Edit Distance of Finite State Transducers

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Abstract

We lift metrics over words to metrics over word-to-word transductions, by defining the distance between two transductions as the supremum of the distances of their respective outputs over all inputs. This allows to compare transducers beyond equivalence.

Two transducers are *close* (*resp.* k-close) with respect to a metric if their distance is finite (*resp.* at most k). Over integer-valued metrics computing the distance between transducers is equivalent to deciding the closeness and k-closeness problems. For common integer-valued edit distances such as, Hamming, transposition, conjugacy and Levenshtein family of distances, we show that the closeness and the k-closeness problems are decidable for functional transducers. Hence, the distance with respect to these metrics is also computable.

Finally, we relate the notion of distance between functions to the notions of diameter of a relation and index of a relation in another. We show that computing edit distance between functional transducers is equivalent to computing diameter of a rational relation and both are a specific instance of the index problem of rational relations.

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1 Introduction

For meaningfully comparing two words (or sequences, vectors, functions, etc.), it is often necessary to have a measure that quantifies their (dis)similarity. It usually consists of associating a nonnegative integer to two words that indicates how different they are from each other. This usually defines a distance between words, the most popular of which are edit distances. It is the minimum number of edit operations required to transform one word into another. These operations typically include inserting or deleting a letter, substituting a letter with another, swapping adjacent letters (transpositions), and cyclic shifts. Edit distances are studied in coding [29, 41], parsing [2], speech recognition [33, 1], molecular biology [18, 24] etc. Interesting combinatorial problems on words such as the computation of longest common subsequences can be reduced to computing edit distances [6]. For a detailed overview of the history and applications of edit distances, see [27].

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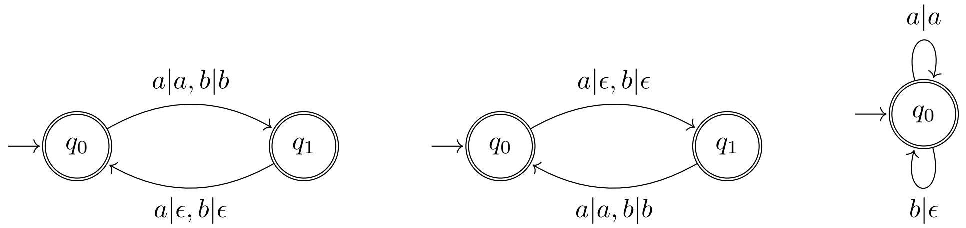
The notion of distance between two words can be lifted naturally to distance between a word and a set of words, or between two sets of words, and so on. There is a long line of research of this kind: computing the edit distance between two languages – usually defined as the smallest distance between any two pairs from the respective sets. It could be between a word and a regular language [42, 4], two regular languages [31], a regular language and itself [25], or a regular language and a context-free language [21]. In all these settings there are efficient algorithms for computing the edit distances.

In this paper we study the distance between two word-to-word functions (transductions) given by finite state transducers, i.e., automata with output. Finite state transducers are used in a variety of software and hardware systems such as encoders, decoders, demuxers, spell checkers, text normalizers, schema translators, template code generators, etc.



(a) $\mathcal{T}_1$. **(b)** $\mathcal{T}_2$. **(c)** $\mathcal{T}_3$.

Figure 1 $\mathcal{T}_1$ outputs letters at the odd positions, $\mathcal{T}_2$ outputs letters at the even positions and $\mathcal{T}_3$ outputs only a's.

Our aim is to develop a framework to meaningfully compare two transductions beyond equivalence. Consider the functions given by the transducers in Figure 1. The transducers $\mathcal{T}_1$ and $\mathcal{T}_2$ output the letters at the odd and even positions respectively, while the transducer $\mathcal{T}_3$ erases b's in the input. If we were to find the odd one among these three functions, arguably $\mathcal{T}_3$ will be picked, with the length of the respective output on any input deviating significantly from that of the others. Our aim is to define a measure that quantifies such distances.

If we have a metric to compare the output words, we can extend it to transductions as follows. The distance between two transductions is the least upper bound of the distances between their respective outputs on any input word. We assume that their domains are the same, and we set the distance to infinity if this is not the case. We say that two transductions are close if their distance is finite, and they are k-close if their distance is at most k. We may simply say that two transducers are close (or k-close) instead, to mean that the transductions defined by these transducers are close (or k-close).

We are interested in the following question: *Given two finite state transducers, are the transductions defined by them close (or simply are the transducers close)?* Clearly, deciding closeness is a boundedness problem. We show closeness as well as k-closeness are decidable for various edit distance metrics, in particular Hamming (letter-to-letter substitutions), transposition (swapping adjacent letters), conjugacy (only cyclic shifts) and Levenshtein family of distances – Longest common subsequence (insertion and deletion), Levenshtein (insertion, deletion and substitution), and Damerau-Levenshtein (insertion, deletion, substitution and adjacent transposition). It turns out that computing distance between transducers is equivalent to deciding closeness and k-closeness over integer-valued metrics (see Proposition 3.6). Hence for the edit distances mentioned above, the distance between transducers is computable.

A related notion is that of *diameter* of a relation. We define it to be the supremum of the distance of every pair in the relation. We are interested in computing the diameter of rational relations over words, that is those given by (not necessarily functional) finite state transducers. A rational relation is said to have *bounded diameter* (*resp.* k-bounded diameter) if the diameter of the relation is finite (*resp.* at most k). It turns out that for every pair of transductions $\mathcal{T}_1$ and $\mathcal{T}_2$ there is a rational relation R such that for every metric, the diameter of R is same as the distance between $\mathcal{T}_1$ and $\mathcal{T}_2$. In fact, the converse is also true by virtue of Nivat's theorem (see Theorem 3.23).

Another related notion is that of the *index of a rational relation in the composition closure of another.* Let R, S be a rational relation over words. The index of R in the composition closure of S is defined to be the smallest integer k such that the relation R is contained in the k-fold composition of S. If such a k exists we say that R has the *finite index property* in the composition closure of S. We show that the finite index property is undecidable for arbitrary rational relations. However, if S is a metrizable relation (see Definition 3.18) w.r.t. the edit distances mentioned above, the index of R in the composition closure of S is computable.

Our decision procedure for k-closeness involves designing a weighted automaton that counts the number of edit operations for transforming one output to the other. We need to check whether there are input instances for which the weight is more than k. We extract a finite state automaton of size exponential in k that achieves this (see Proposition 3.11). This is a generic approach independent of the particular edit operations. However for Hamming and transposition distances, we have a direct polynomial time procedure for deciding k-closeness (see full version).

Recall that deciding closeness of transductions is same as deciding whether the diameter of a rational relation R is bounded. For the latter, consider a transducer recognising R. It turns out that if there are loops in this transducer that produce nonconjugate words (that are not cyclic shifts of each other) then such loops can be iterated to get unbounded diameter/distance. Thus a crucial ingredient in our decision procedure is checking for conjugacy of loops, which is decidable [3]. For boundedness w.r.t. Levenshtein distances, we show that this is also a sufficient condition (see Claim 4.9). For conjugacy distance, we show that the diameter of a rational relation R is bounded if and only if every pair in R is conjugate (see Proposition 4.6). Notice that this is not the case for arbitrary relations. In the case of Hamming distance, which only includes substitutions, we show that it is sufficient to check if the pairs of words generated by the loops after some shifted delay are identical (see Claim 4.11). This also holds true for transposition distance, but additionally, we also need to check if the words are permutations of each other (see Claim 4.12).

1.1 Related Work

The *adjacent functions* in [34] is an analogous definition for closeness between transductions with respect to prefix distance. Two functions $f, g : A^* \to B^*$ are *adjacent* if $\sup\{\, d_p(f(w), g(w)) \mid w \in dom(f) \cap dom(g) \,\} < \infty$. Here, $d_p(u,v) = |u| + |v| - 2\max\{|z| \mid u, v \in zA^*\}$ denotes the *prefix distance* between two words u and v. The adjacency of two rational functions is used in deciding the sequentiality of a function. It is decidable to check if two given rational functions are adjacent or not (Proposition 1 of [34]).

Another problem that is similar in spirit is the *robustness problem.* We say a transducer $\mathcal{T}$ is *robust* w.r.t. a distance d if there is a nontrivial relation R between the distance between two input words (say $d(u,v)$) and distance between their corresponding outputs on $\mathcal{T}$ (say $d(\mathcal{T}(u), \mathcal{T}(v))$). For instance, R could be *Lipschitz continuity* – there is some $k > 0$ such that $d(\mathcal{T}(u), \mathcal{T}(v)) \leq k \cdot d(u, v)$, or *locally Lipschitz continuity* – there exists $b, k > 0$ such that if $d(u, v) < b$ then $d(\mathcal{T}(u), \mathcal{T}(v)) \leq k \cdot d(u, v)$, etc. Sometimes, weaker notions of distance are considered (for instance by dropping the triangle inequality), and respective distances are called *cost* or *similarity* functions. The work [35] solves the locally Lipschitz continuity problem for sequential and unambiguous transducers using reversal bounded counter automata. The problem is shown to be undecidable for Lipschitz continuity even for deterministic transducers and the decidability is shown for the class that has a bound on the delay between input and output words [23].

