Score Design for Multi-Criteria Incentivization

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Abstract

We present a framework for designing scores to summarize performance metrics. Our design has two multi-criteria objectives: (1) improving on scores should improve all performance metrics, and (2) achieving pareto-optimal scores should achieve pareto-optimal metrics. We formulate our design to minimize the dimensionality of scores while satisfying the objectives. We give algorithms to design scores, which are provably minimal under mild assumptions on the structure of performance metrics. This framework draws motivation from real-world practices in hospital rating systems, where misaligned scores and performance metrics lead to unintended consequences.

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1 Introduction

The use of numerical metrics to evaluate performance and guide decision-making is common practice in healthcare, education, business, and public policy. It is common for agencies to design *surrogate scores* that summarize performance metrics, in a way that aligns incentives with performance metrics. Often the scored entities strategically optimize surrogates and end up degrading on metrics, a phenomenon commonly known as *unintended consequences* and pithily conveyed by Goodhart's law [25, 46]:

"When a measure becomes a target, it ceases to be a good measure."

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Agencies thus aim to ensure that optimizing scores leads to improved metrics. As the number of performance metrics can be large in practice [51, 40], agencies must design *succinct* multi-dimensional surrogate scores. We present a framework to study this *minimal design problem*, and propose score designs that prevent unintended consequences.

Our work is directly motivated by real-world examples in safety-critical domains such as healthcare and education, where manifestations of Goodhart's law exemplify the serious ramifications of unintended consequences. When Pacificare, a healthcare provider, incentivized hospitals in 2003 to perform certain medical procedures to improve quality of care, several unrepresented metrics deteriorated [35]. Similar misalignment between performance metrics and score-based hospital ratings, used by the Medicare agency (CMS), has been widely critiqued [47, 11, 33, 1, 44, 3]. Even so, CMS uses these score-based ratings to incentivize hospital policies [13, 18]. Hence, it aims to design scores so that improving on scores also improves all performance metrics. This goal motivates the *improvement objective* in our framework. In a similar vein, rating agencies such as USNews aim to incentivize efficient use of hospital resources through published scores [49]. On multi-dimensional metrics, the efficiency goal [41] naturally translates into the notion of pareto-efficiency, which motivates the *optimality objective* in our framework.

We present a framework for designing scores to summarize performance metrics. We give three natural design restrictions that align with real-world interpretability desiderata [15, 49], and propose score designs that satisfy the multi-criteria objectives under these restrictions. Striving for succinct scores, we formulate our design to minimize the dimensionality of scores. We give polynomial-time algorithms to design these succinct scores, which are provably minimal under mild assumptions on the structure of performance metrics. While existing work on score design for incentivization studies scalar scores [34, 28, 43, 52], we design scores of smallest dimensionality to satisfy the multi-criteria objectives. These objectives are unsatisfiable with scalar scores in general.

1.1 Designing surrogate scores from performance metrics

In our model, the agency aims to design a surrogate score function $S : \mathcal{F} \to \mathcal{S}$ given a set of performance metrics $\mathcal{F}$ of hospitals.

Hospitals report to agencies like CMS and USNews on hundreds of performance metrics such as condition-specific death rates, readmission rates, and percentages of patients receiving satisfactory care [15, 14, 49]. We can denote the values of d metrics of a hospital with a real-valued vector $\boldsymbol{f} \in \mathcal{F} \subseteq \mathbb{R}^d$. Since d is large and metrics can be related through confounding variables [5, 37], the agency wants to summarize the d metrics as k scores with values $\mathcal{S} \subseteq \mathbb{R}^k$, where k is small as possible. For instance, Example 3 suggests that, to summarize COVID and pneumonia death rate metrics, the agency can choose either of the two metrics as the score, so that $k = 1$. Whereas for pneumonia death rate and excess antibiotic use metrics, Example 4 argues that selecting both metrics as scores is necessary, and so $k = 2$.

Surrogate design objectives

Anticipating that the hospital would target the incentives by optimizing the score function S, the agency wants to design S in such a way that optimizing them ensures that the hospital does well on the performance metrics. We formalize this goal with two design objectives, which utilize an ordering on the sets $\mathcal{F}$ and $\mathcal{S}$, denoted by $\succ_{\mathcal{F}}$ and $\succ_{\mathcal{S}}$. The two objectives are motivated from CMS and USNews hospital rating agencies [15, 49].

1. **Improvement objective.** Improving on surrogate scores should result in improving on performance metrics. In particular,

$$\text{for } \boldsymbol{f}, \boldsymbol{f}' \in \mathcal{F}, \quad \text{if } S(\boldsymbol{f}') \succeq_{\mathcal{S}} S(\boldsymbol{f}) \text{ then } \boldsymbol{f}' \succeq_{\mathcal{F}} \boldsymbol{f}. \tag{1}$$

2. **Optimality objective.** Pareto-optimal points of surrogate scores should be pareto-optimal points of performance metrics. In particular,

$$\text{ParetoOpt}(S) \subseteq \text{ParetoOpt}(\mathcal{F}). \tag{2}$$

Throughout the paper, we analyze the setting $\mathcal{F} \subseteq \mathbb{R}^d$ and $\mathcal{S} \subseteq \mathbb{R}^k$ and use elementwise order of vectors for $\succeq_{\mathcal{F}}$ and $\succeq_{\mathcal{S}}$.

Surrogate design restrictions

Due to interpretability and public reporting obligations, rating agencies like CMS and USNews design scores by selecting subsets of the list of performance metrics or by taking weighted averages [14, 15, 16, 17, 49]. Moreover, monotonicity of scores in performance metrics is a desirable property for CMS, as it ensures that a hospital striving to improve all performance metrics sees improved score values [14, 17].

We formulate these requirements as three different restrictions on S. These restrictions impose a linear form on $S : \boldsymbol{f} \mapsto \boldsymbol{A}\boldsymbol{f}$ with $\boldsymbol{A} \in \mathbb{R}^{k \times d}$ satisfying certain structural constraints.

1. **Coordinate Selection (Res-CS).** Each of the k coordinates of scores are chosen from d coordinates of performance metrics. That is, for all $i \in [k]$ there exists $j \in [d]$ such that $S(\boldsymbol{f})_i = \boldsymbol{f}_j$ for all $\boldsymbol{f} \in \mathcal{F}$. Equivalently, $S : \boldsymbol{f} \mapsto \boldsymbol{A}\boldsymbol{f}$ where rows of $\boldsymbol{A}$ are 1-hot vectors.
2. **Linear and Monotone (Res-LM).** The k coordinates of scores are linear combinations of d coordinates of performance metrics, and improving on performance metrics should result in improving on surrogate scores. That is, $S : \boldsymbol{f} \mapsto \boldsymbol{A}\boldsymbol{f}$ where for $\boldsymbol{f}, \boldsymbol{f}' \in \mathcal{F}$, if $\boldsymbol{f}' \geq \boldsymbol{f}$ then $\boldsymbol{A}\boldsymbol{f}' \geq \boldsymbol{A}\boldsymbol{f}$.
3. **Linear (Res-L).** The coordinates of surrogate scores are linear combinations of coordinates of performance metrics. That is, $S : \boldsymbol{f} \mapsto \boldsymbol{A}\boldsymbol{f}$ without any further constraints on $\boldsymbol{A}$.

Minimal design problem

Since the number of performance metrics d can be large [14, 15, 49], a natural goal is to *succinctly* summarize metrics with scores that are accessible to patients and policymakers. This goal of succinctness translates into designing a multi-dimensional function $S : \mathbb{R}^d \to \mathbb{R}^k$ with *the smallest output dimension* k. For a combination of design objective and design restriction, the *minimal design problem* is determining the smallest dimensionality k and providing an algorithm outputs a surrogate score function S with this k.

1.2 Our contributions

In this paper, we study the minimal design problem. Our key contributions are:

1. We formalize surrogate score design for incentivizing multiple criteria, motivated from real-world practices of two hospital rating systems, CMS and USNews.
2. We fully determine the minimal design problems of all combinations of objectives and restrictions introduced in Section 1.1, and propose efficient score design algorithms (Algorithms 1 and 2). We summarize our results in Table 1.

a. We show that the smallest dimensionalities k are dictated by structural properties of the affine hull of performance metrics $\mathcal{F}$.
b. Identifying a relationship between improvement and optimality objectives (Theorem 13), we determine the minimal design problem for simultaneously satisfying both objectives.

Table 1 We list smallest dimensionalities k for the minimal design problem of all combinations of objectives and restrictions. Here columns of $\boldsymbol{Z}$ are an orthonormal basis of the linear subspace associated with r-dimensional affine hull of $\mathcal{F}$. We define the three matrix ranks ConeSubsetRank, ConeGeneratingRank, ConeRank in Theorem 2. For the improvement objective, the listed dimensionalities are also necessary, when $\mathcal{F}$ has non-empty relative interior (Theorem 7).

Restriction	Improvement (§2)	Optimality (§3)	Both (§4)
Res-CS	$\mathsf{ConeSubsetRank}(\boldsymbol{Z})$	r	$\mathsf{ConeSubsetRank}(\boldsymbol{Z})$
Res-LM	$\mathsf{ConeGeneratingRank}(\boldsymbol{Z})$	1	$\mathsf{ConeGeneratingRank}(\boldsymbol{Z})$
Res-L	$\mathsf{ConeRank}(\boldsymbol{Z})$	1	$\mathsf{ConeGeneratingRank}(\boldsymbol{Z})$

1.3 Related work

Recent work has highlighted the plight of score-based incentivization when scores that do not align with performance metrics. In healthcare, design objectives of hospital rating agencies often vary across agencies. Two popular examples are the Medicare agency (CMS), which incentivizes healthcare investment across care metrics through a five-star score [15, 18], and the USNews agency, which promotes highly-specialized medical departments [49]. When hospitals target these score-based ratings, they often degrade on a few performance metrics [35]. For example, CMS's score-based ratings have been found to encourage hospitals to selectively treat patients for minimizing readmission rates [3, 20, 12], and have exacerbated unequal access to healthcare [33, 1, 44]. Such unintended consequences are prevalent in fields that use scores as an incentive mechanism [6], for instance, in standardized testing [35] and financial credit ratings [31, 54, 7, 26].

Our framework extends recent work on score design in principal-agent theory [34, 28, 43, 52, 27, 38, 30, 29, 4, 2] by designing scores for multi-criteria objectives. Kleinberg and Raghavan [34] compare linear with monotone scalar score design for incentivizing *effort* from agents. On a similar front, Haghtalab et al. [28] study scalar score design with a linear threshold restriction. Score design has also been studied through a causality lens to optimize the average treated outcome [52, 27, 38]. Finally, Rolf et al. [43] use noisy score observations to approximate the pareto-frontier of performance metrics. Our framework's optimality objective and design restrictions capture this line of work on scalar scores. However, our improvement objective is a novel contribution, and this objective turns to be unsatisfiable with scalar scores (Theorem 7). Hence, our score design problems are inherently multi-criteria.

Technically, our design algorithms utilize novel techniques to decompose and enclose polyhedral cones, building on work in computational geometry on finding frames of polyhedral cones [21, 39, 53] and enclosing convex hulls [22, 36, 42, 48]. Our definition of ConeRank (Theorem 2) is similar to NonNegativeRank, which is extensively studied in the context of non-negative matrix factorization [23, 24, 19, 50, 36].

1.4 Notation

We represent scalars as $\lambda, c \in \mathbb{R}$, and vectors and matrices as $\boldsymbol{w} \in \mathbb{R}^n, \boldsymbol{W} \in \mathbb{R}^{m\times n}$. We denote the nonnegative orthant with $\mathbb{R}^n_+$. We generally write matrices as a stack of rows, $\boldsymbol{W} = [\boldsymbol{w}_1; \ldots; \boldsymbol{w}_m]$, often denoting the set of rows with W. We say that matrix $\boldsymbol{W}$ (or set

W) generates cone $\mathcal{K}_W$ if $\mathcal{K}_W = \text{Cone}(W) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{\lambda W}, \boldsymbol{\lambda} \in \mathbb{R}^m_+\}$. We denote a vector of zeros (or ones) as $\mathbf{0}_n \in \mathbb{R}^n$ (or $\mathbf{1}_n$), and the n-by-n identity matrix as $\boldsymbol{I}_n$, dropping subscripts when unambiguous.

2 Minimal design problem for improvement objective

We propose a surrogate score design for satisfying the improvement objective under the three design restrictions. Then we illustrate our design strategy on simple examples of performance metrics $\mathcal{F}$, highlighting relationships between the geometry of $\mathcal{F}$ and the succinctness of scores. Finally, we show that our proposed design is minimal under a mild assumption on $\mathcal{F}$, implying that score design for improvement objective is inherently multi-criteria.

