


On Classes of Bounded Tree Rank, Their Interpretations, and Efficient Sparsification

Jakub Gajarský 

University of Warsaw, Poland

Rose McCarty 

School of Mathematics and School of Computer Science,
Georgia Institute of Technology, Atlanta, GA, USA

Abstract

Graph classes of bounded tree rank were introduced recently in the context of the model checking problem for first-order logic of graphs. These graph classes are a common generalization of graph classes of bounded degree and bounded treedepth, and they are a special case of graph classes of bounded expansion. We introduce a notion of decomposition for these classes and show that these decompositions can be efficiently computed. Also, a natural extension of our decomposition leads to a new characterization and decomposition for graph classes of bounded expansion (and an efficient algorithm computing this decomposition).

We then focus on interpretations of graph classes of bounded tree rank. We give a characterization of graph classes interpretable in graph classes of tree rank 2. Importantly, our characterization leads to an efficient sparsification procedure: For any graph class \mathcal{C} interpretable in a graph class of tree rank at most 2, there is a polynomial time algorithm that to any $G \in \mathcal{C}$ computes a (sparse) graph H from a fixed graph class of tree rank at most 2 such that $G = I(H)$ for a fixed interpretation I . To the best of our knowledge, this is the first efficient “interpretation reversal” result that generalizes the result of Gajarský et al. [LICS 2016], who showed an analogous result for graph classes interpretable in classes of graphs of bounded degree.

2012 ACM Subject Classification Theory of computation → Finite Model Theory; Theory of computation → Fixed parameter tractability; Mathematics of computing → Graph theory

Keywords and phrases First-order model checking, structural graph theory, structural sparsity

Digital Object Identifier 10.4230/LIPIcs.ICALP.2024.137

Category Track B: Automata, Logic, Semantics, and Theory of Programming

Related Version *Full Version:* <https://arxiv.org/abs/2404.18904>

Funding *Jakub Gajarský:* Supported by the Polish National Science Centre SONATA-18 grant number 2022/47/D/ST6/03421. Parts of this work were developed while this author received funding from the European Research Council (ERC), grant agreement No 948057 – BOBR.

Rose McCarty: Supported by the National Science Foundation under Grant No. DMS-2202961.

1 Introduction

The graph classes and problems studied in this paper are motivated by considering the first-order (FO) model checking problem for graphs. This problem asks, given a (finite) graph G and sentence φ as input, whether G is a model of φ . This problem is known to be PSPACE-hard, and so we do not expect to obtain a polynomial algorithm solving it. This has motivated the study of the FO model checking problem from the perspective of parameterized complexity, which has led to the discovery of many beautiful connections between structural and algorithmic graph theory and (finite) model theory.

In the parameterized setting (where we consider the size of the formula φ as the parameter), we can easily obtain a brute-force algorithm with runtime $n^{O(|\varphi|)}$, the so-called XP algorithm. However, we are interested in the existence of algorithms with runtime $f(|\varphi|) \cdot n^c$, where c



© Jakub Gajarský and Rose McCarty;

licensed under Creative Commons License CC-BY 4.0

51st International Colloquium on Automata, Languages, and Programming (ICALP 2024).

Editors: Karl Bringmann, Martin Grohe, Gabriele Puppis, and Ola Svensson;

Article No. 137; pp. 137:1–137:20



Leibniz International Proceedings in Informatics

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



is some fixed constant (FPT algorithms). We do not expect to obtain such an algorithm if the input graphs come from the class of all graphs, since this problem is $\text{AW}[*]$ -complete. However, for various structurally restricted graph classes we know that such an algorithm exists. Identifying these graph classes is the topic of a long line of research, and recently a lot of progress has been made towards understanding the boundaries of efficient tractability [2, 4, 5].

For graph classes that admit a model checking algorithm with runtime $f(|\varphi|) \cdot n^c$ one can ask what is the dependence of the runtime on the size of the formula φ . Unfortunately, for most graph classes this dependence is *non-elementary*, that is, $f(k)$ grows like a tower of twos whose height grows with k . By a result of Frick and Grohe [9], we know that this cannot be avoided even when \mathcal{C} is the class of all trees. While this result may seem very limiting, it turns out that the landscape of graph classes that admit an elementary model checking algorithm is surprisingly rich. Classical examples of such graph classes include classes of graphs of bounded degree [9], bounded treedepth, and bounded shrubdepth [11]. (The last two results were obtained in the more general context of MSO model checking.) The interest in this problem was renewed recently when Lampis [15] established elementary model checking for graph classes of bounded pathwidth. After this, Gajarský, Pilipczuk, Sokolowski, Stamoulis and Toruńczyk [13] introduced graph classes of bounded tree rank and proved that these classes also admit (under certain restrictions, see the fourth bullet point below) an elementary FO model checking algorithm.