Frougny and Sakarovitch studied rational relations with bounded delay [20], which is actually our diameter problem for rational relations when the distance over words is measured by their length difference. A problem related to the diameter of a rational relation is *almost reflexivity* of rational relations studied in [11]. A relation $R \subseteq A^* \times A^*$ is k-reflexive, for some integer $k \leq \infty$, if every element u of the domain is at a distance at most k from some element of the range v, with $(u, v) \in R$, and vice versa. The relation R is almost reflexive if $k < \infty$. It is shown undecidable to check if a deterministic rational relation is almost reflexive, or k-reflexive, for any given integer k, with respect to the following – Hamming, prefix, suffix, subword and Levenshtein edit distances. It is shown decidable for synchronized rational relation w.r.t. Hamming distance.

In 1966, Brzozowski raised the question of *finite power property* on regular languages – it takes a regular language L as input and asks whether there exists some positive integer n such that $(L + \epsilon)^n = L^*$. It was solved in 1979 by Hashiguchi [22] and Simon [37], independently. We study the *finite index property* of a rational relation in the iterative composition of another relation. Notice that the finite index property is different from the finite power property in two respects. One, it is over relations and not languages, and secondly and more importantly, the iteration is obtained by relation composition and not concatenation.

1.2 Organisation of the Paper

In § 2, we recall the definitions of finite state transducers, metrics on words and edit distances. In § 3, we define the notion of distance between transducers, the diameter of a rational relation, and the index of a rational relation in another. We also establish the relation between these notions and state our results in this section. In § 4, we give the connections with conjugacy and the proof arguments remaining from § 3. Finally, we conclude in § 5 with a short discussion on future directions. Proofs omitted are provided in the full version.

2 Preliminaries

Let A^* denote the set of all finite words over the alphabet A. We use $|w|$ to denote the length of the word w. Let $w[i\,..\,j]$ denote the factor of w from index i to j where $1 \leq i \leq j \leq |w|$. A transduction is a function from words to words.

2.1 Finite State Transducers

The simplest form of a transducer is a deterministic finite state machine whose each transition and each final state is labelled by a possibly empty *output word*. Formally, a *sequential transducer* $\mathcal{T} = \langle \mathcal{A}, \lambda, o \rangle$ with input alphabet A and output alphabet B is a *deterministic* finite state automaton $\mathcal{A}$ with two associated output functions $\lambda : \Delta \to B^*$ and $o : F \to B^*$ where Δ and F are the set of transitions and the set of accepting states of $\mathcal{A}$ respectively.

On an input word that is accepted by the automaton, we concatenate the output words produced by the transitions in the unique run of the machine and finally append the end-of-input word of the final state to obtain the output of the machine. That is to say, if $\rho = \delta_1 \cdots \delta_n$ is the successful run of $\mathcal{A}$ on a word $w \in A^*$, the *output* of $\mathcal{T}$ on w, denoted by $\mathcal{T}(w)$, is the word $\lambda(\rho) \cdot o(q)$ where $\lambda(\rho) = \lambda(\delta_1) \cdots \lambda(\delta_n)$ and q is accepting state reached by the run. Let $L(\mathcal{A})$ denote the set of words accepted by $\mathcal{A}$, called the *language* of $\mathcal{A}$ or the *domain* of $\mathcal{T}$ (denoted as $dom(\mathcal{T})$). We can see that $\mathcal{T}$ defines a function from $dom(\mathcal{T})$ to B^*. Functions defined by sequential transducers are called *sequential.* In the literature, they are known as *subsequential functions*, introduced by Schützenberger [36]. Transducers given in Figure 1 are sequential.

If we allow the finite state automaton $\mathcal{A}$ to be nondeterministic, then $\mathcal{T}$ no longer defines a function, but a binary relation on $A^* \times B^*$. Such relations are called *rational.* If the relation is a function, then the transducer is called *functional*, and the corresponding functions are called *rational functions.* We can restrict the nondeterminism and still compute all rational functions. A finite state automaton is *unambiguous* if on each input word the machine has at most one run. It is a well-known fact in the theory of transducers that all *rational functions* are computed by finite state transducers whose underlying automata are unambiguous [10]. Such transducers are called *unambiguous transducers.* Clearly sequential functions are a strict subset of rational functions. For instance, the function “output the input word if the last letter of the input is an a, otherwise the empty word” is rational but not sequential.

There exist generalisations of rational functions where the underlying automaton is a two-way finite state automaton or equivalently a finite state automaton with registers (corresponding functions are called *regular* [17, 5]), or two-way finite state automaton with pebbles (polyregular functions [8, 9]). An overview of the classical theory of transducers is given in [19]. In this paper, we restrict our attention to one-way functional transducers.

2.2 Metric on Words, Edit Distances

Simply put, a metric on a set is used to measure distance between any two elements of the set. A *metric on words* over the alphabet A is a function $d : A^* \times A^* \to [0, \infty]$ such that for any words u, v and w in A^*, $d(u, v) = 0 \iff u = v$ (*separation*), $d(u, v) = d(v, u)$ (*symmetry*), and $d(u, v) \leq d(u, w) + d(w, v)$ (*triangle inequality*).

A metric is *integer-valued* if it has range $\mathbb{N} \cup \{\infty\}$. A trivial metric on words is the *discrete metric* – distance between words u and v, denoted by $d_\infty(u, v)$, is 0 if $u = v$ and ∞ otherwise. Another straightforward distance on words is the absolute difference of their lengths (denoted as d_{len}). This is a *pseudo-metric* since the distance between two distinct words can be zero, i.e., does not satisfy the separation property of a metric.

An important class of metrics in the context of word transducers is *edit distances.* Loosely speaking, *edits* are operations that transform words, such as *inserting a letter*, *deleting a letter*, *substitutions (letter-to-letter)*, *adjacent transpositions* (swapping adjacent letters), *left and right shifts* etc. For a fixed set of edit operations C, the edit distance with respect to C between words u and v, is the minimum number of edits in C required to transform u to v if it is possible, and ∞ otherwise. The common edit distances and their corresponding operations are recalled in Table 1. Since many of these operations are obtained by combinations of the others, we can relate these metrics. The notation $d_1 \leq d_2$ is an abbreviation for $d_1(u, v) \leq d_2(u, v)$ for all words u, v. We can also relate the metrics up to boundedness (See [14] for a detailed introduction). Let $\alpha : \mathbb{N} \to \mathbb{N}$ be a *correction* function. Usual examples are increments (e.g. $x \mapsto x + 2$), scaling (e.g. $x \mapsto 2 \cdot x$) etc. We extend α to the domain $\mathbb{N} \cup \{\infty\}$ by letting $\alpha(\infty) = \infty$. We write $d_1 \lesssim d_2$ to mean that there is some α such that

Table 1 Edit Distances.

Edit Distance	Denotation	Allowed Operations
Hamming distance	d_h	letter-to-letter substitutions
Transposition distance	d_t	swapping adjacent letters
Conjugacy distance	d_c	left and right cyclic shifts
Levenshtein edit distance	d_l	insertions, deletions, and substitutions
Longest Common Subsequence	d_{lcs}	insertions and deletions
Damerau-Levenshtein distance	d_{dl}	insertions, deletions, substitutions and adjacent transpositions

$d_1 \leq \alpha \circ d_2$. Clearly, if $d_1 \leq d_2$ then $d_1 \lesssim d_2$. If $d_1 \lesssim d_2$ and $d_2 \lesssim d_1$, we write $d_1 \approx d_2$ (this is known as the cost equivalence or the boundedness equivalence). If two functions f and g are cost-equivalent then f and g are bounded over precisely the same family of subsets (See Proposition 1 of [14]).

▶ **Lemma 2.1.** *The metrics defined in Table 1 are related as follows:*

1. $d_{len} \leq d \leq d_\infty$, *for each edit distance metric* $d \in \{d_l, d_h, d_t, d_c, d_{lcs}, d_{dl}\}$
2. $d_l \approx d_{lcs} \approx d_{dl}$
3. $d_l \leq d_h \lesssim d_t$
4. $d_l \lesssim d_c$
5. d_c *and* d_t *as well as* d_c *and* d_h *are incomparable, i.e.,* $d_h \not\lesssim d_c, d_c \not\lesssim d_h$ *and* $d_t \not\lesssim d_c, d_t \not\lesssim d_c$

3 Distance between Transductions

In this section we define the notion of distance between two rational functions, diameter of a rational relation, and index of a rational relation in another. We establish the relation between these notions and state our results.