We first simplify the improvement objective in Equation (1) to identify geometric objects that represent *movement* and *improvement directions*. Score function $S : \boldsymbol{f} \mapsto \boldsymbol{Af}$ on domain $\mathcal{F}$ satisfies improvement when for all $\boldsymbol{f}, \boldsymbol{f}' \in \mathcal{F}$, if $\boldsymbol{A}(\boldsymbol{f}' - \boldsymbol{f}) \geq \mathbf{0}$ then $(\boldsymbol{f}' - \boldsymbol{f}) \geq \mathbf{0}$. Denoting the *movement directions at center* $\boldsymbol{f}$ with $\mathcal{F}_{\boldsymbol{f}} = \{\boldsymbol{g} = \boldsymbol{f}' - \boldsymbol{f} \in \mathbb{R}^d \mid \text{ for all } \boldsymbol{f}' \in \mathcal{F}\}$, we can rearrange terms to get

$$\text{for all centers } \boldsymbol{f} \in \mathcal{F}, \text{ movement directions } \boldsymbol{g} \in \mathcal{F}_{\boldsymbol{f}}, \quad \text{if } \boldsymbol{Ag} \geq \mathbf{0} \text{ then } \boldsymbol{Ig} \geq \mathbf{0} \tag{3}$$

Here the *set of score improvement directions* is exactly $\mathcal{K}_A^* = \{\boldsymbol{g} \in \mathbb{R}^d \mid \boldsymbol{Ag} \geq \mathbf{0}\}$, which is the dual of polyhedral cone $\mathcal{K}_A$ generated from rows of $\boldsymbol{A}$. Similarly, the *set of metric improvement directions* is $\mathcal{K}_I^* = \{\boldsymbol{g} \in \mathbb{R}^d \mid \boldsymbol{Ig} \geq \mathbf{0}\} = \mathbb{R}^d_+$, which is the dual of polyhedral cone $\mathcal{K}_I = \mathbb{R}^d_+$ generated from rows of $\boldsymbol{I}$. So intuitively, score function $S : \boldsymbol{f} \mapsto \boldsymbol{Af}$ satisfies improvement if and only if every movement direction (in $\mathcal{F}_{\boldsymbol{f}}$) that is a score improvement direction (in $\mathcal{K}_A^*$) is also a metric improvement direction (in $\mathcal{K}_I^*$):

$$S \text{ satisfies improvement} \iff \text{for all } \boldsymbol{f} \in \mathcal{F}, \quad \mathcal{F}_{\boldsymbol{f}} \cap \mathcal{K}_A^* \subseteq \mathcal{K}_I^*. \tag{4}$$

2.1 Design proposal for improvement objective

When performance metrics $\mathcal{F} \subseteq \mathbb{R}^d$ is a full-dimensional set, score design is trivial where the most succinct score design is $S(\boldsymbol{f}) = \boldsymbol{f}$. Note that while performance is measured in many dimensions [51, 40], the number of confounding variables of performance metrics is often smaller due to correlated metrics [5, 37]. This typically induces a low-dimensional structure on $\mathcal{F}$, observed in practice and assumed in theory [6, 8, 5, 37]. We do not assume such low-dimensional structure of $\mathcal{F}$, but the smallest dimensionality k of score function S is impacted by the intrinsic dimension of $\mathcal{F}$. The affine hull of $\mathcal{F}$ is a natural geometric choice to capture its intrinsic dimension.

▶ **Definition 1.** *Define the affine hull of $\mathcal{F}$, $\text{aff}(\mathcal{F})$, as the intersection of all affine subspaces in $\mathbb{R}^d$ containing $\mathcal{F}$. Let $\mathcal{L}$ be the linear subspace associated with $\text{aff}(\mathcal{F})$, i.e. $\mathcal{L}$ is the translation of $\text{aff}(\mathcal{F})$ so that for all centers $\boldsymbol{f} \in \mathcal{F}$, movement directions $\mathcal{F}_{\boldsymbol{f}} \subseteq \mathcal{L}$.*

By utilizing this subspace $\mathcal{L}$ containing all possible movement directions $\mathcal{F}_{\boldsymbol{f}}$, we propose a score design in Algorithm 1 with dimensionalities given in Theorem 2. We introduce three *matrix ranks* – ConeSubsetRank (CSR), ConeGeneratingRank (CGR), and ConeRank (CR) – to characterize the score design dimensionalities for the three respective design restrictions – Coordinate Selection (Res-CS), Linear and Monotone (Res-LM), Linear (Res-L). These three matrix ranks capture the geometric properties of performance metrics $\mathcal{F}$ that dictate the dimensionality of optimal score design for the three restrictions.

▶ **Theorem 2.** *Let columns of $\boldsymbol{Z}$ be an orthonormal basis of linear subspace $\mathcal{L}$ associated with* $\mathrm{aff}(\mathcal{F})$*. For each design restriction, there exists $S : \mathcal{F} \to \mathbb{R}^k$, designed using Algorithm 1, that satisfies the improvement objective with the following dimensionalities.*

	Dimensionality $k \geq$
Res-CS	$\mathsf{ConeSubsetRank}(\boldsymbol{Z}) := \min_q \{q \mid \mathcal{K}_Z = \mathcal{K}_V \text{ for some } \boldsymbol{V} \in \mathbb{R}^{q \times r} \text{ s.t. } \boldsymbol{V} \subseteq \boldsymbol{Z}\}$
Res-LM	$\mathsf{ConeGeneratingRank}(\boldsymbol{Z}) := \min_q \{q \mid \mathcal{K}_Z = \mathcal{K}_V \text{ for some } \boldsymbol{V} \in \mathbb{R}^{q \times r}\}$
Res-L	$\mathsf{ConeRank}(\boldsymbol{Z}) := \min_q \{q \mid \mathcal{K}_Z \subseteq \mathcal{K}_V \text{ for some } \boldsymbol{V} \in \mathbb{R}^{q \times r}\}$

Algorithm 1 Design strategy for improvement objective.

1: Given: performance metrics $\mathcal{F}$ and a design restriction.
2: Find $\boldsymbol{Z}$ whose columns are an orthonormal basis of subspace $\mathcal{L}$ associated with $\mathrm{aff}(\mathcal{F})$.
3: Find $\boldsymbol{V}$ that attains[2] the matrix rank corresponding to the design restriction.
4: Find $\boldsymbol{A}$ that satisfies $\boldsymbol{V} = \boldsymbol{AZ}$ and design $S : \boldsymbol{f} \mapsto \boldsymbol{Af}$.

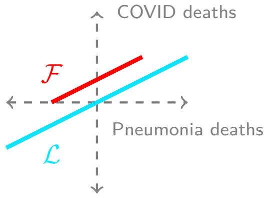
Theorem 2 follows from the following key insight of Equation (4): "for $S : \boldsymbol{f} \to \boldsymbol{Af}$ to satisfy the improvement objective, score improvement directions need to be metric improvement directions **only** for movement directions $\mathcal{F}_{\boldsymbol{f}}$, which are contained in subspace $\mathcal{L}$." In fact, satisfying the improvement objective boils down to ensuring that score improvement directions are a subset of metric improvement directions *in the coefficient space* w.r.t. subspace $\mathcal{L}$. The respective improvement directions $\mathcal{K}_A^*$ and $\mathcal{K}_I^*$ are generated by rows of $\boldsymbol{A}$ and $\boldsymbol{I}$, which have coefficients that are rows of $\boldsymbol{V} = \boldsymbol{AZ}$ and $\boldsymbol{Z}$, where columns of $\boldsymbol{Z}$ are an orthonormal basis of subspace $\mathcal{L}$. It turns out that improvement directions in the coefficient space are precisely the duals $\mathcal{K}_V^*$ and $\mathcal{K}_Z^*$ of polyhedral cones generated from rows of $\boldsymbol{V}$ and $\boldsymbol{Z}$. So to satisfy the improvement objective, we need to ensure $\mathcal{K}_V^* \subseteq \mathcal{K}_Z^*$, or $\mathcal{K}_Z \subseteq \mathcal{K}_V$.

With the three matrix ranks, we capture the additional structure on $\boldsymbol{A}$ imposed by the three design restrictions (Section 1.1). Res-L restriction does not further impose structure on $\boldsymbol{A}$, and so we only need to *enclose* cone $\mathcal{K}_Z$ with $\mathcal{K}_V$. Res-LM restriction further requires function S to be monotone in $\mathcal{F}$, which intuitively means that every metric improvement direction needs to be a score improvement direction, i.e., $\mathcal{K}_Z^* \subseteq \mathcal{K}_V^*$. So to satisfy Res-LM, we must *generate* cone $\mathcal{K}_Z$ with $\mathcal{K}_V$. Finally, Res-CS restriction requires selecting the k score function coordinates from d metrics. In the coefficient space, this requirement means that rows of $\boldsymbol{V}$ are chosen from rows of $\boldsymbol{Z}$ and $\mathcal{K}_V$ generates $\mathcal{K}_Z$. Hence, the three matrix ranks precisely capture structure on $\boldsymbol{A}$ imposed by the improvement objective and the design restrictions. We include the proof of Theorem 2 in Theorem A.1.

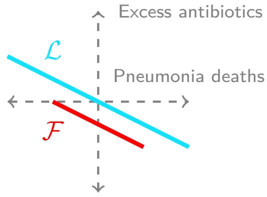
2.2 Geometry of metrics dictates succinctness of scores

We now illustrate Algorithm 1 with several examples of metrics $\mathcal{F}$. We instantiate performance metrics in our examples with familiar notions of hospital metrics, to intuitively bridge our analysis and algorithm with practical score design. In doing so, we discuss how the geometry of $\mathcal{F}$ dictates the shape of polyhedral cone $\mathcal{K}_Z$, influencing the dimensionality of minimal score design for the three design restrictions. Finally, we provide high-level descriptions of techniques to to implement Algorithm 1 efficiently.

[2] For a matrix rank, e.g. CSR, we say that $\boldsymbol{V}$ "attains" it if $\boldsymbol{V} \subseteq \boldsymbol{Z}$ (rows of $\boldsymbol{V}$ are chosen from rows of $\boldsymbol{Z}$), $\mathcal{K}_Z = \mathcal{K}_V$, and the number of rows of $\boldsymbol{V}$ equals $\mathsf{CSR}(\boldsymbol{Z})$.



(a) When the two metrics are correlated (Ex. 3), we can choose either metric in $S : \mathcal{F} \to \mathbb{R}^1$.



(b) When the two metrics are anti-correlated (Ex. 4), we must choose both metrics in $S : \mathcal{F} \to \mathbb{R}^2$.

Figure 1 To design scores for two metrics ($\mathcal{F} \subseteq \mathbb{R}^2$), we can inspect the correlation between metrics – the correlation dictates the succinctness of $S : \mathcal{F} \to \mathbb{R}^k$ for satisfying improvement.

▶ **Example 3** (Two correlated metrics $\implies$ choose either for score design)**.** CMS evaluates hospitals on numerous performance metrics like condition-specific death rates, readmission rates, and safety standards [15]. Often comorbidities of medical conditions can lead to positive correlations between metrics. In the case of two *perfectly* positively correlated metrics, Algorithm 1 suggests to choose either of the two metrics to design $S : \mathcal{F} \to \mathbb{R}^1$.

Consider two metrics – (i) pneumonia death rate and (ii) COVID-19 death rate – that have a positive correlation due to comorbidities. Assume that for a hospital, these two death rates take values $\mathcal{F} = \{\boldsymbol{f} \in \mathbb{R}^2 \mid -f_1 + 2f_2 = 1, -1 \leq f_1 \leq 1\}$, lying in a 1-dimensional affine subspace of $\mathbb{R}^2$ (Figure 1a, red). As the affine hull $\text{aff}(\mathcal{F}) = \{\boldsymbol{f} \mid -f_1 + 2f_2 = 1\}$ is 1-dimensional, the associated linear subspace $\mathcal{L} = \{\boldsymbol{f} \mid -f_1 + 2f_2 = 0\}$ (Figure 1a, blue) containing all movement directions $\mathcal{F}_{\boldsymbol{f}}$ is 1-dimensional. Per Line 2 of Algorithm 1, we arrange an orthonormal basis for $\mathcal{L}$ as columns of $\boldsymbol{Z} \propto \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, whose rows generate the polyhedral cone $\mathcal{K}_Z = \{2\lambda_1 + \lambda_2 \mid \lambda_1, \lambda_2 \geq 0\} = \mathbb{R}_+$. Note that the metric improvement directions in the coefficient space are the dual cone $\mathcal{K}_Z^* = \mathbb{R}_+$.