Classes of bounded tree rank, introduced in [13], can be defined as follows (in the definition, the *depth* of a tree T is the number of edges on a shortest leaf-to-root path in T).

► **Definition 1.** *A graph class \mathcal{C} has tree rank at most d if for every $r \in \mathbb{N}$ there exists a tree T of depth d such that no $G \in \mathcal{C}$ contains T as an r -shallow topological minor¹.*

For example, one can easily see that graph classes of bounded degree have tree rank at most 1, that the class of all trees of height d has tree rank d , and that the class of all trees does not have bounded tree rank.

We briefly summarize some of the basic properties of graph classes of bounded tree rank established in [13]:

- They generalize graph classes of bounded degree, bounded treedepth, and bounded pathwidth.
- They are strictly less general than classes of graphs of bounded expansion.
- The tree rank of a graph class \mathcal{C} is at most d if and only if Splitter has a winning strategy in the so-called “ d -round batched Splitter game” for every $G \in \mathcal{C}$. This game is a natural variant of the Splitter game, which characterizes nowhere dense graph classes and was introduced by Grohe, Kreutzer, and Siebertz in their landmark paper [14].
- If \mathcal{C} is a class of bounded tree rank and there exists an elementary function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that the size of T in Definition 1 is bounded by $f(r)$, then there is an elementary FO model checking algorithm for \mathcal{C} .
- If \mathcal{C} is a monotone graph class of unbounded tree rank, then \mathcal{C} has no FO model checking algorithm with elementary dependence on the size of the formula unless $\text{FPT} = \text{AW}[*]$.

We note that the concepts considered above (bounded degree, bounded treedepth, bounded tree rank, bounded expansion, and nowhere denseness) are all examples of classes of *sparse* graphs. Given how naturally the concept of bounded tree rank fits into the general theory of

¹ We note that this definition of tree rank is seemingly off by 1 from the original definition in [13]. However, we measure the depth of trees differently than in [13], and so the two definitions ultimately coincide.

sparse graphs introduced by Nešetřil and Ossona de Mendez [17], we suspect that it will play an important role in structural and algorithmic graph theory, and that further investigation of its structural and combinatorial properties is desirable.

Since classes of sparse graphs are mostly well-understood (one exception being classes of bounded tree rank), in recent years there has been a trend to study more general classes that can be obtained from classes of sparse graphs by means of *interpretations* or *transductions*, which are graph transformations based in logic. This point of view is also very relevant for classes of bounded tree rank, but we first discuss it in the general setting of sparse graphs. We focus on the simpler setting of interpretations. Essentially, an interpretation I is given by a formula $\psi(x, y)$ (we will actually use a slightly more complicated setting; see Section 2). When applied to a graph H , the result is a new graph $I(H)$ with the same vertex set as H and with edge set $\{uv : H \models \psi(u, v)\}$. This notion generalises easily to graph classes by setting $I(\mathcal{C}) = \{I(H) : H \in \mathcal{C}\}$. Finally, we say that a graph class \mathcal{D} is *interpretable* in a graph class \mathcal{C} if there exists an interpretation I such that $\mathcal{D} \subseteq I(\mathcal{C})$.

As mentioned above, the recent trend is to study graph classes interpretable in sparse (or nowhere dense) classes of graphs. In particular, the model checking problem has been considered extensively, and was recently fully solved for such graph classes.

► **Theorem 2** ([5]). *Let \mathcal{C} be a graph class interpretable in a nowhere dense graph class. Then there exists an FPT model checking algorithm for \mathcal{C} .*

In the context of graph classes interpretable in classes of sparse graphs, it is also natural consider the following problem, which we refer to as the “efficient sparsification problem” or “interpretation reversal problem”. This problem was considered in [12] in the context of graph classes interpretable in classes of graphs of bounded degree.

