The Simultaneous Interval Number

A New Width Parameter that Measures the Similarity to Interval Graphs

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Abstract

We propose a novel way of generalizing the class of interval graphs, via a graph width parameter called simultaneous interval number. This parameter is related to the simultaneous representation problem for interval graphs and defined as the smallest number d of labels such that the graph admits a d-simultaneous interval representation, that is, an assignment of intervals and label sets to the vertices such that two vertices are adjacent if and only if the corresponding intervals, as well as their label sets, intersect. We show that this parameter is NP-hard to compute and give several bounds for the parameter, showing in particular that it is sandwiched between pathwidth and linear mim-width. For classes of graphs with bounded parameter values, assuming that the graph is equipped with a simultaneous interval representation with a constant number of labels, we give FPT algorithms for the clique, independent set, and dominating set problems, and hardness results for the independent dominating set and coloring problems. The FPT results for independent set and dominating set are for the simultaneous interval number plus solution size. In contrast, both problems are known to be W[1]-hard for linear mim-width plus solution size.

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1 Introduction

Interval graphs are among the best-known and most studied graph classes, due to their intuitive representation with an interval intersection model, their rich structure, and many algorithmic advantages. Many problems that are NP-hard on general graphs can be solved in polynomial time on interval graphs. Examples are the coloring problem [35, 37, 47], the dominating set problem [12], and the Hamiltonian cycle problem [46]. Furthermore, due to their definition via interval representations, there are plenty of real-world applications for interval graphs (see [45] for a nice, short overview of such applications).

There are several different ways to generalize the concept of an interval graph. One of these concepts are the so-called *d-interval graphs* where every vertex is represented by a set of d intervals on the real line and two vertices are adjacent if any pair of their intervals intersect. A subclass of these graphs are the *d-track interval graphs* where we have d parallel lines and every vertex is represented by d intervals, one on each line. It is easy to see that any graph is a d-track interval graph (and, thus, a d-interval graph) for some d. Therefore, it makes sense to define the parameters *interval number* $\mathsf{i}(G)$ [34] and *track number* $\mathsf{t}(G)$ [36] as the minimal numbers d such that G is a d-interval (resp. d-track interval) graph. There is some work on these graph classes concerning parameterized complexity [24, 44]. However, most of the classical graph problems are NP-hard for graphs with $\mathsf{i}(G) = 2$ or $\mathsf{t}(G) = 3$ [21, 22]. Furthermore, even the independent set problem and the dominating set problem are W[1]-hard when parameterized by the solution size for graphs with $\mathsf{t}(G) = 2$ [44].

Another way to define a whole family of generalizations of interval graphs comes from the so-called simultaneous representation problems. In this generalization, we are given d interval graphs $G_1, \ldots, G_d$ which may share some vertices and asks for an interval representation that assigns to every vertex in $V(G_1) \cup \cdots \cup V(G_d)$ exactly one interval such that for every $i \in \{1, \ldots, d\}$ two vertices of G_i are adjacent if and only if their intervals intersect. The problem of deciding whether a given set of graphs has such a simultaneous representation was introduced in 2009 by Jampani and Lubiw [41], where they considered chordal graphs, comparability graphs, and permutation graphs, all classes of graphs that can also be defined via certain intersection representations. A year later, the same authors considered the problem of simultaneous interval representations [42]. Since then, there has been several results on the complexity of this problem for different classes of graphs [4, 7, 58].

An equivalent definition for a simultaneous interval representation can be given as follows: For some interval model we add additional label sets in the form of subsets of $\{1, \ldots, d\}$ and two vertices belonging to two intervals are adjacent if these intervals intersect and the intersection of their label sets is non-empty. This definition leads to an intuitive application in scheduling, where each of the labels $1, \ldots, d$ represents some machine and an interval represents a job with its processing window (the interval) and the set of machines needed to perform the job (the label set). An independent set in such a graph would then represent a conflict-free schedule of a subset of jobs.

Similar to d-interval graphs and d-track interval graphs, any graph can be defined as a d-simultaneous interval graph for some d. Thus, we can introduce the *simultaneous interval number* $\mathsf{si}(G)$ as the smallest number d for which G is a d-simultaneous interval graph. Many width parameters are unbounded for interval graphs, as these tend to grow with the clique number (for example treewidth/pathwidth is unbounded for interval graphs). Furthermore, even width parameters that can be bounded for dense graphs, such as cliquewidth or twin-width, are unbounded for interval graphs [8, 31]. On the other hand, those parameters that are bounded for interval graphs, such as linear mim-width or tree-independence number

Table 1 Parameterized complexity summary. Abbreviations mean ind → independent, dom → dominating, W[1] → W[1]-hard, W[2] → W[2]-hard, pNPh → para-NP-hard, tree-α → tree-independence number. Green results are given in this paper. Hardness results for problems with given solution size k means that the problem is hard when parameterized by $p+k$. For space reasons, we omitted the $\mathcal{O}$ and $\mathcal{O}^*$ notations in the running time bounds.

problem\parameter	$p = \mathsf{si}(G)$	$p = \mathsf{linear\text{-}mim}(G)$	$p = \mathsf{tree}\text{-}\alpha(G)$	$p = \mathsf{t}(G)$
clique	$p2^{2^p+2p}$	pNPh [60]	pNPh [23]	pNPh [26]
clique of size k	2^{kp}	?	2^{k^p} [14, 19]	$p^k k^{k+2}$ [24]
coloring	pNPh	pNPh [28]	pNPh [28]	pNPh [28]
k-coloring	k^{kp}	n^{kp} [32]	k^{k^p} [14, 19]	pNPh [22]
ind set	n^p	W[1]/n^{2p} [25, 39]	n^p [19, 62]	pNPh [22]
ind set of size k	2^{kp}	W[1]/n^{2p} [25, 39]	?	W[1] [24]
dom set	n^{2p}	W[1]/n^{2p} [25, 39]	pNPh [3, 16]	pNPh [22]
dom set of size k	2^{kp}	W[1]/n^{2p} [25, 39]	W[2] [49]	W[1] [24]
ind dom set	W[1]/n^{2p}	W[1]/n^{2p} [25, 39]	?	pNPh [22]
ind dom set of size k	n^{2p}	W[1]/n^{2p} [25, 39]	?	W[1] [24]

(see [19, 40]), do not properly reflect the structural advantages of interval graphs. Many of the problems that are easy for interval graphs, such as coloring or independent set, are either para-NP-hard or W[1]-hard (see Table 1). Furthermore, the maximum clique problem is para-NP-hard when parameterized by one of those parameters, even though the structure of the maximal cliques is very restricted for interval graphs.

When parameterized by the simultaneous interval number, however, the maximum clique problem becomes FPT, as we will show. In addition, some of the problems that are W[1]-hard when parameterized by linear mim-width plus solution size, such as independent set and dominating set (see [25, 39]), are FPT when parameterized by simultaneous interval number plus solution size. Therefore, we argue that the simultaneous interval number is a strong candidate to fill the gap in describing graphs with a structure similar to interval graphs.

