensional co-boundary expansion no Cheeger-type inequality can be shown, whereas such a relation is immediate from Cheeger's inequality in case of up-walk. Nevertheless, [28] obtained (under some minor assumptions) an extension of Cheeger's inequality on the m-dimensional boundary expansion. Finally, [10] showed that the operator norm of the difference between up-down and down-up walks between two consecutive levels is within an $\mathcal{O}(k)$ factor of link expansion. In contrast, no such relation between up-Laplacian, down-Laplacian (see [28] for definition) and link expansion is known.

1.1 Additional Related Works

The work [8] generalized the Laplacian of graphs to hypergraphs by expressing the graph Laplacian in terms of a non-linear diffusion process. They showed an analogue of Cheeger's inequality relating the expansion of the hypergraph to the second smallest eigenvalue of the Laplacian. Yoshida [30] introduced the notion of submodular transformations and extended the notions of degree, cut, expansion, and Laplacian to them. They derived the Cheeger's inequality in this setting. This generalizes Cheeger's inequality on graphs and hypergraphs (as in [8]) while showing similar inequalities for entropy.

There are also several works exploring Cheeger-like inequalities for simplicial complexes. Parzanchevski, Rosenthal, and Tessler [27] defined the notion of Cheeger constant $h(X)$ for a simplicial complex, a generalization of the sparsity of a graph. The quantity $h(X)$ is the minimum over all partitions of the vertex set V into k sets the fraction of k-dimensional faces present crossing the partition compared to the maximum possible k-dimensional faces crossing the partition. They also showed that for simplicial complex X with a complete skeleton $h(X) \geqslant \lambda(X)$ where $\lambda(X)$ is the link expansion of the simplicial complex. Gundert and Szedlák [13] extended this result to any simplicial complex. Very recently, Jost and Zhang [18] extended the Cheeger-like inequality for bipartiteness ratio[5] on graphs due to Trevisan [29] to a cohomology based definition of bipartiteness ratio for simplicial complexes.

[5] The bipartiteness ratio of G is defined as $\beta_G = \min_{S \subseteq V, L \sqcup R = S} \frac{2\partial(L) + 2\partial(R) + \partial(S)}{\mathsf{vol}_G(S)}$.

In the case of an HDX, Bafna, Hopkins, Kaufmann, and Lovett [6] consider high-dimensional walks (a generalization of swap walks and up-down walks) on levels $i < k$. They then relate the (non-)expansion of a link[6] of a level-j face (with $j \leqslant i$) in the graph corresponding to the walk and level-j approximate eigenvalue of the walk. Here λ_j is the level-j approximate eigenvalue of a high-dimensional walk M if there is a function f_j such that $\|\mathsf{M} f_j - \lambda_j f_j\| \leqslant O(\sqrt{\gamma})\, \|f_j\|$ and $f_i = \mathsf{U}^{i-j} g$ where $g \in \mathbb{R}^{X(j)}$.

1.2 Preliminaries

1.2.1 Simplicial Complexes

▶ **Definition 5.** *A simplicial complex X is a set system that consists of a ground set V and a downward closed collection of subsets of V, i.e., if $s \in X$ and $t \subseteq s$ then $t \in X$. The sets in X are called the faces of X.*

We define a level/slice $X(l)$ of the simplicial complex X as $X(l) = \{s \in X || s| = l\}$. Note that for the simplicial complex corresponding to the hypergraph, the top level $X(k)$ is the set of k-uniform hyperedges and the ground set of vertices[7] is denoted by $X(1)$. By convention we have that $X(0) = \{\emptyset\}$. Similarly, we define $X(\leqslant l) = \{s \in X || s| \leqslant l\}$.

We call a simplicial complex X as k-dimensional if k is the smallest integer for which $X(\leqslant k) = X$.[8] A k-dimensional simplicial complex X is a pure simplicial complex if for all $s \in X$ there exists $t \in X(k)$ such that $s \subseteq t$.

▶ Remark 6. We note that our definition of dimension deviates slightly from the standard definition. In the standard definition, the dimension is the cardinality of the largest face minus 1.

Given a k-uniform hypergraph $H = (V, E)$, we obtain a pure simplicial complex X where the ground set is V and downward close the set system E of hyperedges. Given a distribution Π_k on the hyperedges, we have an induced distribution Π_l on sets s in level $X(l)$ given by $\Pi_l(s) = \frac{1}{\binom{k}{l}} \sum_{e \in E | s \subseteq e} \Pi_k(e)$. We refer to the joint distribution as $\Pi = (\Pi_k, \Pi_{k-1}, \ldots, \Pi_1)$. If the input hypergraph is unweighted, then we take the distribution Π_k to be the uniform distribution on $X(k)$. We thus obtain a weighted simplicial complex (X, Π). We refer to (X, Π) as the (weighted[9]) simplicial complex *induced* by (H, Π_k).

▶ **Lemma 7** (Folkore)**.** *For any two non-negative integers $m \leqslant l$ and any $s \in X(m)$, we have that $\sum_{t \in X(l) | t \supseteq s} \Pi_l(t) = \binom{l}{m} \Pi_m(s)$.*

In this work, we consider a notion of expansion for weighted simplicial complexes called link expansion. To that end, we first define the notion of a link of a complex and its skeleton.

▶ **Definition 8.** *For a simplicial complex X and some $s \in X$, X_s denotes the link complex of s defined by $X_s = \{t \setminus s | s \subseteq t \in X\}$. The skeleton of a link X_s for a face $s \in X(\leqslant k-2)$ (where k is the size of the largest face) denoted by $G(X_s)$ is a weighted graph with vertex set $X_s(1)$, edge set $X_s(2)$ and weights proportional to Π_2.*

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6 [6] uses a different (albeit related) notion of the link of a face $\sigma \in X(j)$. There, the link of a face σ is the set of level-i faces containing σ.

