Recognition and Proper Coloring of Unit Segment Intersection Graphs

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Abstract

In this work, we concern ourselves with the fine-grained complexity of recognition and proper coloring problems on highly restricted classes of geometric intersection graphs of "thin" objects (i.e., objects with unbounded aspect ratios). As a point of motivation, we remark that there has been significant interest in finding algorithmic lower bounds for classic decision and optimization problems on these types of graphs, as they appear to escape the net of known planar or geometric separator theorems for "fat" objects (i.e., objects with bounded aspect ratios). In particular, letting n be the order of a geometric intersection graph, and assuming a geometric ply bound, per what is known as the "square root phenomenon", these separator theorems often imply the existence of $\mathcal{O}\left(2^{\left(\sqrt{n}\right)}\right)$ algorithms for problems ranging from finding proper colorings to finding Hamiltonian cycles. However, in contrast, it is known for instance that no $2^{o(n)}$ time algorithm can exist under the Exponential Time Hypothesis (ETH) for proper 6-coloring intersection graphs of line segments embedded in the plane (Biró et. al.; *J. Comput. Geom.* **9**(2); pp. 47–80; 2018).

We begin by establishing algorithmic lower bounds for proper k-coloring and recognition problems of intersection graphs of line segments embedded in the plane under the most stringent constraints possible that allow either problem to be non-trivial. In particular, we consider the class UNIT-PURE-k-DIR of unit segment geometric intersection graphs, in which segments are constrained to lie in at most k directions in the plane, and no two parallel segments are permitted to intersect. Here, under the ETH, we show for every $k \geq 3$ that no $2^{o\left(\sqrt{n/k}\right)}$ time algorithm can exist for either recognizing or proper k-coloring UNIT-PURE-k-DIR graphs of order n. In addition, for every $k \geq 4$, we establish the same algorithmic lower bound under the ETH for the problem of proper $(k-1)$-coloring UNIT-PURE-k-DIR graphs when provided a list of segment coordinates specified using $\mathcal{O}(n \cdot k)$ bits witnessing graph class membership. As a consequence of our approach, we are also able to show that the problem of properly 3-coloring an arbitrary graph on m edges can be reduced in $\mathcal{O}\left(m^2\right)$ time to the problem of properly $(k-1)$-coloring a UNIT-PURE-k-DIR graph. Finally, we consider a slightly less constrained class of geometric intersection graphs of lines (of unbounded length) in which line-line intersections must occur on any one of $(r=3)$ parallel planes in $\mathbb{R}^3$. In this context, for every $k \geq 3$, we show that no $2^{o(n/k)}$ time algorithm can exist for proper k-coloring these graphs unless the ETH is false.

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1 Introduction

The notion of a geometric intersection graph, where vertices correspond to geometric shapes embedded in $\mathbb{R}^n$, for some $n \in \mathbb{N}_{>0}$, and edges encode their intersections, provides a direct bridge between topology and intuitive Euclidean geometry. In particular, fundamental graph and complexity theoretic questions concerning these objects tend to have answers with distinctly "physical" implications.

Nice examples of this phenomena come from the *graph recognition problem* of deciding if a given graph can be realized as a particular type of geometric intersection graph. With regard to positive results, we can note the proof by Koebe that every planar graph is realizable as an intersection graph of disks [32], as well as Chalopin & Gonçalves' proof [12] of Scheinerman's conjecture [61] that every planar graph is likewise realizable as an intersection graph of segments. We can also observe a proof by Pach & Tóth [52] that, of the $2^{\binom{n}{2}}$ graphs on n labeled vertices, at least $2^{\left(\left(\frac{3}{4}\right)\cdot\binom{n}{2}\right)}$ and at most $2^{\left(\left(\frac{3}{4}+o(1)\right)\cdot\binom{n}{2}\right)}$ can be realized as the intersection graphs of n Jordan curves in $\mathbb{R}^2$ (i.e., "string" graphs [19, 35, 62]). However, while it is known to be decidable whether a particular graph is a string graph [51, 59, 60], the problem was also shown to be NP-hard [33] and eventually NP-complete [58]. Similar hardness results exist for recognizing disk graphs [27], unit disk graphs [9], as well as angle-constrained segment [34, 37] and unit segment graphs [47].

Going further, we can ask questions that pertain to the nature of the realizations that are possible for a geometric intersection graph. Here, letting the *ply* or *thickness* of a geometric system correspond to the maximum number of objects intersecting at a common point, a rather remarkable finding has been that results analogous to the Lipton-Tarjan separator theorem [41] for planar graphs likewise exist for (typically bounded ply) geometric intersection graphs of "fat" objects (i.e., objects with bounded aspect ratios) [6, 20, 22, 46, 63], and in a much more limited sense, for "thin" objects (i.e., objects with unbounded aspect ratios) such as string graphs [23, 24, 25, 39, 45]. This, in turn, often leads to a "square root phenomena" for geometric intersection graphs, once again like that observed in planar graphs [43], of subexponential (e.g., $2^{\mathcal{O}\left(\sqrt{n}\right)}$) algorithms for problems ranging from independent set to Hamiltonian cycle where, say, only a $2^{\mathcal{O}(n)}$ algorithm might be known in the general case (see, e.g., [2, 6, 8, 15, 21, 24, 31, 42, 44, 49, 53]).

In this work, we concern ourselves with further exploring the gap between what is known concerning geometric separator theorems for intersection graphs of "fat" and "thin" objects. In particular, we proceed by deriving new algorithmic lower bounds for fundamental graph class recognition and proper coloring problems on geometric intersection graphs of "thin" objects under the most stringent possible constraints. We remark that proper coloring problems are of particular interest in this context, as a proper k-coloring of a given geometric intersection graph can only exist if the graph has ply-at-most-k.

Towards this objective, we first consider the problems of recognizing and proper k-coloring geometric intersection graphs in the class UNIT-PURE-k-DIR (generalizing graph classes discussed in ref. [11, 13, 50]), consisting of all geometric intersection graphs of unit length straight line segments, lying in at most k directions in the plane, where all parallel segments are disjoint. Subsequently, to obtain tighter lower bounds for related geometric intersection graphs of uniform length line segments, we consider the complexity of proper k-coloring geometric intersection graphs of lines (of unbounded length), in which all line-line intersections are required to occur on any one of three parallel planes in $\mathbb{R}^3$.

As a high level summary of our findings, extending a result of Mustaţă & Pergel [47] that recognizing UNIT-PURE-2-DIR graphs is NP-complete, and a result of Barish & Shibuya [4] that finding a proper 3-coloring of a UNIT-PURE-4-DIR is NP-complete, assuming the

Exponential Time Hypothesis (ETH) of Impagliazzo & Paturi [29] we show in part that: for every $k \geq 3$, no $2^{o\left(\sqrt{n/k}\right)}$ time algorithm can exist for either recognizing or proper k-coloring order n UNIT-PURE-k-DIR graphs (Theorem 1); and for every $k \geq 4$, no $2^{o\left(\sqrt{n/k}\right)}$ time algorithm can exist for finding a proper $(k-1)$-coloring of a UNIT-PURE-k-DIR graph, even when provided a list of segment coordinates, specified using $\mathcal{O}(n \cdot k)$ bits, witnessing graph class membership (Theorem 2). Here, as a partial consequence of these efforts, we are also able to extend a result of Barish & Shibuya [4] that the problem of proper 3-coloring an arbitrary graph on m edges can be reduced in $\mathcal{O}\left(m^2\right)$ time[2] to the problem of properly 3-coloring a UNIT-PURE-4-DIR graph. In particular, we show for every $k \geq 4$ that the same proper 3-coloring problem can likewise be reduced to a proper $(k-1)$-coloring problem for UNIT-PURE-k-DIR graphs in $\mathcal{O}\left(m^2 \cdot k\right)$ time (Corollary 1).

Finally, assuming the ETH, for every $k \geq 3$, we show that no $2^{o(n/k)}$ time algorithm can exist for proper k-coloring order n geometric intersection graphs of lines in which we require that line-line intersections must occur on any one of three parallel planes in $\mathbb{R}^3$ (Theorem 3).