Our Contribution. We introduce a new graph width parameter, the *simultaneous interval number*, in Section 2. This parameter is compared to most of the other common width parameters such as treewidth, cliquewidth, or mim-width in Section 3, where we also give several bounds involving the order and the size of the graph, the edge clique cover number, the clique number, and other width parameters. In Section 4 we show that the computation of the simultaneous interval number is NP-hard. Furthermore, we give results on the parameterized complexity of several graph problems, such as clique (Section 5), coloring (Section 6), and variants of the independent set and dominating set problems (Section 7). For an overview of these results see Table 1. Proofs omitted due to lack of space can be found in the full version [1].

Definitions and Notation. Unless stated otherwise, all the graphs considered are simple, finite, non-empty and undirected. Given a graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. Often we will denote the number of vertices of graph, i.e., $|V(G)|$, as n and the number of edges, i.e., $|E(G)|$, as m. A *matching* in a graph is a set of pairwise disjoint edges; a matching is *induced* if no two vertices belonging to different edges of the matching are adjacent.

Next we define the term *class of intersection graphs*. Such a graph class $\mathcal{C}$ can be defined via a family $S_{\mathcal{C}}$ of sets whose elements are also families of sets. For the sake of convenience, we assume that $S_{\mathcal{C}}$ contains a set family that contains a non-empty set. A *$\mathcal{C}$-representation* of a graph G is a mapping $R : V(G) \to \mathcal{F}$ where $\mathcal{F} \in S_{\mathcal{C}}$ such that $xy \in E(G)$ if and only if $R(x) \cap R(y) \neq \emptyset$. We call $\mathcal{F}$ the *ground set family* of R. By definition, $\mathcal{C}$ consists precisely of graphs G having a $\mathcal{C}$-representation.

The class of *chordal graphs* is defined via the set $S_{\mathcal{C}}$ that contains for every tree the set of its subtrees. For the class of *interval graphs*, the set $S_{\mathcal{C}}$ contains only the one set family, namely the set of all open intervals of the real line. For any interval representation R of graph G, we define $\ell(v)$ and $r(v)$ to be the left and right endpoints of the interval $R(v)$.

A graph G is a *bipartite graph* if its vertex set can be partitioned into two independent sets A and B. Furthermore, a bipartite graph is *complete bipartite* if every vertex of A is adjacent to every vertex of B. A graph is a *split graph* if its vertex set can be partitioned into a clique and an independent set. A graph is a *complete split graph* if there exists a partition in which every vertex of the independent set is adjacent to all the vertices of the clique. A graph is *C_4-free* if it does not contain an induced cycle of length 4.

2 Simultaneous Representations and Simultaneous Interval Number

In [41, 43], Jampani and Lubiw introduce the concept of *simultaneous representations* as well as the *simultaneous representation problem*. This concept was then taken up by Bok and Jedličková [7] who give the following definition:

▶ **Definition 2.1.** *Let $\mathcal{C}$ be a class of intersection graphs. Graphs $G_1, \dots G_d \in \mathcal{C}$ are* simultaneously $\mathcal{C}$-representable *if there exist $\mathcal{C}$-representations $R_1, \dots, R_d$ of $G_1, \dots G_d$ with a common ground set family $\mathcal{F} \in S_{\mathcal{C}}$ such that*

$$\forall i, j \in \{1, \dots, d\}, \forall v \in V(G_i) \cap V(G_j) \colon R_i(v) = R_j(v).$$

*In particular, we say that $G = G_1 \cup \ldots \cup G_d$ is a d-*simultaneous $\mathcal{C}$-graph.

For convenience of notation, we will oftentimes use the following equivalent definition of a simultaneous representation.

▶ **Definition 2.2.** *Let $d \in \mathbb{N}$, let G be a graph, and let $L : V(G) \to \mathcal{P}(\{1, \dots, d\})$ be a labeling of the vertices of G. Furthermore, let $G' \in \mathcal{C}$ with $V(G) = V(G')$ and $E(G) \subseteq E(G')$ be a graph with a $\mathcal{C}$-representation R. We say that (R, L) is a d-*simultaneous $\mathcal{C}$-representation *of G if it holds that $vw \in E(G)$ if and only if $R(v) \cap R(w) \neq \emptyset$ and $L(v) \cap L(w) \neq \emptyset$.*

Note that this definition allows the *empty set* as a label set. Obviously, any vertex with an empty label set is isolated. Therefore, the graphs admitting a 0-simultaneous $\mathcal{C}$-representation are exactly the edgeless graphs.

▶ **Observation 2.3.** *Let $\mathcal{C}$ be a class of intersection graphs. Let the graphs $G_1, \dots, G_d \in \mathcal{C}$ be simultaneously $\mathcal{C}$-representable with $\mathcal{C}$-representations $R_1, \dots, R_d$ with a common ground set family $\mathcal{F}$. Let $G := G_1 \cup \cdots \cup G_d$ and let $R \colon V(G) \to \mathcal{F}$ be defined as $R(v) := R_i(v)$ for any i with $v \in V(G_i)$. Let L be the labeling given by $L(v) = \{i \colon v \in G_i\}$ for all $v \in V(G)$. Then (R, L) is a d-simultaneous $\mathcal{C}$-representation of G.*

Figure 1 Two forbidden induced subgraphs of interval graphs with 2-simultaneous interval representations. Yellow intervals have label set $\{1\}$, blue intervals have label set $\{2\}$ and black intervals have label set $\{1, 2\}$. Note that the representation of the 4-cycle can be extended to a 2-simultaneous interval representation of cycles of arbitrary length.

This observation implies that every d-simultaneous $\mathcal{C}$-graph has a d-simultaneous $\mathcal{C}$-representation. However, the converse is not true in general.[1] However, if we exclude empty label sets and unused labels, then there is an analogous result to Observation 2.3.

▶ **Observation 2.4.** *Let (R, L) be a d-simultaneous $\mathcal{C}$-representation of a graph G with $L(v) \neq \emptyset$ for all $v \in V(G)$ and such that for all $i \in \{1, \ldots, d\}$ there exists a vertex v with $i \in L(v)$. Let G_i be the subgraph of G induced by the vertex set $\{v : i \in L(v)\}$ and let R_i be the restriction of R to $V(G_i)$. Then the graphs $G_1, \ldots, G_d$ are simultaneously $\mathcal{C}$-representable with $\mathcal{C}$-representations $R_1, \ldots, R_d$.*

A vertex with an empty label set would have to be considered as a vertex that is in none of the graphs of a simultaneous representation. However, this technical addition to the definition is very useful to address the issue of isolated vertices and leads to more compact statements and simpler proofs. For all of the classes considered here, it is always possible to represent isolated vertices without the empty label set. For example, for interval graphs we can always represent such a vertex with an interval that intersects nothing else. However, in general we cannot assume that this is possible for any class of intersection graphs (see Footnote 1).

▶ **Theorem 2.5.** *For every class of intersection graphs $\mathcal{C}$, every graph G has an $|E(G)|$-simultaneous $\mathcal{C}$-representation.*

In particular, this theorem holds for the class of intervals graphs, motivating the following definition.