7 We shall often simply write v for a face $\{v\} \in X(1)$

8 We shall often write $X(\leqslant k)$ for X to stress the fact that X is k-dimensional

9 Whenever it is clear from the context, we use X in place of (X, Π) for the sake of brevity.

▶ **Definition 9** (γ-HDX, [19, 11]). *A simplicial complex $X(\leqslant k)$ is a γ-High Dimensional Expander (γ-HDX) if for all $s \in X(\leqslant k-2)$, the second singular value of the adjacency matrix of the graph $G(X_s)$ (denoted by $\sigma_2(G(X_s))$) satisfies $\sigma_2(G(X_s)) \leqslant \gamma$. We refer to $1-\gamma$ as the link-expansion of X.*

▶ **Definition 10** (Weighted inner product). *Given two functions $f, g \in \mathbb{R}^S$, i.e., $f, g : S \to \mathbb{R}$ and a measure μ on S, we define the weighted inner product of these functions as, $\langle f, g \rangle_\mu = \mathbb{E}_{s\sim\mu}[f(s)g(s)] = \sum_{s\in S} f(s)g(s)\mu(s)$. We drop the subscript μ from $\langle \cdot, \cdot \rangle_\mu$ whenever μ is clear from context.*

▶ Remark. In this paper, we will use the weighted inner product between two functions $f, g \in \mathbb{R}^{X(m)}$ on levels $X(m)$ of the simplicial complex X under consideration and with the measure Π_m, unless otherwise specified. In particular, for any linear operator $\mathsf{A} : \mathbb{R}^{X(m)} \to \mathbb{R}^{X(l)}$ the adjoint $\mathsf{A}^\dagger$ and the i-th largest singular value $\sigma_i(\mathsf{A})$ are with respect to this inner-product.

1.2.2 Walks on a Simplicial Complex

▶ **Definition 11** (Up and Down operators). *Given a simplicial complex (X, Π), we define the up operator $\mathsf{U}_i : \mathbb{R}^{X(i)} \to \mathbb{R}^{X(i+1)}$ that acts on a function $f \in \mathbb{R}^{X(i)}$ as*

$$[\mathsf{U}_i f](s) = \mathop{\mathbb{E}}_{s' \in X(i), s' \subseteq s}[f(s')] = \frac{1}{i+1} \sum_{x \in s} f(s \setminus \{x\})$$

and the down operator $\mathsf{D}_{i+1} : \mathbb{R}^{X(i+1)} \to \mathbb{R}^{X(i)}$ that acts on a function $g \in \mathbb{R}^{X(i+1)}$ as

$$[\mathsf{D}_{i+1} g](s) = \mathop{\mathbb{E}}_{s' \sim \Pi_{i+1}, s' \supset s}[g(s')] = \frac{1}{i+1} \sum_{x \notin s} g(s \cup \{x\}) \frac{\Pi_{i+1}(s \cup \{x\})}{\Pi_i(s)}\,.$$

As a consequence of the definition of the up and down operators, the following holds.

▶ **Lemma 12** (Folklore). $\mathsf{U}_i^\dagger = \mathsf{D}_{i+1}$.

The up operator, U_i, can be thought of as defining a random walk moving from $X(i+1)$ to $X(i)$ where a subset of size i is selected uniformly for a given face $s \in X(i+1)$. Similarly, the down operator D_{i+1} can be thought of as defining a random walk moving from $X(i)$ to $X(i+1)$ where a superset $s' \in X(i+1)$ of size $i+1$ is selected for a given face $s \in X(i)$ with probability $\frac{\Pi_{i+1}(s')}{\Pi_i(s)}$. This leads us to the following definition.

▶ **Definition 13.** *Given a simplicial complex (X, Π) and its two levels $X(m)$, $X(l)$, we define a bipartite graph on $X(m) \cup X(l)$ as $B_{m,l} = (X(m) \cup X(l), E_{m,l}, w_{m,l})$ where $E_{m,l} = \{\{s,t\} \,|\, s \in X(m), t \in X(l),$ and $s \subseteq t\}$ and $m \leqslant l$. The weight of an edge $\{s,t\}$ where $s \in X(m)$ and $t \in X(l)$ is given by $w_{m,l}(s,t) = \binom{k}{l}\Pi_l(t)$.*

As we will show in Fact 16, in the random walk on $B_{m,l}$ the block corresponding to the transition from a vertex in $X(m)$ to a vertex in $X(l)$ is the up walk (i.e., the down operator) and the block corresponding to the transition from a vertex in $X(l)$ to a vertex in $X(m)$ is the down-walk (i.e., the up operator).

Now, we define the $B^{(2)}_{m,l}$ graph such that the random walk on it corresponds to the two-step walk starting from vertices in $X(m)$ on $B_{m,l}$, i.e., the random walk on $B^{(2)}_{m,l}$ corresponds to an up-walk followed by a down-walk. Fact 17 shows that this correspondence indeed holds.

▶ **Definition 14.** *Given a simplicial complex* (X, Π) *and its two levels* $X(m)$, $X(l)$ *with* $m \leqslant l$, *we define a graph on* $X(m)$ *as* $B^{(2)}_{m,l} = (X(m), E^{(2)}_{m,l}, w^{(2)}_{m,l})$ *where*

$$E^{(2)}_{m,l} = \{\{s,t\} \,|\, s,t \in X(m) \text{ and } \exists s' \in X(l) \text{ such that } s' \supseteq s \cup t\}\,.$$

The weight of an edge $\{s,t\}$ *where* $s,t \in X(m)$ *is given by* $w^{(2)}_{m,l}(s,t) = \sum_{s' \supseteq s\cup t} w_{m,l}(s,s') = \binom{k}{l} \sum_{s' \supseteq s \cup t} \Pi_l(s')$. *The normalized adjacency matrix corresponding to* $B^{(2)}_{m,l}$ *is denoted by* $\mathsf{A}^{(2)}_{m,l}$.