2 Elaboration concerning motivation

An important point of motivation for the current work comes from a question posed by Miltzow – in the workshop on Graph Classes, Optimization, and Width Parameters (GROW) list of open problems [56] – concerning the lower bound complexity under the ETH of proper k-coloring geometric intersection graphs of unit segments in the plane.

As noted by Miltzow, the difficulty in answering this question has been due, in part, to the failure to extend geometric separator theorems for bounded ply intersection graphs of "fat" objects to bounded ply intersection graphs of "thin" objects. Here, this gap becomes readily apparent when looking at algorithmic lower bounds for proper k-coloring problems. In particular, specifying $k \in \Theta(n^\alpha)$ for some $0 \leq \alpha \leq 1$, Biró et. al. [6] was able to show the existence of a $2^{\mathcal{O}\left(\sqrt{n} \cdot k \cdot \ln n\right)}$ algorithm for finding a proper k-coloring of intersection graphs of disks or other "fat" objects, while also showing that the existence of a $2^{o\left(\sqrt{n/k}\right)}$ algorithm would refute the ETH. On the other hand, Biró et. al. [6] was also able to establish that no $2^{o(n)}$ algorithm can exist for proper 6-coloring 2-DIR graphs (though not PURE-2-DIR graphs) assuming the ETH, even if all segments are constrained to lie at angles of 0 and $\frac{\pi}{2}$ radians in the plane. This latter result was later strengthened and extended by Bonnet & Rzążewski [8], who were able to establish a $2^{o(n)}$ (resp. $2^{o\left(n^{2/3}\right)}$) lower bound for proper k-coloring 2-DIR graphs (resp. 3-DIR graphs with unit length segments) for any constant $k \geq 4$, as well as the existence of a subexponential time $2^{\tilde{\mathcal{O}}\left(n^{2/3}\right)}$ algorithm for finding a proper $(k = 3)$-coloring.

To elaborate on our choice to focus on the class UNIT-PURE-k-DIR of geometric intersection graphs, this first of all represents a limit case for geometric intersection graphs of "thin" objects embedded in the plane with orientation and length constraints. More specifically, let "string" [19, 35, 62] be the earlier defined class of geometric intersection graphs of Jordan curves, let "CONV" [33, 36, 57, 64] be the class of all geometric intersection graphs of convex shapes, let "SEG" [19, 33, 34, 37, 38] be the class of all geometric intersection graphs of straight line segments in the plane, let k-DIR [34, 37, 38] be a subclass of "SEG" where segments must lie in at most k directions, and let PURE-k-DIR [34, 37, 38] be a subclass of k-DIR where all parallel segments are required to be disjoint. We can now observe

[2] This was erroneously reported to be an $\mathcal{O}(m)$ time reduction in Barish & Shibuya [4].

that, for every $k \geq 1$, PURE-k-DIR $\subsetneq$ k-DIR $\subsetneq$ SEG $\subsetneq$ CONV $\subsetneq$ STRING [19, 37], that k-DIR $\subsetneq$ $(k+1)$-DIR [37], and that PURE-k-DIR $\subsetneq$ PURE-$(k+1)$-DIR [37]. Second of all, as intersections between parallel segments are prohibited for this graph class, an analysis of proper k-coloring problems for this graph class requires fundamentally different techniques that of either Biró et. al. [6] or Bonnet & Rzążewski [8], and we considered this to be an interesting challenge.

To elaborate on our choice to consider geometric intersection graphs of lines, in which we require that line-line intersections must occur on any one of three parallel planes in $\mathbb{R}^3$, this class of graphs can be understood as a weak-as-possible generalization of the type of geometric intersection graphs of unit segments in the plane for which we were able to obtain results. Here, we hope our ability to rule out the existence of $2^{o(n/k)}$ time algorithms for proper k-coloring order n instances of these graphs under the ETH, for all $k \geq 3$, can be extended to establish the same algorithmic lower bound for intersection graphs of unit segments in the plane for some $k \geq 3$.

Finally, we remark that the problem of proper k-coloring UNIT-PURE-k-DIR graphs has practical application to problems ranging from realistic instances of the frequency assignment problem [1, 16, 26], to the design of Very Large Scale Integration (VLSI) circuits [17, 48] (e.g., where overlapping wires (segments) must be assigned to distinct circuit layers abstracted as colors). To briefly elaborate on the former case, the frequency assignment problem asks one to assign a sparse set of frequency bands (colors) to a set of antennas (vertices) in an *interference graph*, where we have that two vertices are adjacent if and only if they correspond to antennas spaced closely enough to interfere when emitting within the same frequency band (e.g., within $\approx 50 - 100$ kHz [16]). Here, in a realistic scenario, any such interference graph will correspond to a geometric intersection graph between radiation emission patterns for antennas, which in many cases (e.g., coastal radio stations) are optimized to be narrow oriented cones approximating bounded length segments.

3 Preliminaries & clarifications

3.1 Graph theoretic terminology

All graphs in this work should be considered to be simple (i.e., loop and multi-edge free), undirected, and unweighted. Concerning basic graph theoretic terminology, we will generally follow Diestel [18], or where appropriate, Bondy & Murty [7]. However, for some brief clarifications, recall that a graph is k-connected if there exist k vertex disjoint simple paths between all pairs of vertices, and recall that a graph is *planar* if it admits an embedding in the plane without edge crossings. In addition, recall that a *proper k-coloring* for a graph is an assignment of $\leq k$ colors to the graph's vertices under the constraint that no two adjacent vertices have an identical coloration, and that the *chromatic number* for a graph is the minimum value of k for which it admits a proper k-coloring.

Concerning less common terminology, when we *identify* a vertex v_a with a vertex v_b, we delete v_a and v_b and create a new vertex v_c adjacent to any vertex formerly adjacent to v_a or v_b. Additionally, when we refer to a vertex v in a drawing or embedding of a graph G as a *metavertex* (e.g., corresponding to a clique of some size), it should be understood that v corresponds to an induced subgraph H of G, where every vertex in G adjacent to v is adjacent to each vertex of H. Here, we can also identify pairs of metavertices by identifying each pair of equivalent vertices in an isomorphism between the subgraphs they correspond to.