► **Problem 1.** *Let \mathcal{C} be a graph class interpretable in a nowhere dense graph class. Show that there exists a nowhere dense graph class \mathcal{D} , an interpretation I , and a polynomial time algorithm that given $G \in \mathcal{C}$ as input computes a graph $H \in \mathcal{D}$ such that $G = I(H)$.*

It is well-known (and easy to argue, see for instance [12]) that solving Problem 1 would lead to an alternative proof of Theorem 2. Indeed, before the result of [5] was established, this was considered the main line of attack on the model checking problem on graph class interpretable in a nowhere dense graph class. However, Problem 1 turned out to be very challenging, and despite considerable effort essentially no success has been achieved in solving it. Instead, Theorem 2 was proved by adapting the techniques for classes of sparse graphs from [14] to the dense setting. Despite this, Problem 1 remains of considerable interest, and solving it would likely lead to many new insights on the structure of graph classes interpretable in classes of sparse graphs.

Coming back to classes of bounded tree rank, it is natural to try to obtain an elementary analogue of Theorem 2.

► **Problem 2.** *Let \mathcal{C} be a graph class interpretable in a graph class of bounded tree rank. Show that there exists an elementary FPT model checking algorithm for \mathcal{C} .*

In light of the results of [13], it is possible that one needs to put some extra restriction on \mathcal{C} – perhaps requiring that the size of trees avoided as r -shallow topological minors is bounded by an elementary function of r . However, it is currently unknown whether this is necessary, even for the original theorem from [13].

In relation to interpretations of graph classes of bounded tree rank, the authors of [13] introduced the more general graph classes of bounded *rank*, and conjectured that these are precisely the interpretations of colored graph classes of bounded tree rank. In order to attack this conjecture, or Problem 2, it is natural to consider the following variant of Problem 1.

► **Problem 3.** *Let \mathcal{C} be a graph class interpretable in a graph class of bounded tree rank. Show that there exists a graph class \mathcal{D} of bounded tree rank, an interpretation I , and a polynomial time algorithm that given $G \in \mathcal{C}$ as input computes a graph $H \in \mathcal{D}$ such that $G = I(H)$.*

Our contribution

For sparse graphs we introduce the notion of the (r, m) -rank of a vertex. Roughly speaking, the (r, m) -rank of a vertex $v \in V(G)$ is a positive integer that measures how complicated the r -neighborhood of v is with respect to the parameter m . We also allow the (r, m) -rank to be ∞ if the r -neighborhood of v is too complicated. Then the (r, m) -ranking of a graph G is the function $f : V(G) \rightarrow \mathbb{N} \cup \{\infty\}$ that assigns to each vertex of G its (r, m) -rank. This definition has the advantage of being much more localized than Definition 1 of tree rank. Yet we prove that it can also be used to define tree rank, as follows.

- **Theorem 3.** *For any $d \in \mathbb{N}$ and any graph class \mathcal{C} , the following are equivalent:*
- *The tree-rank of \mathcal{C} is at most d .*
 - *For every $r \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that every vertex of every graph in \mathcal{C} has (r, m) -rank at most d .*

Vertex rankings provide a natural notion of decomposition for graph classes of bounded tree rank. We will show that this decomposition can be computed in elementary FPT runtime with respect to the parameters r and m (see Theorem 14). Moreover, the notion of (r, m) -ranking (and the FPT algorithm computing it) very naturally extends to graph classes of bounded expansion.

- **Theorem 4.** *The following are equivalent for any graph class \mathcal{C} :*
- *The class \mathcal{C} is of bounded expansion.*
 - *For every $r \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that every vertex of every graph in \mathcal{C} has finite (r, m) -rank, that is, has (r, m) -rank not equal to ∞ .*

We note that the (r, m) -rank of a vertex is preserved under graph automorphisms. This fact that rankings are “canonical” is one of the key advantages of Theorem 4.

One of the main motivations for proving alternate characterizations of sparse graph classes (like Theorems 3 and 4) is the efficient sparsification problems discussed earlier (Problems 1 and 3). In particular, we hope that this definition of (r, m) -rank can be generalized to accommodate for interpretations of classes of bounded expansion. While we cannot yet take care of this more general case, we are able to use the insights obtained from (r, m) -rank to find the following structural characterization of interpretations of classes of tree rank 2.