▶ **Definition 15** (Up-Down Walk, [19, 20])**.** *For positive integers* $m \leqslant l$, *let* $\mathsf{D}_{m,l}$ *and* $\mathsf{U}_{l,m}$ *denote the products,* $\mathsf{D}_{m+1}\mathsf{D}_{m+2}\ldots\mathsf{D}_{l-1}\mathsf{D}_l$ *and* $\mathsf{U}_{l-1}\mathsf{U}_{l-2}\ldots\mathsf{U}_{m+1}\mathsf{U}_m$ *respectively. We denote the following walk between* $X(m)$ *and* $X(l)$ *as* $\mathsf{N}_{m,l}$,

$$\mathsf{N}_{m,l} = \begin{bmatrix} 0 & \mathsf{D}_{m,l} \\ \mathsf{U}_{l,m} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathsf{D}_{m,l} \\ \mathsf{D}^{\dagger}_{m,l} & 0 \end{bmatrix},$$

where the second equality is due to Lemma 12. The up-down walk on $X(m)$ *through* $X(l)$ *is a random walk on* $X(m)$ *whose transition matrix (denoted by* $\mathsf{N}^{(2)}_{m,l}$*) is given by* $\mathsf{N}^{(2)}_{m,l} = \mathsf{D}_{m,l}\mathsf{U}_{l,m}$.

▶ **Fact 16.** *The transition matrix for random walk on* $B_{m,l}$ *is* $\mathsf{N}_{m,l}$.

▶ **Fact 17.** *The transition matrix for random walk on* $B^{(2)}_{m,l}$ *is* $\mathsf{D}_{m,l}\mathsf{U}_{l,m}$.

1.2.3 Notations

We use $[n]$ for the set $\{1, 2, \ldots, n\}$ and $A \sqcup B$ for disjoint union of sets A and B.

2 Computing Sparse Cut in Hypergraphs

Theorem 18 shows an analogue of Cheeger's inequality based on the eigenvalues of up-down walks $\mathsf{N}_{1,l}$.

▶ **Theorem 18.** *Let* $H = (V,E)$ *be a* k*-uniform hypergraph such that the induced simplicial complex* X *has a up-down walk* $\mathsf{N}^{(2)}_{1,l}$ *such that* $\lambda_2(\mathsf{N}_{1,l}) = 1 - \varepsilon$ *for some* $\varepsilon > 0$ *and some* $l \in \{2, 3, \ldots, k\}$. *Then* $\frac{\varepsilon}{k} \leqslant \phi_H \leqslant 4\sqrt{\varepsilon}$. *Furthermore there is an algorithm which on input* H, *outputs a set* $S \subset V$ *such that* $\phi_H(S) \leqslant 4\sqrt{\varepsilon}$ *in* $\mathsf{poly}(|V|, |E|)$ *time where* poly *is a polynomial.*

Fact 19 will allow us to work with $\mathsf{D}_{1,2}$ instead of $\mathsf{N}_{1,l}$ for some $l \in \{3, 4, \ldots, k\}$.

▶ **Fact 19** (Folklore)**.** *Let* $\mathsf{A} \in \mathbb{R}^{n\times m}$, $\mathsf{B} \in \mathbb{R}^{m\times p}$ *and* σ_i *denote the* i^{th} *singular value. Then, we have*

$$\sigma_i(\mathsf{AB}) \leqslant \sigma_1(\mathsf{A})\sigma_i(\mathsf{B}) \text{ and } \sigma_i(\mathsf{AB}) \leqslant \sigma_i(\mathsf{A})\sigma_1(\mathsf{B}),$$

for $i = 1, \ldots, r$, *where* $r = \mathsf{rank}(\mathsf{AB})$.

▶ **Corollary 20.** *If* $\sigma_2(\mathsf{D}_{1,l}) = 1 - \varepsilon$ *for an arbitrary* $l \in \{2, 3, \ldots, k\}$, *we have that* $\sigma_2(\mathsf{D}_{1,2}) \geqslant 1 - \varepsilon$.

Proof. The proof follows by using Fact 19 and writing $\mathsf{D}_{1,l} = \mathsf{D}_{1,2}\mathsf{D}_{2,l}$ to get

$$\sigma_2(\mathsf{D}_{1,l}) = \sigma_2(\mathsf{D}_{1,2}\mathsf{D}_{2,l}) \overset{\text{Fact 19}}{\leqslant} \sigma_2(\mathsf{D}_{1,2})\sigma_1(\mathsf{D}_{2,l}) = \sigma_2(\mathsf{D}_{1,2})\,,$$

where the last equality holds since $\sigma_1(\mathsf{D}_{2,l}) = 1$. ◀

Next, we show that we can use this information about $\sigma_2(\mathsf{D}_{1,2})$ to compute a set $S \subset V$ such that its expansion in the graph $B_{1,2}^{(2)}$ is at most $2\sqrt{\varepsilon}$.