3.1 Comparing Transducers

We lift a metric over words to the class of word-to-word functions as follows.

▶ **Definition 3.1** (Metric on transductions). *Let d be a metric on words over the alphabet B. Given two partial functions $\mathcal{T}, \mathcal{S} : A^* \to B^*$, we define*

$$d(\mathcal{T}, \mathcal{S}) = \begin{cases} \sup\{\, d(\mathcal{T}(w), \mathcal{S}(w)) \mid w \in dom(\mathcal{T})\} & \text{if } dom(\mathcal{T}) = dom(\mathcal{S}) \\ \infty & \text{otherwise} \end{cases}$$

▶ **Proposition 3.2.** *d is a metric on transductions.*

▶ Remark 3.3. We can define a notion of distance between word-to-word relations in the above manner, however this distance will not be a metric. In particular $d(R, R)$ will not be 0 for a relation R that is not a (partial) function.

▶ **Example 3.4.** Consider the sequential transducers $\mathcal{T}_1$ and $\mathcal{T}_2$ in Figure 1. The transducers $\mathcal{T}_1$ and $\mathcal{T}_2$ output the letters at the odd and even positions respectively. For any input word u, $||\mathcal{T}_1(u)| - |\mathcal{T}_2(u)|| \leq 1$. Hence $d_{len}(\mathcal{T}_1, \mathcal{T}_2) = 1$. For input word $(ab)^n$ where $n > 1$, the outputs produced by $\mathcal{T}_1$ and $\mathcal{T}_2$ are a^n and b^n respectively. Since n substitutions are required to convert a^n to b^n, $d_l(a^n, b^n) = n$. Therefore, $d_h(\mathcal{T}_1, \mathcal{T}_2) = \infty$ as well as $d_l(\mathcal{T}_1, \mathcal{T}_2) = \infty$.

Table 2 Problems about distance between two transducers w.r.t. the metric d.

Problem	Input	Question
Distance Problem	transducers $\mathcal{T}, \mathcal{S}$	$d(\mathcal{T}, \mathcal{S})$?
Closeness Problem	transducers $\mathcal{T}, \mathcal{S}$	Is $d(\mathcal{T}, \mathcal{S}) < \infty$?
k-Closeness Problem	integer k, transducers $\mathcal{T}, \mathcal{S}$	Is $d(\mathcal{T}, \mathcal{S}) \leq k$?

▶ **Example 3.5.** The sequential transducer $\mathcal{T}_4$ in Figure 2 replaces each block of 0's by a single 0 and each block of 1's by a single 1. Similarly, $\mathcal{T}_5$ substitutes a block of 0's by a single 1 and a block of 1's by a single 0. The output words produced by the transducers on any input word is an alternate sequence of 0's and 1's. If $\mathcal{T}_4$ outputs 010, then $\mathcal{T}_5$ produces its complement, i.e., 101. The Hamming distance between $\mathcal{T}_4$ and $\mathcal{T}_5$ is ∞, but the Levenshtein distance is 2.

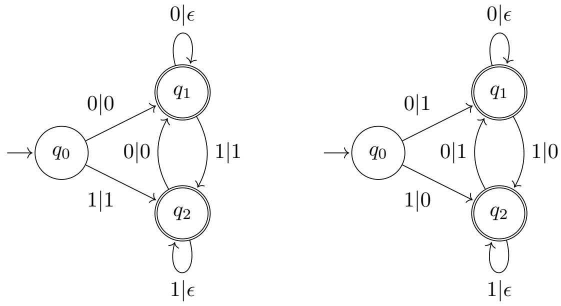


Figure 2 $\mathcal{T}_4$ (left) outputs 0 & 1 for each block of 0's & 1's *resp.* whereas $\mathcal{T}_5$ (right) outputs 1 & 0 for each block of 0's & 1's *resp.*

Let d be a distance on words. The value $d(\mathcal{T}, \mathcal{S})$ is an upper bound on how dissimilar the outputs of transducers $\mathcal{T}$ and $\mathcal{S}$ can be on any input. It is natural to ask the computational and boundedness problems given in Table 2.

Closeness and k-closeness are respectively a boundedness and an upper bound problem on distance.

▶ **Proposition 3.6.** *Let d be an integer-valued metric. The distance problem w.r.t. d is computable if and only if k-closeness and closeness problems w.r.t. d are decidable.*

Proof. Clearly, if we can compute the distance w.r.t. d then we can decide k-closeness as well as closeness. For the other direction, given two transducers, we first check if they are close and if it is we perform an exponential search – check if they are k-close for $k = 2^0, 2^1, 2^2, \ldots$ till it fails and subsequently perform a binary search on the interval $[2^n, 2^{n+1}]$, $n \in \mathbb{N}$ that contains the distance. ◀

We say two transducers $\mathcal{T}$ and $\mathcal{S}$ are *close* (*resp.* k-*close*, for $k \geq 0$) w.r.t. d if $d(\mathcal{T}, \mathcal{S}) < \infty$ (*resp.* $d(\mathcal{T}, \mathcal{S}) \leq k$). Closeness with respect to the discrete metric d_∞ is precisely the equivalence problem. Closeness w.r.t. the length metric d_{len} can be characterised in terms of delay as follows.

▶ **Proposition 3.7.** *Given two transducers $\mathcal{T}_1, \mathcal{T}_2$ with identical domain, $d_{len}(\mathcal{T}_1, \mathcal{T}_2)$ is finite iff there exists a $k \in \mathbb{N}$ such that on any input word w, the difference in lengths of the partial outputs of $\mathcal{T}_1, \mathcal{T}_2$ on any prefix of w is bounded by k.*

In the case of edit distances, closeness means that the output of $\mathcal{T}_1$ can be converted to the output of $\mathcal{T}_2$ by doing a bounded number of edits.

▶ Remark 3.8. From Definition 3.1, it is easy to verify that Lemma 2.1 holds for transducers as well. If $d_1 \lesssim d_2$, then it is easy to see that if transducers $\mathcal{T}_1$ and $\mathcal{T}_2$ are *not* close w.r.t. d_1, then they are not close w.r.t. d_2 either.

The problems in Table 2 for unambiguous transducers with identical domains can be reduced to that for sequential transducers by considering the cartesian product of the unambiguous transducers. Given two unambiguous transducers $\mathcal{T}_1$ and $\mathcal{T}_2$, we obtain the sequential transducers $\mathcal{T}_1'$ and $\mathcal{T}_2'$ as follows. The input automata for $\mathcal{T}_1'$ and $\mathcal{T}_2'$ are the same, call it $\mathcal{A}$, which is the cartesian product of the input automata of $\mathcal{T}_1$ and $\mathcal{T}_2$. By treating the transitions of the cartesian product as the input alphabet, we get input determinism. The output functions of $\mathcal{T}_1'$ and $\mathcal{T}_2'$ are lifted from T_1 and T_2 respectively.

▶ **Proposition 3.9.** *Let d be a distance on words. For each pair of unambiguous transducers $\mathcal{T}_1$ and $\mathcal{T}_2$ with identical domain, there exist a DFA $\mathcal{A}$ and output functions λ_1', o_1' and λ_2', o_2' such that $d(\mathcal{T}_1, \mathcal{T}_2) = d(\mathcal{T}_1', \mathcal{T}_2')$ where the sequential transducer $\mathcal{T}_i' = \langle \mathcal{A}, \lambda_i', o_i' \rangle$, $i \in \{1, 2\}$. Furthermore, the size of the automaton $\mathcal{A}$ is polynomial in the size of $\mathcal{T}_1$ and $\mathcal{T}_2$.*

Given two transductions $\mathcal{T}$ and $\mathcal{S}$, we define a distance function that maps each word w to the distance between their outputs on w.

▶ **Definition 3.10** (Distance function). *The* distance function $f^d_{\mathcal{T},\mathcal{S}} : A^* \to \mathbb{N} \cup \{\infty\}$ *of $\mathcal{T}$ and $\mathcal{S}$ is $f^d_{\mathcal{T},\mathcal{S}}(w) = d(\mathcal{T}(w), \mathcal{S}(w))$ if $w \in dom(\mathcal{T}) \cap dom(\mathcal{S})$; otherwise $f^d_{\mathcal{T},\mathcal{S}}(w) = \infty$.*

Transducers $\mathcal{T}$ and $\mathcal{S}$ are close w.r.t. a metric d if their domains are the same and their distance function $f^d_{\mathcal{T},\mathcal{S}}$ is *limited* (i.e.,$< \infty$ on its domain). Similarly k-closeness w.r.t. d of $\mathcal{T}$ and $\mathcal{S}$ reduces to k-limitedness of $f^d_{\mathcal{T},\mathcal{S}}$. Limitedness problems are well-studied in the context of weighted automata [28, 12]. Therefore, when the distance function $f^d_{\mathcal{T},\mathcal{S}}$ is computable by a $(\min, +)$-automaton, the distance between $\mathcal{T}$ and $\mathcal{S}$ is computable due to Proposition 3.6.