▶ **Definition 2.6.** *Let G be a graph. The* simultaneous interval number $\mathsf{si}(G)$ *of G is the smallest integer d such that there exists a d-simultaneous interval representation of G.*

As observed before, the graphs with simultaneous interval number 0 are exactly the edgeless graphs. Furthermore, the graphs with simultaneous interval number at most 1 are exactly the interval graphs, and the class of graphs with the simultaneous interval number equal to 2 contains some asteroidal triples and all cycles (see Figure 1).

In the following, we show some bounds on the simultaneous interval number. The first result is implied directly by Theorem 2.5.

▶ **Corollary 2.7.** *For any graph G it holds that $\mathsf{si}(G) \leq |E(G)|$.*

Next we show that this bound is tight, up to a constant factor.

[1] As an example, we consider the class $\mathcal{K}$ of complete graphs which can be represented as intersection graphs via the set $S_{\mathcal{K}} = \{\{1\}\}$. The n-vertex edgeless graph has a 1-simultaneous $\mathcal{K}$-representation where all vertices are labeled with the empty set. However, it is not a d-simultaneous $\mathcal{K}$-graph for any d.

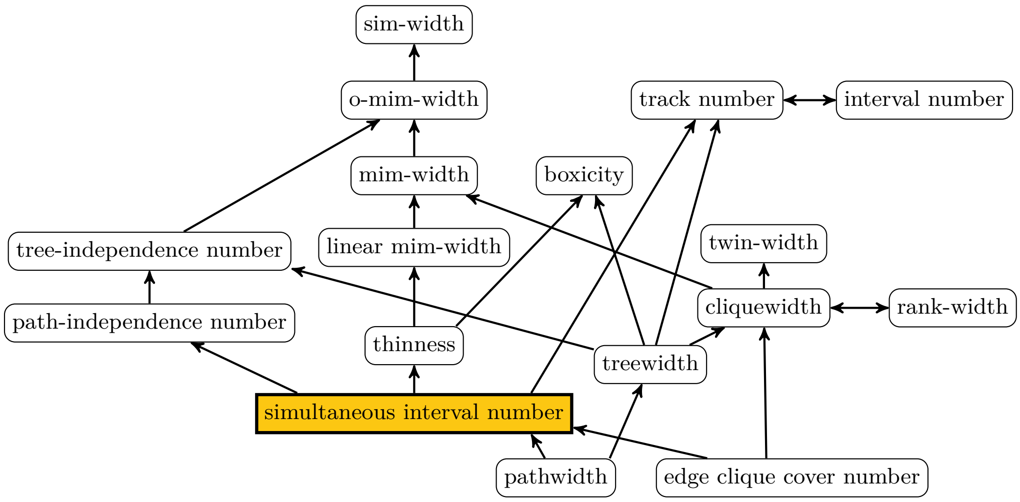

Figure 2 Diagram illustrating the relations between different graph width parameters. A directed edge from parameter P to parameter Q means that a bounded value of P implies a bounded value for Q. If a directed path from P to Q is missing, then parameter Q is unbounded for the graphs of bounded P.

▶ **Theorem 2.8.** *Let G be a complete 3-partite graph with parts of equal size. Then,* $\mathsf{si}(G) = \frac{1}{9}|V(G)|^2 = \frac{1}{3}|E(G)|$.

▶ **Theorem 2.9.** *Let $G = (V, E)$ be a bipartite graph with a bipartition $V = X \dot{\cup} Y$. Then* $\mathsf{si}(G) \leq \min\{|X|, |Y|\}$. *This bound is tight for complete bipartite graphs.*

The *complement of a matching* is a graph obtained from a complete graph of even order n by removing from it $\frac{n}{2}$ pairwise disjoint edges.

▶ **Lemma 2.10.** *If G is the complement of a matching with n vertices, then* $\mathsf{si}(G) \geq \log_2(n-1)$.

We will see later, in Lemma 5.5, that this bound is tight.

3 Placing si(G) in the Zoo of Graph Width Parameters

In this section we compare the simultaneous interval number to several other graph width parameters. See Figure 2 for an overview. A verification of the figure can be found in the full version [1].

3.1 Lower Bounds

It is easy to see that d-simultaneous interval graphs are d-track interval graphs. This implies the following result.

▶ **Theorem 3.1.** *Every graph satisfies* $\mathsf{t}(G) \leq \mathsf{si}(G)$.

The concept of *thinness* was introduced by Mannino et al. [51].

▶ **Definition 3.2** (Thinness). *The* thinness $\mathsf{thin}(G)$ *of a graph G is the smallest integer k such that there is a partition $\{V_1, \ldots, V_k\}$ of $V(G)$ and a vertex ordering $(v_1, \ldots, v_n)$ of G fulfilling that for any three vertices v_a, v_b, v_c with $a < b < c$ and $v_a, v_b \in V_i$ for some i it holds that $v_b v_c \in E(G)$ if $v_a v_c \in E(G)$.*

▶ **Theorem 3.3.** *For any graph G it holds that* $\mathsf{thin}(G) \leq 2^{\mathsf{si}(G)}$.

Complements of matchings with n edges have thinness n [13]. We will later see in Lemma 5.5 that the simultaneous interval number of such a graph is $\mathcal{O}(\log n)$. This implies that the bound given in Theorem 3.3 is asymptotically sharp. Bipartite permutation graphs and, hence, also complete bipartite graphs have thinness at most 2 [10]. As we have seen in Theorem 2.9, the simultaneous interval number of complete bipartite graphs is unbounded. Therefore, this class shows that bounded thinness does not imply bounded simultaneous interval number.

The concept of a linearized version of mim-width was introduced by Vatshelle [60] as mim-width using a caterpillar decomposition. This concept has since been called *linear mim-width* (for example by Golovach et al. [30]).

▶ **Definition 3.4** (Linear mim-width). *Given a graph G and a vertex ordering $\sigma = (v_1, \ldots, v_n)$ of G, we define the quantity* $\mathsf{linear\text{-}mim}(G, \sigma, i)$ *for $1 \leq i \leq n$ to be the maximum size of an induced matching in the bipartite graph that contains all the edges of G between the two sets $\{v_1, \ldots, v_i\}$ and $\{v_{i+1}, \ldots, v_n\}$. We define* $\mathsf{linear\text{-}mim}(G, \sigma) := \max_{i \in \{1,\ldots,n\}} \mathsf{linear\text{-}mim}(G, \sigma, i)$. *The* linear mim-width *of G, denoted* $\mathsf{linear\text{-}mim}(G)$, *is defined as the minimum value* $\mathsf{linear\text{-}mim}(G, \sigma)$ *among all vertex orderings σ of G.*

It was shown by Bonomo and de Estrada [9] that for any graph G it holds that $\mathsf{linear\text{-}mim}(G) \leq \mathsf{thin}(G)$. Combining this with Theorem 3.3 we see that bounded simultaneous interval number also implies bounded linear mim-width. Moreover, using a more direct argumentation, the lower bound on the simultaneous interval number given by the logarithm of the linear mim-width can be improved to a linear lower bound.

▶ **Theorem 3.5.** *For any graph G it holds that* $\mathsf{linear\text{-}mim}(G) \leq \mathsf{si}(G)$.