▶ **Lemma 21.** *If $\sigma_2(\mathsf{D}_{1,2}) = 1 - \varepsilon$ for some $\varepsilon \in (0, 1)$, then there exists a set $S \subseteq X(1)$ such that $\phi_{B_{1,2}^{(2)}}(S) \leqslant 2\sqrt{\varepsilon}$. Furthermore, there is a $\mathsf{poly}(|V_{B_{1,2}^{(2)}}|, |E_{B_{1,2}^{(2)}}|)$ time algorithm to compute such a set S.*

A natural choice for our set S with low conductance in input hypergraph is this set S guaranteed by Fiedler's algorithm for which $\phi_{B_{1,2}^{(2)}}(S)$ is small. We show in Lemma 22 that $B_{1,2}^{(2)}$ is a weighted graph where the weight of an edge between two distinct vertices in $X(1)$ is the multiplicity of that edge in the construction of $B_{1,2}^{(2)}$ graph. We note that a hyperedge e, induces a clique on the vertices in the hyperedge e, in the $B_{1,2}^{(2)}$ graph. This is commonly known as the clique expansion of the hypergraph.

▶ **Lemma 22.** *For any k-uniform hypergraph $H = (V, E)$, let X be the induced simplicial complex and let $\{s, t\}$ be an edge in $B_{m,l}^{(2)}$ with $s, t \in X(m)$. Then $w(s, t) = \binom{k-|s\cup t|}{l-|s\cup t|} \sum_{e\in E|s\cup t\subseteq e} \Pi_k(e)$ and $\mathsf{deg}_{B_{m,l}^{(2)}}(s) = \binom{l}{m}^2 \frac{\binom{k}{l}}{\binom{k}{m}} \sum_{e\in E|e\supseteq s} \Pi_k(e)$.*

Now in Lemma 23, we show how the weight of edges cut in the boundary of the weighted graph $B_{1,2}^{(2)}$ and the input hypergraph are related.

▶ **Lemma 23.** *Given a set $S \subset X(1)$ we have*

$$(k-1)\Pi_k\left(\partial_H(S)\right) \leqslant w(\partial_{B_{1,2}^{(2)}}(S)) \,.$$

Proof. By Lemma 22, $B_{1,2}^{(2)}$ is a weighted graph where the weight $w(i, j)$ of an edge $\{i, j\}$ where $i \neq j$ is given by $w(i, j) = \sum_{e\in E|\{i,j\}\subseteq e} \Pi_k(e)$. Therefore, to compute $w(\partial_{B_{1,2}^{(2)}}(S))$ we sum over all $i \in S$ and $j \in V \setminus S$, the number of hyperedges containing $\{i, j\}$, i.e.,

$$w(\partial_{B_{1,2}^{(2)}}(S)) = \sum_{i\in S, j\in V\setminus S} \sum_{\substack{e\in H\\ e\supseteq\{i,j\}}} \Pi_k(e) = \sum_{e\in H} \sum_{\substack{i\in S,j\in V\setminus S\\ \{i,j\}\subseteq e}} \Pi_k(e),$$

where the last equality in the equation above follows by exchanging the order of summation. Now, we note that the number of $\{i, j\} \subseteq e$ where $i \in S$ and $j \in V \setminus S$ is non-zero if and only if $e \in \partial_H(S)$, and hence,

$$w(\partial_{B_{1,2}^{(2)}}(S)) = \sum_{e\in\partial_H(S)} \sum_{\substack{i\in S,j\in V\setminus S\\ \{i,j\}\subseteq e}} \Pi_k(e). \tag{1}$$

Now, let $e \cap S = \{i_1, i_2, \ldots, i_t\}$ for some $t \in \{1, 2, \ldots, k-1\}$. For the lower bound, we note that the number of $\{i, j\} \subseteq e$ where $i \in S$ and $j \in V \setminus S$ is $t(k-t)$. Therefore, for some $e \in \partial_H(S)$, we have the minimum value of $t(k-t)$ as $k-1$ and hence u eqn. (1) to get,

$$w(\partial_{B_{1,2}^{(2)}}(S)) \geqslant \sum_{e\in\partial_H(S)} (k-1)\Pi_k(e) = (k-1)\Pi_k\left(\partial_H(S)\right) \left(\Pi_k(\partial_H(S)) = \sum_{e\in\partial_H(S)} \Pi_k(e)\right).$$

◀

We now show an upper bound for the boundary of $B_{1,l}^{(2)}$ in terms of the boundary of H.

▶ **Lemma 24.** *For any l, such that $2 \leqslant l \leqslant k$, Given a set $S \subset X(1)$ we have*

$$w(\partial_{B^{(2)}_{1,l}}(S)) \leqslant \binom{k}{l}\binom{l}{2}\Pi_k\left(\partial_H(S)\right).$$

Next, in Lemma 25, we will use these bounds to analyze the expansion of this set S in the input hypergraph.

▶ **Lemma 25.** *For an arbitrary set $S \subset X(1)$, we have that $\phi_H(S) \leqslant 2\phi_{B^{(2)}_{1,2}}(S)$.*

Proof. We start by comparing the numerator in the expressions for expansion of the given arbitrary set S in original hypergraph $|\partial_H(S)|$ and in the $B^{(2)}_{1,2}$ graph, i.e., $w(\partial_{B^{(2)}_{1,2}}(S))$. Using Lemma 23 we have that, $\Pi_k\left(\partial_H(S)\right) \leqslant \frac{1}{(k-1)} \cdot w(\partial_{B^{(2)}_{1,2}}(S))$.