- **Theorem 5.** *The following are equivalent for any graph class \mathcal{C} :*
- *The class \mathcal{C} is interpretable in a graph class of tree rank 2.*
 - *The class \mathcal{C} is a perturbation of a locally almost near-covered graph class.*

We defer the definition of locally almost near-covered graph classes to Section 5. Our characterization leads to the following algorithmic sparsification result. For technical reasons we restrict ourselves to classes of graphs interpretable in classes of tree rank 2 for which the bounds in the definition of tree rank can be efficiently computed. We call such classes *efficiently bounded* (see Section 5.1) for precise definition.

- **Theorem 6.** *Let \mathcal{C} be a graph class interpretable in an efficiently bounded graph class of tree rank 2. Then there exists a graph class \mathcal{D} of tree rank 2, an interpretation I , and a polynomial time algorithm that given $G \in \mathcal{C}$ as input computes a graph $H \in \mathcal{D}$ such that $G = I(H)$.*

We remark that the degree of the polynomial in Theorem 6 depends on the class \mathcal{C} .

While at first sight Theorem 6 might seem like a modest improvement on the analogous result of [12] for graph classes of bounded degree (or equivalently, classes of tree rank at most 1), this is the first progress on the challenging Problem 1 since its introduction that applies to graph classes of unbounded treewidth. (For graph classes of bounded treewidth such a result is given in [18].) Moreover, our proof introduces new techniques and ideas that may be useful for solving the sparsification problem (Problem 1) in greater generality.

In particular, as part of the proof of Theorem 5 we prove a lemma (Lemma 23) about the behaviour of the k -near-twin relation on graphs that do not contain a half-graph as a semi-induced subgraph. This lemma may be of independent interest in the context of (monadically) stable graph classes, which play a prominent role in recent developments establishing connections between (finite) model theory and algorithmic graph theory [4, 6].

Related work

In [13] it was shown that classes of tree rank at most d can be characterized using the *batched Splitter game* with d rounds. Roughly speaking, in this game two players take turns, one of them trying to simplify the graph, and the other trying to keep the graph as complicated as possible. The result of [13] states that the player trying to simplify the graph (this player is called Splitter) wins in at most d rounds.

While this characterization is very nice and useful, it is a characterization in terms of a dynamic process, and as such it does not directly provide us with a useful notion of decomposition (in the sense of giving us a concrete compact object on which one can design algorithms). One could of course consider using as a decomposition the game tree arising from a winning play by Splitter, but this tree has size of order n^d . It is known that this can be circumvented by combining the Splitter game with *sparse neighborhood covers* as introduced in [14], but working with the resulting object is technically demanding. Compared to this, vertex rankings give us a static decomposition on which one can use bottom-up inductive arguments and can design algorithms.

For graph classes of bounded expansion, vertex rankings of a graph (which certify that it has only vertices of finite rank) are closely related to strong coloring orders [23] and admissibility orders [7]. These are total orders on the vertex set of a graph that satisfy certain properties, and which have proven to be very useful for showing properties of graphs of bounded expansion. The main idea used in the definition of rankings, when one checks whether a removal of a small set of vertices can separate a vertex v from the set of previously processed vertices, has been used before (see Appendix A1 in [22]). This was again in the context of defining a suitable total order on the vertex set of a graph. Compared to total orders, our definition of rankings essentially leads to a pre-order on the vertex set of a graph, and therefore it can capture situations when two vertices are equally complicated (have the same rank).

Regarding the sparsification problem, in [12] it was shown that there is a polynomial time algorithm for sparsifying graphs from graph classes that are interpretable in classes of bounded degree. Another result related to efficient sparsification is [8], where the authors showed how to efficiently interpret the class of map graphs in a nowhere dense class of graphs. Finally, the already mentioned result [18] establishes a sparsification result for graph classes interpretable in graph classes of bounded treewidth, extending earlier results in [19].

Outline of our approach

We now briefly outline the approach used to prove Theorems 5 and 6. This approach builds on the ideas used in [12] to sparsify classes of graphs interpretable in graph classes of bounded degree. The key notion behind this result was that of *k-near-twin* vertices. We say that

two vertices u, v of G are k -near-twins if $|N^G(v) \Delta N^G(u)| \leq k$, i.e. if they have the same neighborhoods with at most k exceptions. The idea behind the proof given in [12] is that graphs from graph classes interpretable in classes of bounded degree have a simple structure – for such graphs we can find a small k such that the k -near-twin relation is an equivalence on $V(G)$ with a bounded number of classes. This is then easily used to find a bounded number of *flips* (edge complementations between two subsets of $V(G)$) to produce a sparse graph H from which G can be recovered by an interpretation.