To satisfy improvement objective under a design restriction, we need to find matrix $\boldsymbol{V}$ that attains the corresponding matrix rank. For all three matrix ranks, the cone $\mathcal{K}_V$ generated by rows of $\boldsymbol{V}$ needs to *enclose* cone $\mathcal{K}_Z$. Equivalently, in the coefficient space, score improvement directions $\mathcal{K}_V^*$ need to be a subset of metric improvement directions $\mathcal{K}_Z^*$. The choice of $\boldsymbol{V} = [2] \in \mathbb{R}^{1\times 1}$ yields the desired property $\mathcal{K}_Z \subseteq \mathcal{K}_V$. In fact, we get $\mathcal{K}_Z = \mathcal{K}_V$ and $\boldsymbol{V} \subseteq \boldsymbol{Z}$, and so all three matrix ranks have value 1.

Finally, we can recover $\boldsymbol{A} = [1, 0]$ such that $\boldsymbol{V} = \boldsymbol{A}\boldsymbol{Z}$, and design $S(\boldsymbol{f}) = [1, 0] \cdot \boldsymbol{f} = f_1$. It is easy to verify that this S satisfies the improvement objective (we could also have chosen $\boldsymbol{V} = [1]$ previously to design $S(\boldsymbol{f}) = [0, 1] \cdot \boldsymbol{f} = f_2$). Hence, when the two metrics are perfectly positively correlated, choosing one for score design suffices.

▶ **Example 4** (Two anti-correlated metrics $\implies$ must choose both for score design)**.** Performance metrics used by CMS can also be negatively correlated when a hospital must balance its effort to simultaneously improve all metrics. In the case of two *perfectly* negative correlated metrics, Algorithm 1 suggests to use both metrics to design $S : \mathcal{F} \to \mathbb{R}^2$, as no 1-dimensional score function can satisfy improvement objective.

Consider two metrics – (i) pneumonia death rate and (ii) excessive antibiotic use – that have a negative correlation as improving on one degrades the other. Assume that these two metrics take values $\mathcal{F} = \{\boldsymbol{f} \in \mathbb{R}^2 \mid -f_1 - 2f_2 = 1, -1 \leq f_1 \leq 1\}$, lying in a 1-dimensional affine subspace of $\mathbb{R}^2$ (Figure 1b, red). Similar to Example 3, the subspace $\mathcal{L} = \{\boldsymbol{f} \mid -f_1 + 2f_2 = 0\}$ (Figure 1b, blue) associated to $\text{aff}(\mathcal{F})$ is 1-dimensional. But the rows of orthonormal basis $\boldsymbol{Z} \propto \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ generate cone $\mathcal{K}_Z = \{2\lambda_1 - \lambda_2 \mid \lambda_1, \lambda_2 \geq 0\} = \mathbb{R}$, which contains a linear subspace within. This means that the metric improvement directions in the coefficient space are the dual cone $\mathcal{K}_Z^* = \{\mathbf{0}\}$, i.e., there are no non-trivial directions to simultaneously improve both metrics.

To satisfy improvement objective, score improvement directions in the coefficient space $\mathcal{K}_V^*$ need to be a subset of metric improvement directions $\mathcal{K}_Z^* = \{\mathbf{0}\}$, or equivalently $\mathcal{K}_Z \subseteq \mathcal{K}_V$. Hence, we choose $\boldsymbol{V} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \propto \boldsymbol{Z}$ with 2 rows. Note that $\boldsymbol{V}$ with just 1 row would generate either cone $\mathbb{R}_+$ or cone $-\mathbb{R}_+$, and fail to enclose cone $\mathcal{K}_Z = \mathbb{R}$. Hence, all three matrix ranks have value 2 even though all movement directions $\mathcal{F}_{\boldsymbol{f}}$ lie in a 1-dimensional subspace $\mathcal{L}$.

Finally, we can recover $\boldsymbol{A} = \boldsymbol{I}_2$ such that $\boldsymbol{V} = \boldsymbol{A}\boldsymbol{Z}$ and design the trivial $S(\boldsymbol{f}) = \boldsymbol{f}$. Due to the perfect negative correlation in metrics, we must choose both in the score design.

▶ **Example 5** (Restriction with monotonicity $\Longrightarrow$ higher dimensionality)**.** When the number of metrics is large, understanding correlations among them can be unintuitive. Hence, we rely on structure of polyhedral cones for score design, specifically improvement directions of scores $\mathcal{K}_V^*$ and metrics $\mathcal{K}_Z^*$ (in the coefficient space). We find that score function dimensionality k under Res-CS and Res-LM restrictions can be much larger than under Res-L, as $\mathsf{CSR}, \mathsf{CGR} \gg \mathsf{CR}$.

Consider the case of four metrics where two of them balance the other two, i.e., a toy example where performance metrics take values $\mathcal{F} = \text{aff}(\mathcal{F}) = \{\boldsymbol{f} \in \mathbb{R}^4 \mid [1, -1, 1, -1] \cdot \boldsymbol{f} = 0\}$. Here the four metrics lie in a 3-dimensional linear subspace of $\mathbb{R}^4$ and $\mathcal{F} = \text{aff}(\mathcal{F}) = \mathcal{L}$. Hence, three orthonormal vectors in $\mathbb{R}^4$ form a basis of $\mathcal{L}$ such that the rows of $\boldsymbol{Z}$ generate the "square" cone $\mathcal{K}_Z$ in $\mathbb{R}^3$ (Figure 2a, red):

$$\boldsymbol{Z} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \in \mathbb{R}^{4\times 3}.$$

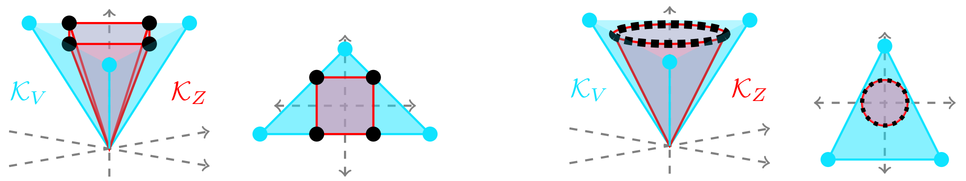
For Res-CS and Res-LM restrictions, we need to find matrix $\boldsymbol{V}$ such that $\mathcal{K}_V = \mathcal{K}_Z$. As all rows of $\boldsymbol{Z}$ are *extreme rays* of $\mathcal{K}_Z$, matrix $\boldsymbol{V}$ must have four rows $\boldsymbol{V} = \boldsymbol{I}_4\boldsymbol{Z}$ (any $\boldsymbol{V}$ with fewer rows would not *generate* the square cone). Hence, $\mathsf{CSR}(\boldsymbol{Z}) = \mathsf{CGR}(\boldsymbol{Z}) = 4$. But for Res-L restriction that does not require monotonicity, rows of $\boldsymbol{V}$ need only ensure $\mathcal{K}_Z \subseteq \mathcal{K}_V$. The following matrix $\boldsymbol{V}$ with three rows that generates a "triangular" cone $\mathcal{K}_V$ (Figure 2a, blue) *enclosing* the square cone $\mathcal{K}_Z$:

$$\boldsymbol{V} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & -1 \\ 1 & -3 & -1 \end{bmatrix} \qquad \text{and so } \boldsymbol{V} = \boldsymbol{A}\boldsymbol{Z} \text{ with } \boldsymbol{A} = \frac{1}{4} \cdot \begin{bmatrix} 3 & 3 & -1 & -1 \\ 3 & -3 & -1 & 5 \\ -3 & 3 & 5 & -1 \end{bmatrix}.$$

Generally, CSR and CGR can be much larger than CR (Figure 2b). Since these three matrix ranks describe the dimensionality under the three restrictions (Theorem 2), restrictions that require monotonicity (Res-CS, Res-LM) lead to higher dimensionality in score design compared to Res-L. In other words, allowing negative values in matrix $\boldsymbol{A}$ can significantly reduce dimensionality of score design.

▶ Remark 6 (Competing metric improvement directions $\Longrightarrow$ higher dimensionality under Res-CS). When rows of $\boldsymbol{Z}$ generate cone $\mathcal{K}_Z$ that is *pointed*[3], we get $\mathsf{CSR}(\boldsymbol{Z}) = \mathsf{CGR}(\boldsymbol{Z})$. But when

[3] A cone $\mathcal{K}$ is pointed if for all nonzero $\boldsymbol{x} \in \mathcal{K}$, we have $-\boldsymbol{x} \notin \mathcal{K}$. It is called *non-pointed* otherwise.



(a) Rows of $\boldsymbol{Z}$ are extreme rays of the generated "square" cone $\mathcal{K}_Z$. The square cone can be enclosed by a "triangular" cone $\mathcal{K}_V$.

(b) All rows of $\boldsymbol{Z} \in \mathbb{R}^{d\times 3}$ are extreme rays of the generated "circular" cone $\mathcal{K}_Z$. The circular cone can be enclosed by a "triangular" cone $\mathcal{K}_V$.

Figure 2 Side and top views of cones $\mathcal{K}_Z$ (red) generated by rows of $\boldsymbol{Z}$, whose columns are orthonormal basis of 3-dimensional subspace $\mathcal{L}$. As CSR and CGR require *generating* $\mathcal{K}_Z$ with $\mathcal{K}_V$, the matrix ranks depend on the number of extreme rays of $\mathcal{K}_Z$, which can be much higher than $\dim \operatorname{aff}(\mathcal{F}) = 3$. On the other hand, CR only requires *enclosing* $\mathcal{K}_Z$ with $\mathcal{K}_V$; and so is independent of the number of extreme rays.

cone $\mathcal{K}_Z$ that is *non-pointed* , we get $\mathsf{CSR}(\boldsymbol{Z}) > \mathsf{CGR}(\boldsymbol{Z})$. $\mathcal{K}_Z$ can be non-pointed when improving one metric degrades another, i.e., when metric improvement directions compete among themselves. In this setting, dimensionality under Res-CS is higher than that under Res-LM (see Example A.2).

Efficiently implementing Algorithm 1

Our proposed design strategy in Algorithm 1 can be efficiently implemented with algorithms that utilize the geometry of metrics $\mathcal{F}$. Elementary linear algebra operations can implement Lines 2 and 4 of Algorithm 1, i.e., finding orthonormal basis $\boldsymbol{Z}$ and recovering $\boldsymbol{A}$ from $\boldsymbol{V} = \boldsymbol{A}\boldsymbol{Z}$. It is also possible to efficiently implement Line 3, to find matrix $\boldsymbol{V}$ that attain the matrix ranks – ConeSubsetRank, ConeGeneratingRank, and ConeRank [32]. We briefly discuss algorithms for Line 3, thus ensuring that the full Algorithm 1 can be efficiently implemented. These algorithms leverage a key property of polyhedral cones, *pointedness*.

When the cone $\mathcal{K}_Z$ generated from rows of $\boldsymbol{Z}$ is pointed, we can easily find $\boldsymbol{V}$ that attains the matrix ranks. For ConeSubsetRank, we can keep the rows of $\boldsymbol{Z}$ that are extreme rays of the polyhedral cone $\mathcal{K}_Z$, as extreme rays minimally generate a pointed cone [9, Prop. 26.5.4]. ConeGeneratingRank turns out to be the same as ConeSubsetRank, as every extreme ray of $\mathcal{K}_Z$ is a row of matrix $\boldsymbol{Z}$ [9, Prop. 26.5.4]. For ConeRank, the matrix $\boldsymbol{V}$ attaining it must generate $\mathcal{K}_V$ that encloses $\mathcal{K}_Z$. An intuitive procedure can find this $\boldsymbol{V}$: can scale rows of $\boldsymbol{Z}$ to lie on a hyperplane, and find a simplex that encloses the convex hull of scaled rows [22].

When the cone $\mathcal{K}_Z$ is non-pointed, the cone contains a linear subspace within. Here we can utilize the unique Minkowski decomposition of polyhedral cones into two orthogonal components: the maximal linear subspace within, and a pointed remnant [45, Sec. 8.2]. Then, for all three matrix ranks, we can generate/enclose non-pointed cone $\mathcal{K}_Z$, by generating/enclosing the two orthogonal components separately.