Next, we compare the denominators in the respective expression for expansions, i.e., $\mathsf{vol}_H(S)$ and $\mathsf{vol}_{B^{(2)}_{1,2}}(S)$. For the hypergraph, by definition we have that $\mathsf{vol}_H(S) = \sum_{i\in S}\mathsf{deg}(i)$. By Lemma 22 we have

$$\mathsf{vol}_{B^{(2)}_{1,2}}(S) = \sum_{i\in S}\mathsf{deg}_{B^{(2)}_{1,2}}(i) = \sum_{i\in S}\binom{2}{1}^2\frac{k(k-1)}{2k}\mathsf{deg}_H(i) = 2(k-1)\mathsf{vol}_H(S).$$

Now, putting everything together, we have

$$\phi_H(S) = \frac{\Pi_k\left(\partial_H(S)\right)}{\mathsf{vol}_H(S)} = 2(k-1)\frac{\Pi_k\left(\partial_H(S)\right)}{\mathsf{vol}_{B^{(2)}_{1,2}}(S)} \leqslant 2\cdot\frac{(k-1)}{(k-1)}\cdot\frac{w(\partial_{B^{(2)}_{1,2}}(S))}{\mathsf{vol}_{B^{(2)}_{1,2}}(S)} = 2\phi_{B^{(2)}_{1,2}}(S)\,. \blacktriangleleft$$

▶ **Lemma 26.** *For an arbitrary set $S \subset X(1)$, we have that $\phi_H(S) \geqslant \frac{2}{k}\phi_{B^{(2)}_{1,l}}(S)$.*

Proof of Theorem 18. First, we note by Fact 51, $1-\varepsilon \leqslant \sqrt{1-\varepsilon} \leqslant \sqrt{\lambda_2(\mathsf{N}^{(2)}_{1,l})} = \sigma_2(\mathsf{D}_{1,l})$.

Now, using Corollary 20 we conclude that $\sigma_2(\mathsf{D}_{1,2}) = 1-\varepsilon' \geqslant 1-\varepsilon$ for some $\varepsilon' \leqslant \varepsilon$. Further, in Lemma 21, we show that we can use this information about the spectrum of $\mathsf{D}_{1,2}$ to compute a set $S \subset V$ such that its expansion in the graph $B^{(2)}_{1,2}$ is at most $2\sqrt{\varepsilon}$. We fix this as the set S we return in our sparse cut. In Lemma 25 we show that expansion of this set S in the input hypergraph is at most $2\phi_{B^{(2)}_{1,2}}(S)$ and hence

$$\phi_H(S) \leqslant 2\phi_{B^{(2)}_{1,2}}(S) \leqslant 4\sqrt{\varepsilon}\,.$$

Now, by Fact 17 the matrices $\mathsf{N}^{(2)}_{1,l}$ and $\mathsf{A}^{(2)}_{1,l}$ are similar and hence have the same eigenvalues and therefore by Cheeger's inequality, we have $\phi_{B^{(2)}_{1,l}} \geqslant \frac{\varepsilon}{2}$. Therefore by Lemma 26, we have

$$\phi_H \geqslant \frac{2}{k}\phi_{B^{(2)}_{1,l}} \geqslant \frac{\varepsilon}{k}. \qquad \blacktriangleleft$$

3 An expanding hypergraph with walks having small spectral gap

3.1 Splittability of a Hypergraph

In this section, we consider a "non-lazy" version of the up-down walk. While typically, for a walk on the graph to be non-lazy, we require that there be no transition from a vertex to itself, we obtain the swap walks by imposing an even stronger condition where we don't allow any face to have a transition to another face with a non-empty intersection with the starting face.

▶ **Definition 27** (Swap walk, [2, 9]). *Given a k-dimensional simplicial complex (X, Π), for non-negative integers m, l such that $l + m \leqslant k$ we define the swap walk denoted by $\mathsf{S}_{m,l} : \mathbb{R}^{X(l)} \to \mathbb{R}^{X(m)}$ that acts on a $f \in \mathbb{R}^{X(l)}$ as,*

$$[S_{m,l}f](s) = \mathop{\mathbb{E}}_{s' \sim \Pi_{m+l} | s' \supseteq s} f(s' \setminus s).$$

▶ **Lemma 28** ([2]). $\mathsf{S}^{\dagger}_{m,l} = \mathsf{S}_{l,m}$.

Again, the swap walk $\mathsf{S}_{m,l}$ can be thought of as defining a random walk moving from $X(m)$ to $X(l)$ where we first move from $s \in X(m)$ to a superset $s'' \in X(m+l)$ with probability $\frac{\Pi_{m+l}(s'')}{\Pi_m(s)}$ and then determistically move to $s' = s'' \setminus s$, i.e., we move from face $s \in X(m)$ to a disjoint face $s' \in X(l)$ with probability $\frac{\Pi_{m+l}(s \sqcup s')}{\Pi_m(s)}$. This leads us to the following definition for swap graphs.

▶ **Definition 29** (Swap graph, Section 6 in [2]). *Given a simplicial complex (X, Π) and its two levels $X(m), X(l)$, the swap graph (denoted by $G_{m,l}$) is defined as a bipartite graph $G_{m,l} = (X(m) \cup X(l), E(m,l), \mathrm{w}_{m,l})$ where the weight function is defined as, $\mathrm{w}_{m,l}(s,t) = \frac{\Pi_{m+l}(s \sqcup t)}{\binom{m+l}{m}}$ and $E(m,l) = \{\{s,t\} \, | s \in X(m), t \in X(l),$ and $s \sqcup t \in X(m+l)\}$.*

The random walk matrix corresponding to these walks denoted by $\mathsf{W}_{m,l}$ is a matrix of size $(|X(m)| + |X(l)|) \times (|X(m)| + |X(l)|)$ and is given by,

$$\mathsf{W}_{m,l} = \begin{bmatrix} 0 & \mathsf{S}_{m,l} \\ \mathsf{S}_{l,m} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathsf{S}_{m,l} \\ \mathsf{S}^{\dagger}_{m,l} & 0 \end{bmatrix}, \tag{2}$$

where the last equality is a consequence of Lemma 28.