A *tree decomposition* of a graph G is a pair $(T, \{X_t\}_{t \in V(T)})$ consisting of a tree T and a mapping asigning to each node $t \in V(T)$ a set $X_t \subseteq V(G)$ (called a *bag*) such that the following conditions are satisfied: (i) the union of all the bags equals $V(G)$, (ii) for every edge $uv \in E(G)$ there exists a bag X_t such that $u, v \in X_t$, and (iii) for every vertex $v \in V(G)$ the bags containing v form a subtree of T. A *path decomposition* of G is a tree decomposition of G such that T is a path. For simplicity, we will denote a path decomposition simply by the corresponding sequence $\mathcal{P} = (X_1, \ldots, X_k)$ of bags. Note also that in this case, condition (iii) simplifies to: for every vertex $v \in V(G)$ the bags containing v form a consecutive subsequence of $\mathcal{P}$. The *width* of a tree decomposition is the maximal size of its bags minus 1. The *treewidth* of a graph G, denoted by $\mathsf{tw}(G)$, is the minimum width of a tree decomposition of G. The *pathwidth*, denoted by $\mathsf{pw}(G)$, is defined analogously, with respect to path decompositions. Yolov [62] and independently Dallard et al. [19] introduced the parameter called tree-independence number (or *α-treewidth*). The *independence number* of a tree decomposition $(T, \{X_t\}_{t \in V(T)})$ of a graph G is defined as the maximum cardinality of an independent set I in G such that there exists a bag X_t with $I \subseteq X_t$, or, equivalently, the maximum, over all bags X_t, of the independence number of the subgraph of G induced by the bag X_t. The *tree-independence number* of a graph G, denoted by $\mathsf{tree}\text{-}\alpha(G)$, is defined as the minimum independence number of a tree decomposition of G. We define the *path-independence number*, denoted by $\mathsf{path}\text{-}\alpha(G)$, analogously, with respect to path decompositions.

We now prove a characterization of the path-independence number, which relies on the concept of the *intersection* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \cap G_2$ and defined as the graph $(V_1 \cap V_2, E_1 \cap E_2)$. In the proof we will use the following two known facts about path decompositions and interval graphs (see [6, 27]):

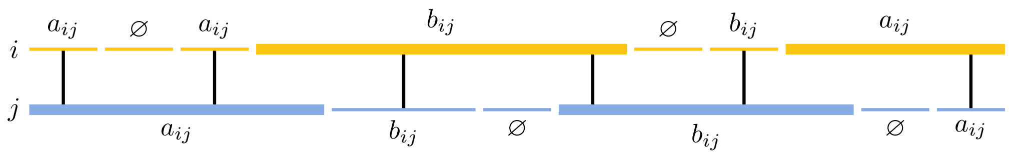

Figure 3 Illustration of the proof of Theorem 3.9. The thick intervals mark the active intervals. A black edge between two intervals means that the corresponding vertices are adjacent in G. The symbol ∅ means that the corresponding vertex has neither label a_{ij} nor the label b_{ij}. However, the vertex will have other labels.

- Let G be a graph, let $\mathcal{P}$ be a path decomposition of G, and let $S \subseteq V(G)$ be a set of vertices of G such that for every two vertices $u, v \in S$ there exists a bag X_i of P such that $u, v \in X_i$. Then there exists a bag X_j of $\mathcal{P}$ such that $S \subseteq X_j$.
- A graph G is an interval graph if and only if it admits a path decomposition in which each bag is a clique in G.

▶ **Theorem 3.6.** *Let G be a graph. Then, the path-independence number of G equals the minimum integer $k \geq 0$ such that G is the intersection of an interval graph and a graph with independence number at most k.*

With a similar approach as that used to prove Theorem 3.6, it can be proved that the tree-independence number of a graph G equals the minimum integer $k \geq 0$ such that G is the intersection of a chordal graph and a graph with independence number at most k.

Theorem 3.6 has the following consequence.

▶ **Corollary 3.7.** *Every graph G satisfies* $\mathsf{path}\text{-}\alpha(G) \leq \mathsf{si}(G)$.

Note that complements of matchings have independence number 2 and, thus, also path-independence number at most 2. Due to Lemma 2.10, they form a class of graphs with bounded path independence number but unbounded simultaneous interval number.

In the spirit of Dallard et al. [18], we can also show that graphs with bounded simultaneous interval number are (pw, ω)-bounded (and consequently (tw, ω)-bounded; note that Chaplick et al. [14] refer to the same property as the *clique-treewidth property*). A graph class $\mathcal{G}$ is said to be (pw, ω)*-bounded* (resp., (tw, ω)*-bounded*) if there is a function f such that for all graphs $G \in \mathcal{G}$ and all induced subgraphs G' of G, it holds that $\mathsf{pw}(G') \leq f(\omega(G'))$ (resp., $\mathsf{tw}(G') \leq f(\omega(G'))$), where $\omega(G')$ is the clique number of G'.

▶ **Theorem 3.8.** *Every graph G satisfies* $\mathsf{pw}(G) \leq \mathsf{si}(G)\omega(G) - 1$.

3.2 Upper Bounds

We begin our discussion on upper bounds by proving that bounded pathwidth implies bounded simultaneous interval number.

▶ **Theorem 3.9.** *Every graph G satisfies* $\mathsf{si}(G) \leq \mathsf{pw}(G)^2 + \mathsf{pw}(G)$.

Proof. Let $k := \mathsf{pw}(G) + 1$. Consider a path decomposition $\mathcal{P}$ of G with maximal bag size k. It is easy to see that we can transform $\mathcal{P}$ in such a way that every bag has size k. Furthermore, we can ensure that two consecutive bags differ only in two vertices, i.e., both vertices are part of exactly one of the two bags and all the other vertices are part of both bags or of none of them. This can be done by adding a sequence of new bags between two old ones in which the vertices are removed and introduced one by one. Let $\mathcal{P}' = (X_1, \ldots, X_p)$ be the resulting path decomposition of G. Now there exists a mapping $f : V(G) \to \{1, \ldots, k\}$ such that every bag of $\mathcal{P}'$ contains a vertex v with $f(v) = i$ for all $i \in \{1, \ldots, k\}$. For every vertex of G, we define the interval $R(v)$ as $(a - \varepsilon, b + \varepsilon)$ where $0 < \varepsilon < \frac{1}{2}$, a is the smallest index such that X_a contains v and b is the largest index such that X_b contains v. It follows that the intervals of two vertices have a non-empty intersection if and only if these vertices are part of a common bag. Therefore, intervals of vertices with the same f-value have an empty intersection.