2.3 Proposed design is minimal

Theorem 2 states that dimensionalities determined by the three matrix ranks – ConeSubsetRank, ConeGeneratingRank, and ConeRank – are sufficient for score design. It turns out that these dimensionalities are also necessary under a mild assumption on $\mathcal{F}$ (Theorem 7). Hence, Theorems 2 and 7 together imply that the *three matrix ranks exactly determine the minimal design problem for improvement objective.*

▶ **Theorem 7.** *Assume metrics $\mathcal{F} \subseteq \mathbb{R}^d$ have non-empty relative interior with respect to* $\operatorname{aff}(\mathcal{F})$*. Then the listed dimensionalities k in Theorem 2 are necessary.*

We briefly discuss the implication of metrics $\mathcal{F}$ having non-empty relative interior on satisying the improvement objective. Such a set $\mathcal{F}$ contains a center $\boldsymbol{f}^* \in \mathcal{F}$ where every direction in subspace $\mathcal{L}$ is a positively-scaled movement direction from $\mathcal{F}_{\boldsymbol{f}^*}$. Intuitively, all score improvement directions are movement directions in the coefficient space. As a result, we get an equivalence between satisfying improvement in the ambient space and the coefficient space, i.e., satisfying improvement in Equation (4) is equivalent to satisfying $\mathcal{K}_Z \subseteq \mathcal{K}_V$. See Theorem A.3 for the proof.

▶ Remark 8. In Figure 3 we illustrate examples of $\mathcal{F}$ and their relative interior. $\mathcal{F}$ having non-empty relative interior is a reasonable condition in practice, as performance metrics used by rating agencies are often correlated and not isolated points [6, 15, 49, 8, 37, 5]. For instance, CMS uses percentage-rate-based metrics, such as condition-specific death rates, readmission rates, and screening rates [15, 14]. This leads to real-valued metrics $\mathcal{F} = [0, 1]^d$, which has non-empty relative interior. We note that, when the relative interior is *empty*, dimensionality k significantly less than listed values in Theorem 2 can suffice (Proposition A.5).

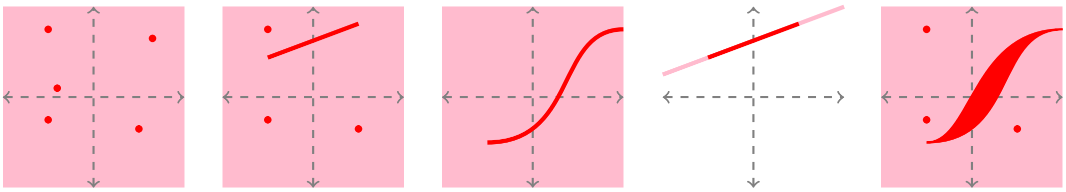


Figure 3 Examples of $\mathcal{F} \subseteq \mathbb{R}^2$. The left three have empty relative interior, whereas the right two have non-empty relative interior with respect to $\operatorname{aff}(\mathcal{F})$, which is lightly shaded.

▶ Remark 9 (Choice of affine subspace and orthonormal basis). Our design strategy in Algorithm 1 can use *any* orthonormal basis $\boldsymbol{Z}$ of the linear subspace $\mathcal{L}_{\mathcal{H}}$ associated with *any* affine subspace $\mathcal{H}$ containing metrics $\mathcal{F}$. To design the *minimal* $S : \mathcal{F} \to \mathbb{R}^k$, we pick *any* orthonormal basis of subspace $\mathcal{L}$ associated with affine hull $\mathcal{H} = \operatorname{aff}(\mathcal{F})$. This follows from Lemma A.4, which states that three matrix ranks are (1) invariant to the choice of orthonormal basis for a fixed subspace $\mathcal{L}_{\mathcal{H}}$, and (2) minimized with the choice of $\mathcal{H} = \operatorname{aff}(\mathcal{F})$.

3 Minimal design problem for optimality objective

We propose a surrogate score design for satisfying the optimality objective and discuss the minimality of our proposed design. We use the standard definition of pareto-optimality.

▶ **Definition 10.** *Point $\boldsymbol{f} \in \mathcal{F}$ is pareto-optimal for maximizing S if no other point in $\mathcal{F}$ both improves $S(\boldsymbol{f})$ in all coordinates and strictly improves $S(\boldsymbol{f})$ in at least one coordinate.*

$$\operatorname{ParetoOpt}(S) := \left\{ \boldsymbol{f} \in \mathcal{F} \mid \textit{for all } \boldsymbol{f}' \in \mathcal{F},\ \textit{either } S(\boldsymbol{f}') \not\geq S(\boldsymbol{f}) \textit{ or } S(\boldsymbol{f}') = S(\boldsymbol{f}) \right\}.$$

We write $\operatorname{ParetoOpt}(\mathcal{F})$ *to denote the pareto-optimal points in $\mathcal{F}$ w.r.t. the identity map.*

We simplify the optimality objective in Equation (2) – $\operatorname{ParetoOpt}(S) \subseteq \operatorname{ParetoOpt}(\mathcal{F})$ – using movement directions $\mathcal{F}_{\boldsymbol{f}}$ at center $\boldsymbol{f}$, score improvement directions $\mathcal{K}_A^*$, and metric improvement directions $\mathcal{K}_I^*$. Intuitively, score function $S : \boldsymbol{f} \mapsto \boldsymbol{A}\boldsymbol{f}$ satisfies optimality if and only if movement directions $\mathcal{F}_{\boldsymbol{f}}$ that are *non-strict score improvement directions* are also *non-strict metric improvement directions*:

$$\text{Optimality} \iff \left\{\boldsymbol{f} \in \mathcal{F} \mid \mathcal{F}_{\boldsymbol{f}} \subseteq (\mathcal{K}_A^*)^{\mathsf{c}} \cup \ker \boldsymbol{A}\right\} \subseteq \left\{\boldsymbol{f} \in \mathcal{F} \mid \mathcal{F}_{\boldsymbol{f}} \subseteq (\mathcal{K}_I^*)^{\mathsf{c}} \cup \ker \boldsymbol{I}\right\}. \quad (5)$$

3.1 Design proposal for optimality objective

We propose a score design in Algorithm 2 with dimensionalities given in Theorem 11. We note that dimensionalities for score design are much smaller for the optimality objective than for the improvement objective (Theorem 2). Specifically, for Res-LM and Res-L restrictions, a 1-dimensional score function $S : \mathcal{F} \to \mathbb{R}$ suffices to satisfy optimality whereas multi-dimensional function S is necessary for improvement (Theorem 7). This suggests that the optimality objective is significantly weaker than the improvement objective.

▶ **Theorem 11.** *For each design restriction, there exists $S : \mathcal{F} \to \mathbb{R}^k$, designed using Algorithm 2, that satisfies the optimality objective with the following dimensionalities.*

	Dimensionality $k \geq$
Res-CS	$\dim \operatorname{aff}(\mathcal{F})$
Res-LM	1
Res-L	1

Algorithm 2 Design strategy for optimality objective.

Given: $\mathcal{F}$ and a design restriction.
if Design restriction is Res-LM or Res-L **then**
  Design $S(\boldsymbol{f}) = \boldsymbol{a} \cdot \boldsymbol{f}$ with any positive vector $\boldsymbol{a}$.
else if Design restriction is Res-CS **then**
  Find $\boldsymbol{Z}$ whose columns are an orthonormal basis of subspace $\mathcal{L}$ associated with $\operatorname{aff}(\mathcal{F})$.
  Let $\boldsymbol{V}$ be linearly independent rows of $\boldsymbol{Z}$.
  Find $\boldsymbol{A}$ that satisfies $\boldsymbol{V} = \boldsymbol{A}\boldsymbol{Z}$ and design $S : \boldsymbol{f} \mapsto \boldsymbol{A}\boldsymbol{f}$.

For Res-LM and Res-L restrictions, the minimal design is straightforward: design $S : \boldsymbol{f} \mapsto \boldsymbol{a} \cdot \boldsymbol{f}$ using any vector $\boldsymbol{a} > \boldsymbol{0}$ [55]. For Res-CS restriction, we utilize an isomorphism between movement directions $\mathcal{F}_{\boldsymbol{f}}$ and their coefficients $\mathcal{C}_{\boldsymbol{f}} \subseteq \mathbb{R}^r$ w.r.t. orthonormal basis $\boldsymbol{Z} \in \mathbb{R}^{d \times r}$ of subspace $\mathcal{L}$ associated with r-dimensional $\operatorname{aff}(\mathcal{F})$. The columns of $\boldsymbol{Z}$ span subspace $\mathcal{L}$ and its rows correspond to coordinates of movement directions $\mathcal{F}_{\boldsymbol{f}}$. Using this isomorphism, choosing r linearly independent rows of $\boldsymbol{Z}$ as rows of $\boldsymbol{V}$ suffices to satisfy the optimality objective. As $\boldsymbol{V} \subseteq \boldsymbol{Z}$, we can find $\boldsymbol{A} \in \mathbb{R}^{r \times d}$ with 1-hot rows such that $\boldsymbol{V} = \boldsymbol{A}\boldsymbol{Z}$, and design $S : \boldsymbol{f} \mapsto \boldsymbol{A}\boldsymbol{f}$ that satisfies the Res-CS restriction. We include the proof in Theorem A.6.

3.2 Discussion of minimality of proposed design

While our proposed design for improvement objective is minimal when $\mathcal{F}$ has non-empty relative interior (Theorem 7), our design for the optimality objective is *not necessarily* minimal under the same condition on $\mathcal{F}$. The challenge is that ParetoOpt($\mathcal{F}$), the optimal trade-off surface [10], depends on the boundary of $\mathcal{F}$. To demonstrate this, we give three examples of d-dimensional $\mathcal{F}$ with non-empty relative interior – for one of the examples dimensionality $k = \dim \operatorname{aff}(\mathcal{F})$ is necessary for satisfying optimality under Res-CS, whereas for the other two examples, a 1-dimensional S suffices. See Proposition A.7 for the proof.

▶ **Proposition 12.** *Consider designing $S : \mathcal{F} \to \mathbb{R}^k$ to satisfy optimality objective.*

1. *For $\mathcal{F} = \left\{\boldsymbol{f} \in \mathbb{R}^d \mid \|\boldsymbol{f}\|_1 \leq 1\right\}$, $k \geq 1$ is necessary and sufficient for all design restrictions.*
2. *For $\mathcal{F} = \left\{\boldsymbol{f} \in \mathbb{R}^d \mid \|\boldsymbol{f}\|_2 \leq 1\right\}$, $k \geq 1$ is necessary and sufficient for all design restrictions.*
3. *For $\mathcal{F} = \left\{\boldsymbol{f} \in \mathbb{R}^d \mid \|\boldsymbol{f}\|_\infty \leq 1\right\}$, $k \geq d$ is necessary and sufficient for Res-CS. Moreover, $k \geq 1$ is necessary and sufficient for the Res-LM and Res-L restrictions.*

4 Minimal design problem for both objectives simultaneously

So far we have separately analyzed the minimal design problems for improvement and optimality objectives. We now give results for simultaneously satisfying both objectives.

First, we establish a relationship between the improvement and optimality objectives. This result holds even for score functions S that are not linear in $\mathcal{F}$.

▶ **Theorem 13.** *Let $S : \mathcal{F} \to \mathbb{R}^k$ be monotone in $\mathcal{F}$. If S satisfies improvement, then S satisfies optimality.*

Proof. Let score function $S : \mathcal{F} \to \mathbb{R}^k$ be monotone in $\mathcal{F}$ and satisfy improvement. Hence, for all $\boldsymbol{f}, \boldsymbol{f}' \in \mathcal{F}$ we have $S(\boldsymbol{f}') \geq S(\boldsymbol{f}) \iff \boldsymbol{f}' \geq \boldsymbol{f}$, i.e., the function S preserves the ordering on set $\mathcal{F}$. We prove by contradiction that such an S satisfies optimality. Assume that $\boldsymbol{f}^* \in \text{ParetoOpt}(S)$ but $\boldsymbol{f}^* \notin \text{ParetoOpt}(\mathcal{F})$. That is, there exists $\boldsymbol{f} \in \mathcal{F}$ such that $\boldsymbol{f} \geq \boldsymbol{f}^*$ and $\boldsymbol{f} \neq \boldsymbol{f}^*$. Because S preserves the ordering, it must be that $S(\boldsymbol{f}) \geq S(\boldsymbol{f}^*)$ and $S(\boldsymbol{f}) \neq S(\boldsymbol{f}^*)$, which means that $\boldsymbol{f}^* \notin \text{ParetoOpt}(S)$ and contradicts our assumption. ◀

We utilize Theorem 13 to design S that simultaneously satisfies both objectives. As S is monotone in $\mathcal{F}$ under Res-CS and Res-LM restrictions, it suffices to design S that satisfies the improvement objective. We include the proof in Corollary A.8.