Arora, Barak, and Steurer [5] introduced the notion of the threshold rank of a graph.

▶ **Definition 30** (Threshold rank of a graph, [5]). *Given a weighted graph $G = (V, E, w)$ and its normalized random walk matrix W such that $\lambda_n(\mathsf{W}) \leqslant \lambda_{n-1}(\mathsf{W}) \leqslant \ldots \leqslant \lambda_1(\mathsf{W}) = 1$ and a threshold $\tau \in (0,1]$, we define the τ-threshold rank of the graph G (denoted by $\mathsf{rank}_{\geqslant\tau}(\mathsf{W})$) as $\mathsf{rank}_{\geqslant\tau}(\mathsf{W}) = |\{i | \lambda_i(\mathsf{W}) \geqslant \tau\}|$.*

[2] proposed an analogue of the threshold rank for hypergraphs called (τ, r)-splittability by considering specific sets of swap walks given by the following class of binary tree.

▶ **Definition 31** (k-splitting tree, Section 7 in [2]). *A binary tree $\mathcal{T}$ given with its labeling is called a k-splitting tree if*

- *$\mathcal{T}$ has exactly k leaves.*
- *The root of $\mathcal{T}$ is labeled with k and all other vertices in $\mathcal{T}$ are labeled with a positive integer.*
- *All the leaves are labeled with 1.*
- *The label of every internal node of $\mathcal{T}$ is the sum of the labels of its two children.*

Now, we define a set of swap walks and its threshold rank based on a k-splitting tree $\mathcal{T}$.

▶ **Definition 32** (Swap graphs in a tree, Section 7 in [2]). *For a simplicial complex $X(\leqslant k)$ and a k-splitting tree $\mathcal{T}$, we consider all swap graphs (denoted by $\mathsf{Swap}(\mathcal{T}, X)$) from $X(a)$ to $X(b)$ where a and b are labels of a non-leaf node in $\mathcal{T}$. Further, we extend the definition of threshold rank as*

$$\mathsf{rank}_{\geqslant\tau}(\mathsf{Swap}(\mathcal{T}, X)) = \max_{G \in \mathsf{Swap}(\mathcal{T}, X)} \mathsf{rank}_{\geqslant\tau}(G).$$

Finally, define (τ, r)-splittability by considering all such sets of swap walks.

▶ **Definition 33** ((τ, r)-splittability, Definition 7.2 in [2])**.** *A k-uniform hypergraph with an induced simplicial complex $X(\leqslant k)$ is said to be (τ, r)-splittable if there exists some k-splittable tree $\mathcal{T}$ such that $\mathsf{rank}_{\geqslant \tau}(\mathsf{Swap}(\mathcal{T}, X)) \leqslant r$.*

3.2 The main results

In Theorem 34, we show an example of an expanding hypergraph such that for all m, l such that $m + l \leqslant k$ the swap walk from $X(m)$ to $X(l)$ in the corresponding simplicial complex has its top r singular values as 1 (for $r \approx n/k$) if either $m, l \geqslant 2$ or $m = k - l$.

▶ **Theorem 34.** *For any positive integers r, k with $r \geqslant 2, k \geqslant 3$, there exists an k-uniform hypergraph H on $n(= r(k-1)+1)$ vertices such that $\phi_H \geqslant \frac{1}{k}$ and for any m, l such that $m + l \leqslant k$, if either $m, l \geqslant 2$ or $m = k - l$ then $\lambda_r(G_{m,l}) = \sigma_r(\mathsf{S}_{m,l}) = 1$, where $\mathsf{S}_{m,l}$, $G_{m,l}$ are the swap walk and the swap graph on the induced simplicial complex X, respectively.*

Now, Corollary 35 is a simple consequence of Theorem 34 and the definition of splittability.

▶ **Corollary 35.** *For any positive integers r, k with $r \geqslant 2, k \geqslant 3$, there exists an k-uniform hypergraph H on $n(= r(k-1)+1)$ vertices, such that $\phi_H \geqslant \frac{1}{k}$ and the induced simplicial complex X is not (τ, r)-splittable for all $\tau \in [-1, 1]$.*

We were also able to show that in the above example, for all m, l such that $2 \leqslant m < l \leqslant k$, the up-down walk from $X(m)$ to $X(l)$ has its top r singular value as 1 (for $r \approx n/k$).

▶ **Theorem 36.** *For any positive integers r, k with $r \geqslant 2, k \geqslant 3$, there exists a k-uniform hypergraph H on $n(= r(k-1)+1)$ vertices such that $\mathsf{rank}_{\geqslant \tau}\left(\mathsf{N}^{(2)}_{m,l}\right) \geqslant r$ for all $\tau \in [-1, 1]$ but $\phi_H \geqslant \frac{1}{k}$.*

We use the following construction to show Theorem 34, Corollary 35 and Theorem 36.

▶ **Construction 37.** *Take the vertex set of the hypergraph $H(V, E)$ to be $V = [n]$ where $n = r(k-1)+1$ and the edge set $E = \{e_1, e_2, \ldots, e_r\}$ where $e_i = \{0, (k-1)(i-1)+1, \ldots, (k-1)i\}$. Let X be the simplicial complex induced by H and $\mathsf{S}_{m,l}, \mathsf{N}_{m,l}$ be the corresponding walk matrices.*

▶ Remark 38. Remark 1.9 of [23] considers all hypergraphs whose edges intersected at most $k-2$ vertices to show a separation between co-boundary expansion and hypergraph expansion. Here, we consider a sub-class of such hypergraphs with edges intersecting exactly one vertex. Although the second singular value of the up-down walks and co-boundary expansion may seem related, a relation between them is not known. Also, the way in which [23] bounds the co-boundary expansion is similar to how we bound the spectrum of the up-down walks. However, here, we also prove that the threshold rank (for any threshold) can be made arbitrarily large while having the same bound on the hypergraph expansion.