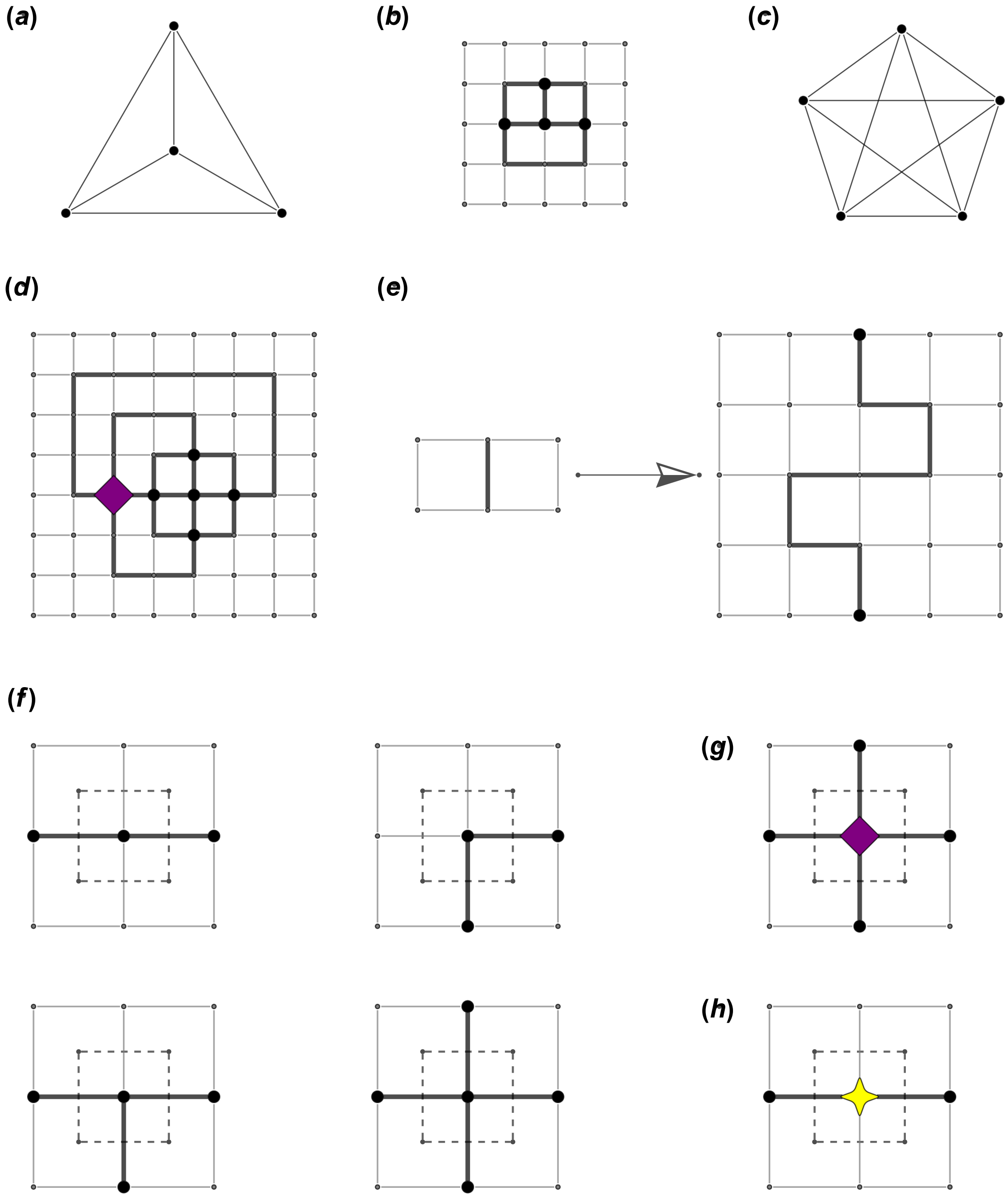

Figure 1 Illustrations of orthogonal integer lattice embeddings of graphs, where larger (black) vertices and (highlighted black) edges indicate the vertices and polylines for the embedding, respectively; **(a)** the complete graph K_4; **(b)** orthogonal integer lattice embedding of K_4; **(c)** the complete graph K_5; **(d)** orthogonal integer lattice embedding of K_5 with a lone polyline crossing indicated by a (purple) diamond polygon; **(e)** scheme for the enlargement and modification of an orthogonal integer lattice embedding to ensure all polylines have horizontal segments; **(f)** all local vertex and polyline configurations (up to rotation and reflection) in an orthogonal integer lattice embedding of a 2-connected graph of maximum vertex degree ≤ 4, where cells around each lattice vertex are indicated by a (dashed) box; **(g)** a cell containing a polyline crossing indicated by a (purple) diamond polygon; **(h)** a cell marked with a (yellow) concave diamond marker designating it to be identified with a "color change" gadget.

3.2 Exponential Time Hypothesis (ETH)

Recalling that k-SAT is the problem of deciding the satisfiability of a Boolean expression in conjunctive normal form where each clause contains at most k literals, the *Exponential Time Hypothesis* (ETH) of Impagliazzo & Paturi [29] can be defined as follows:

▶ **Definition 1.** *Exponential Time Hypothesis (ETH) [29]. Assuming $k \geq 3$, letting n and m be the number of variables and clauses for an instance of k-SAT, and letting $s_k = inf\{\delta : k$-SAT can be solved in $2^{(\delta \cdot n)} \cdot poly\,(m)$ time$\}$, it holds that $s_k > 0$.*

3.3 Linear time orthogonal integer lattice embeddings of graphs

Let G be an arbitrary not-necessarily-planar graph of maximum vertex degree ≤ 4, with vertex set V_G and edge set E_G. An orthogonal integer lattice embedding (or drawing) $\mathcal{Q}$ of G places each vertex at a distinct integral coordinate, and represents each edge $v_i \leftrightarrow v_j \in E_G$ as a *polyline* consisting of a polygonal chain of axis-parallel unit length horizontal and vertical segments – corresponding to a simple path in the integer lattice into which G is embedded – connecting v_i and v_j in $\mathcal{Q}$. If G is non-planar, then we necessarily must allow for polyline crossings in the embedding $\mathcal{Q}$. Here, a *bend* in $\mathcal{Q}$ corresponds to an instance where a horizontal and a vertical segment meet in a polyline (i.e., an instance where two unit segments meet at an angle of $\frac{\pi}{2}$ radians). We remark that it is possible to find an orthogonal integer lattice embedding for G on a square integer lattice, of total area $\mathcal{O}\left(|V_G|^2\right)$, in $\mathcal{O}\left(|V_G|\right)$ time via either the method of Papakostas & Tollis [54] or Biedl & Kant [5].

For illustrative examples, we refer the reader to Fig. 1(a–d), where in Fig. 1(a) (resp. Fig. 1(c)) we show an instance of the complete graph K_4 (resp. K_5), and in Fig. 1(b) (resp. Fig. 1(d)) we show an orthogonal integer lattice embedding of the graph in a 5×5 (resp. 8×8) integer lattice. For the Fig. 1(d) embedding of the graph K_5, we can observe that one polyline crossing occurs, where we indicate this crossing via a (purple) diamond polygon. Additionally, in Fig. 1(f) we show all possible local polyline configurations in a cell, up to rotation and reflection, under the assumption that the embedded graph is at least 2-connected.

4 Recognition and proper coloring of UNIT-PURE-k-DIR graphs

In this section, we establish Theorem 1 through Theorem 3, as well as Corollary 1. Concerning Theorem 1, we briefly remark that previous reductions for proving the NP-hardness of recognizing "string" graphs [33], k-DIR and PURE-k-DIR graphs [34, 37], and UNIT-PURE-2-DIR graphs [47], have generally followed a strategy of giving an $\mathcal{O}\left(n^2\right)$ reduction from a variant of planar 3-SAT already admitting a subexponential time algorithm. Accordingly, we required a different approach for establishing our claims.

▶ **Theorem 1.** *Unless the ETH is false, for any $k \geq 3$, no $2^{o\left(\sqrt{n/k}\right)}$ time algorithm can exist for either recognizing or proper k-coloring order n UNIT-PURE-k-DIR graphs.*

Proof. We proceed by giving an $\mathcal{O}\left(n^2 \cdot k\right)$ reduction from the problem of finding a proper 3-coloring of a 2-connected graph with n vertices and maximum vertex degree ≤ 4. Here, it is known that no $2^{o(n)}$ time algorithm can exist for this problem unless the ETH fails (see, e.g., "Lemma 2.1" of [14]). In particular, let G and H be a pair of graphs with vertex sets V_G and V_H, respectively, where $n = |V_G|$, G is an arbitrary 4-regular graph, and H is a graph constructable from G in $\mathcal{O}\left(n^2 \cdot k\right)$ time with $|V_H| = \mathcal{O}\left(n^2 \cdot k\right)$ vertices. We will