However, there are distance functions that are not computable by weighted automata. Let $A = \{a, b\}$. Consider the sequential transducers $\mathcal{T}_1, \mathcal{T}_2 : A^* \to A^*$ with the domain a^*b^* defining the functions $a^pb^q \mapsto a^p$, $a^pb^q \mapsto a^q$ respectively ($\mathcal{T}_1$ outputs the a's and erases the b's, $\mathcal{T}_2$ erases a's and renames the b's as a's). It is easily checked that their distance function w.r.t. the Levenshtein family ($d \in \{d_l, d_{lcs}, d_{dl}\}$) is $f^d_{\mathcal{T}_1,\mathcal{T}_2} : a^pb^q \mapsto |p - q|$.

If $f : A^* \to \mathbb{N} \cup \{\infty\}$ is a function computed by weighted automata ($(\min, +)$ or $(\max, +)$ or B-automata [13]), then $L_{f \leq k} = \{w \in A^* \mid f(w) \leq k\}$ is regular for each $k \leq \mathbb{N}$. Hence the function $f^d_{\mathcal{T}_1,\mathcal{T}_2}$ is not realised by any of them (consider the language $L_{f^d_{\mathcal{T}_1,\mathcal{T}_2} \leq k}$). In fact, it can be shown that the function $f^d_{\mathcal{T}_1,\mathcal{T}_2}$ is not computed even upto boundedness [15].

To compute k-closeness w.r.t. any of the edit distances, it is not necessary to compute the distance function precisely. The k-approximation of the distance function $f^d_{\mathcal{T},\mathcal{S}}$ is the function $\lceil f^d_{\mathcal{T},\mathcal{S}} \rceil^{\leq k} : w \mapsto f^d_{\mathcal{T},\mathcal{S}}(w)$ if $f^d_{\mathcal{T},\mathcal{S}}(w) \leq k$ and ∞ otherwise.

▶ **Proposition 3.11.** *If $\mathcal{T}$ and $\mathcal{S}$ are close w.r.t. the length metric, then the approximation $\lceil f^d_{\mathcal{T},\mathcal{S}} \rceil^{\leq k}$ for a metric $d \in \{d_l, d_{lcs}, d_{dl}, d_h, d_t, d_c\}$ is computed by a distance automaton for each $k \in \mathbb{N}$.*

To check if $\mathcal{T}$ and $\mathcal{S}$ are k-close, we check if they have the same domain and they are close w.r.t. the length metric (otherwise they are neither close nor k-close). If so, we check if the domain of T is same as the domain of $\lceil f^d_{\mathcal{T},\mathcal{S}} \rceil^{\leq k}$. Thus we get:

▶ **Corollary 3.12.** *Let d be any metric from Table 1, and $\mathcal{T}$ and $\mathcal{S}$ be any functional transducers. It is decidable if $\mathcal{T}$ and $\mathcal{S}$ are k-close w.r.t. d.*

Table 3 Problems about diameter of a rational relation w.r.t. the metric d.

Problem	Input	Question
Diameter Problem	rational relation R	$dia_d(R)$?
Bounded Diameter Problem	rational relation R	Is $dia_d(R) < \infty$?
k-Bounded Diameter Problem	integer k, rational relation R	Is $dia_d(R) \leq k$?

3.2 Diameter of a Rational Relation

▶ **Definition 3.13** (Diameter of a Rational Relation w.r.t. a distance d). *The diameter of a rational relation R with respect to a distance d, denoted by $dia_d(R)$, is the supremum of the distance of the related words in R.*

$$dia_d(R) = \sup\{\, d(u,v) \mid (u,v) \in R \,\}$$

Similar to the questions asked in Table 2, we can ask the questions given in Table 3 about diameter of a rational w.r.t. a metric d. We say a rational relation has *bounded (resp. k-bounded) diameter w.r.t. a distance d* if the diameter of the relation w.r.t. d is finite (*resp.* $\leq k$). A rational relation with *bounded delay* is precisely those relations with bounded diameter w.r.t. a length metric. Relations with 0-delay are called *length-preserving relations* [16] where any two related words are of equal length. It is decidable to check if a rational relation has bounded delay or 0-delay [20].

Relations bounded w.r.t. the discrete metric are simply those with only identical pairs. It is decidable to determine if a rational relation R is identity. First, check if R is length-preserving. If so, we can construct a letter-to-letter transducer for R based on Eilenberg and Schützenberger's theorem [16] stating that a length-preserving rational relation over $A^* \times B^*$ is a rational subset of $(A \times B)^*$, or equivalently, it can be realised by a letter-to-letter transducer whose transitions are labelled with elements of $A \times B$. Finally, validate this transducer for identity by examining the labels of each transition.

3.3 Index of a Rational Relation in a Composition Closure

Consider two rational relations R over $A^* \times B^*$ and S over $B^* \times C^*$. The composition $S \circ R$ over $A^* \times C^*$ is defined by $(S \circ R)(u) = S(R(u)) = \bigcup_{v \in R(u)} S(v)$.

▶ **Definition 3.14** (Composition closure of a Rational Relation). *Let S be a rational relation over $A^* \times A^*$. Let $S^{(n)}$ denote the composition of S with itself $n \geq 0$ times ($S^{(0)}$ is taken to be the identity relation), and let $S^{\leq(n)}$ denotes the composition of S with itself at most n times, i.e., $S^{\leq(n)} = S^{(0)} \cup S^{(1)} \cup \cdots S^{(n)}$.*

The composition closure *of S, denoted as $S^{(*)}$, is defined as $S^{(*)} = \bigcup_{i \geq 0} S^{(i)}$.*

Notice that we use parenthesis around the superscript to indicate that the base operation is composition, and not concatenation.

▶ **Definition 3.15** (Index of a Rational Relation in a Composition Closure). *Let S be a rational relation over $A^* \times A^*$. An* index *of a rational relation R in the composition closure of S, denoted as $Index(R, S)$, is the smallest integer k such that R is contained in $S^{\leq(k)}$.*

▶ **Example 3.16.** Consider a relation S over $\{a,b\}^* \times \{a,b\}^*$ that deletes the first a if exists on any input. Fix an integer $k > 0$ and let R be the relation that deletes the first k a's from the input if exists. The index of R in $S^{(*)}$ is k since for any input word $u \in A^*$, $R(u) \in S^{\leq(k)}(u)$.

Table 4 Problems about the index of a rational relation in the composition closure of another.

Problem	Input	Question
Index Problem	rational relation R, S	$\text{Index}(R, S)$?
Bounded (or Finite) Index Problem	rational relation R, S	Is $\text{Index}(R, S) < \infty$?
k-Bounded Index Problem	integer k, rational relation R, S	Is $\text{Index}(R, S) \leq k$?

Consider another relation R' that deletes all a's from the input. Since $R'(a^{k+1}) \notin S^{\leq(k)}(a^{k+1})$ for any $k > 0$, the index of R' in $S^{(*)}$ is ∞.

As seen in the case of the distance and diameter problem, we can ask questions in Table 4 about the index of a rational relation in the composition closure of a relation. We say a rational relation R has *bounded (resp. k-bounded)* index in the composition closure of a rational relation S if the index of R in $S^{(*)}$ is finite (*resp.* $\leq k$).

Deciding the boundedness of the index problem for an arbitrary rational relation is difficult.

▶ **Lemma 3.17.** *It is undecidable to check if a rational relation has a bounded index in the composition closure of an arbitrary rational relation.*

However, we show that the index problem is decidable w.r.t. a large class of rational relations defined below.

▶ **Definition 3.18** (Metrizable Relation). *Let S be a rational relation over $A^* \times A^*$. Let $d_S : A^* \times A^* \to \mathbb{N} \cup \{\infty\}$ be the distance between two vertices in the graph of S, i.e., for any two words u and v, $d_S(u, v)$ is the smallest i such that $v \in S^{(i)}(u)$, and ∞ otherwise.*

*We say S is a d-*metrizable relation *for a metric d if $d_S \approx d$.*

▶ **Proposition 3.19.** *Let R be a rational relation and S be a d-metrizable relation for an integer-valued metric d for which $d_{len} \lesssim d$. If boundedness of diameter w.r.t. d is decidable for a rational relation, then Index(R, S) is computable.*

Proof. Similar to distance problem, the index problem is computable iff bounded index and k-bounded index problems are decidable. For a rational relation R and d-metrizable relation S, we show that $\text{Index}(R, S) < \infty$ iff $dia_d(R) < \infty$ as follows.

$$\begin{aligned}
dia_d(R) < \infty &\iff \exists k \in \mathbb{N} \text{ s.t. } \forall (u, v) \in R, d(u, v) \leq k \\
&\iff \exists k' \in \mathbb{N} \text{ s.t. } \forall (u, v) \in R, d_S(u, v) \leq k' && (\text{Since } d_S \approx d) \\
&\iff \forall (u, v) \in R, v \in S^{\leq(k')}(u) && (\text{Definition of } d_S) \\
&\iff \text{Index}(R, S) < \infty
\end{aligned}$$

Therefore, if the boundedness of diameter w.r.t. d is decidable for a rational relation, then we can decide if $\text{Index}(R, S) < \infty$. If so, then it suffices to decide if $\text{Index}(R, S) \leq k$ for $k = 0, 1, \ldots$ and output the smallest k as the index of R in the composition closure of S.