First, we show that any swap walk $\mathsf{S}_{l,k-l}$ has $\sigma_i = 1$, for any $i \in [r]$.

▶ **Lemma 39.** *Given a hypergraph as per Construction 37,*
we have that $\lambda_r(G_{1,k-1}) = \sigma_r(\mathsf{S}_{1,k-1}) = \sigma_r(\mathsf{S}_{k-1,1}) = 1$.

Proof. Firstly, using Fact 52 and eqn. (2) we have $\lambda_i(G_{1,k-1}) = \sigma_i(\mathsf{S}_{1,k-1}), \forall i \in [r]$.

We note that for any $i \in [r]$, the edge $\{\{(k-1)(i-1)\}, e_i \setminus \{(k-1)(i-1)\}\}$ is the only edge in $G_{1,k-1}$ (and $G_{k-1,1}$) incident on the vertices $\{(k-1)(i-1)\}, e_i \setminus \{(k-1)(i-1)\}$. Again, $G_{1,k-1}$ has r connected components, and hence $\lambda_r(G_{1,k-1}) = \sigma_r(\mathsf{S}_{1,k-1}) = \sigma_r(\mathsf{S}_{k-1,1}) = 1$. ◀

▶ **Lemma 40.** *Given a hypergraph as per Construction 37, and for any $m, l \geqslant 2$ such that $m + l \leqslant k$, we have that $\lambda_r(G_{m,l}) = \sigma_r(\mathsf{S}_{m,l}) = 1$.*

▶ **Lemma 41.** *Given a hypergraph as per Construction 37 and an arbitrary set $S \subseteq V$ where $\mathsf{vol}_H(S) \leqslant \mathsf{vol}_H(V)/2$, we have that $\phi_H(S) \geqslant \frac{1}{k}$.*

Proof. We consider an arbitrary (non-empty) set $S \subset V$ such that $\mathsf{vol}_H(S) \leqslant \mathsf{vol}_H(V)/2$. Let $|S \cap e_1| = t_1, |S \cap e_2| = t_2, \dots, |S \cap e_r| = t_r$ and let $t = t_1 + t_2 + \dots t_r$. We note that $\mathsf{vol}_H(V) = r(k-1) + r$ where $r(k-1)$ is the contribution from the vertices in $V \setminus \{0\}$ and we have a contribution of r from the vertex $\{0\}$. Next, we will precisely compute the expansion $\phi_H(S)$. We will break into cases depending upon whether $\{0\} \in S$ or $\{0\} \notin S$.

First, consider the case where $\{0\} \in S$. We note that in this case, $t_i \geqslant 1, \forall i \in [r]$. In this case, we have that $|\{i | t_i = k\}| < r/2$. This is because otherwise $\mathsf{vol}_H(S) \geqslant r + \frac{r}{2}(k-1) > \frac{rk}{2}$ which contradicts $\mathsf{vol}_H(S) \leqslant \mathsf{vol}_H(V)/2$. Thus, $|\{i | t_i < k\}| \geqslant r/2$ and hence $\partial_H(S) \geqslant r/2$. Next we have that $\mathsf{vol}_H(S) = r + \sum_{i=1}^{r}(t_i - 1) = t_1 + t_2 + \dots t_r = t$. Using $\mathsf{vol}_H(S) \leqslant \mathsf{vol}_H(V)/2$, we have that $t \leqslant rk/2$ and we get

$$\phi_H(S) = \frac{|\partial_H(S)|}{\mathsf{vol}_H(S)} \geqslant \frac{r}{2t} \geqslant \frac{1}{k}.$$

Next, we consider the case where $\{0\} \notin S$. Let $t^+ = |\{i\} \, | t_i > 0|$. Since $\{0\} \notin S$, we know that $t_i < k, \forall i \in [r]$ and hence the number of edges in the boundary of S is exactly t^+. Moreover we can bound the volume of S as $\mathsf{vol}_H(S) \leqslant t^+(k-1)$ and hence we have

$$\phi_H(S) = \frac{|\partial_H(S)|}{\mathsf{vol}_H(S)} \geqslant \frac{t^+}{t^+(k-1)} \geqslant \frac{1}{k}.$$ ◀

Proof of Theorem 34. Immediate from Lemma 41, Lemma 40, and Lemma 39. ◀

Proof of Corollary 35. Consider the hypergraph H (and the induced simplicial complex) guaranteed by Theorem 34. Fix any $\tau \in [-1, 1]$ and any k-splitting tree $\mathcal{T}$. We note $G_{l,k-1} \in \mathsf{Swap}(\mathcal{T}, X)$ for some $l \in [k-1]$ as children of the root of $\mathcal{T}$ must be labeled l and $k - l$ for some l. Note that we have $\lambda_r(G_{l,k-l}) = 1$. Hence, we have $\mathsf{rank}_{\geqslant\tau}(\mathsf{Swap}(\mathcal{T}, X)) \geqslant \mathsf{rank}_{\geqslant\tau}(G_{l,k-l}) \geqslant r$. Since, $\mathsf{rank}_{\geqslant\tau}(\mathsf{Swap}(\mathcal{T}, X)) \geqslant r$ for any k-splitting tree $\mathcal{T}$, therefore (X, Π) is not (τ, r)-splittable for any $\tau \in [-1, 1]$. ◀

▶ **Lemma 42.** *Given a hypergraph as per Construction 37, and any $m, l \in [k]$ such that $2 \leqslant m \leqslant l$, we have that $\lambda_r(\mathsf{N}^{(2)}_{m,l}) = 1$.*

Proof of Theorem 36. Immediate from Lemma 41 and Lemma 42. ◀

4 An expanding hypergraph with low link expansion

In Theorem 43, we show that there is a family of expanding k-uniform hypergraphs H with the induced simplicial complex having low link expansion.