It remains to show that we can label the vertices with at most $k \cdot (k-1)$ labels in such a way that the defined intervals form a simultaneous interval representation of G. For every set $\{i, j\} \subseteq \{1, \ldots, k\}$ with $i \neq j$, we introduce labels a_{ij} and b_{ij}. Note that $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$. In the following we describe a procedure how to label the vertices of G (see Figure 3 for an illustration). During that labeling procedure, we will always have one *active vertex* $\hat{v}$ and one *active label* $c_{ij} \in \{a_{ij}, b_{ij}\}$. To define the first active vertex let x be the vertex with $f(x) = i$ whose interval ends first and let y be the vertex with $f(y) = j$ whose interval ends first. Without loss of generality, we may assume that $r(x) < r(y)$. We define the first active vertex $\hat{v}$ to be y. The first active label c_{ij} is a_{ij}. The active vertex $\hat{v}$ gets the label c_{ij}. For all vertices z with $f(z) \in \{i, j\} \setminus f(\hat{v})$ and $\ell(\hat{v}) < r(z) < r(\hat{v})$, we add c_{ij} to $L(z)$ if and only if $\hat{v}z \in E(G)$. Now consider the vertex w with $f(w) \in \{i, j\} \setminus f(\hat{v})$ and $\ell(w) < r(\hat{v}) < r(w)$. Vertex w becomes the new active interval. If $\hat{v}w \in E(G)$, then the active label stays the same, otherwise the new active label becomes the other one. In any case w gets the new active label. Note that $L(\hat{v}) \cap L(w) \neq \emptyset$ if and only if $\hat{v}w \in E(G)$. We repeat this procedure until the end of the interval representation. Furthermore, we repeat the whole procedure for all sets $\{i, j\} \subseteq \{1, \ldots, k\}$. In the end, we obtain a d-simultaneous interval representation (R, L) of G where $d = 2\binom{k}{2} = k(k-1) = \mathsf{pw}(G)^2 + \mathsf{pw}(G)$. ◀

Observe at this point that bounded simultaneous interval number does not imply bounded pathwidth as is proven by the class of interval graphs.

An *edge clique cover* of a graph G is a set $\mathcal{K}$ of cliques of G such that every edge of G is contained in some clique of $\mathcal{K}$. We denote by $\mathsf{ecc}(G)$ the *edge clique cover number* of G, that is, the minimum size of an edge clique cover of G.

▶ **Lemma 3.10.** *Let $\mathcal{C}$ be a class of intersection graphs. Let G be a graph, let $d \geq 0$ be an integer, and let R be a $\mathcal{C}$-representation of some graph $F \in \mathcal{C}$. Then, there exists a d-simultaneous $\mathcal{C}$-representation (R, L) of G if and only if there exists a graph H with $\mathsf{ecc}(H) \leq d$ and G is the intersection of F and H.*

Lemma 3.10 implies the following.

▶ **Corollary 3.11.** *Let $\mathcal{C}$ be a class of intersection graphs, let G be a graph, and let $d \geq 0$ be an integer. Then, G has an d-simultaneous $\mathcal{C}$-representation if and only if G is the intersection of a graph in $\mathcal{C}$ and a graph with edge clique cover number at most d.*

Lemma 3.10 also implies the following strengthening of Theorem 2.5.

▶ **Theorem 3.12.** *For every class of intersection graphs $\mathcal{C}$, every graph G has an $\mathsf{ecc}(G)$-simultaneous $\mathcal{C}$-representation.*

▶ **Corollary 3.13.** *Every graph G satisfies $\mathsf{si}(G) \leq \mathsf{ecc}(G)$.*

Interval graphs, and in particular paths, have unbounded edge clique cover number. Thus, bounded simultaneous interval number does not imply bounded edge clique cover number.

The bound given by Corollary 3.13 is tight. Let us denote by K_n^p the complete multipartite graph on p partite sets of the same size n and by $\lambda(n)$ the largest size of a family of mutually orthogonal Latin squares of order n. It is known that $\lambda(n) \leq n-1$ and that equality holds if and only if there exists a projective plane of order n. Thus $\lambda(q) = q-1$ if q is a prime power, but in general the exact computation of the value of $\lambda(n)$ is difficult. Park, Kim, and Sano showed in [55] that for any two integers p and n such that $3 \leq p \leq \lambda(n)+2$, the edge clique cover number of K_n^p equals n^2. Taking $p = 3$, we thus obtain, by combining Theorem 2.8 and Corollary 3.13, that for the complete 3-partite graph G with parts of equal size, we have $\mathsf{si}(G) = \mathsf{ecc}(G) = \frac{|V(G)|^2}{9}$.

4 Complexity of Computing the Simultaneous Interval Number

Using the characterization from Definition 2.2, we can state three natural recognition problems for d-simultaneous $\mathcal{C}$-representations.

▶ **Problem 1** (Simultaneous $\mathcal{C}$-Representation Problem).
Input: *A graph G and a labeling $L : V(G) \to \mathcal{P}(\{1, \ldots, d\})$ of G.*
Question: *Does there exist a d-simultaneous $\mathcal{C}$-representation (R, L) of G?*

By Observations 2.3 and 2.4, Problem 1 is a generalization of the simultaneous representation problems by Jampani and Lubiw [41].

In the second problem we are given the graph and some representation and want to find a suitable labeling.

▶ **Problem 2** (Simultaneous Labeling Problem Given a $\mathcal{C}$-Representation).
Input: *A graph G and a $\mathcal{C}$-representation R of a graph F with $V(G) = V(F)$ and $E(G) \subseteq E(F)$.*
Question: *What is the smallest number $d \in \mathbb{N}$ such that there exists a d-simultaneous $\mathcal{C}$-representation (R, L) of G?*

In the third version, we are given just a graph and wish to compute the smallest number of labels needed for the graph to have a d-simultaneous $\mathcal{C}$-representation.

▶ **Problem 3** (Generalized Simultaneous $\mathcal{C}$-Representation Problem).
Input: *A graph G.*
Question: *What is the smallest number $d \in \mathbb{N}$ such that there exists a d-simultaneous $\mathcal{C}$-representation of G?*

Recall the definition of a class of intersection graphs given on p. 4.

▶ **Theorem 4.1.** *The Simultaneous Labeling Problem Given a $\mathcal{C}$-Representation is* NP*-hard for any class of intersection graphs $\mathcal{C}$, even if all sets in the given $\mathcal{C}$-representation pairwise intersect.*

▶ **Theorem 4.2.** *The Generalized Simultaneous $\mathcal{C}$-Representation Problem is* NP*-hard for every class of intersection graphs that is a subclass of the class of C_4-free graphs and contains the class of complete split graphs.*

▶ **Corollary 4.3.** *Let $\mathcal{C}$ be the class of interval graphs or the class of chordal graphs. Then, the Generalized Simultaneous $\mathcal{C}$-Representation Problem is* NP*-hard.*

▶ **Corollary 4.4.** *It is* NP*-hard to compute the simultaneous interval number of a graph G.*

5 Cliques

In this section, we focus on the *Maximum Clique* problem: Given a graph $G = (V, E)$, compute a largest clique in G. The problem can be naturally generalized to the weighted case, where the input graph is equipped with a vertex weight function $w : V \to \mathbb{Q}_+$ and the task is to find a clique C in G maximizing its weight, $w(C)$, defined as the sum of the weights of the vertices in C.