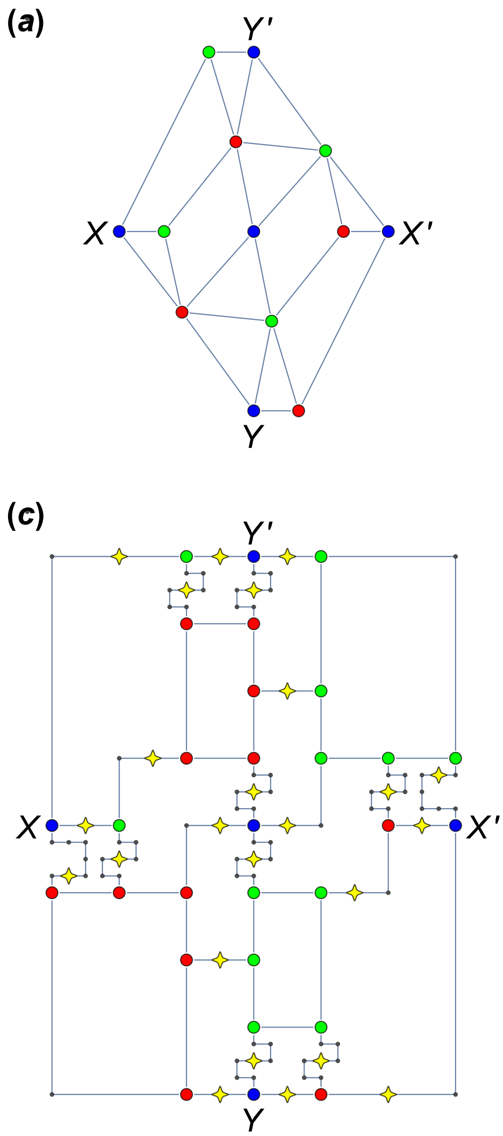

Figure 2 Illustration and example proper colorings of a novel proper 3-coloring planarization gadget, and scheme for its adaption as the gadget Υ in the context of an orthogonal integer lattice embedding; **(a)** the order 13 and size 24 planarization gadget, where vertices labeled X and X' (resp. Y and Y') are required to have the same coloration, with an example proper 3-coloring where vertices labeled X and Y are assigned the same color; **(b)** an alternative 3-coloration of the gadget, where the vertices labeled X and Y are assigned distinct colors; **(c)** orthogonal 25×33 integer lattice embedding of a modification of the gadget shown in (a,b) – denoted Υ – where some vertices in the planarization gadget from (a,b) are replaced with connected subgraphs, where such subgraphs are colored in accordance with the proper 3-coloring from the illustration of the gadget in (a), and where exactly one cell along each polyline between distinct subgraphs is marked for identification with the "color change" gadget; **(d)** another coloring of Υ in accordance with the proper 3-coloring of the planarization gadget from (b).

show for every $k \geq 3$ that G admits a proper 3-coloring if and only if H is proper k-colorable, or equivalently in this specific context, realizable as a UNIT-PURE-k-DIR graph. From this we will deduce that a $2^{o\left(\sqrt{n/k}\right)}$ algorithm for either recognizing or proper k-coloring a UNIT-PURE-k-DIR graph would refute the ETH.

Provided the instance of the aforementioned graph G, we begin by computing an orthogonal integer lattice embedding $\mathcal{Q}_1$ for G, of total area $\mathcal{O}\left(n^2\right)$, in $\mathcal{O}\left(n\right)$ time via either the method of Papakostas & Tollis [54] or Biedl & Kant [5] (see, e.g., Section 3.3 for an elaboration on these embeddings). In this context, we define a *cell* in an orthogonal integer lattice embedding $\mathcal{Q}_1$ to be a square area of volume 1 centered on each lattice point in $\mathcal{Q}_1$. For instance, the boundaries of this square area for a lattice point at coordinates $(x, y) \in \mathbb{Z}^2$ would be given by the coordinates $\left(\left(x - \frac{1}{2}, y - \frac{1}{2}\right), \left(x + \frac{1}{2}, y - \frac{1}{2}\right), \left(x + \frac{1}{2}, y + \frac{1}{2}\right), \left(x - \frac{1}{2}, y + \frac{1}{2}\right)\right)$. Letting the Manhattan distance between a pair of cells correspond to the Manhattan distance between their respective lattice points, we also define the *von Neumann neighborhood* of a cell as the set containing both the cell itself as well as its four neighbors at distance 1. For a cell with a centerpoint at some coordinate $(i, j) \in \mathbb{Z}^2$, we refer to its distance 1 neighbors at coordinates $(i, j+1)$, $(i-1, j)$, $(i+1, j)$, and $(i, j-1)$ as being to the North, West, East, and South, respectively.

We next construct an orthogonal integer lattice embedding $\mathcal{Q}_3$ from $\mathcal{Q}_1$ – via an intermediate graph $\mathcal{Q}_2$ potentially containing polyline crossings – satisfying the dual constraints that: (constraint 1) each polyline originally in $\mathcal{Q}_1$ has at least one horizontal segment marking the position of a "color change" gadget with a (yellow) concave diamond marker (e.g, as shown in Fig. 1(h)); and (constraint 2) each polyline crossing is replaced with an orthogonal integer lattice embedding of a gadget such that $\mathcal{Q}_3$ has no polyline crossings, and in addition, its corresponding graph is proper 3-colorable if and only if the graph corresponding to the embedding $\mathcal{Q}_1$ is proper 3-colorable.

To address (constraint 1), we begin by checking if every polyline contains at least one horizontal segment, and subsequently marking this horizontal segment if it exists. For any remaining polylines consisting of only vertical segments, we can perform the operation shown in Fig. 1(e) to introduce and subsequently mark a horizontal segment. We note that this may require expanding the embedding $\mathcal{Q}_1$ by moving each point at a position (x, y) to a position $(2x, 4y)$, treating expanded polyline edges as chains of unit length segments. Here, we call the resulting embedding $\mathcal{Q}_2$.

To address (constraint 2), let Ψ be the subset of cells in $\mathcal{Q}_2$ corresponding to the type of polyline crossing indicated by the (purple) diamond polygon in Fig. 1(g). If $\Psi = \emptyset$, we can specify $\mathcal{Q}_3 = \mathcal{Q}_2$. If $\Psi \neq \emptyset$, we proceed by substituting each polyline crossing with an orthogonal integer lattice embedding of a gadget, denoted Υ, having the same properties as the novel proper 3-coloring planarization gadget shown in Fig. 2(a,b) (note that Fig. 2(a) and Fig. 2(b) are identical aside from having distinct vertex colorations), where these properties are given by the following lemma:

▶ **Lemma 1.** *The proper 3-coloring planarization gadget shown in Fig. 2(a,b) has the following properties: (1) its chromatic number is 3; (2) any proper 3-coloring will assign identical colors to the vertices labeled X and X' as well as the vertices labeled Y and Y'.*

Proof. Let W be a graph isomorphic to the 13 vertex and 24 edge gadget shown in Fig. 2(a,b). To establish properties (1) and (2), it suffices to first evaluate the chromatic polynomial $P\left(W, k\right)$ of the Fig. 2(a,b) gadget, check that $P\left(W, 2\right) = 0$, observe that $P\left(W, 3\right) = 12$, and then inspect each of the 12 possible proper 3-colorings to confirm property (2). Here, noting that there are at most $3^{13} = 1594323$ possible vertex colorings to check, we used brute force methods to enumerate all 12 possible proper 3-colorings. We briefly remark that, while a more insightful proof is possible, we were unable to find one of a short enough length to include in the current context. ◀

In particular, we specify Υ as the 25×33 cell construction shown in Fig. 2(c,d) (note that Fig. 2(c) and Fig. 2(d) are identical aside from having distinct vertex colorations). As in the case of the Fig. 2(a,b) graph, and as we will see in Lemma 2, the vertices labeled X and X' (respectively, Y and Y') in the graph corresponding to the Fig. 2(c,d) construction are likewise forced to be the same in any proper 3-coloring. Here, to replace polyline crossings in $\mathcal{Q}_2$ with Υ, we can expand the embedding $\mathcal{Q}_2$ by moving each point at a position (x, y) to a position $(25x, 33y)$, then replace each 25×33 block of cells centered on a polyline crossing with Υ. Observe that this will serve to ensure that the outgoing polylines to the North, West, East, and South cells in the von Neumann neighborhood of each cell in Ψ are connected to the Υ gadget vertices labeled Y', X, X', and Y, respectively. In this context, we let Φ correspond to all cells marking the position of a "color change" gadget with a (yellow) concave diamond marker in $\mathcal{Q}_3$, where Φ includes the markers shown in Fig. 2(c,d).