Since $dia_d(R) < \infty$ and $d_{len} \lesssim d$, the rational relation R has a bounded delay. Similarly, S also has a bounded delay since for all $(u, v) \in S$, $d_S(u, v) = 1 \Rightarrow \exists k \in \mathbb{N}$ s.t. $d(u, v) \leq k$ (since $d_S \approx d$) $\Rightarrow \exists k' \in \mathbb{N}$ s.t. $d_{len}(u, v) \leq k'$ (since $d_{len} \lesssim d$). Since S has bounded delay, for any $k \in \mathbb{N}$, $S^{(k)}$ also has bounded delay. It is known that emptinesss and set difference of two rational relations with bounded delay is decidable (Corollary 2 of [20]). For any $k \in \mathbb{N}$, deciding $\text{Index}(R, S) \leq k$ reduces to checking if $R \subseteq S^{\leq(k)}$ (or equivalently, $R \setminus S^{\leq(k)} = \emptyset$), and hence decidable. ◀

A close and (almost) dual notion is that of a metric that defines a rational relation.

▶ **Definition 3.20** (Rationalizable Distance). *A distance d on words is* rationalizable *if the relation $S_d = \{(u,v) \mid d(u,v) = 1\}$, called the* distance relation *of d, is rational.*

▶ **Example 3.21.** Consider the hamming distance d_h. We can construct a rational relation $S_h = \{(u,v) \mid u$ and v differ only in exactly one position$\}$. For example, let $A = \{a,b\}$ and $S_h(aba) = \{bba, aaa, abb\}$. For this, construct a transducer that nondeterministically chooses a position and replaces the input letter with other letters in the alphabet. Similarly, the distance relation of the length metric $S_{len} = \{(u,v) \mid ||u| - |v|| = 1\}$ is also rational.

In fact, we have the following result about the rationalizability of edit distances referred in Table 1.

▶ **Proposition 3.22.** *Every edit distance $d \in \{d_l, d_h, d_t, d_c, d_{lcs}, d_{dl}\}$ is rationalizable.*

3.4 Reductions between Distance, Diameter and Index Problems

We show that the distance problem of two rational functions is mutually reducible to the diameter problem of a rational relation, which in turn is mutually reducible to the index problem of a rational relation in the composition closure of a metrizable relation. Thus, their closeness and boundedness problems are also interreducible.

The correspondence between distance and diameter follows from Nivat's theorem:

▶ **Theorem 3.23** ([32]). *Let A and B be alphabets. The following conditions are equivalent.*
1. *R is a rational relation over $A^* \times B^*$.*
2. *There exist an alphabet C, two alphabetic morphisms $\phi : C^* \to A^*$ and $\psi : C^* \to B^*$ and a regular language $L \subset C^*$ such that $R = \{(\phi(w), \psi(w)) \mid w \in L\}$*

From Proposition 3.9 and (2) $\Rightarrow$ (1) in the above theorem, it follows that distance of two rational functions reduces to the diameter of a rational relation. Now, given a rational relation R, we can create two functional transducers $\mathcal{T}_1$ and $\mathcal{T}_2$ in the following way. The domain for these transducers corresponds to the set L in Theorem 3.23. For each transition in $\mathcal{T}_1$ and $\mathcal{T}_2$ that involves an input alphabet symbol σ, we set the outputs to be $\phi(\sigma)$ and $\psi(\sigma)$ in Theorem 3.23, respectively. Consequently, $\mathcal{T}_1$ and $\mathcal{T}_2$ consist of the sets $\{\phi(w) \mid w \in L\}$ and $\{\psi(w) \mid w \in L\}$ respectively. Since the domain of these transducers is identical, the distance between $\mathcal{T}_1$ and $\mathcal{T}_2$ with respect to any distance d, $d(\mathcal{T}_1, \mathcal{T}_2) = \sup\{\, d(\phi(w), \psi(w)) \mid w \in L\,\}$, that is equivalent to the diameter of R w.r.t. the distance d.

The correspondence between diameter and index for rationalizable distances is stated in the following proposition.

▶ **Proposition 3.24.** *The diameter of a rational relation R w.r.t. a rationalizable distance d is equal to the index of the rational relation R in the composition closure of the distance relation of d.*

Proof. Assume that the diameter of a relation R w.r.t. a distance d is ∞. We claim that the index of R in $S_d^{(*)}$ is also ∞ where S_d is the distance relation of d. Suppose not, i.e., let $k < \infty$ be the index of R in $S_d^{(*)}$. Thus, $\forall (u,v) \in R$, $v \in S_d^{\leq (k)}(u)$. Since S_d is the distance relation of d, $\forall (u,v) \in R$, $d(u,v) \leq k$. However, this contradicts the fact that $dia_d(R) = \infty$. Hence, the index of R in $S_d^{(*)}$ is infinite. Similarly, we can prove the other direction. Now, suppose the diameter of R w.r.t. d is finite, i.e.,

$$\begin{aligned}
\text{diameter of } R \text{ w.r.t. } d \text{ is } k < \infty &\iff \forall (u,v) \in R, d(u,v) \leq k \\
&\iff \forall (u,v) \in R, v \in S_d^{\leq (k)}(u) \\
&\iff \text{ index of } R \text{ in } S_d^{(*)} \text{ is } k.
\end{aligned}$$ ◀

3.5 Decidability Results

We study the problems stated in Tables 2, 3 and 4 and show that they are decidable for the metrics in Table 1. The index problems stated in Table 4 are undecidable in general (see Lemma 3.17), but is decidable for d-metrizable relations for metrics d given in Table 1.

Recall that, w.r.t. a metric d, distance problem is computable if and only if both closeness and k-closeness are decidable (see Proposition 3.6). We have shown that the k-closeness is decidable for all the metrics in Table 2 (Corollary 3.12). Hence to show the decidability of all the problems in Table 2, it suffices to show the decidability of the closeness problem. Furthermore, thanks to the inter-reductions described above (see § 3.4), the decidability of Table 3 follows as well as the decidability for the problems in Table 4 for the rationalizable distance. Moreover, the index of a rational relation in the composition closure of a d-metrizable relation for a metric d given in Table 2 is computable by Proposition 3.19.

It only remains to prove that closeness is decidable for edit distances in Table 1. This is stated below, and proved in the following section.

▶ **Theorem 3.25.** *Let d be any metric from Table 1, and $\mathcal{T}$ and $\mathcal{S}$ be any functional transducers. It is decidable if $\mathcal{T}$ and $\mathcal{S}$ are close w.r.t. d.*

4 Closeness for Edit Distances

In this section, we show that closeness is decidable for all the edit distances in Table 1. The first step is to check if the domain of the transducers are the same. This reduces to checking the equivalence of the underlying automata. For sequential transducers, the underlying automaton is a DFA, while for unambiguous transducers, the underlying automaton is an unambiguous NFA. Checking the equivalence of two unambiguous automata can be done in polynomial time [39], while it is PSPACE in the case of ambiguous automata [40]. Therefore, from now on, we assume that the domains of the transducers given as input to the closeness problem are identical.

Proposition 3.9 allows us to state the distance and closeness problems more abstractly in terms of an automaton over pairs of words. The proposition asserts that distance and closeness problems of two given sequential or unambiguous transducers $\mathcal{T}_1$ and $\mathcal{T}_2$ can be reduced to the corresponding problem for a DFA $\mathcal{A}$ with two sets of output functions λ_1, o_1 and λ_2, o_2. We can combine the output functions to output a pair of words. That is to say, let $\lambda : \Delta \to B^* \times B^*$ be defined as $\lambda(\delta) = (\lambda_1(\delta), \lambda_2(\delta))$, where $\delta \in \Delta$ and Δ is the set of transitions of $\mathcal{A}$. Similarly let $o(p) = (o_1(p), o_2(p))$, where $p \in F$ and F is the set of accepting states of $\mathcal{A}$. Henceforth, we can assume that we are given a DFA $\mathcal{A}$ with the output functions λ and o, denoted as the sequential transducer $\mathcal{T}$. Since the input words are inconsequential for computing the distance, we can convert the transducer $\mathcal{T}$ to an automaton $\mathcal{A}$ that accepts a set of pairs of output words over $B^* \times B^*$, i.e., $L(\mathcal{A}) = \{(u, v) \in B^* \times B^* \mid (u, v) = \mathcal{T}(w), w \in dom(\mathcal{T})\}$. Clearly, transducers $\mathcal{T}_1$ and $\mathcal{T}_2$ are close w.r.t. d if and only if there exist an integer $k \geq 0$ such that $\forall (u, v) \in L(\mathcal{A})$, $d(u, v) \leq k$.