▶ **Theorem 5.1.** *A graph G has at most $2^{2^{\mathsf{si}(G)}} \cdot n$ many maximal cliques.*

Proof. Let $d = \mathsf{si}(G)$ and fix a d-simultaneous interval representation (R, L) of G. Let C be a maximal clique of G. There exists a point p on the real line that is contained in any interval of the vertices contained in C. Furthermore, for every subset $S \subseteq \{1, \ldots, d\}$, if there is any vertex $u \in C$ such that $L(u) = S$, then the clique C contains all the vertices v whose label set is exactly S and whose interval $R(v)$ contains p. There are at most n points on the real line such that the sets of intervals containing these points are pairwise incomparable with respect to inclusion. These are always points before the endpoint of some interval. For each of those points we have to decide for every subset of $\{1, \ldots, d\}$ if vertices having this subset as label set are contained in the maximal cliques. There are 2^d many subsets. Therefore, there are 2^{2^d} different decisions and, thus, there are at most $2^{2^d} n$ many maximal cliques. ◀

▶ **Theorem 5.2.** *Given a graph G with n vertices and a d-simultaneous interval representation of G, the maximal cliques of G can be enumerated in time $\mathcal{O}(d \cdot 2^{2^d + 2d} \cdot n \log n)$.*

This result implies directly that we can compute a maximum-weight clique of a graph G within the same time bound.

▶ **Corollary 5.3.** *Given a vertex-weighted graph G with n vertices and a d-simultaneous interval representation of G, we can find a maximum weight clique of G in time $\mathcal{O}(d \cdot 2^{2^d + 2d} \cdot n \log n)$.*

Tsukiyama et al. [59] gave an algorithm that generates all maximal cliques in time $\mathcal{O}(n^3 \mu)$ where μ is the number of maximal cliques. Using this algorithm, we can drop the requirement in Theorem 5.2 and Corollary 5.3 that the input graph is given together with a d-simultaneous interval representation.

▶ **Theorem 5.4.** *Given a vertex-weighted graph G with n vertices, we can find a list of all maximal cliques and a maximum weight clique of G in time $\mathcal{O}(2^{2^{\mathsf{si}(G)}} \cdot n^3)$.*

Let us remark that a faster dependency on n (although still slower than quadratic in n) could be obtained by using some of the more recent maximal clique enumeration algorithms (see, e.g., [15]).

Note that the unweighted maximum clique problem is already NP-hard for 2-unit interval graphs and 3-track interval graphs [26] while it is polynomial-time solvable for 2-track interval graphs. However, there is an FPT algorithm for the clique problem on d-interval graphs when parameterized by d plus solution size [24].

Next we prove that the bound given in Theorem 5.1 is tight. To this end, we consider complements of matchings. As we have seen in Lemma 2.10, the simultaneous interval number of those graphs is at least $\log_2(n-1)$ where n is the number of vertices. Here, we show that this bound is tight. Let M_m be the complement of a matching with m edges. Gregory and Pullman [33] showed that $\lim_{m\to\infty} \frac{\mathsf{ecc}(M_m)}{\log_2(m)} = 1$. As we have seen in Corollary 3.13, it holds that $\mathsf{si}(G) \leq \mathsf{ecc}(G)$. This implies the following result.

▶ **Lemma 5.5.** *For any* $\varepsilon > 0$*, there exists some* $n' \in \mathbb{N}$ *such that for all even* $n \geq n'$*, the following holds: If* G *is the complement of a matching with* n *vertices, then* $\mathsf{si}(G) \leq (1+\varepsilon)\log_2 n$*.*

Using this result, we are able to prove that the bound given in Theorem 5.1 is tight.

▶ **Theorem 5.6.** *For any* ε *with* $0 < \varepsilon < 1$ *and any* $k \in \mathbb{N}$*, there is an infinite family* $\mathcal{F}$ *of graphs such that any graph* $G \in \mathcal{F}$ *with* n *vertices has at least* $2^{2^{(1-\varepsilon)\mathsf{si}(G)}} \cdot n^k$ *many maximal cliques.*

This result shows that the bound on the running time of our approach for the Maximum Clique problem cannot be significantly improved. Furthermore, the following result shows that the Maximum Clique problem cannot be solved with a single-exponential FPT algorithm parameterized by the simultaneous interval number.

▶ **Theorem 5.7.** *Unless* P = NP*, for any fixed* $k \in \mathbb{N}$ *there is no algorithm that solves the Maximum Clique problem on complements of cubic graphs with* n *vertices in time* $2^{\mathcal{O}(\mathsf{si})}n^k$*.*

Note that the above result does not rule out the possibility that it may be possible to solve the Maximum Clique problem in time $2^{\mathcal{O}(d)}n^k$ when a d-simultaneous interval representation of the graph is given.

As graphs with bounded simultaneous interval number are (pw, ω)-bounded (Theorem 3.8), we can use the results from Chaplick et al. [14, Theorem 11] to show that the clique problem admits an FPT algorithm when parameterized by the simultaneous interval number plus solution size.

▶ **Theorem 5.8.** *Given an* n*-vertex graph* G *and an integer* k*, it can be determined in time* $2^{\mathcal{O}(\mathsf{si}(G)k)}n$ *whether* G *contains a clique of size* k*.*

6 Coloring

Circular-arc graphs have linear mim-width at most 2 [2, Lemma 4], path-independence number at most 2 [53, proof of Theorem 4.5] and track number at most 2. Since the Coloring problem is NP-hard on circular-arc graphs [28], the same holds for graphs whose linear mim-width, path-independence number, and track number are at most 2. This result does not transfer directly to the simultaneous interval number, as the simultaneous interval number of complements of matchings and, thus, of circular-arc graphs is unbounded, due to Lemma 2.10. Nevertheless, we can adapt a proof for the NP-hardness of the Coloring problem on circular-arc graphs given by Marx [52] to the case of graphs of simultaneous interval number 2. This proof uses the following definitions and results.

▶ **Problem 4** (Disjoint Paths).
Input: *Directed graphs* G *and* H *on the same vertex set.*
Question: *Are there paths* P_e *in* G *for each* $e \in E(H)$ *such that these paths are edge disjoint and path* P_e *together with edge* e *forms a directed cycle?*

Given a directed graph $G = (V, E)$, the *in-degree* (resp. *out-degree*) of a vertex $v \in V$ in G is the number of directed edges $(x, y) \in E$ such that $v = y$ (resp. $v = x$), and the *degree* $d_G(v)$ of v in G is the number of directed edges $(x, y) \in E$ such that $v \in \{x, y\}$. A directed graph $G = (V, E)$ is *Eulerian* if for each vertex $v \in V$, the in-degree of v equals its out-degree.

▶ **Theorem 6.1** (Vygen [61]). *The Disjoint Paths problem remains* NP*-complete even if G is acyclic and $G+H$ is Eulerian.*

▶ **Lemma 6.2** (Marx [52]). *If $G+H$ is Eulerian and G is acyclic, then every solution of the Disjoint Path problem given G and H uses every edge of G.*

▶ **Lemma 6.3.** *The Disjoint Paths problem remains* NP*-complete even if G is acyclic, $G+H$ is Eulerian, and every vertex in H has degree at most one.*

▶ **Theorem 6.4.** *The Coloring problem is* NP*-complete on graphs G with* $\mathsf{si}(G) \leq 2$ *even if a 2-simultaneous interval representation of G is given.*

Proof. We adapt a proof given by Marx [52] to establish NP-completeness of the Coloring problem on circular-arc graphs. Let (G, H) be an instance of the Disjoint Paths problem such that G is acyclic, $G+H$ is Eulerian, and $d_H(v) \leq 1$ for all $v \in V(G)$. Let $k = |E(H)|$.