Next, for each cell $c \notin \Phi$ containing an endpoint of a polyline, we replace a $\left(\frac{103}{50}\right) \times 3$ distortion of the cell with an appropriate version of the embedded "color copying" gadget shown in Fig. 3. We note that (purple) nodes in this gadget represent metavertices corresponding to cliques of size $(k-3)$. More specifically, for a given cell $c \notin \Phi$, letting ζ_c be an instance of the Fig. 3 graph, we generate a graph ζ'_c by: (case N) deleting the vertices $\{N_1, N_2\}$ and all vertices embedded above (i.e., with a larger y coordinate) if and only if c *does not* have an outgoing polyline segment to the North in its von Neumann neighborhood; (case W) deleting the vertices $\{W_1, W_2\}$ and all vertices embedded to the left (i.e., with a smaller x coordinate) if and only if c does not have an outgoing polyline segment to the West in its von Neumann neighborhood; (case E) deleting the vertices $\{E_1, E_2\}$ and all vertices embedded to the right (i.e., with a larger x coordinate) if and only if c does not have an outgoing polyline segment to the East in its von Neumann neighborhood; and (case S) deleting the vertices $\{S_1, S_2\}$ and all vertices embedded below (i.e., with a smaller y coordinate) if and only if c does not have an outgoing polyline segment to the South in its von Neumann neighborhood. To complete the construction of H, we then embed ζ' on the aforementioned $\left(\frac{103}{50}\right) \times 3$ distortion of the cell $c \notin \Phi$ in exactly the manner shown in Fig. 3, embed an instance η_c of the "color change" gadget shown in Fig. 4 on a $\left(\frac{103}{50}\right) \times 3$ distortion of each cell $c \in \Phi$ – with (purple) metavertices corresponding to cliques of size $(k-3)$ as in Fig. 3 – and identify any vertices or metavertices from the borders of adjacent cells mapped to the same coordinates (see Section 3.1 for an elaboration on metavertex identification).

We can now observe the following lemma:

▶ **Lemma 2.** *The graph H admits a proper k-coloring if and only if the graph G admits a proper 3-coloring.*

Proof. Assume the definitions previously given in the proof argument for Theorem 1.

First consider the case where G is planar, and no copies of the gadget Υ were embedded during the process of generating $\mathcal{Q}_3$ from $\mathcal{Q}_1$. Here, we have that H will be constructed from the embedding $\mathcal{Q}_3$ by: (1) replacing exactly one cell falling along each polyline at a position where exactly two polyline segment endpoints coincide with the "color change" gadget; and (2) replacing all remaining cells hosting polyline segments with a "color copying" gadget η in such a manner that the vertices labeled X_1 and metavertices labeled X_2 in Fig. 3 will be present on the North, West, East, and South boundaries of each cell's von Neumann neighborhood if and only if the cell has a polyline segment egressing from its North,

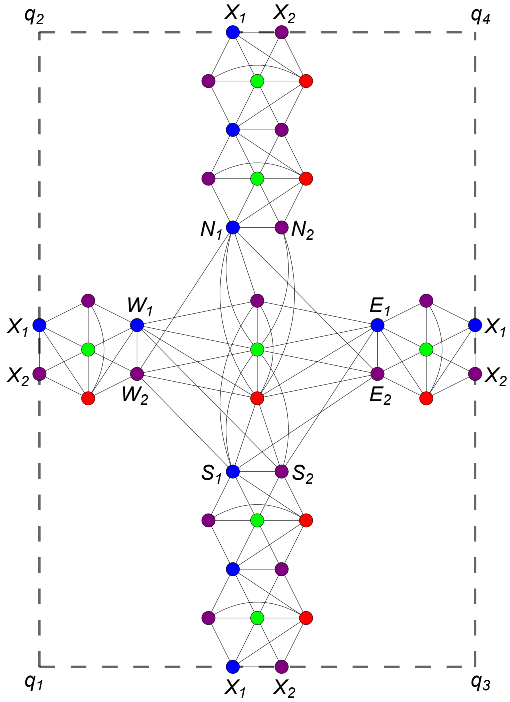

Figure 3 Illustration of the "color copying" gadget – and scheme for the gadget's placement on a "distorted" $\left(\frac{103}{50}\right) \times 3$ cell (dashed box) – where (purple) vertices (e.g., labeled X_2) are metavertices corresponding to cliques of size $(k-3)$.

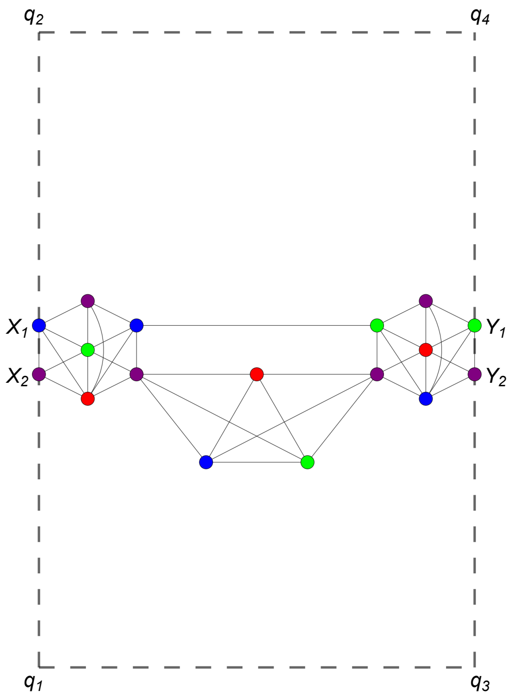

Figure 4 Illustration of the "color change" gadget – and scheme for the gadget's placement on a "distorted" $\left(\frac{103}{50}\right) \times 3$ cell (dashed box) – where (purple) vertices (e.g., labeled X_2 or Y_2) are metavertices corresponding to cliques of size $(k-3)$.

West, East, and South boundaries, respectively. Accordingly, each cell in $\mathcal{Q}_3$ corresponding to a vertex in G of degree d will, in turn, correspond to a specific "color copying" gadget with d copies of the vertices labeled X_1 and X_2. Furthermore, all but one of the cells corresponding to part of a polyline connecting vertices in G will, on some pair of boundaries, have exactly two copies of vertices labeled X_1 and two copies of metavertices labeled X_2, with the remaining cell corresponding to a "color change" gadget.

To now show that H admits a proper k-coloring if and only if the planar instance of G admits a proper 3-coloring, it suffices to observe for any proper k-coloring of H that: (requirement 1) the vertices labeled X_1 in every "color copying" gadget must have the same coloration; and (requirement 2) the vertices labeled X_1 and Y_1 in every "color change" gadget must have a distinct coloration.

Concerning (requirement 1), consider first the case where $k = 3$. Here, for every possible instance of the gadget ζ'_c, we can check the chromatic polynomial $P\left(\zeta'_c, k\right)$ with parameter k, or enumerate all possible proper 3-colorings via brute force, observe that $P\left(\zeta'_c, 2\right) = 0$ and $P\left(\zeta'_c, 3\right) \neq 0$ (e.g., $\zeta_c = \zeta'_c \implies P\left(\zeta'_c, 3\right) = 384$), and furthermore check that the constraint is satisfied in each of these cases. To address cases where $k \geq 4$, it suffices to observe that, for any instance of ζ'_c (recalling that the metavertices shown in Fig. 3 correspond to cliques of size $(k-3)$), every vertex will necessarily belong to a clique of size k. Accordingly, if (1) is satisfied in the case where $k = 3$, it will likewise be satisfied in the case where $k = 4$ and we are forced to assign the additional color to the single vertex corresponding to each metavertex, satisfied in the case where $k = 5$ and we are required to assign the two additional colors to the two vertices corresponding to each metavertex, and by induction, satisfied for every $k \geq 3$.

Concerning (requirement 2), we proceed in a similar manner. In particular, letting η_n be an instance of the "color change" gadget in the case where $k = 3$, we observe that $P\left(\eta_n, 2\right) = 0$ and $P\left(\eta_n, 3\right) = 144$. With exactly the same inductive argument used to address (requirement 1), we can then show that (requirement 2) will hold for every $k \geq 3$.