Conjugacy of words plays an important role in closeness problems. A pair of words (u, v) is *conjugate* if there exist words x and y (possibly empty) such that $u = xy$ and $v = yx$ or equivalently, u and v are cyclic shifts of one another. For example, $(aaab, aaba)$ is a conjugate pair where $x = a$ and $y = aab$. Conjugacy relation is an equivalence relation on the set of words. A set of pairs is *conjugate* if each pair in the set is conjugate.

▶ **Lemma 4.1.** *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two sequential transducers that define a function from A^* to B^*. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are close w.r.t. a metric $d \in \{d_l, d_h, d_t, d_c, d_{lcs}, d_{dl}\}$, then every loop in the trim automaton over $B^* \times B^*$, that accepts set of all pairs of output words of $\mathcal{T}_1$ and $\mathcal{T}_2$ on any input, generates only conjugate pair of words.*

Proof. This proof is adaptation of a related result in [3]. Let $\mathcal{A}$ be a trim automaton that realises the pair of output words of transducers $\mathcal{T}_1$ and $\mathcal{T}_2$ on any input. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are close w.r.t. d, there exist an integer $k \geq 0$ such that $\forall (u,v) \in L(\mathcal{A})$, $d(u,v) \leq k$. Let (u,v) be a pair labelled in a loop rooted at some state q. Hence (u^ℓ, v^ℓ) for each $\ell \geq 0$ is also a pair in a loop rooted at q. We can safely assume that $|u| = |v|$, otherwise the edit distance will be unbounded as each iteration will increase the edit distance by a difference in length of u and v (Item 1 of Lemma 2.1).

Since $\mathcal{A}$ is trimmed, there exists a path from an initial state q_0 to q and from q to a final state q_f. Let (α_0, β_0) be a pair labelled in a path from q_0 to q, and let (α_1, β_1) be a pair labelled in a path from q to q_f. Thus, pair $(\alpha_0, \beta_0)(u^\ell, v^\ell)(\alpha_1, \beta_1)$ belongs to $L(\mathcal{A})$ where $\ell = 2^k$ (some value much larger than k). Since ℓ is much larger than k and $d(\alpha_0 u^\ell \alpha_1, \beta_0 v^\ell \beta_1) \leq k$, there exist large portions of u's and v's that match. Therefore, we can infer that u is a factor of vv, and v is a factor of uu.

Since v is an infix of uu, the following holds. There exist words x, y, p and q such that $v = xy$ and $u = px = yq$. Since $|u| = |v|$, length of p and length of y are the same, that implies $p = y$ (since $u = px = yq$). Therefore, $u = yx$. Hence u and v are conjugate words. Since the pair (u,v) was arbitrary, any pair generated by a loop in $\mathcal{A}$ is conjugate. ◀

4.1 Closeness w.r.t. Levenshtein distances and Conjugacy

In this subsection, we decide closeness w.r.t. Levenshtein family of distances – Levenshtein, Damerau-Levenshtein, and LCS distances – and conjugacy distance. Levenshtein family of distances are all equivalent with respect to closeness problems by Lemma 2.1 and Remark 3.8.

We have already seen that given two unambiguous transducers $\mathcal{T}_1$ and $\mathcal{T}_2$ with identical domains, there exists an automaton $\mathcal{A}$ over $B^* \times B^*$ that accepts a set of all pairs of output words of $\mathcal{T}_1$ and $\mathcal{T}_2$ on any input. Thus, we can state the distance and closeness problems in terms of rational expressions over $B^* \times B^*$.

We define *pairs over the alphabet* B to be the set $B^* \times B^*$ with the pointwise concatenation $(u,v) \cdot (u', v') = (u \cdot u', v \cdot v')$. A *rational expression of pairs* over the alphabet B is a rational expression over the alphabet $\{(b, b') \mid b, b' \in (B \cup \{\epsilon\})\}$ that generates a subset of pairs over B. From the automaton $\mathcal{A}$ over $B^* \times B^*$, using state elimination method ([26], Lecture 9), we can construct the rational expression of pairs E for the output pairs generated by the transducer $\mathcal{T}_1$ and $\mathcal{T}_2$ on any input. We can lift the metric d to expressions by letting $d(E) = \sup \{d(u,v) \mid (u,v) \in L(E)\}$. Clearly $d(E) = d(\mathcal{T}_1, \mathcal{T}_2)$. Thus, the distance and closeness problems of sequential and unambiguous transducers reduce to the corresponding problems for a rational expression of pairs. Henceforth we assume that we are given a rational expression of pairs.

In the context of conjugacy distance, the closeness of a rational expression necessarily implies that every pair in the expression is conjugate. Otherwise, if there exists a pair $(u,v) \in L(E)$ such that u is not conjugate to v, then $d_c(u,v) = \infty$, thus $d_c(E) = \infty$. In fact, this is also a sufficient condition. The proof relies on the results from [3] that studies the conjugacy of rational expression over pairs of words. It crucially uses the notion of a *common witness* of a set of pairs.

▶ **Definition 4.2** (Common Witness of a Set of Pairs). *A* witness *of pair of conjugate words (u,v) is a word z such that either $uz = zv$ (called an* inner witness*) or $zu = vz$ (called an* outer witness*). A* common witness *of a set of pairs is a word z such that either z is an inner witness of every pair in the set, or z is an outer witness of every pair in the set.*

Lyndon and Schützenberger gave a characterisation of conjugacy of a pair of words, stated as a pair of words is conjugate if and only if it has both inner and outer witness (Proposition 1.3.4 of [30]). In [3], it is generalised to a set of pairs as follows.

▶ **Theorem 4.3** ([3]). *Let $M = (\alpha_0, \beta_0)G_1^*(\alpha_1, \beta_1) \cdots G_k^*(\alpha_k, \beta_k)$ be a set of pairs where $G_1, \ldots, G_k$, $k > 0$ are arbitrary sets of pairs of words, and $(\alpha_0, \beta_0), \ldots, (\alpha_k, \beta_k)$ are arbitrary pairs of words. The set M is conjugate iff M has a common witness.*

Existence of a common witness bounds the conjugacy distance of an expression as follows.

▷ Claim 4.4. If a rational expression over pairs E has a common witness z, then $d_c(E) \leq |z|$.

Proof. Since E has a common witness, either $\forall (u, v) \in L(E)$, $uz = zv$, or $\forall (u, v) \in L(E)$, $zu = vz$. WLOG, assume that $\forall (u, v) \in L(E)$, $uz = zv$. Now, for any pair $(u, v) \in L(E)$:

1. If $|u| > |z|$, then z is a prefix of u and suffix of v and hence $(u, v) = (zp, pz)$ for some word $p \in A^*$. Therefore $d_c(u, v) \leq |z|$ since v can be obtained by $|z|$ left cyclic shifts of u.
2. Otherwise, when $|u| \leq |z|$, the number of cyclic shifts required to transform u to v (note that u and v are conjugate since they have a witness) is less than $|u| \leq |z|$. ◁

A rational expression is *sumfree* if it does not use sum (i.e., $+$). In [3], it is shown that if a common witness exists, it is computable for a sumfree rational expression over pairs of words. It is folklore that every rational expression is equivalent to a sum of sumfree expressions [3]. The proposition below implies that to show closeness for a sum of sumfree expressions, it suffices to show closeness for each of its constituent sumfree expressions.

▶ **Proposition 4.5.** *Let $E = E_1 + \cdots + E_k, k \geq 1$ be a rational expression of pairs. Then $d(E) = \max(d(E_1), \ldots, d(E_k))$ for all word metrics d.*

An expression is conjugate if every pair generated by the expression is conjugate. The following proposition characterises closeness w.r.t. conjugacy distance.

▶ **Proposition 4.6.** *A rational expression over pairs of words is close w.r.t. conjugacy distance if and only if the expression is conjugate. Furthermore, the closeness w.r.t. conjugacy distance is decidable.*

Proof. One direction is trivial. Assume E to be an arbitrary rational expression of pairs and is conjugate. Let $E = E_1 + E_2 + \cdots + E_k$ where $E_1, E_2, \ldots, E_k$ are sumfree expressions. Since E is conjugate, each of its sumfree constituents E_i for $1 \leq i \leq k$ is also conjugate. Using Theorem 4.3, each E_i has a common witness, say z_i. From Claim 4.4, $d_c(E_i) \leq z_i$. Therefore, $d_c(E)$ is close w.r.t. conjugacy distance by Proposition 4.5. Hence, to decide closeness of E w.r.t. conjugacy distance, it suffices to check if E is conjugate. This reduces to checking if a common witness exists for each sumfree constituent. It is shown to be decidable in [3]. ◀

Now consider the case of Levenshtein distances. From Lemma 4.1, if an expression is close w.r.t. Levenshtein distances, it is necessary that every pair generated by a Kleene star in the expression needs to be conjugate. Using common witness, we show that it is also a sufficient condition.