Let $v_1, \ldots, v_n$ be a topological sort of G. For every edge $(v_i, v_j) \in E(G)$ we construct an interval (i, j) with label set $\{1\}$. Note that $i < j$, due to the property of the topological sort. For every edge $(v_i, v_j) \in E(H)$ we may assume that $i > j$ since otherwise there is no path from v_j to v_i in G. We add the intervals $(0, j)$ and $(i, n+1)$ with label set $\{1, 2\}$ and the interval (j, i) with label set $\{2\}$. We call the resulting 2-simultaneous interval graph G'.

We claim that (G, H) is a yes instance of the disjoint path problem if and only if G' can be colored with k colors. First assume that (G, H) is a yes instance. Fix a solution, that is, paths P_e in G for each $e \in E(H)$ such that these paths are edge-disjoint and path P_e together with edge e forms a directed cycle. By Lemma 6.2, the solution covers all the edges with k directed cycles. Let C be the ℓ-th cycle in the solution. For every edge $(v_i, v_j) \in E(C) \cap E(G)$ we color the corresponding interval (i, j) that has label $\{1\}$ with color ℓ. For the edge $(v_i, v_j) \in E(C) \cap E(H)$ we color with color ℓ the intervals $(0, j)$ and $(i, n+1)$ that have label set $\{1, 2\}$ as well as the interval (j, i) that has label set $\{2\}$. This leads to a proper coloring of G' since the only intervals with the same color that intersect each other do not share a label and, thus, their corresponding vertices are not adjacent.

Now assume the graph G' can be properly colored with k colors. As all the k intervals with label set $\{1, 2\}$ that start in 0 pairwise intersect, they have different colors. Now consider the subgraph of G induced by the intervals containing label 2. Since every vertex in H has degree at most one, whenever an interval ends before point $n+1$, there is no other interval that ends at this point. Furthermore, there is exactly one interval that starts at this point. This implies that every point p in the interval $(0, n+1)$ in which no interval ends belongs to exactly k intervals. Consequently, any two intervals such that the second one starts where the first one ends must have the same color. This implies, in particular, that the two intervals with label set $\{1, 2\}$ representing the same edge of H have the same color.

Now consider all the intervals that contain the label 1. There are k of those intervals that start in point 0. If exactly j of those intervals end in point i, then there are exactly j intervals that start in i, due to the Eulerian property of $G+H$. Thus, any non-integer point in $(0, n+1)$ is contained in exactly k intervals. This also implies that for any of those points there is an interval with color $d \in \{1, \ldots, k\}$. Therefore, the intervals with color d represent a directed cycle in $G+H$ containing exactly one edge of H.Thus, (G, H) is a yes instance of the disjoint path problem. ◀

As any class of graphs with bounded simultaneous interval number are (pw, ω)-bounded (Theorem 3.8), we can use the results from Chaplick et al. [14, Theorem 12] to show that the List k-Coloring problem admits an FPT algorithm when parameterized by k plus the simultaneous interval number.

▶ **Theorem 6.5.** *Given a graph G, we can solve the List k-Coloring problem on G in time* $k^{\mathcal{O}(\mathsf{si}(G)k)}n$.

7 Domination and Independent Sets

The Dominating Set problem and the Independent Set problem can be solved in polynomial time on interval graphs [29, 57]. However, when we parameterize these problems by the solution size and linear mim-width they are W[1]-hard [39]. If we parameterize the Dominating Set problem by tree-α and the solution size then it is W[2]-hard [49]. In contrast, when the problems are parameterized by simultaneous interval number and the solution size, then bounded-search-tree methods lead to FPT-algorithms.

▶ **Theorem 7.1.** *Given a graph G with n vertices and a d-simultaneous interval representation of G, we can decide whether G has a dominating set of size at most k or an independent set of size k in time $\mathcal{O}(2^{kd} \cdot n)$.*

Using a technique due to Fomin et al. [25], in [39] Jaffke et al. showed that a whole range of domination-type problems (including dominating and independent set) are W[1]-hard when parameterized by mim-width and solution size. While that approach cannot be easily adapted for the simultaneous interval number, it is possible to show that at least one of these problems is W[1]-hard when parameterized just by si.

▶ **Problem 5.** *Independent Dominating Set Problem (IDSP)*

Instance: *A graph G and an integer k.*

Question: *Does there exist a set X of at most k vertices of G such that $G[X]$ is edgeless and $N_G[X] = V(G)$?*

The results in [25] use a reduction from the *Multicolored Clique problem* (MCP), a technique popularized by Fellows et al. [24]. We will use a reduction from the *Multicolored Independent Set problem*.

▶ **Problem 6.** *Multicolored Independent Set Problem (MISP)*

Instance: *A graph G with a proper coloring of k colors.*

Question: *Is there an independent set I in G such that I contains exactly one vertex of each color?*

The MCP (and thus the MISP) was shown to be W[1]-hard when parameterized by solution size by Pietrzak [56] and by Fellows et al. [24]. In fact, in [17, 50] the authors show the following result under the assumption of the Exponential Time Hypothesis (ETH) which asserts that solving n-variable 3-SAT requires time $2^{\Omega(n)}$ (see [38]).

▶ **Theorem 7.2** (Cygan et al. [17], Lokshtanov et al. [50])**.** *Assuming the Exponential Time Hypothesis, there is no $f(k)n^{o(k)}$ time algorithm for the MCP (MISP) for any computable function f.*

For an instance G of the MCP we can assume that all color classes are of the same size q, since adding isolated vertices does not affect the existence or nonexistence of a multicolored clique. A similar assumption can be made for the MISP. For each color class $i \in \{1, \ldots, k\}$, we denote the vertices in the class by $v_1^i, \ldots, v_q^i$.

We will show that the IDSP is W[1]-hard when parameterized by the simultaneous interval number. To this end we will construct a reduction from the MISP in the following way. Let G together with a vertex partition $V(G) = V_1 \dot\cup \ldots \dot\cup V_k$ be an instance of the MISP, where $V_i = \{v_1^i, \ldots, v_q^i\}$ for all $i \in \{1, \ldots, k\}$.