As we have now seen that (requirement 1) and (requirement 2) will be satisfied for every $k \geq 3$, this yields the lemma in the case where G is planar.

In the case where G is non-planar, it suffices to observe that the Υ construction simply replaces certain vertices in the Fig. 2(a,b) proper 3-coloring planarization gadget with connected subgraphs $s_1, s_2, \ldots$, then places the "color change" gadget on polylines if and only if they connect distinct subgraphs. Accordingly, following our earlier argument, the "color change" gadget will conceptually force each of the embedded vertices in the same subgraph s_i in Υ to have an identical coloration, and each pair of adjacent subgraphs s_i and s_j in Υ to have distinct colorations. In observation of Lemma 1, this yields the current lemma in the case where G is non-planar. ◀

We can also observe that the construction of H from an initial order n graph G takes at most $\mathcal{O}\left(n^2\right)$ time as a consequence of the orthogonal integer lattice embedding $\mathcal{Q}_1$ having at most $\mathcal{O}\left(n^2\right)$ cells hosting at least one endpoint of a polyline, where the embedding of the "color copying" and "color change" gadgets on each cell hosting a polyline then increases the time complexity of the construction to $\mathcal{O}\left(n^2 \cdot k\right)$. This together with Lemma 2 implies that, as a consequence of the fact that no $2^{o\left(\sqrt{n_G}\right)}$ time algorithm can exist for proper 3-coloring an order n_G instance of the graph G under the ETH, we likewise have that no $2^{o\left(\sqrt{n_H} \cdot k\right)}$ time algorithm can exist for proper k-coloring an order n_H instance of the constructed graph H unless the ETH is false.

The subsequent step of this proof argument, which is simultaneously the most technically difficult and easiest to describe, is to show that H can be realized as a UNIT-PURE-k-DIR graph if and only if H admits a proper k-coloring. Here, we can begin by observing that, due to the requirement no parallel segments may intersect, any proper k-coloring for a UNIT-PURE-k-DIR can be understood as an assignment of embedding angles for the segments of a geometric system of intersecting unit segments witnessing graph class membership.

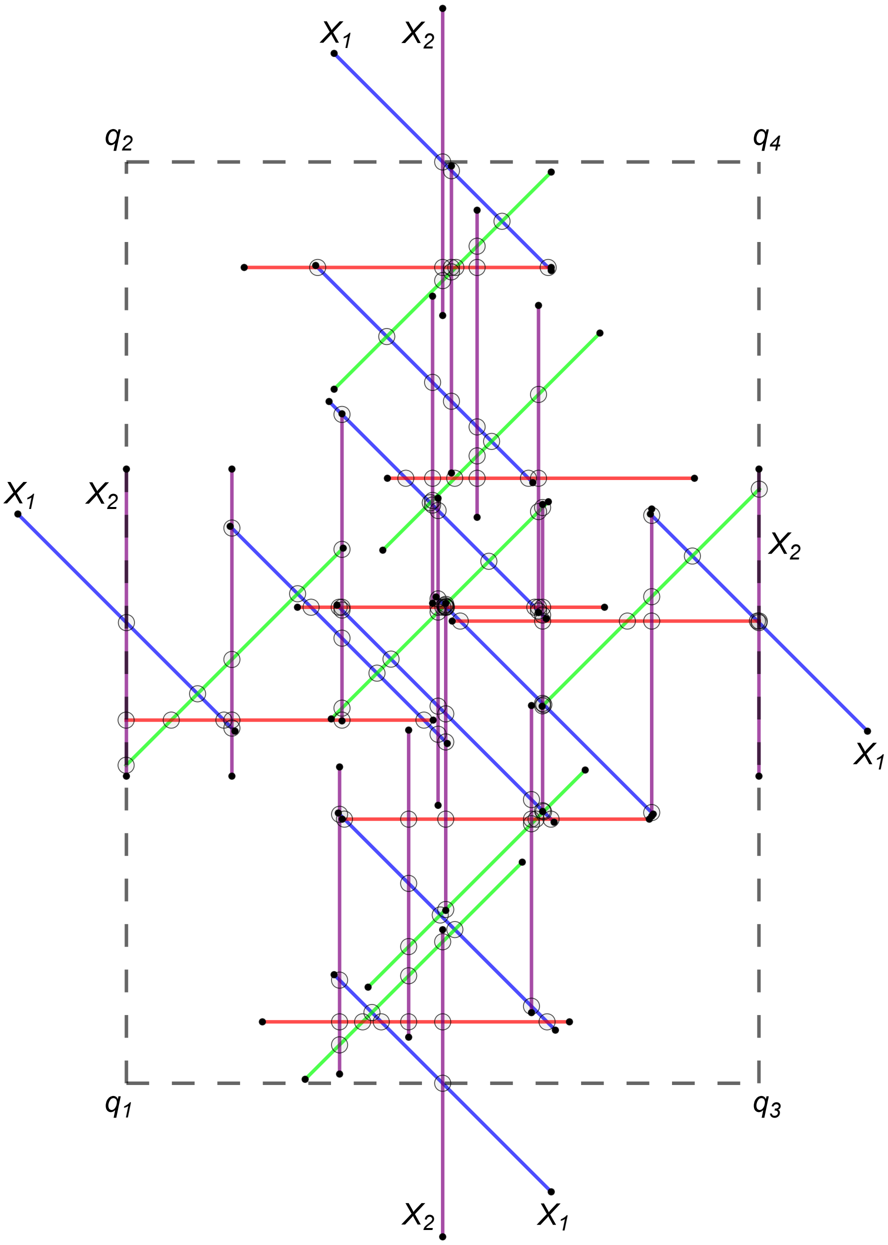

Figure 5 UNIT-PURE-k-DIR realization (for every $k \geq 3$) of the "color copying" gadget – and scheme for the gadget's placement on a "distorted" $\left(\frac{103}{50}\right) \times 3$ cell (dashed box) – where (purple) segments (e.g., labeled X_2) correspond to $(k-3)$ overlapping segments embedded in the plane at distinct infinitesimal perturbations of $\epsilon_1 < \epsilon_2 < \ldots < \epsilon_{(k-3)}$ from $\frac{\pi}{2}$ radians, and where segments corresponding to vertices labeled X_1 in Fig. 3 are embedded in the plane at an angle of $-\frac{\pi}{4}$ radians (respectively, $\frac{\pi}{4}$ radians in a reflection of the embedding across the y-axis); segment-segment intersections are denoted with a (hollow) circle; note that no endpoint of a line segment is ever embedded along another line segment.