▶ **Corollary 14.** *Let columns of $\boldsymbol{Z}$ be an orthonormal basis of linear subspace $\mathcal{L}$ associated with $\text{aff}(\mathcal{F})$. For each design restriction, there exists score function $S : \mathcal{F} \to \mathbb{R}^k$ that simultaneously satisfies improvement and optimality objectives with following dimensionalities.*

	Dimensionality $k \geq$
Res-CS	$\mathsf{ConeSubsetRank}(\boldsymbol{Z})$
Res-LM	$\mathsf{ConeGeneratingRank}(\boldsymbol{Z})$
Res-L	$\mathsf{ConeGeneratingRank}(\boldsymbol{Z})$

Moreover, for Res-CS and Res-LM restrictions, the score design is minimal when $\mathcal{F}$ has non-empty relative interior.

▶ Remark 15. For simultaneously satisfying both objectives under Res-L restriction, dimensionality $k = \mathsf{CR}(\boldsymbol{Z})$ is necessary, when $\mathcal{F}$ has non-empty relative interior (Theorem 7). Corollary 14 states that $k = \mathsf{CGR}(\boldsymbol{Z})$ is sufficient, and $\mathsf{CGR} \gg \mathsf{CR}$ in general (Example 5). We leave to future work to close this gap between necessary and sufficient dimensionality.

5 Conclusion

We propose a framework to design succinct scores to summarize performance metrics $\mathcal{F}$, and give polynomial-time algorithms that design scores that are provably minimal under mild assumptions on $\mathcal{F}$. Two future directions are to design scores: (1) when metrics takes discrete high-dimensional values, (2) using incomplete, noisy high data from historical samples of metric values, and (3) when metrics have a non-linear structure. On a technical note, it remains to identify structural properties of $\mathcal{F}$ and corresponding minimal designs for the optimality objective. Designing minimal scores for simultaneously satisfying both objectives under linear restriction is also an open direction.

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A Omitted Proofs

A.1 Minimal design problem for improvement objective

▶ **Theorem A.1** (Theorem 2). *Let columns of $\boldsymbol{Z}$ be an orthonormal basis of linear subspace $\mathcal{L}$ associated with $\operatorname{aff}(\mathcal{F})$. For each design restriction, there exists $S : \mathcal{F} \to \mathbb{R}^k$, designed using Algorithm 1, that satisfies the improvement objective with the following dimensionalities.*

	Dimensionality $k \geq$
Res-CS	$\mathsf{ConeSubsetRank}(\boldsymbol{Z}) := \min_q \{q \mid \mathcal{K}_Z = \mathcal{K}_V \textit{ for some } \boldsymbol{V} \in \mathbb{R}^{q\times r} \textit{ s.t. } \boldsymbol{V} \subseteq \boldsymbol{Z}\}$
Res-LM	$\mathsf{ConeGeneratingRank}(\boldsymbol{Z}) := \min_q \{q \mid \mathcal{K}_Z = \mathcal{K}_V \textit{ for some } \boldsymbol{V} \in \mathbb{R}^{q\times r}\}$
Res-L	$\mathsf{ConeRank}(\boldsymbol{Z}) := \min_q \{q \mid \mathcal{K}_Z \subseteq \mathcal{K}_V \textit{ for some } \boldsymbol{V} \in \mathbb{R}^{q\times r}\}$

Proof. We give a proof for the Res-CS restriction; proofs for the other two restrictions are similar. We show that, if $k \geq \mathsf{CSR}(\boldsymbol{Z})$, then there exists $S(\boldsymbol{f}) = \boldsymbol{A}\boldsymbol{f}$ satisfying improvement and Res-CS.

Let columns of $\boldsymbol{Z} \in \mathbb{R}^{d\times r}$ be an orthonormal basis of r-dimensional linear subspace $\mathcal{L}$ associated with $\text{aff}(\mathcal{F})$. The definition of CSR states that $k \geq \mathsf{CSR}(\boldsymbol{Z})$ when there exists $\boldsymbol{V} \in \mathbb{R}^{k\times r}$ such that (i) $\boldsymbol{V} \subseteq \boldsymbol{Z}$ and (ii) $\mathcal{K}_Z = \mathcal{K}_V$. Property (i) means that $\boldsymbol{V} = \boldsymbol{A}\boldsymbol{Z}$ for some $\boldsymbol{A} \in \mathbb{R}^{k\times d}$ with 1-hot rows, and so $S(\boldsymbol{f}) = \boldsymbol{A}\boldsymbol{f}$ satisfies the Res-CS restriction. Property (ii) implies that $\mathcal{K}_Z \subseteq \mathcal{K}_V$, and so S satisfies improvement:

$$\mathcal{K}_Z \subseteq \mathcal{K}_V \xLeftrightarrow{\text{Lem. B.2}} \mathcal{L} \cap \mathcal{K}_A^* \subseteq \mathcal{K}_I^* \xRightarrow{\text{Def. 1}} \begin{array}{c}\text{for all } \boldsymbol{f} \in \mathcal{F},\\ \mathcal{F}_{\boldsymbol{f}} \cap \mathcal{K}_A^* \subseteq \mathcal{K}_I^*\end{array} \xLeftrightarrow{\text{Eq. 4}} \text{Improvement.} \quad (6)$$

The proof of Lemma B.2 uses $\boldsymbol{V} = \boldsymbol{A}\boldsymbol{Z}$, and the projection of rows of $\boldsymbol{A}$ and $\boldsymbol{I}_d$ in subspace $\mathcal{L}$ using orthonormal basis $\boldsymbol{Z}$. ◀

▶ **Example A.2** (Competing metric improvement directions $\Longrightarrow$ dimensionality for Res-CS > Res-LM)**.** When cone $\mathcal{K}_Z$ generated by rows of $\boldsymbol{Z}$ is non-pointed, we have $\mathsf{CSR}(\boldsymbol{Z}) > \mathsf{CGR}(\boldsymbol{Z})$, implying that the score design dimensionality is higher under Res-CS restriction than under Res-LM. The cone $\mathcal{K}_Z$ can be non-pointed in the presence of competing metric improvement directions, i.e., when improving on one metric degrades another. A non-pointed $\mathcal{K}_Z$ results in a gap between $\mathsf{CSR}(\boldsymbol{Z})$ and $\mathsf{CGR}(\boldsymbol{Z})$.

Consider 8 metrics lying in a 5-dimensional subspace, which has the following orthonormal basis (arranged as columns of $\boldsymbol{Z}$):

$$\boldsymbol{Z} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \in \mathbb{R}^{8\times 5}.$$

The rows generate a 5-dimensional cone $\mathcal{K}_Z$ with two orthogonal parts: (i) a 2-dimensional linear subspace due to the first 4 metrics, and (ii) a 3-dimensional "square" pointed cone due to the last 4 metrics, as visualized in Figure 4. Since $\mathcal{K}_Z$ contains a 2-dimensional linear subspace within, it is a non-pointed cone.

A matrix $\boldsymbol{V}$ that attains $\mathsf{CSR}(\boldsymbol{Z})$ must have rows of $\boldsymbol{V}$ chosen from rows of $\boldsymbol{Z}$ and $\mathcal{K}_Z = \mathcal{K}_V$. Excluding any row of $\boldsymbol{Z}$ shrinks the generated cone – excluding any row of the first 4 generates a halfspace rather than the 2-dimensional subspace, and excluding any row of the last 4 does not generate the "square" pointed cone. So $\mathsf{CSR}(\boldsymbol{Z}) = 8$. On the other hand, a matrix $\boldsymbol{V}$ that attains $\mathsf{CGR}(\boldsymbol{Z})$ need not have rows of $\boldsymbol{V}$ chosen from rows of $\boldsymbol{Z}$; $\boldsymbol{V}$ must only satisfy $\mathcal{K}_Z = \mathcal{K}_V$. We need all last 4 rows to generate the "square" cone, but there exists 3 points (the blue and two bottom black points) whose nonnegative combinations generate the 2-dimensional linear subspace. So $\mathsf{CGR}(\boldsymbol{Z}) = 7$.

▶ **Theorem A.3** (Theorem 7)**.** *Assume metrics $\mathcal{F} \subseteq \mathbb{R}^d$ have non-empty relative interior with respect to* $\text{aff}(\mathcal{F})$*. Then the listed dimensionalities k in Theorem 2 are necessary.*

Proof. We give a proof for the Res-CS restriction; proof for the other two restrictions are similar. We show that, when $\mathcal{F}$ has non-empty relative interior, we get:

$$\text{for all } \boldsymbol{f} \in \mathcal{F}, \quad \mathcal{F}_{\boldsymbol{f}} \cap \mathcal{K}_A^* \subseteq \mathcal{K}_I^* \implies \mathcal{L} \cap \mathcal{K}_A^* \subseteq \mathcal{K}_I^*. \quad (7)$$

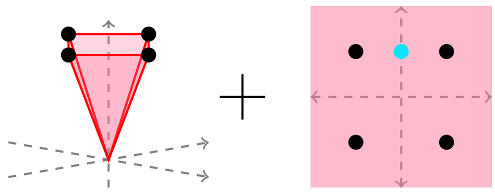

Figure 4 A 5-dimensional non-pointed cone $\mathcal{K}_Z$ with two orthogonal components: a 2-dimensional linear subspace, and a 3-dimensional "square" pointed cone.

By adding this implication to Equation (6), we prove that, when $\mathcal{F}$ has non-empty relative interior, a score function S satisfies the improvement objective and Res-CS restriction *if and only if* $k \geq \mathsf{CSR}(\boldsymbol{Z})$.

We now prove the implication in Equation (7). Let $\boldsymbol{x} \in \mathcal{L} \cap \mathcal{K}_A^*$. Since $\mathcal{F}$ has non-empty relative interior, there exists $\boldsymbol{f}^*$ in the relative interior. Lemma B.3 states that, as $\boldsymbol{x} \in \mathcal{L}$, there exists $a > 0$ such that $a\boldsymbol{x} \in \mathcal{F}_{\boldsymbol{f}^*}$. Since $\boldsymbol{x}$ is in cone $\mathcal{K}_A^*$ as well, we have $a\boldsymbol{x} \in \mathcal{K}_A^*$. Hence, $a\boldsymbol{x} \in \mathcal{F}_{\boldsymbol{f}^*} \cap \mathcal{K}_A^*$. According to the premise of Equation (7), we know that $\mathcal{F}_{\boldsymbol{f}^*} \cap \mathcal{K}_A^* \subseteq \mathcal{K}_I^*$, and so $a\boldsymbol{x} \in \mathcal{K}_I^*$. As $a > 0$, we get $\boldsymbol{x} \in \mathcal{K}_I^*$, completing the proof. ◀

▶ **Lemma A.4.** *Given affine subspace $\mathcal{H}$ containing $\mathcal{F}$, the matrix ranks are invariant to the choice of orthonormal basis of $\mathcal{L}_{\mathcal{H}}$. Moreover, among all affine subspaces containing $\mathcal{F}$, the matrix ranks are smallest for $\mathcal{H} = \mathrm{aff}(\mathcal{F})$.*

Proof. We give a proof for CSR, proofs for the other two matrix ranks are similar.

1. We first give a geometric interpretation for invariance to choice of orthonormal basis of $\mathcal{L}_{\mathcal{H}}$. Then we give an algebraic proof.

   Geometric interpretation. For any matrix $\boldsymbol{W}$, note that $\mathsf{CSR}(\boldsymbol{W})$ is the minimum cardinality of a subset V of W (set of rows of $\boldsymbol{W}$), such that cone $\mathcal{K}_V$ encloses $\mathcal{K}_W$. By rotating rows of $\boldsymbol{W}$ without altering the column span of $\boldsymbol{W}$, although the row vectors W change, the *relative position of them with respect to each other is the same*. So the cone generated by the rotated vectors is just a rotation of cone $\mathcal{K}_W$. As a result, the minimum cardinality of a subset of rotated vectors (to enclose the rotated cone) is unchanged, and so $\mathsf{CSR}(\boldsymbol{W})$ is unchanged.