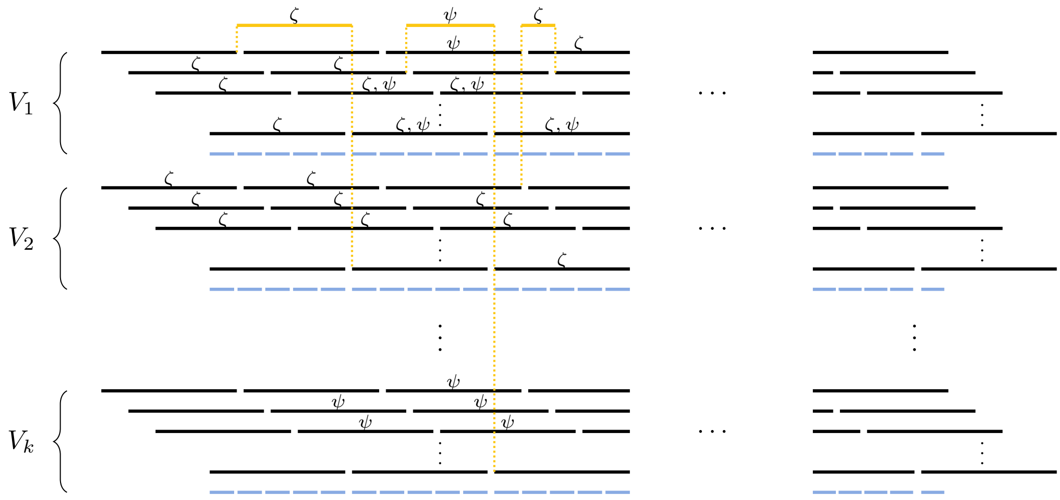

Figure 4 The yellow intervals represent the edges of G, the black intervals are the intervals of the W^i_j. The blue intervals are in S_i. Each of the rows marked V_i represent that vertex set of G. For visual reasons the intervals belonging to the V_i have not been shifted by ϵ as in the definition. For the same reason, we define $\zeta := k+1$ and $\psi := k+2$. The labels of the edge intervals are denoted completely above these. Each of the other intervals also contains the label i if it is associated with V_i. The intervals on the right have not been labeled.

Let $E(G) = \{e_1, \ldots, e_m\}$. We will now define a $(k+2)$-simultaneous interval graph G' together with its $(k+2)$-simultaneous interval representation. For each vertex $v^i_j \in V(G)$ we will define a collection of $m+1$ (open) intervals (see Figure 4)

$$W^i_j := \left\{ \left(\gamma - 1 + (j-1)\frac{1}{q} + i\epsilon, \gamma + (j-1)\frac{1}{q} + i\epsilon \right) : \gamma \in \{1, \ldots, m+1\} \right\},$$

where $\epsilon \ll \frac{1}{kq}$, i.e., all k shifts in sum are much smaller than one interval of a W^i_j.

We will denote the γ-th interval of v^i_j as $I^i_j(\gamma)$. These intervals will be referred to as the *vertex intervals*. Note that none of these intervals have common left endpoints or common right endpoints. Furthermore, for each of the V_i we add an additional collection of $2mq + 2$ intervals

$$S_i := \left\{ \left(\frac{q-1}{q} + \gamma\frac{1}{q} + i\epsilon, 1 + \gamma\frac{1}{q} + i\epsilon \right) : \gamma \in \{0, \ldots, 2mq+1\} \right\},$$

where again $\epsilon \ll \frac{1}{qk}$.

Finally, we add further intervals for each edge in G. Let $e_\gamma = v^i_a v^j_b$ be an edge with $v^i_a \in V_i$, $v^j_b \in V_j$. W.l.o.g. we may assume that $a \le b$ and if $a = b$, then $i < j$. We add an interval of the form $I(e_\gamma) = (r(I^i_a(\gamma)), \ell(I^j_b(\gamma+1)))$. These intervals will be referred to as the *edge intervals*. As none of the intervals of different vertices have common endpoints, we can be sure that each of these edge intervals has strictly positive length. In the following, we will frequently identify the intervals and vertices of G' in order to simplify the notation.

In the next step, we assign a label set to each of the intervals in order to construct a simultaneous interval graph. To each interval in S_i we assign the label set $\{i\}$ and to each $I(e_\gamma)$ we assign the label set $\{k+1\}$ if γ is odd and $\{k+2\}$ if γ is even.

Before we label the vertex intervals, we need the following observation which follows easily from the definitions above.

▶ **Observation 7.3.** *Any interval $I^i_j(\gamma)$ intersects at most two edge intervals and these intervals have distinct labels.*

Any interval of a W^i_j is given at least the label i. Let $I^i_j(\gamma)$ be one of the intervals representing the vertices of G. If $I^i_j(\gamma)$ does not intersect any edge interval such that one of the endpoints of that edge is contained in V_i, then $L(I^i_j(\gamma)) = \{i\}$. If $I^i_j(\gamma)$ intersects some edge interval such that one of that edges endpoints is contained in V_i but is not identical to v^i_j, then we add the label of that edge to $L(I^i_j(\gamma))$. If $I^i_j(\gamma)$ intersects some edge interval such that one of that edges' endpoints is identical to v^i_j, then $L(I^i_j(\gamma))$ does not contain the label of that edge. Note that these last two rules cannot lead to a contradiction, due to Observation 7.3. Therefore, any interval $I^i_j(\gamma)$ has label set either $\{i\}$, $\{i, k+1\}$, $\{i, k+2\}$ or $\{i, k+1, k+2\}$.

▶ **Lemma 7.4.** *Any minimum independent dominating set of G' must contain all the vertices corresponding to the intervals in the set $W^1_{j_1} \cup \ldots \cup W^k_{j_k}$ for some set of indices $\{j_1, \ldots, j_k\}$.*

▶ **Lemma 7.5.** *The vertices belonging to $W := W^1_{j_1} \cup \ldots \cup W^k_{j_k}$ form an independent dominating set of G' if and only if $C := \{v^1_{j_1}, \ldots, v^k_{j_k}\}$ is a multicolored independent set of G.*

Combining Lemmas 7.4 and 7.5 with the fact that MISP is $\mathsf{W}[1]$-hard when parameterized by the solution size and Theorem 7.2 we get the following result.

▶ **Theorem 7.6.** *The IDSP is $\mathsf{W}[1]$-hard when parameterized by the simultaneous interval number even if a $\mathsf{si}(G)$-simultaneous interval representation is given. Furthermore, assuming the ETH, there is no $f(\mathsf{si})n^{o(\mathsf{si})}$-time algorithm for the ISDP for any computable function f.*

Note that this reduction cannot be easily adapted to show that independent dominating set is $\mathsf{W}[1]$-hard when parameterized by the simultaneous interval number *and* the solution size k, as the minimum size of an independent dominating set in G' is of the order $\Omega(km)$.

8 Conclusion

While we have presented some algorithmic properties for graphs of bounded simultaneous interval number, many open problems still remain. First and foremost is the computation of si. Unsurprisingly, the computation of si is NP-hard, however, we are not aware of any results regarding the decision problem whether si is at most some fixed value d. It still remains to be seen whether there exists a computable function f and an FPT or an XP algorithm that for a given graph G and integer d, either correctly determines that $\mathsf{si}(G) > d$ or computes an $f(d)$-simultaneous interval representation of G. Such FPT algorithms are known for treewidth [5, 48] and cliquewidth [54], and XP algorithms are known for the tree-independence number [20, 62].

Furthermore, the complexity status of many important problems is still open when parameterized by si, for example that of independent set and dominating set (see Table 1). Regarding the obtained FPT results, it remains to be shown whether the running times are best possible. Especially in the case of the clique problem, there is still a large discrepancy between the achieved running time and the lower bound.

The simultaneous representation problem has also been considered for chordal graphs [41]. This imposes the question whether similar results can be made for a *simultaneous chordal number*. In fact, some of the results given here for the simultaneous interval number can be directly translated for the simultaneous chordal number as well. However, as the Dominating Set problem is $\mathsf{W}[2]$-hard for chordal graphs, the FPT algorithm for that problem given in Theorem 7.1 does not carry over.

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