In the case of classes of graphs interpretable in graph classes of tree rank 2, the structure of the k -near-twin relation is much more complicated. In what follows we will actually focus on analysing this relation on graphs from graph classes interpreted in classes of tree rank 2 by an *interpretation of bounded range*. This is an interpretation in which the formula $\psi(x, y)$ has the property that there is a number b such that for every G and $u, v \in V(G)$ we have that if $\text{dist}_G(u, v) > b$, then $G \not\models \psi(u, v)$. In other words, such interpretation will never create edges between vertices that are far apart in G . The case of interpretations of bounded range forms the technical core of our approach, since the reduction from the general case to the case of bounded range can be achieved by using existing tools (see Section 5.3). For the rest of this overview, let us fix a graph class \mathcal{C} interpretable in a class of graphs of tree rank at most 2 by an interpretation of bounded range.

We now proceed with analysing the k -near-twin relation on a graph G from \mathcal{C} . It is useful to think of this relation in terms of the k -near-twin graph of G , denoted by $NT_k(G)$. This graph has the same vertex set as G , and two vertices are adjacent in $NT_k(G)$ if they are k -near-twins in G . In the case of interpretations of bounded degree, this graph was a collection of a small number of cliques. In our case of interpretations of classes of tree rank 2, the connected components of the graph $NT_k(G)$ are not cliques, and there can be arbitrarily many of them. Moreover, the connected components of $NT_k(G)$ can have arbitrarily large diameter – this is important because if we could find a bound d such that the diameter of all connected components of $NT_k(G)$ was at most d , then all vertices in any component C would be kd -near-twins, and this could be exploited for sparsification. In our proofs of Theorems 5 and 6 we overcome all these difficulties. The key insight (Lemma 23) is that even though the connected components of $NT_k(G)$ can have arbitrarily large diameter, we can nevertheless guarantee that any two vertices in the same connected component are k' -near-twins, for k' depending only on k and the order of largest half-graph in G . Using this, we sparsify G as follows: For a suitably chosen k , we consider $NT_k(G)$ and create a partition \mathcal{F} of $V(G)$ by putting two vertices in the same part if they are in the same connected component of $NT_k(G)$. The aforementioned Lemma 23 then guarantees that the vertices in the same part A of \mathcal{F} are pairwise k' -near-twins. We then create a sparse graph $\mathcal{S}(G)$ from G as follows: If A and B are two large parts of \mathcal{F} such that there are almost all edges between A and B (see the proof of Lemma 33 for precise meaning of “large” and “almost all”), we complement the adjacency between them, create new vertices v_A and v_B adjacent to all vertices of A and B , respectively, and create an edge between v_A and v_B . The introduction of new vertices v_A, v_B guarantees that we can recover G from $\mathcal{S}(G)$ by a simple interpretation. The technical part of the proof is establishing that $\mathcal{S}(G)$ comes from a fixed class \mathcal{D} of graphs of tree rank 2 which depends only on \mathcal{C} .

Organisation of the paper

After preliminaries in the next section, we introduce vertex rankings in Section 3, and then we relate them to graph classes of bounded tree rank and show how to compute them in FPT runtime. In Section 4 we prove a lemma about the behaviour of the k -near-twin relation in

graphs excluding arbitrarily large half-graphs; a crucial tool for the last section. In Section 5 we prove our main results – characterization and a sparsification algorithm for graph classes interpretable in classes of tree rank 2.

Due to space restrictions, the proofs of some statements could not be included in the conference version of the paper. Such statements are marked with [*] and their proofs can be found in the full version of the paper.

2 Preliminaries

Graph theory. We use $[k]$ to denote the set $\{1, \dots, k\}$. We use mostly standard graph theoretic notation. Let G be a graph. We write $N_r^G(v)$ for the *closed* r -neighborhood of a vertex v , that is, $N_r^G(v)$ is set of all vertices that are reachable from v by a path with at most r edges, including v . The *distance* between vertices u and v , denoted by $\text{dist}_G(u, v)$, is the minimum number of edges in a path between u and v . The *radius* of a graph is smallest integer r so that there exists a vertex that has distance at most r from every other vertex. Given a graph G and a set $X \subseteq V(G)$, we write $G[X]$ for the subgraph of G induced on X , and $G - X$ for the subgraph of G induced on $V(G) \setminus X$.