   Algebraic argument. Let columns of $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ be two sets of orthonormal basis of $r_{\mathcal{H}}$-dimensional $\mathcal{L}_{\mathcal{H}}$. We will show that $\mathsf{CSR}(\boldsymbol{Z}_1) = \mathsf{CSR}(\boldsymbol{Z}_2)$. The two orthonormal bases have the same column span, and are rotations/reflections of each other. So there exists orthogonal matrix $\boldsymbol{Q} \in \mathbb{R}^{r_{\mathcal{H}} \times r_{\mathcal{H}}}$ such that $\boldsymbol{Z}_1 = \boldsymbol{Z}_2\boldsymbol{Q}$ and $\boldsymbol{Z}_1\boldsymbol{Q}^\top = \boldsymbol{Z}_2$.
   We prove that $\mathsf{CSR}(\boldsymbol{Z}_1) \leq \mathsf{CSR}(\boldsymbol{Z}_2)$. Let $\mathsf{CSR}(\boldsymbol{Z}_2) = k^*$. Then there exists $\boldsymbol{V}_2 \in \mathbb{R}^{k^* \times r_{\mathcal{H}}}$ such that $\boldsymbol{V}_2 \subseteq \boldsymbol{Z}_2$ and $\mathcal{K}_{Z_2} \subseteq \mathcal{K}_{V_2}$. These two properties mean that $\boldsymbol{V}_2 = \boldsymbol{A}\boldsymbol{Z}_2$ for some $\boldsymbol{A}$ with 1-hot rows, and $\boldsymbol{Z}_2 = \boldsymbol{B}\boldsymbol{V}_2$ for some nonnegative $\boldsymbol{B}$. Multiplying with $\boldsymbol{Q}$ on the right, we get $\boldsymbol{V}_2\boldsymbol{Q} = \boldsymbol{A}\boldsymbol{Z}_2\boldsymbol{Q}$ and $\boldsymbol{Z}_2\boldsymbol{Q} = \boldsymbol{B}\boldsymbol{V}_2\boldsymbol{Q}$. Therefore, $\boldsymbol{V}_1 = \boldsymbol{V}_2\boldsymbol{Q} \in \mathbb{R}^{k^* \times r_{\mathcal{H}}}$ has the properties $\boldsymbol{V}_1 \subseteq \boldsymbol{Z}_1$ and $\mathcal{K}_{Z_1} \subseteq \mathcal{K}_{V_1}$. This proves that $\mathsf{CSR}(\boldsymbol{Z}_1) \leq \mathsf{CSR}(\boldsymbol{Z}_2)$. With a symmetric argument, we also get $\mathsf{CSR}(\boldsymbol{Z}_1) \geq \mathsf{CSR}(\boldsymbol{Z}_2)$.
2. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two non-empty affine subspaces containing $\mathcal{F}$ such that $\mathcal{H}_1 \subseteq \mathcal{H}_2$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be linear subspaces corresponding to $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Since $\mathcal{H}_1 \subseteq \mathcal{H}_2$ and for any $\boldsymbol{f} \in \mathcal{H}_1$ we can write $\mathcal{L}_1 = \mathcal{H}_1 - \boldsymbol{f}$ and $\mathcal{L}_2 = \mathcal{H}_2 - \boldsymbol{f}$, we find that $\mathcal{L}_1 \subseteq \mathcal{L}_2$. According to statement (1), CSR is invariant to the choice of orthonormal basis of linear subspace. Hence, pick columns of $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ as orthonormal basis of $\mathcal{L}_1$ and $\mathcal{L}_2$

respectively, such that columns of $\boldsymbol{Z}_2$ are a superset of columns of $\boldsymbol{Z}_1$. In the definition of CSR, adding vectors to $\boldsymbol{Z}_1$ only increases the number of constraints to satisfy, and so CSR can only grow. Hence, $\mathsf{CSR}(\boldsymbol{Z}_1) \leq \mathsf{CSR}(\boldsymbol{Z}_2)$.

Since $\operatorname{aff}(\mathcal{F})$ is the unique intersection of all affine subspaces containing $\mathcal{F}$, we have $\operatorname{aff}(\mathcal{F}) \subseteq \mathcal{H}$ for every affine subspace $\mathcal{H}$ containing $\mathcal{F}$. Thus, $\mathsf{CSR}(\boldsymbol{Z}) \leq \mathsf{CSR}(\boldsymbol{Z}_{\mathcal{H}})$, where columns of $\boldsymbol{Z}$ and $\boldsymbol{Z}_{\mathcal{H}}$ are orthonormal basis of linear subspaces corresponding to $\operatorname{aff}(\mathcal{F})$ and $\mathcal{H}$ respectively. ◀

▶ **Proposition A.5.** *For each design restriction, there exists $\mathcal{F} \subseteq \mathbb{R}^d$ with $\dim \operatorname{aff}(\mathcal{F}) = d$ and empty relative interior such that there exists function $S : \mathcal{F} \to \mathbb{R}$ that satisfies improvement objective.*

Proof. We first give an example of $\mathcal{F} \subseteq \mathbb{R}^2$, and show that there exists $S : \mathcal{F} \to \mathbb{R}$ that satisfies improvement and the Res-CS restriction. So S will also satisfy the other two design restrictions.

Consider $\mathcal{F} = \{(0,0),(1,1),(2,3)\} \subseteq \mathbb{R}^2$ and let $\boldsymbol{A} = [1,0] \in \mathbb{R}^{1\times 2}$. We now argue that $S(\boldsymbol{f}) = \boldsymbol{A}\boldsymbol{f}$ satisfies the improvement objective. For metric pairs

$$(\boldsymbol{f}', \boldsymbol{f}) \in \{((1,1),(0,0)),((2,3),(1,1)),((2,3),(0,0))\}$$

we have $\boldsymbol{A}\boldsymbol{f}' \geq \boldsymbol{A}\boldsymbol{f}$ and $\boldsymbol{f}' \geq \boldsymbol{f}$. Hence, improvement objective holds for these pairs. Whereas for metric pairs

$$(\boldsymbol{f}', \boldsymbol{f}) \in \{((0,0),(1,1)),((1,1),(2,3)),((0,0),(2,3))\}$$

the left-hand side of the implication ($\boldsymbol{A}\boldsymbol{f}' \geq \boldsymbol{A}\boldsymbol{f}$) is not true. And so improvement objective holds for these pairs *vacuously*. Thus for all $\boldsymbol{f}, \boldsymbol{f}' \in \mathcal{F}$ if $\boldsymbol{A}\boldsymbol{f}' \geq \boldsymbol{A}\boldsymbol{f}$ then $\boldsymbol{f}' \geq \boldsymbol{f}$.

We now give a counterexample of $d+1$ points in $\mathcal{F} \subseteq \mathbb{R}^d$. Let $\boldsymbol{f}^{(0)} = \boldsymbol{0}_d$ and $\boldsymbol{f}^{(1)} = \boldsymbol{1}_d$. For $i = 2, \ldots, d$, construct $\boldsymbol{f}_j^{(i)} = \left(\boldsymbol{f}_j^{(i-1)}\right)^2 + j$ for each coordinate $j \in [d]$. For example, the construction in $\mathbb{R}^4$ is:

$$\mathcal{F} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 11 \\ 19 \\ 29 \end{pmatrix}, \begin{pmatrix} 26 \\ 123 \\ 364 \\ 845 \end{pmatrix} \right\}$$

Points $\boldsymbol{f}^{(1)}, \ldots, \boldsymbol{f}^{(d)}$ are linearly independent, and so $\dim \operatorname{span}(\mathcal{F}) = d$. Let $\boldsymbol{A} = [1, 0, \ldots, 0] \in \mathbb{R}^{1\times d}$. Following a similar argument as the $d = 2$ case, we find that $S(\boldsymbol{f}) = \boldsymbol{A}\boldsymbol{f}$ satisfies the improvement objective (with dimensionality $k = 1$). ◀

A.2 Minimal design problem for optimality objective

▶ **Theorem A.6** (Theorem 11). *For each design restriction, there exists $S : \mathcal{F} \to \mathbb{R}^k$, designed using Algorithm 2, that satisfies the optimality objective with the following dimensionalities.*

	Dimensionality $k \geq$
Res-CS	$\dim \operatorname{aff}(\mathcal{F})$
Res-LM	1
Res-L	1

Proof. For the last two design restrictions, the minimal design is straightforward. Using any vector $\boldsymbol{a} > \boldsymbol{0}$ of positive entries, design $S : \boldsymbol{f} \mapsto \boldsymbol{a} \cdot \boldsymbol{f}$ [55]. Clearly, S is linear in $\boldsymbol{f}$. To see that S is also monotone, fix $\boldsymbol{f}, \boldsymbol{f}' \in \mathcal{F}$ such that $\boldsymbol{f} \geq \boldsymbol{f}'$. Taking inner product with positive vector $\boldsymbol{a}$, we get $\boldsymbol{a}\cdot\boldsymbol{f} \geq \boldsymbol{a}\cdot\boldsymbol{f}'$. To see that optimality objective is satisfied, fix $\boldsymbol{f}^* \in \text{ParetoOpt}(S)$. Since S is 1-dimensional, by definition of ParetoOpt(S), we have $\boldsymbol{a}\cdot\boldsymbol{f}^* \geq \boldsymbol{a}\cdot\boldsymbol{f}$ for all $\boldsymbol{f}\in\mathcal{F}$. Since $\boldsymbol{a}$ only has positive elements, for any $\boldsymbol{f}\in\mathcal{F}$ either $\boldsymbol{f}^*=\boldsymbol{f}$ or there exists $j\in[d]$ such that $\boldsymbol{f}^*_j > \boldsymbol{f}_j$. Therefore, $\boldsymbol{f}^*\in\text{ParetoOpt}(\mathcal{F})$.

Res-CS restriction. We now give a design for the Res-CS restriction. We first simplify the optimality objective – $\text{ParetoOpt}(S)\subseteq\text{ParetoOpt}(\mathcal{F})$ using movement directions $\mathcal{F}_{\boldsymbol{f}} = \left\{\boldsymbol{g}=\boldsymbol{f}'-\boldsymbol{f}\in\mathbb{R}^d \mid \text{ for all } \boldsymbol{f}'\in\mathcal{F}\right\}$, definitions of dual cones $\mathcal{K}^*_A$ and $\mathcal{K}^*_I$, and $\ker\boldsymbol{A} = \left\{\boldsymbol{x}\in\mathbb{R}^d \mid \boldsymbol{Ax}=\boldsymbol{0}\right\}$. We rewrite ParetoOpt$(S)$ as follows:

$$\begin{aligned}\text{ParetoOpt}(S) &= \{\boldsymbol{f}\in\mathcal{F} \mid \text{for all } \boldsymbol{g}\in\mathcal{F}_{\boldsymbol{f}},\ \text{either } \boldsymbol{Ag}\not\geq\boldsymbol{0} \text{ or } \boldsymbol{Ag}=\boldsymbol{0}\}\\ &= \left\{\boldsymbol{f}\in\mathcal{F} \mid \mathcal{F}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_A)^{\mathsf{c}}\cup\ker\boldsymbol{A}\right\}.\end{aligned}$$

Similarly, $\text{ParetoOpt}(\mathcal{F}) = \left\{\boldsymbol{f}\in\mathcal{F} \mid \mathcal{F}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_I)^{\mathsf{c}}\cup\ker\boldsymbol{I}\right\}$. Thus we get:

$$\text{Optimality} \iff \left\{\boldsymbol{f}\in\mathcal{F} \mid \mathcal{F}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_A)^{\mathsf{c}}\cup\ker\boldsymbol{A}\right\} \subseteq \left\{\boldsymbol{f}\in\mathcal{F} \mid \mathcal{F}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_I)^{\mathsf{c}}\cup\ker\boldsymbol{I}\right\}. \tag{Eq. 5}$$

We now identify an isomorphism between movement directions $\mathcal{F}_{\boldsymbol{f}}$ in the ambient space and the coefficient space. Let columns of $\boldsymbol{Z}\in\mathbb{R}^{d\times r}$ be an orthonormal basis of r-dimensional linear subspace $\mathcal{L}$ associated with aff$(\mathcal{F})$. Fix any $\boldsymbol{f}\in\mathcal{F}$. Denote with $\mathcal{C}_{\boldsymbol{f}}\in\mathbb{R}^r$ the set of coefficients of $\mathcal{F}_{\boldsymbol{f}}$ w.r.t. orthonormal basis $\boldsymbol{Z}$, i.e., $\mathcal{C}_{\boldsymbol{f}} = \boldsymbol{Z}^\top(\mathcal{F}_{\boldsymbol{f}})$. This introduces an isomorphism between the sets $\mathcal{F}_{\boldsymbol{f}}$ and $\mathcal{C}_{\boldsymbol{f}}$, i.e., for every $\boldsymbol{g}\in\mathcal{F}_{\boldsymbol{f}}$ these exists unique $\boldsymbol{d}\in\mathcal{C}_{\boldsymbol{f}}$ such that $\boldsymbol{g}=\boldsymbol{Zd}$. With $\boldsymbol{V}=\boldsymbol{AZ}$, we have four equivalences:

$$\begin{aligned}\boldsymbol{Ag}\geq\boldsymbol{0} \iff \boldsymbol{Vd}\geq\boldsymbol{0} \qquad &\text{and} \qquad \boldsymbol{Ag}=\boldsymbol{0} \iff \boldsymbol{Vd}=\boldsymbol{0},\\ \boldsymbol{g}\geq\boldsymbol{0} \iff \boldsymbol{Zd}\geq\boldsymbol{0} \qquad &\text{and} \qquad \boldsymbol{g}=\boldsymbol{0} \iff \boldsymbol{Zd}=\boldsymbol{0}.\end{aligned}$$

Lemma B.4 uses these equivalences to state that for any $\boldsymbol{f}\in\mathcal{F}$, we have

$$\mathcal{F}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_A)^{\mathsf{c}}\cup\ker\boldsymbol{A} \iff \mathcal{C}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_V)^{\mathsf{c}}\cup\ker\boldsymbol{V} \tag{8}$$

$$\mathcal{F}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_I)^{\mathsf{c}}\cup\ker\boldsymbol{I} \iff \mathcal{C}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_Z)^{\mathsf{c}}\cup\ker\boldsymbol{Z}. \tag{9}$$

We further simplify the optimality objective (Equation (5)):

$$\text{Optimality} \iff \left\{\boldsymbol{f}\in\mathcal{F} \mid \mathcal{F}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_A)^{\mathsf{c}}\cup\ker\boldsymbol{A}\right\} \subseteq \left\{\boldsymbol{f}\in\mathcal{F} \mid \mathcal{F}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_I)^{\mathsf{c}}\cup\ker\boldsymbol{I}\right\} \tag{10}$$

$$\iff \left\{\boldsymbol{f}\in\mathcal{F} \mid \mathcal{C}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_V)^{\mathsf{c}}\cup\ker\boldsymbol{V}\right\} \subseteq \left\{\boldsymbol{f}\in\mathcal{F} \mid \mathcal{C}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_Z)^{\mathsf{c}}\cup\ker\boldsymbol{Z}\right\} \tag{11}$$

where Equation (11) follows from Lemma B.4.