▶ **Theorem 43.** *Let n, k be any positive integers such that $n \geqslant 3k$ and $k \geqslant 3$, there exists a k-uniform hypergraph H on $n + k - 2$ vertices such that the link expansion of the induced simplicial complex X is at most $1 - \cos\frac{2\pi}{n}$ and the expansion of H is at least $\frac{1}{(3k)^k}$.*

Construction 44 is a k-hypergraph with $n + k - 2$ vertices such that its expansion is $\frac{1}{(3k)^k}$ while the link expansion for the induced simplicial complex is $1 - \cos\frac{2\pi}{n}$.

▶ **Construction 44.** *Take the vertex set of the hypergraph $H(V,E)$ to be $V = [n+k-2]$ and the edge set $E = \binom{[n]}{k} \cup \{e \cup \{n+1,\dots,n+k-2\} \,|e \in C_n\}$ where $C_n = \{\{i,i+1\} \,|i \in [n-1]\} \cup \{\{n,1\}\}$, i.e., C_n is the set of edges in a cycle on $[n]$. Let X be the simplicial complex induced by H.*

The idea behind this construction is to have the cycle C_n as the link of $\{n+1,...,n+k-2\}$ while adding sufficient edges to make the hypergraph into an expanding hypergraph.

▶ **Lemma 45.** *For any n,k such that $n \geqslant 3k$ and $k \geqslant 3$, the hypergraph H as defined in Construction 44 has expansion $\phi_H \geqslant \frac{1}{(3k)^k}$.*

We now show that the simplicial complex X is not a γ-HDX (refer to Definition 9). For this we consider the face $\tau = \{n+1, n+2, \dots, n+k-2\}$ and the link complex X_τ.

By definition of X_τ and our construction in Construction 44, the two-dimensional link complex X_τ is the downward closure of C_n. Hence, the corresponding skeleton graph $G(X_\tau)$ is the cycle on $[n]$.

▶ **Fact 46** (Folklore). *The second singular value of the normalized adjacency matrix of an n-cycle is $\cos \frac{2\pi}{n}$.*

Therefore, we have the following lemma by Definition 9.

▶ **Lemma 47.** *X has link expansion at most $1 - \cos \frac{2\pi}{n}$.*

Theorem 43 follows directly from Lemma 47 and Lemma 45.

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A Additional Preliminaries

Linear Algebra

We recall a few facts and definitions from linear algebra.

▶ **Fact 48** ([16]). *Let V, W be two vector spaces with inner products $\langle \cdot,\cdot\rangle_V$, $\langle \cdot,\cdot\rangle_W$. If $A : V \to W$ be a linear operator, then there exists a unique linear operator $B : W \to V$ such that $\langle \mathsf{A}f, g\rangle_W = \langle f, \mathsf{B}g\rangle_V$. If $v \in V$ then there exists a unique linear operator $C : V \to \mathbb{R}$ such that $Cu = \langle v, u\rangle_V$ for any $u \in V$.*

▶ **Definition 49.** *Given a linear operator $\mathsf{A} : V \to W$ between two vector spaces V and W with inner products $\langle \cdot,\cdot\rangle_V$ and $\langle \cdot,\cdot\rangle_W$ defined on them, the adjoint of A is defined as the (unique) linear operator $\mathsf{A}^\dagger : W \to V$ such that $\langle \mathsf{A}f, g\rangle_W = \left\langle f, \mathsf{A}^\dagger g\right\rangle_V$ for any $f \in V$ and $g \in W$. Furthermore, given any $v \in V$ we define $v^\dagger : V \to \mathbb{R}$ as the linear operator which satisfies $v^\dagger u = \langle v, u\rangle_V$ for any $u \in V$.*

It can be easily verified that most properties of the transpose of an operator also hold for the adjoint, e.g., $(\mathsf{A}^\dagger)^\dagger = \mathsf{A}$, $(\mathsf{AB})^\dagger = \mathsf{B}^\dagger\mathsf{A}^\dagger$, etc.

▶ **Definition 50.** *Given a linear operator $\mathsf{A} : V \to W$ between two inner product spaces V and W a singular value σ is a non-negative real number such that there exists $v \in V$ and $w \in W$ which satisfy $\mathsf{A}v = \sigma w$ and $w^\dagger\mathsf{A} = \sigma v^\dagger$. The vectors v and w are called the right and left singular vectors, respectively, associated with the singular value σ. We denote the i-th largest singular value of A by $\sigma_i(\mathsf{A})$.*

▶ **Fact 51.** *Let V,W be two inner product spaces, and $\mathsf{A} : V \to W$ be a linear operator. Then the eigenvalues $\lambda_i(\mathsf{A}^\dagger\mathsf{A})$ are non-negative. Furthermore, the singular values $\sigma_i(\mathsf{A}) = \sqrt{\lambda_i(\mathsf{A}^\dagger\mathsf{A})}$.*

▶ **Fact 52.** *Let V,W be two inner product spaces and $\mathsf{A} : V \to W$ be a linear operator and let B be defined by the expression,*

$$\mathsf{B} = \begin{bmatrix} 0 & \mathsf{A} \\ \mathsf{A}^\dagger & 0 \end{bmatrix}$$

then for any $i \in \{1, \ldots, r\}$, $\sigma_i(\mathsf{A}) = \lambda_i(\mathsf{B})$ where $r = \mathsf{rank}(\mathsf{A})$.