By a *tree* we mean a connected acyclic graph with a specified root vertex. The *depth* (respectively, *height*) of a tree T is the maximum number of edges (respectively, vertices) on any leaf-to-root path in T .

Let G be a graph, and let A and B be subsets of $V(G)$ with $A \cap B = \emptyset$ or $A = B$. By *flipping the edges between A and B* , we mean removing all edges uv in G with $u \in A$ and $v \in B$, and adding all new edges of the form uv where $u \in A$, $v \in B$, $u \neq v$, and $uv \notin E(G)$. For $k \in \mathbb{N}$ and a graph G , a *k -flip* of G is any graph that can be obtained from G by considering a partition \mathcal{F} of $V(G)$ with $|\mathcal{F}| \leq k$, and flipping the edges between some pairs of parts of \mathcal{F} (that is, for each pair of parts A and B of \mathcal{F} , we may choose whether or not to flip the edges between A and B). We say that a graph class \mathcal{C} is a *perturbation* of a graph class \mathcal{D} if there exists k such that every $G \in \mathcal{C}$ is a k -flip of some $H \in \mathcal{D}$.

Let S be a set of vertices of a graph G . Let \mathcal{F}_S be the partition of $V(G)$ such that each $v \in S$ is in its own part and all vertices in $V(G) \setminus S$ are partitioned according to their adjacency to S (so vertices with the same neighbors in S are in the same part). An *S -flip* of G is any graph G' obtained by flipping the edges between some pairs of parts of \mathcal{F}_S .

Shallow topological minors and bounded tree rank. Let H be a graph. An *$\leq r$ -subdivision* of H is any graph that can be obtained from H by replacing each edge uv of H by a path with endpoints u and v and with at most r internal vertices (so that all of the paths are internally disjoint). We call the original vertices of H the *principal* vertices of the $\leq r$ -subdivision. We say that H is an *r -shallow topological minor*² of a graph G if G contains a subgraph that is isomorphic to an $\leq r$ -subdivision of H .

► **Definition 7.** We define $T_{d,m}$ to be the tree of depth d in which every non-leaf vertex has exactly m children.

It is easily seen that the definition of graph classes of bounded tree rank given in Definition 1 is equivalent to the following:

² We note that this definition differs slightly from the standard definition, which says that H is a r -shallow minor of G if there is a subgraph of G isomorphic to a graph obtained from H by subdividing its edges at most $2r$ times.

► **Definition 8.** A graph class \mathcal{C} has tree rank at most d if for every $r \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that no $G \in \mathcal{C}$ contains $T_{d,m}$ as an r -shallow topological minor.

Strong coloring numbers, admissibility, and bounded expansion. Let G be a graph, and let \leq be an order on its vertex set. Fix a number $r \in \mathbb{N}$. For two vertices v and w of G , we say that w is *strongly r -reachable*³ from v (with respect to \leq) if $w \geq v$ and there is a path from v to w of length at most r in G such that all vertices on this path apart from v and w are smaller than v in \leq . The *strong r -coloring number* of G , denoted by $\text{scol}_r(G)$, is the minimum over all orderings \leq of $V(G)$, of the maximum number of vertices that are strongly r -reachable from a single vertex v of G .

Similarly as above, let G be a graph, and let \leq be an order on its vertex set. Fix a number $r \in \mathbb{N}$. The *r -backconnectivity* of a vertex v of G is the maximum number of paths of length at most r in G that start in v , end at vertices $w \geq v$, and are vertex-disjoint except for their common endpoint v . The *r -admissibility* of G , denoted by $\text{adm}_r(G)$, is the minimum over all orderings \leq of $V(G)$, of the maximum r -backconnectivity of a vertex v of G .

Classes of graphs of *bounded expansion* were introduced by Nešetřil and Ossona de Mendez as one of the key notions of their general theory of sparsity [17]. We do not provide the original definition of bounded expansion; instead we use the following characterizations.