Now, we choose r linear independent rows of $\boldsymbol{Z}$ to create $\boldsymbol{V}\in\mathbb{R}^{r\times r}$. Since $\boldsymbol{Z}$ has orthonormal columns, we have $\ker\boldsymbol{V}=\ker\boldsymbol{Z}=\{\boldsymbol{0}\}$. Moreover, we have $\boldsymbol{V}\subseteq\boldsymbol{Z}$, implying $\mathcal{K}_V\subseteq\mathcal{K}_Z$ and $\mathcal{K}^*_Z\subseteq\mathcal{K}^*_V$ (Lemma B.1). This shows that $\mathcal{K}^*_Z\cup(\ker\boldsymbol{Z})^{\mathsf{c}}\subseteq\mathcal{K}^*_V\cup(\ker\boldsymbol{V})^{\mathsf{c}}$. As a result, $(\mathcal{K}^*_V)^{\mathsf{c}}\cup\ker\boldsymbol{V}\subseteq(\mathcal{K}^*_Z)^{\mathsf{c}}\cup\ker\boldsymbol{Z}$. Hence, for any $\boldsymbol{f}\in\mathcal{F}$ for which $\mathcal{C}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_V)^{\mathsf{c}}\cup\ker\boldsymbol{V}$, we also have $\mathcal{C}_{\boldsymbol{f}}\subseteq(\mathcal{K}^*_Z)^{\mathsf{c}}\cup\ker\boldsymbol{Z}$. This shows that Equation (11) holds with the proposed choice of $\boldsymbol{V}$. As $\boldsymbol{V}=\boldsymbol{AZ}$ for $\boldsymbol{A}$ with 1-hot rows, this design satisfies optimality and Res-CS restriction. ◀

▶ **Proposition A.7** (Proposition 12). *Consider designing $S:\mathcal{F}\to\mathbb{R}^k$ to satisfy optimality objective.*

1. *For $\mathcal{F}=\left\{\boldsymbol{f}\in\mathbb{R}^d \mid \|\boldsymbol{f}\|_1\leq 1\right\}$, $k\geq 1$ is necessary and sufficient for all design restrictions.*
2. *For $\mathcal{F}=\left\{\boldsymbol{f}\in\mathbb{R}^d \mid \|\boldsymbol{f}\|_2\leq 1\right\}$, $k\geq 1$ is necessary and sufficient for all design restrictions.*
3. *For $\mathcal{F}=\left\{\boldsymbol{f}\in\mathbb{R}^d \mid \|\boldsymbol{f}\|_\infty\leq 1\right\}$, $k\geq d$ is necessary and sufficient for Res-CS. Moreover, $k\geq 1$ is necessary and sufficient for the Res-LM and Res-L restrictions.*

Proof. Theorem 11 states $k \geq 1$ is sufficient for Res-LM and Res-L restrictions for any $\mathcal{F}$; trivially, $k \geq 1$ is necessary. So, we prove the claims for the Res-CS restriction. For the stated sets $\mathcal{F}$, we determine $\mathrm{ParetoOpt}(\mathcal{F})$ and discuss choice of S to satisfy $\mathrm{ParetoOpt}(S) \subseteq \mathrm{ParetoOpt}(\mathcal{F})$.

We denote the d coordinates of metric value $\boldsymbol{f} \in \mathcal{F}$ with $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_d$. Let $\boldsymbol{e}_j$ be the j^{th} canonical basis vector of $\mathbb{R}^d$. We denote the unit ℓ_p-norm ball with $\mathbb{B}_p^d = \left\{\boldsymbol{f} \in \mathbb{R}^d \mid \|\boldsymbol{f}\|_p \leq 1\right\}$.

1. Let $\mathcal{F} = \mathbb{B}_1^d$, the unit ℓ_1-norm ball centered at the origin. Note that the j^{th} coordinate of metric value $\boldsymbol{f}_j$ is maximized when $\boldsymbol{f} = \boldsymbol{e}_j$. So vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ are pareto-optimal w.r.t. $\mathcal{F}$. In fact, all vectors on the surface of $\mathbb{B}_1^d$ in the nonnegative orthant are pareto-optimal w.r.t. $\mathcal{F}$. That is, $\mathrm{ParetoOpt}(\mathcal{F}) = \left\{\boldsymbol{f} \in \mathbb{R}_+^d \mid \mathbf{1}_d \cdot \boldsymbol{f} = 1\right\}$.
   We choose any coordinate $j \in [d]$ and design 1-dimensional $S(\boldsymbol{f}) = \boldsymbol{f}_j$. Since $\mathcal{F}$ is the unit ℓ_1-norm ball, $\mathrm{ParetoOpt}(S) = \{\boldsymbol{e}_j\}$, which a subset of $\mathrm{ParetoOpt}(\mathcal{F})$ as $\mathbf{1}_d \cdot \boldsymbol{e}_j = 1$. Hence, this design with dimensionality $k = 1$ satisfies the optimality objective under Res-CS restriction.
   Trivially, $k \geq 1$ is necessary as well.
2. Let $\mathcal{F} = \mathbb{B}_2^d$, the unit L_2-ball centered at the origin. Note that the j^{th} coordinate of metric value $\boldsymbol{f}_j$ is maximized when $\boldsymbol{f} = \boldsymbol{e}_j$. So vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ are pareto-optimal w.r.t. $\mathcal{F}$. In fact, all vectors on the unit shell in the nonnegative orthant are pareto-optimal w.r.t. $\mathcal{F}$. That is, $\mathrm{ParetoOpt}(\mathcal{F}) = \mathbb{S}_2^{d-1} \cap \mathbb{R}_d^+ = \mathbb{S}_2^{d-1} \cap \mathcal{K}_I$ where $\boldsymbol{I}$ is the identity matrix.
   We can similarly determine pareto-optimal points w.r.t. $S(\boldsymbol{f}) = \boldsymbol{A}\boldsymbol{f}$. Let $\boldsymbol{A}$ have k rows $\boldsymbol{A} = [\boldsymbol{a}_1; \ldots; \boldsymbol{a}_k] \in \mathbb{R}^{k \times d}$. The i^{th} coordinate of S is maximized when $\boldsymbol{f} = \frac{\boldsymbol{a}_i}{\|\boldsymbol{a}_i\|_2}$. So vectors $\frac{\boldsymbol{a}_1}{\|\boldsymbol{a}_1\|_2}, \ldots, \frac{\boldsymbol{a}_k}{\|\boldsymbol{a}_k\|_2}$ are pareto-optimal w.r.t. S. In fact, all vectors on the unit shell and cone $\mathcal{K}_A$ generated by rows of $\boldsymbol{A}$ are pareto-optimal w.r.t. S. That is, $\mathrm{ParetoOpt}(S) = \mathbb{S}_2^{d-1} \cap \mathcal{K}_A$.
   So S satisfies optimality if $\mathbb{S}_2^{d-1} \cap \mathcal{K}_A \subseteq \mathbb{S}_2^{d-1} \cap \mathcal{K}_I$. Any matrix $\boldsymbol{A} \subseteq \boldsymbol{I}_d$ implies $\mathcal{K}_A \subseteq \mathcal{K}_I$. Hence, we can choose any coordinate $j \in [d]$ and construct 1-dimensional $S(\boldsymbol{f}) = \boldsymbol{f}_j$. This design with dimensionality $k = 1$ satisfies the optimality objective under Res-CS restriction.
   Trivially, $k \geq 1$ is necessary as well.
3. Let $\mathcal{F} = \mathbb{B}_\infty^d$, the unit L_∞-ball centered at the origin. It is easy to see that $\mathrm{ParetoOpt}(\mathcal{F}) = \{\mathbf{1}_d\}$, a singleton set.
   Under the Res-CS restriction, $S : \mathcal{F} \to \mathbb{R}^k$ is such that $S(\boldsymbol{f}) = [\boldsymbol{f}_{i_1}; \ldots; \boldsymbol{f}_{i_k}]$ where the every index $i_j \in [d]$. Let I be the set of unique indices. We will now show that if $k < d$, then there does not exist score function S that satisfies optimality. Since $k < d$, we have $|I| < d$. The point $\boldsymbol{f} \in \mathbb{B}_\infty^d$ is pareto-optimal w.r.t. S if $\boldsymbol{f}_i = 1$ for every $i \in I$. Precisely, $\mathrm{ParetoOpt}(S) = \left\{\boldsymbol{f} \in [-1, 1]^d \mid \boldsymbol{f}_i = 1 \text{ for all } i \in I\right\}$. Since there exists $j \in [d]$ that is not in I, $\mathrm{ParetoOpt}(S)$ contains points with $\boldsymbol{f}_j = -1$. Hence, $\mathrm{ParetoOpt}(S)$ is not a subset of $\mathrm{ParetoOpt}(\mathcal{F})$. Therefore, for $\mathcal{F} = \mathbb{B}_\infty^d$ and $k < d$ it is not possible to design $S : \mathcal{F} \to \mathbb{R}^d$ that satisfies optimality objective under Res-CS restriction.
   Trivially, $k = d$ is sufficient to satisfy the optimality objective under Res-CS restriction: design $S(\boldsymbol{f}) = \boldsymbol{f}$. Hence, $k \geq d$ is both necessary and sufficient when $\mathcal{F} = \mathbb{B}_\infty^d$. ◀

A.3 Minimal design problem for both objectives simultaneously

▶ **Corollary A.8.** *Let columns of $\boldsymbol{Z}$ be an orthonormal basis of linear subspace $\mathcal{L}$ associated with $\mathrm{aff}(\mathcal{F})$. For each design restriction, there exists score function $S : \mathcal{F} \to \mathbb{R}^k$ that simultaneously satisfies improvement and optimality objectives with following dimensionalities.*

	Dimensionality $k \geq$
Res-CS	ConeSubsetRank($\boldsymbol{Z}$)
Res-LM	ConeGeneratingRank($\boldsymbol{Z}$)
Res-L	ConeGeneratingRank($\boldsymbol{Z}$)

Moreover, for Res-CS and Res-LM restrictions, the score design is minimal when $\mathcal{F}$ has non-empty relative interior.