▷ Claim 4.7. If a rational expression of pairs E has a common witness z, then $d_l(E) \leq 2|z|$.

Proof. The proof is similar to Claim 4.4. Since E has a common witness, either $\forall (u, v) \in L(E)$, $uz = zv$, or $\forall (u, v) \in L(E)$, $zu = vz$. WLOG, assume that $\forall (u, v) \in L(E)$, $uz = zv$. For any pair $(u, v) \in L(E)$, $|u| = |v|$ since $uz = zv$. There are two cases, either $|u| > |z|$ or $|u| \leq |z|$. If $|u| > |z|$, then z is a prefix of u and suffix of v and hence $(u, v) = (zp, pz)$ for some word p. Therefore, $d_l(E) \leq 2|z|$ by deleting z in the beginning and insert z at the end of u. Suppose $|u| \leq |z|$, the number of edits required to transform u to v is less than $|u| + |v| \leq 2|u| \leq 2|z|$. ◁

▶ **Proposition 4.8.** *Closeness of a rational expression w.r.t. Levenshtein distance is decidable.*

Proof. Given an arbitrary rational expression, there is an equivalent sum of sumfree expression. From Proposition 4.5, to show closeness for a sum of sumfree expressions, it suffices to show closeness for each of its constituent sumfree expressions. The general form of a sumfree expression $E = (\alpha_0, \beta_0)E_1^*(\alpha_1, \beta_1)\cdots E_k^*(\alpha_k, \beta_k)$ where $k \in \mathbb{N}$, for $0 \le j \le k$, (α_j, β_j) is a (possibly empty) pair of words, and for each $1 \le i \le k$, E_i is a sumfree expression.

▷ Claim 4.9. A sumfree expression $E = (\alpha_0, \beta_0)E_1^*(\alpha_1, \beta_1)\cdots E_k^*(\alpha_k, \beta_k)$ is close w.r.t. Levenshtein distance if and only if each E_i^* for $1 \le i \le k$ is conjugate.

Proof. From Lemma 4.1, if E is close w.r.t. Levenshtein edit distance then each E_i^* is conjugate. For the other direction, if each E_i^* is conjugate, then each E_i^* has a common witness, say z_i, by Theorem 4.3. From Claim 4.7, $d_l(E_i^*) \le 2|z_i|$. Further, $d_l(E) \le \sum_{j\in\{0\ldots k\}} d_l(\alpha_j, \beta_j) + \sum_{i\in\{1\ldots k\}} d_l(E_i^*) = \sum_{j\in\{0\ldots k\}} d_l(\alpha_j, \beta_j) + 2\sum_{i\in\{1\ldots k\}} z_i$, hence finite. This implies that if each E_i^* in E is conjugate, then $d_l(E)$ is finite. ◁

Therefore, checking the closeness of a rational expression w.r.t. Levenshtein distances reduces to checking the existence of a common witness of each Kleene star in its sumfree constituents, and thus decidable. ◀

For a sumfree rational expression, a witness, if exists, can be computed in polynomial time [3], and thus closeness w.r.t. Levenshtein and conjugacy distances are decidable in polynomial time. However, converting a rational expression to a sum of sumfree rational expressions can cause an exponential blow-up both in the number of summands and the size of each summand [3].

4.2 Closeness w.r.t. Hamming and Transposition distances

▶ **Theorem 4.10.** *Closeness w.r.t. Hamming and Transposition distance are decidable for functional transducers.*

Given two functional transducers, check if their domains are the same. If not, the distance is ∞ hence they are not close. Assume they have an identical domain. By Proposition 3.9, it suffices to consider two sequential transducers with a common underlying DFA. Let $\mathcal{T}_1 = \langle \mathcal{A}, \lambda_1, o_1 \rangle$ and $\mathcal{T}_2 = \langle \mathcal{A}, \lambda_2, o_2 \rangle$ be two sequential transducers. WLOG, we make the following assumptions.

1. (Property $\star$) Automaton $\mathcal{A}$ is trimmed, i.e., all states are accessible (reachable from the initial state) and coaccessible (from each state there is a path to some final state).
2. (Property ‡) $\mathcal{T}_1$ and $\mathcal{T}_2$ produce output words of identical length; otherwise the Hamming as well as transposition distance will be ∞. We can check this property: rename all the output letters in $\mathcal{T}_1$ and $\mathcal{T}_2$ to a and check their equivalence.
3. The delay between partial outputs of $\mathcal{T}_1$ and $\mathcal{T}_2$ is at most $k \in \mathbb{N}$ (By Proposition 3.7).

Let Q and $F \subseteq Q$ be the set of states and final states of $\mathcal{A}$ respectively, and let $q_0 \in Q$ be the initial state. For states $p, q \in Q$, Let $M_{p,q}$ be the set of pairs (u, v) such that there is a run ρ from p to q and $u = \lambda_1(\rho)$ and $v = \lambda_2(\rho)$. Extending this notation, for a state $q_f \in F$, let M'_{q,q_f} be the set of pairs (u, v) such that $u = u' \cdot o_1(q_f), v = v' \cdot o_2(q_f)$ and $(u', v') \in M_{q,q_f}$.

Let q be a state of the automaton. If (α, β) and (α', β') are two pairs in $M_{q_0,q}$, then $|\alpha| - |\beta| = |\alpha'| - |\beta'|$, or else one of the pairs in $\{(\alpha\alpha'', \beta\beta''), (\alpha'\alpha'', \beta'\beta'')\}$ will have different lengths, where (α'', β'') is some pair in M_{q,q_f}, for some $q_f \in F$, guaranteed by Property ($\star$).

Therefore with each state q, we can associate the delay of a run reaching it, called the *delay at* q, denoted by ∂_q, as $|\alpha| - |\beta|$. Clearly $\partial_q \leq k$. By a symmetric argument, if (α, β) and (α', β') are two pairs in M_{q,q_f}, where q_f is some final state, then $|\alpha| - |\beta| = |\alpha'| - |\beta'| = -\partial_q$. This also implies that for all $(u, v) \in M_{q,q}$, $|u| = |v|$.

For each state q, either $M_{q,q} = \{(\epsilon, \epsilon)\}$, or $M_{q,q}$ is infinite. Let q be a state for which $M_{q,q}$ is nonempty. For a delay $\partial \in \mathbb{Z}$, a pair $(u, v) \in M_{q,q}$ where $n = |u| > \partial$, we define the interior of the pair (u, v) as

$$\mathit{interior}_\partial(u, v) = \begin{cases} (u[1 \ldots n - \partial], v[\partial + 1 \ldots n]) & \text{if } \partial \geq 0 \\ (u[\partial + 1 \ldots n], v[1 \ldots n - \partial]) & \text{if } \partial < 0 \end{cases}$$

For example, $\mathit{interior}_1(abc, def) = (ab, ef)$ and $\mathit{interior}_{-1}(abc, def) = (bc, de)$. We also define the *Left-Border* and *Right-Border* of the pair (u, v) as

$$\mathit{lborder}_\partial(u, v) = \begin{cases} v[1 \ldots \partial] & \text{if } \partial \geq 0 \\ u[1 \ldots \partial] & \text{if } \partial < 0 \end{cases} \qquad \mathit{rborder}_\partial(u, v) = \begin{cases} u[n - \partial + 1 \ldots n] & \text{if } \partial \geq 0 \\ v[n - \partial + 1 \ldots n] & \text{if } \partial < 0 \end{cases}$$

▷ Claim 4.11. Hamming distance between $\mathcal{T}_1$ and $\mathcal{T}_2$ is unbounded if and only if there exists a state $q \in Q$ and $(u, v) \in M_{q,q}$ such that $|u| = |v| > \partial_q$, and $u' \neq v'$ where $(u', v') = \mathit{interior}_{\partial_q}(u, v)$.

Proof. The Figure 3 depicts the situation described by (2).

($\leftarrow$): Assume there exists a state $q \in Q$ and $(u, v) \in M_{q,q}$ such that $|u| = |v| > \partial_q$, and $u' \neq v'$ where $(u', v') = \mathit{interior}_{\partial_q}(u, v)$. Let $(\alpha_0, \beta_0) \in M_{q_0,q}$ and $(\alpha_1, \beta_1) \in M_{q,q_f}$. Consider the pair $(u_i = \alpha_0 u^i \alpha_1, v_i = \beta_0 v^i \beta_1)$, $i \geq 1$ (shown in Figure 4). Since $u' \neq v'$, we can deduce that $d_h(u_i, v_i) \geq i$. Hence $d_h(\mathcal{T}_1, \mathcal{T}_2) = \infty$.