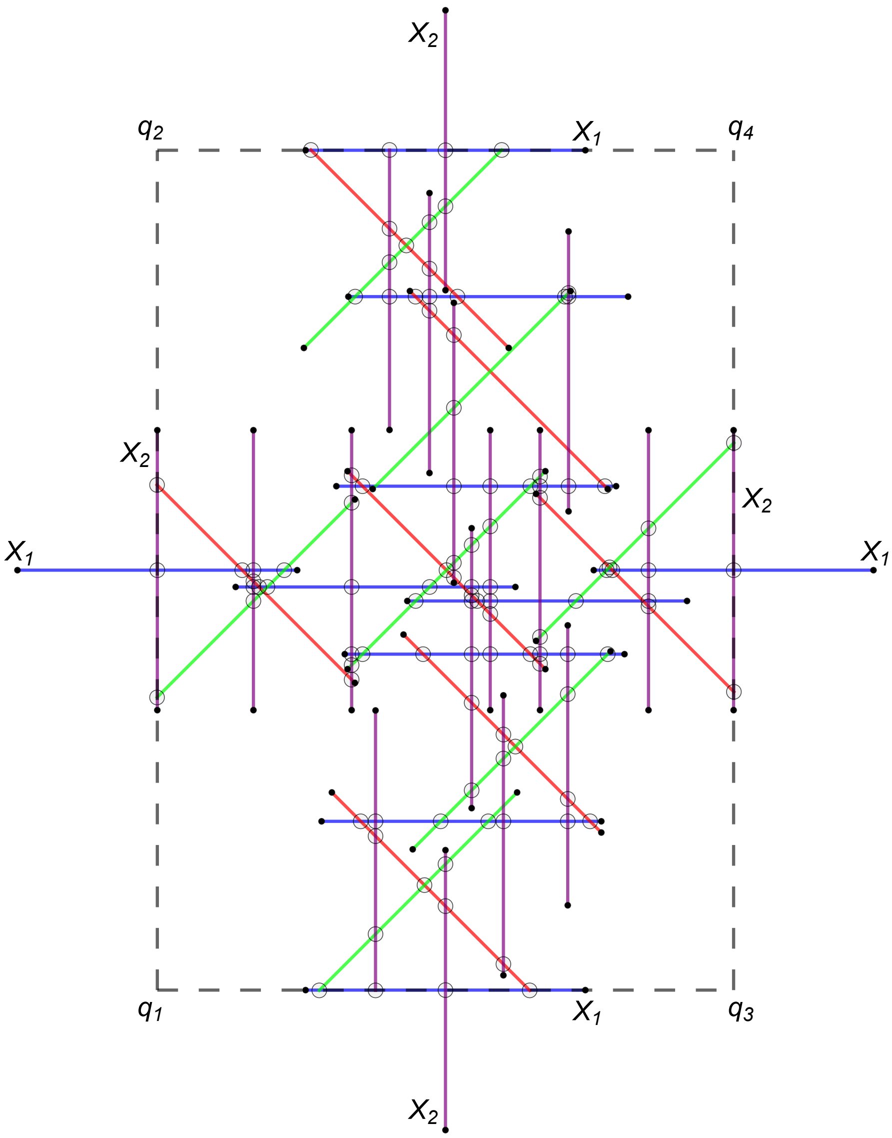

Figure 6 Alternative UNIT-PURE-k-DIR realization (for every $k \geq 3$) of the "color copying" gadget from the Theorem 1 proof argument – and scheme for the gadget's placement on a "distorted" $\left(\frac{103}{50}\right) \times 3$ cell (dashed box) – where (purple) segments (e.g., labeled X_2) correspond to $(k-3)$ overlapping segments embedded in the plane at distinct infinitesimal perturbations of $\epsilon_1 < \epsilon_2 < \ldots < \epsilon_{(k-3)}$ from $\frac{\pi}{2}$ radians, and segments corresponding to vertices labeled X_1 in Fig. 3 are embedded in the plane at an angle of 0 radians; segment-segment intersections are denoted with a (hollow) circle; note that no endpoint of a line segment is ever embedded along another line segment.

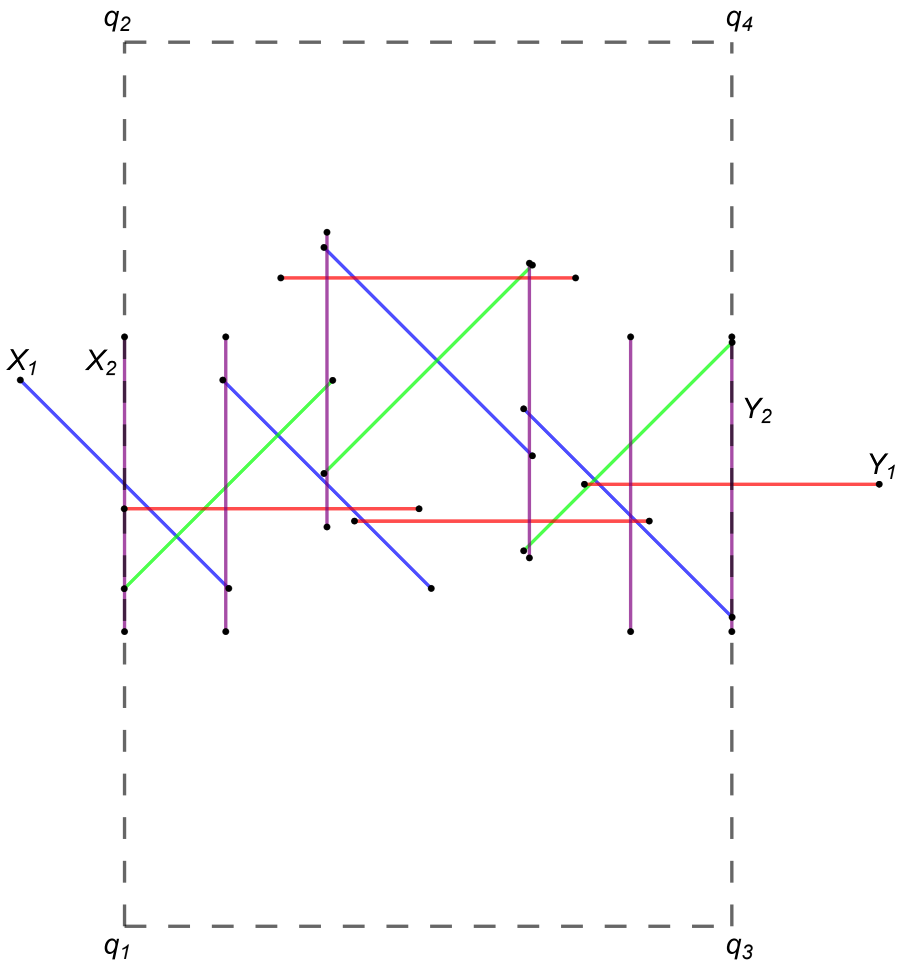

Figure 7 UNIT-PURE-k-DIR realization (for every $k \geq 3$) of the "color change" gadget from the Theorem 1 proof argument – and scheme for the gadget's placement on a "distorted" $\left(\frac{103}{50}\right) \times 3$ cell (dashed box) – where (purple) segments (e.g., labeled X_2) correspond to $(k-3)$ overlapping segments embedded in the plane at distinct infinitesimal perturbations of $\epsilon_1 < \epsilon_2 < \ldots < \epsilon_{(k-3)}$ from $\frac{\pi}{2}$ radians, and where segments corresponding to vertices labeled X_1 and Y_1 in Fig. 4 are, respectively, embedded in the plane at angles of $-\frac{\pi}{4}$ and 0 radians, $\frac{\pi}{4}$ and 0 radians after reflection across x-axis, 0 and $\frac{\pi}{4}$ radians after reflection across y-axis, and 0 and $-\frac{\pi}{4}$ radians after reflection across both the x-axis and y-axis; note that no endpoint of a line segment is ever embedded along another line segment.

We now observe that the Fig. 3 "color copying" gadget can be realized as a UNIT-PURE-k-DIR graph with three distinct segment angles for the vertices labeled X_1 on each of the four borders of the "distorted" (i.e., non-square) $\left(\frac{103}{50}\right) \times 3$ cell embedding the gadget. In particular, we refer the reader to the UNIT-PURE-k-DIR realizations of this gadget shown in Fig. 5 and Fig. 6, where in both cases (purple) segments (e.g., labeled X_2) correspond to $(k-3)$ overlapping segments embedded in the plane at distinct infinitesimal perturbations of $\epsilon_1 < \epsilon_2 < \ldots < \epsilon_{(k-3)}$ from $\frac{\pi}{2}$ radians. Note that, as no endpoint of a segment is ever embedded along another line segment, these infinitesimal perturbations can always be chosen to be small enough to not change the set of segment-segment intersections. For Fig. 5, we can observe that segments corresponding to vertices labeled X_1 in Fig. 3 are embedded in the plane at an angle of $-\frac{\pi}{4}$ radians (respectively, $\frac{\pi}{4}$ radians in a reflection of the embedding across the y-axis). For Fig. 6, we can observe that segments corresponding to vertices labeled X_1 in Fig. 3 are embedded in the plane at an angle of 0 radians. This covers each of the three necessary cases.