Proof. For the first two restrictions (Res-CS and Res-LM), S is monotone in $\mathcal{F}$. So, Theorems 2 and 13 immediately give the design for simultaneously satisfying both objectives with dimensionality $k = \mathsf{CSR}(\boldsymbol{Z})$ and $\mathsf{CGR}(\boldsymbol{Z})$ respectively. Theorem 7 proves the minimality of this design. The design for Res-LM restriction also applies for the Res-L restriction. ◀

B Technical Lemmas

▶ **Lemma B.1.** *For two polyhedral cones $\mathcal{K}_1$ and $\mathcal{K}_2$, we have $\mathcal{K}_1 \subseteq \mathcal{K}_2 \iff \mathcal{K}_2^* \subseteq \mathcal{K}_1^*$.*

Proof. Since the two cones are polyhedral, they are closed and convex. For any closed and convex cone $\mathcal{K}$, the dual of its dual cone is the cone itself: $\mathcal{K}^{**} = \mathcal{K}$. The result then follows from the fact that for any two convex cones $\mathcal{K}_1 \subseteq \mathcal{K}_2 \implies \mathcal{K}_2^* \subseteq \mathcal{K}_1^*$ [10, Sec. 2.6.1]. ◀

▶ **Lemma B.2.** *Let $\mathcal{L} \subseteq \mathbb{R}^d$ be an r-dimensional linear subspace, and let columns of $\boldsymbol{Z} \in \mathbb{R}^{d\times r}$ be an orthonormal basis of $\mathcal{L}$. Let $\mathcal{K}_{A_1}$ and $\mathcal{K}_{A_2}$ be cones in $\mathbb{R}^d$ generated by rows of matrices $\boldsymbol{A}_1 \in \mathbb{R}^{m_1\times d}$ and $\boldsymbol{A}_2 \in \mathbb{R}^{m_2\times d}$ respectively. With $\boldsymbol{V}_1 = \boldsymbol{A}_1\boldsymbol{Z}$ and $\boldsymbol{V}_2 = \boldsymbol{A}_2\boldsymbol{Z}$, we have,*

$$\mathcal{L} \cap \mathcal{K}_{A_1}^* \subseteq \mathcal{K}_{A_2}^* \iff \mathcal{K}_{V_1}^* \subseteq \mathcal{K}_{V_2}^* \iff \mathcal{K}_{V_2} \subseteq \mathcal{K}_{V_1}.$$

Proof. We can simplify this condition $\mathcal{L} \cap \mathcal{K}_{A_1}^* \subseteq \mathcal{K}_{A_2}^*$ further by expressing vectors in the basis $\boldsymbol{Z}$.

First, every $\boldsymbol{x} \in \mathcal{L}$ has a unique representation in the basis $\boldsymbol{Z}$. That is, $\boldsymbol{x} = \boldsymbol{Z}\boldsymbol{c}$ for some $\boldsymbol{c} \in \mathbb{R}^r$. Second, every d-dimensional row $\boldsymbol{a}$ of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ can be written as $\boldsymbol{a}^{\|} + \boldsymbol{a}^{\perp}$, where $\boldsymbol{a}^{\|} = \boldsymbol{a}\boldsymbol{Z}\boldsymbol{Z}^\top \in \mathcal{L}$ and $\boldsymbol{a}^{\perp} = \boldsymbol{a}(\boldsymbol{I} - \boldsymbol{Z}\boldsymbol{Z}^\top) \in \mathcal{L}^{\perp}$. Therefore, $\boldsymbol{A}_1 = \boldsymbol{A}_1^{\|} + \boldsymbol{A}_1^{\perp}$ where $\boldsymbol{A}_1^{\|} = \boldsymbol{A}_1\boldsymbol{Z}\boldsymbol{Z}^\top + \boldsymbol{A}_1(\boldsymbol{I} - \boldsymbol{Z}\boldsymbol{Z}^\top)$. Note that $\boldsymbol{A}_1^{\perp}\boldsymbol{Z} = \boldsymbol{0}_{m_1\times r}$. Similarly we can decompose the matrix $\boldsymbol{A}_2 = \boldsymbol{A}_2^{\|} + \boldsymbol{A}_2^{\perp}$. Denote the coefficients as $\boldsymbol{V}_1 = \boldsymbol{A}_1\boldsymbol{Z}$ and $\boldsymbol{V}_2 = \boldsymbol{A}_2\boldsymbol{Z}$. Using these simplifications, we get:

$$\begin{aligned}
\mathcal{L} \cap \mathcal{K}_{A_1}^* \subseteq \mathcal{K}_{A_2}^* &\iff \text{for all } \boldsymbol{x} \in \mathcal{L}, \boldsymbol{A}_1\boldsymbol{x} \geq \boldsymbol{0} \implies \boldsymbol{A}_2\boldsymbol{x} \geq \boldsymbol{0} && (12)\\
&\iff \text{for all } \boldsymbol{c} \in \mathbb{R}^r, \boldsymbol{A}_1\boldsymbol{Z}\boldsymbol{c} \geq \boldsymbol{0} \implies \boldsymbol{A}_2\boldsymbol{Z}\boldsymbol{c} \geq \boldsymbol{0} && (13)\\
&\iff \text{for all } \boldsymbol{c}, (\boldsymbol{A}_1^{\|} + \boldsymbol{A}_1^{\perp})\boldsymbol{Z}\boldsymbol{c} \geq \boldsymbol{0} \implies (\boldsymbol{A}_2^{\|} + \boldsymbol{A}_2^{\perp})\boldsymbol{Z}\boldsymbol{c} \geq \boldsymbol{0} && (14)\\
&\iff \text{for all } \boldsymbol{c}, \boldsymbol{V}_1\boldsymbol{Z}^\top\boldsymbol{Z}\boldsymbol{c} \geq \boldsymbol{0} \implies \boldsymbol{V}_2\boldsymbol{Z}^\top\boldsymbol{Z}\boldsymbol{c} \geq \boldsymbol{0} && (15)\\
&\iff \text{for all } \boldsymbol{c}, \boldsymbol{V}_1\boldsymbol{c} \geq \boldsymbol{0} \implies \boldsymbol{V}_2\boldsymbol{c} \geq \boldsymbol{0} && (16)\\
&\iff \mathcal{K}_{V_1}^* \subseteq \mathcal{K}_{V_2}^* && (17)\\
&\iff \mathcal{K}_{V_2} \subseteq \mathcal{K}_{V_1}. && (18)
\end{aligned}$$

where the last equivalence follows from Lemma B.1. ◀

▶ **Lemma B.3.** *Let $\mathcal{L}$ be the linear subspace corresponding to* $\mathrm{aff}(X)$. *For any $\boldsymbol{x}^*$ in the relative interior of X and any $\boldsymbol{x} \in \mathcal{L}$, there exists $a > 0$ such that $a\boldsymbol{x} \in X_{\boldsymbol{x}^*}$.*

Proof. We use the definition of relative interior. Since $\boldsymbol{x}^*$ is in relative interior of X, there exists $R > 0$ such that $(\boldsymbol{x}^* + R \cdot \mathbb{B}_2^d) \cap \operatorname{aff}(X) \subseteq X$. Centering the sets at $\boldsymbol{x}^*$, there exists $R > 0$ such that $R \cdot \mathbb{B}_2^d \cap \operatorname{aff}(X)_{\boldsymbol{x}^*} \subseteq X_{\boldsymbol{x}^*}$. We note that $\mathcal{L} = \operatorname{aff}(X)_{\boldsymbol{x}^*}$.

Let $\boldsymbol{x} \in \mathcal{L}$. If $\boldsymbol{x} = \boldsymbol{0}$ then we are done as $a\boldsymbol{x} = \boldsymbol{0} \in X_{\boldsymbol{x}^*}$ for any $a > 0$. If $\boldsymbol{x}$ is nonzero, then we can normalize it so that $\tilde{\boldsymbol{x}} = R \cdot \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \in R \cdot \mathbb{B}_2^d \cap \mathcal{L}$. From the definition of relative interior, we get that $\tilde{\boldsymbol{x}} \in X_{\boldsymbol{x}^*}$. Thus for any nonzero $\boldsymbol{x} \in \mathcal{L}$ there exists $a = R/\|\boldsymbol{x}\|$ such that $a\boldsymbol{x} \in X_{\boldsymbol{x}^*}$. ◀

▶ **Lemma B.4.** *Let $\mathcal{L}$ be the linear subspace corresponding to r-dimensional $\operatorname{aff}(X) \subseteq \mathbb{R}^d$, and let columns of $\boldsymbol{Z} \in \mathbb{R}^{d \times r}$ be an orthonormal basis of $\mathcal{L}$. For any $\boldsymbol{x} \in X$, denote with $\mathcal{C}_{\boldsymbol{x}} \subseteq \mathbb{R}^r$ the preimage of $X_{\boldsymbol{x}}$ under the orthonormal basis $\boldsymbol{Z}$. Let $\mathcal{K}_A \subseteq \mathbb{R}^d$ be generated by rows of $\boldsymbol{A} \in \mathbb{R}^{m \times d}$, and let $\boldsymbol{V} = \boldsymbol{AZ}$. Then for every $\boldsymbol{f} \in \mathcal{F}$,*

$$X_{\boldsymbol{x}} \cap \mathcal{K}_A^* \cap (\ker \boldsymbol{A})^{\mathsf{c}} = \emptyset \iff \mathcal{C}_{\boldsymbol{x}} \cap \mathcal{K}_V^* \cap (\ker \boldsymbol{V})^{\mathsf{c}} = \emptyset.$$

Proof. Note that for every $\boldsymbol{x} \in X$, the linear subspace spanned by the set $X_{\boldsymbol{x}}$ is $\mathcal{L}$, and columns of $\boldsymbol{Z}$ are an orthonormal basis of $\mathcal{L}$. That is, for every $\boldsymbol{y} \in X_{\boldsymbol{x}}$ these exists unique $\boldsymbol{d} \in \mathcal{C}_{\boldsymbol{x}}$ such that $\boldsymbol{y} = \boldsymbol{Zd}$. Moreover, we can decompose rows of $\boldsymbol{A}$ in the linear subspace $\mathcal{L}$ and its orthogonal complement $\mathcal{L}^{\perp}$, as in proof of Lemma B.2. We decompose $\boldsymbol{A} = \boldsymbol{AZZ}^{\top} + \boldsymbol{A}(\boldsymbol{I}_d - \boldsymbol{ZZ}^{\top})$.

We use these decomposition results to prove the desired result. We first prove the forward direction by contradiction. Let $\boldsymbol{x} \in X$ and assume that $X_{\boldsymbol{x}} \cap \mathcal{K}_A^* \cap (\ker \boldsymbol{A})^{\mathsf{c}} = \emptyset$. Now assume that there exists $\boldsymbol{d} \in \mathcal{C}_{\boldsymbol{x}} \cap \mathcal{K}_V^* \cap (\ker \boldsymbol{V})^{\mathsf{c}}$. So $\boldsymbol{Vd} \geq \boldsymbol{0}$ and $\boldsymbol{Vd} \neq \boldsymbol{0}$, implying that $\boldsymbol{AZd} \geq \boldsymbol{0}$ and $\boldsymbol{AZd} \neq \boldsymbol{0}$. Hence, there exists $\boldsymbol{y} = \boldsymbol{Zd} \in X_{\boldsymbol{x}}$ such that $\boldsymbol{y} \in \mathcal{K}_A^*$ and $\boldsymbol{y} \in (\ker \boldsymbol{A})^{\mathsf{c}}$. This contradicts our assumption that $X_{\boldsymbol{x}} \cap \mathcal{K}_A^* \cap (\ker \boldsymbol{A})^{\mathsf{c}} = \emptyset$, and so we must have $\mathcal{C}_{\boldsymbol{x}} \cap \mathcal{K}_V^* \cap (\ker \boldsymbol{V})^{\mathsf{c}} = \emptyset$.

We also prove the backward direction by contradiction. Let $\boldsymbol{x} \in X$ and assume that $\mathcal{C}_{\boldsymbol{x}} \cap \mathcal{K}_V^* \cap (\ker \boldsymbol{V})^{\mathsf{c}} = \emptyset$. Now assume that there exists $\boldsymbol{y} \in X_{\boldsymbol{x}} \cap \mathcal{K}_A^* \cap (\ker \boldsymbol{A})^{\mathsf{c}}$. So $\boldsymbol{Ay} \geq \boldsymbol{0}$ and $\boldsymbol{Ay} \neq \boldsymbol{0}$. Using decomposition of rows of $\boldsymbol{A}$ and $\boldsymbol{y}$ in the basis $\boldsymbol{Z}$, we get that $\boldsymbol{Ay} = \boldsymbol{AZd}$ where $\boldsymbol{y} = \boldsymbol{Zd}$ for $\boldsymbol{d} \in \mathcal{C}_{\boldsymbol{x}}$. So there exists $\boldsymbol{d} \in \mathcal{C}_{\boldsymbol{x}}$ such that $\boldsymbol{AZd} \geq \boldsymbol{0}$ and $\boldsymbol{AZd} \neq \boldsymbol{0}$. Since $\boldsymbol{V} = \boldsymbol{AZ}$, we get that there exists $\boldsymbol{d} \in \mathcal{C}_{\boldsymbol{x}} \cap \mathcal{K}_V^* \cap (\ker \boldsymbol{V})^{\mathsf{c}}$. This contradicts our assumption that $\mathcal{C}_{\boldsymbol{x}} \cap \mathcal{K}_V^* \cap (\ker \boldsymbol{V})^{\mathsf{c}} = \emptyset$, and so we must have $X_{\boldsymbol{x}} \cap \mathcal{K}_A^* \cap (\ker \boldsymbol{A})^{\mathsf{c}} = \emptyset$. ◀