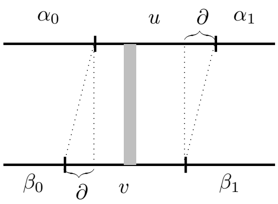


Figure 3 An edit in the interior of u and v.

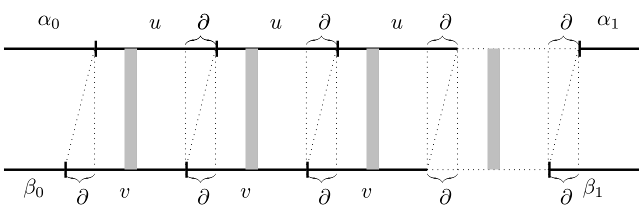


Figure 4 Words that require an arbitrarily large number of edits.

($\rightarrow$): Assume $d_h(\mathcal{T}_1, \mathcal{T}_2) = \infty$. Assume $\mathcal{A}$ has n states and the maximum length of an output produced on any transition or at the end-of-input is ℓ. Choose a run ρ of $\mathcal{A}$ such that the distance between the outputs produced on $\rho = \delta_1 \cdots \delta_m$, $m > 0$ is at least $((k+2)n+1)\ell$. We can associate each edit in $\lambda_1(\rho)$ with the transition δ_i such that the edit happens in

$\lambda_1(\delta_i)$. Since there are $((k+2)n+1)\ell$ edits, there are at least $(k+2)n+1$ transitions in ρ whose output words are edited. Associate each transition with its source state. By pigeonhole principle, there is a state q such that $\rho = \rho_1 \cdot \rho_2 \cdot \rho_3$ where

1. ρ_1 is a run from the initial state q_0 to q,
2. ρ_2 is a run from q to itself,
3. ρ_3 is a run from q to a final state q_f, and
4. there are at least $(k+1)$ edits in the factor $\lambda_1(\rho_2)$.

Let $u = \lambda_1(\rho_2)$ and $v = \lambda_2(\rho_2)$. Clearly $|u| = |v|$ and $|u| \geq (k+1)$. Since the edits in u are at least $k+1$, there is a position on which the pair $\mathit{interior}_{\partial_q}(u,v)$ differ. ◁

Next we show closeness w.r.t. transposition distance. We write $u \equiv v$ to denote that words u and v are permutations of each other. The *alphabetic vector* of a word over the alphabet A, denoted by $\vec{u}$, is the sequence $(|w|_{a_i})_{a_i \in A}$ for some fixed ordering of A. It is easy to observe that two words are permutations of each other if their alphabetic vectors are the same.

▷ Claim 4.12. Transposition distance between $\mathcal{T}_1$ and $\mathcal{T}_2$ is unbounded if and only if one of the following holds

1. There is a pair $(u,v) \in M'_{q_0,q_f}$, $q_f \in F$ such that $u \not\equiv v$.
2. There exists a state $q \in Q$ and $(u,v) \in M_{q,q}$ such that $|u| = |v| > \partial_q$, and $u' \neq v'$ where $(u',v') = \mathit{interior}_{\partial_q}(u,v)$.
3. There exists a state $q \in Q$ such that $M_{q,q}$ is infinite, and for each pair $(u,v) \in M_{q,q}$ of length at least $|\partial_q|$, $\mathit{interior}_{\partial}(u,v)$ is identical. Further, there are pairs $(u,v) \in M_{q,q}$ and $(\alpha,\beta) \in M_{q_0,q}$ (*resp.* M_{q,q_f}) such that: If $\partial_q \geq 0$, then $\alpha \not\equiv \beta \cdot \mathit{lborder}(u,v)$ (*resp.* $\mathit{rborder}(u,v) \cdot \alpha \not\equiv \beta$), and if $\partial_q < 0$, then $\alpha \cdot \mathit{lborder}(u,v) \not\equiv \beta$ (*resp.* $\alpha \not\equiv \mathit{rborder}(u,v) \cdot \beta$).

Proof. ($\leftarrow$): It is obvious that if Item 1 is true, then the transposition distance between $\mathcal{T}_1$ and $\mathcal{T}_2$ is unbounded. Therefore we assume that the output pairs of the transducers are permutations of each other. For Item 2, the proof is the same as in Claim 4.11. Next we consider Item 3. The cases are symmetric. Assume that there exist a pair $(u,v) \in M_{q,q}$, $(\alpha,\beta) \in M_{q_0,q}$, and WLOG $\partial_q \geq 0$ such that $\alpha \not\equiv \beta \cdot \mathit{lborder}(u,v)$. Let (α',β') be some pair in M_{q,q_f}. Consider the pair $(u_i = \alpha u^i \alpha', v_i = \beta v^i \beta')$, $i \geq 1$.

Let $(x,x) = \mathit{interior}_{\partial_q}(u,v)$, $z_1 = \mathit{lborder}(u,v)$, $z_2 = \mathit{rborder}(u,v)$. By assumption $\alpha \not\equiv \beta z_1$, and hence $z_2\alpha' \not\equiv \beta'$. Since interior of (u,v) is (x,x), we can deduce that $\alpha z_2 \alpha' \equiv \beta z_1 \beta'$. Therefore $\vec{\alpha} - \vec{\beta z_1} = \vec{z_2\alpha'} - \vec{\beta'}$. This means that the transpositions have to cancel out the differences in the vectors at each end of the word. We can prove by induction that it requires at least $|x|$ transpositions to mitigate a difference of 1, while keeping the alphabetic vector of the middle portion the same. Hence we deduce that $d_t(u_i,v_i) \geq i$.

($\rightarrow$): If $d_t(\mathcal{T}_1,\mathcal{T}_2) \in \infty$, either there is a pair of outputs (u,v) such that $d_t(u,v) = \infty$ (This is Item 1), or all the output pairs are permutations of each other and there is an infinite set of pairs $S = \{(u_i,v_i) \mid i > 0\}$ such that $d_t(u_i,v_i) \geq i$.

In the latter case, we show that either Item 2 or Item 3 holds. We say the set S is *error-bounded* if there is an $r > 0$ such that u_i and v_i differ in at most r positions. Clearly, there are sets with bounded errors on which d_t is infinite. We do case analysis.

If there is an infinite set of pairs $S = \{(u_i,v_i) \mid i > 0\}$ such that $d_t(u_i,v_i) \geq i$ that is *not* error-bounded, we proceed as in the proof of Claim 4.11 and obtain Item 2 by pigeonhole principle.

If the set of all output pairs is error-bounded, then clearly for states q such that $M_{q,q}$ is infinite, the interior of all the sufficiently large pairs in $M_{q,q}$ are identical. Moreover since the output pairs are permutations of each other there is a state q such that $|M_{q,q}| = \infty$ and there is a partial run from q_0 to q (or a partial run from q to q_f) whose output words are not permutations of each other. ◁

Claim 4.11 and Claim 4.12 can be verified for $\mathcal{T}_1$ and $\mathcal{T}_2$ in polynomial time. Thus, closeness of sequential and unambiguous transducers w.r.t. hamming and transposition distance is decidable in polynomial time.

5 Discussion and Conclusion

It is shown that distance between two rational functions w.r.t. common edit distances is computable. The related notions of diameter of a rational relation, and the index of a rational relation in the composition closure of another are also computable. We leave open the question of finding the precise computational complexity of the problems in Tables 2, 3 and 4.

The current decision procedure for closeness w.r.t. conjugacy and Levenshtein family of distances proceeds through the analysis of rational expressions. One could directly work on automata, but it is not enough to check for the conjugacy of simple cycles, as there can be complex strongly connected components. In such cases, a decidability proof for conjugacy can be achieved by utilizing Simon's factorization forests [38] and checking the conjugacy of the factorization trees inductively. Sumfree expressions are doing this in essence, circumventing the need to construct the transition monoids.

Lifting these notions to infinite words, and two-way transducers is an immediate next step. Distance between one-way transducers could be seen as the diameter of a rational relation obtained by the cartesian product. However, when the transducers $\mathcal{T}, \mathcal{S}$ are two-way or polyregular, the relation $\{(\mathcal{T}(w), \mathcal{S}(w)) \mid w \in \mathit{dom}(\mathcal{T})\}$ need not be rational. It remains to develop techniques for checking the conjugacy of non-rational relations.

An interesting question is: given two functional transducers $\mathcal{T}_1$ and $\mathcal{T}_2$ with bounded distance, does there exist a transducer $\mathcal{T}$ such that $\mathcal{T}_2$ is equivalent to a cascading composition of $\mathcal{T}_1$ and $\mathcal{T}$? This is often called the *repair problem* and is well-studied between two regular languages [7].

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