Similarly, the Fig. 4 "color change" gadget can be realized as a UNIT-PURE-k-DIR graph with all possible combinations of three distinct segment angles for the vertices labeled X_1 and Y_1 on West and East border, respectively, of the "distorted" (i.e., non-square) $\left(\frac{103}{50}\right) \times 3$ cell embedding the gadget, with the segments labeled X_2 and Y_2 embedded in the same manner as the (purple) segments in Fig. 5 and Fig. 6. While we are unable to show all possible UNIT-PURE-k-DIR realizations of this gadget due to space constraints, we refer the reader to Fig. 7 for an illustration of the case where segments corresponding to the vertices labeled X_1 and Y_1 in Fig. 4 are embedded in the plane at angles of $-\frac{\pi}{4}$ and 0 radians, respectively.

Putting everything together yields that, assuming the ETH, no $2^{o\left(\sqrt{n/k}\right)}$ time algorithm can exist for recognizing order n UNIT-PURE-k-DIR graphs. It remains to deduce that this directly implies no $2^{o\left(\sqrt{n/k}\right)}$ time algorithm can exist for proper k-coloring an order n UNIT-PURE-k-DIR graph under the ETH. Here, simply observe that if such a $2^{o\left(\sqrt{n/k}\right)}$ time coloring algorithm $\mathcal{A}$ exists, there will likewise exist some upperbound for its run time of the form $2^{\left(\sqrt{n} \cdot k - \epsilon\right)}$ for some constant $\epsilon \in \mathbb{R}_{>0}$ and n sufficiently large. Accordingly, we could simply run $\mathcal{A}$ for $2^{\left(\sqrt{n} \cdot k - \epsilon\right)}$ time steps on the graph H, and if no proper k-coloring is found, determine that no such proper k-coloring exists. However, as H has a proper k-coloring if and only if it is a UNIT-PURE-k-DIR graph, this implies the existence of a $2^{o\left(\sqrt{n/k}\right)}$ time algorithm for recognizing H as a UNIT-PURE-k-DIR graph. Therefore, by our earlier arguments, no such algorithm $\mathcal{A}$ can exist under the ETH. ◀

▶ **Theorem 2.** *For any $k \geq 4$, provided a UNIT-PURE-k-DIR graph and a list of segment coordinates specified using $\mathcal{O}(n \cdot k)$ bits witnessing graph class membership, no $2^{o\left(\sqrt{n/k}\right)}$ time algorithm can exist for finding a proper $(k-1)$-coloring of the graph unless the ETH is false.*

Proof Sketch. Recall that the Theorem 1 proof argument realized the Fig. 4 "color change" gadget as a UNIT-PURE-k-DIR graph. Here, we can instead realize the Fig. 4 "color change" gadget as a UNIT-PURE-$(k+1)$-DIR graph in which we allow for one additional segment embedded in the plane at an angle of $\frac{\pi}{20}$ radians. While we omit the details due to space constraints, briefly, this can be shown to allow for the vertices labeled X_1 and Y_1 in Fig. 4 to correspond to unit segments having the same $\frac{\pi}{4}$, $-\frac{\pi}{4}$, or 0 radian angle. Accordingly, there will no longer be a correspondence between segment angles and color assignments in a proper $(k-1)$-coloring. This then allows one to show that the proper $(k-1)$-coloring problem remains hard even when provided a list of segment coordinates. Finally, as the reduction given in Theorem 1 can be performed in $\mathcal{O}\left(n^2 \cdot k\right)$ time, this yields the current theorem. ◀

▶ **Corollary 1.** *For each $k \geq 4$, the problem of properly 3-coloring an arbitrary m edge graph can be reduced in $\mathcal{O}\left(m^2\right)$ time to properly $(k-1)$-coloring a UNIT-PURE-k-DIR graph.*

Proof. Let Q be an arbitrary graph with vertex set V_Q and edge set E_Q, where $|E_Q| = m$. Generate a graph G with $2m$ edges by replacing every vertex $v_i \in V_Q$ of degree d with a cycle of length d, doing so in such a manner that exactly one vertex in the cycle is adjacent to each distinct neighbor of $v_i \in V_Q$. Assign a unique "cycle label" to the vertices in each generated cycle. We can now generally proceed along the lines of the Theorem 2 proof argument to reduce the problem of properly 3-coloring G to properly $(k-1)$-coloring a

UNIT-PURE-k-DIR graph, with the exception that we do not place the "color change" gadget on polylines corresponding to edges connecting vertices with the same "cycle label" in G. It now suffices to observe that all vertices with the same "cycle label" will be forced to have an identical coloration. ◀

▶ **Theorem 3.** *Unless the ETH is false, for any $k \geq 3$, no $2^{o(n/k)}$ time algorithm can exist for proper k-coloring order n geometric intersection graphs of lines where line-line intersections are constrained to occur on any one of three parallel planes in $\mathbb{R}^3$.*

Proof Sketch. For every $k \geq 3$, we proceed via reduction from the problem of proper edge k-coloring a proper 3-colorable simple undirected k-regular graph G. Briefly, by a reduction of Holyer [28] in the case where $k = 3$ (where Brooks' theorem [10] implies proper 3-colorability), a reduction of Leven & Galil [40] in cases where $k \geq 4$, and by invoking the sparsification lemma of Impagliazzo et. al. [30], it is straightforward to rule out the existence of a $2^{o(n/k)}$ algorithm for finding edge proper k-colorings of these graphs under the ETH.

To begin, generate an embedding $\mathcal{Q}$ of G on three parallel planes in $\mathbb{R}^3$ by embedding each vertex v at a coordinate $\{x, y, z\}$ in $\mathcal{Q}$, where $z = 1$, 2, or 3, if the proper 3-coloring of G places v in the first, second, or third (arbitrarily ordered) color classes, respectively. Next, generate a new embedding $\mathcal{Q}'$ from $\mathcal{Q}$ by perturbing only the x and y coordinates of vertices to place them in general position, guaranteeing that no four points are concyclic in $\mathbb{R}^3$, and thus, that no two edges in G will fall along the same hyperplane in the embedding $\mathcal{Q}'$ unless they intersect at a common vertex point. Here, treating edges in $\mathcal{Q}'$ as line segments with endpoints at vertices, we can replace each line segment with a line containing the segment while ensuring that line-line intersections must occur at vertex positions on any one of three parallel planes in $\mathbb{R}^3$. In this context, letting $\mathcal{L}$ be the set of all lines generated in this manner from the edges in the embedding $\mathcal{Q}'$, it will accordingly be the case that the geometric intersection graph of these lines will correspond to a line graph for G.

Putting everything together, as finding a proper k-coloring of a line graph for G is equivalent to finding an edge proper k-coloring of G, and as we have shown that no $2^{o(n/k)}$ time algorithm can exist for edge proper k-coloring G, this yields the current theorem. ◀

5 Concluding remarks

Should it happen to be the case that no $2^{o(n/k)}$ algorithm exists under the ETH for recognizing or proper k-coloring a UNIT-PURE-k-DIR graph for some $k \in \mathbb{N}_{>2}$ – which would answer the open question of Miltzow discussed in Section 2 [56] – it seems unlikely that it will be possible to prove this via a straightforward modification of our approach in Theorem 1. In particular, recall that in our proof argument for Theorem 1, we make use of an orthogonal integer lattice embedding algorithm for a graph of maximum degree ≤ 4, then proceed by substituting cells in at most a constant factor expansion of this embedding with different UNIT-PURE-k-DIR subgraphs. Here, while there again exist linear time algorithms for computing these embeddings [5, 54], for an order n graph there can be $\mathcal{O}\left(n^2\right)$ cells hosting polylines (or polyline intersections). Furthermore, we remark that finding an embedding of a degree ≤ 4 order n graph minimizing the total length of all polyline segments is NP-hard [55] and, unless $P = NP$, inapproximable within a factor of $\mathcal{O}\left(n^{1/2-\epsilon}\right)$ [3].

A possible path forward would be to: (1) find a problem on a class of graphs that does not admit a $2^{o(n/k)}$ algorithm under the ETH, and (2) show that graphs in this class admit linear time computable orthogonal integer lattice embeddings where the total length of all polylines is asymptotically $\mathcal{O}(n)$. However, we know of no such problem.

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