► **Theorem 9** ([23]). A class \mathcal{C} of graphs has bounded expansion if and only if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $G \in \mathcal{C}$ and $r \in \mathbb{N}$, we have $\text{scol}_r(G) \leq f(r)$.

► **Theorem 10** ([7]). A class \mathcal{C} of graphs has bounded expansion if and only if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $G \in \mathcal{C}$ and $r \in \mathbb{N}$, we have $\text{adm}_r(G) \leq f(r)$.

Logic, interpretations and locality. We assume familiarity with first-order logic and basic notions related to it, such as signatures, quantifier rank, and so on. We model graphs as a structure with one binary irreflexive symmetric relation E . We work with colored graphs, and we model colors as unary predicates.

Interpretations are logic-based transformations that allow us to create new structures from old ones. A (simple) *interpretation* $I = (\psi, \delta)$ consists of two formulas $\psi(x, y)$ and $\delta(x)$. When applied to a graph G , an interpretation defines a new graph $I(G)$ with $V(I(G)) = \{v \in V(G) : G \models \delta(v)\}$ and $E(I(G)) = \{uv : u, v \in V(I(G)), u \neq v, \text{ and } G \models \psi(u, v)\}$. (Here we assume that $\psi(x, y)$ is symmetric and irreflexive, that is, for all G and $u, v \in V(G)$ we have $G \models \psi(u, v)$ if and only if $G \models \psi(v, u)$, and also for each $v \in V(G)$ we have that $G \not\models \psi(v, v)$ so that the resulting graph is undirected and does not contain loops.)

For a less general version of interpretation that uses only one formula $\psi(x, y)$, we sometimes use the notation $\psi(G)$ to denote the graph on the same vertex set as G and with $E(\psi(G)) := \{uv : u \neq v \text{ and } G \models \psi(u, v)\}$. For a graph class \mathcal{C} , we say that \mathcal{C} is *interpretable* in a graph class \mathcal{D} if there exists an interpretation I such that $\mathcal{C} \subseteq I(\mathcal{D})$.

A crucial role in this paper will be played by interpretations of bounded range. We say that a formula $\psi(x, y)$ is of *range b* if for all graphs G and all $u, v \in V(G)$ we have that $\text{dist}_G(u, v) > b$ implies $G \not\models \psi(u, v)$. This means that if the formula ψ is used in an interpretation, it will not create edges between vertices that were at distance more than b in the original graph. We say that an interpretation $I = (\psi, \delta)$ is of *bounded range* if there exists b such that ψ is of range b .

³ We remark that many authors use the order \leq in the opposite direction in the definition of strong reachability, that is, they require $w \leq v$ and that the path from v to w goes through vertices larger than v in \leq . The definition we chose will be convenient later on.

When working with graphs, we often mark vertices of a graph G with (new) unary predicates and call the resulting graph \widehat{G} . Formally, this means that we are extending the signature of G , and that the formulas that we later evaluate on \widehat{G} are assumed to be over this new signature. To make the exposition more streamlined, we always do this implicitly.

Locality-based methods are commonly used in relation with first-order logic; probably the most commonly used such result is Gaifman's locality theorem [10]. We do not need the full statement of Gaifman's theorem, only the following lemma, which is its simple corollary.

► **Lemma 11.** *For every formula $\psi(x, y)$, there exist an integer r and a formula ψ' with the following property: Every graph G can be equipped with two unary predicates to obtain \widehat{G} such that for any $u, v \in V(G)$ and any $S \subseteq V(G)$ that contains $N_r^G(u) \cup N_r^G(v)$, we have $G \models \psi(u, v) \iff \widehat{G}[S] \models \psi'(u, v)$.*

For completeness we explain how Lemma 11 follows from Gaifman's theorem. This theorem tells us that for a given formula $\psi(x, y)$ there exists another formula $\alpha(x, y)$ and sentences τ_1, \dots, τ_k such that $\psi(x, y)$ can be written as a boolean combination of $\alpha(x, y), \tau_1, \dots, \tau_k$. Importantly, the formula $\alpha(x, y)$ has the property that there exists r such that for any G and any $u, v \in V(G)$ we have $G \models \alpha(u, v)$ if and only if $G[N_r^G(u) \cup N_r^G(v)] \models \alpha(u, v)$. For any G we can evaluate all sentences τ_1, \dots, τ_k on G , and then the boolean combination of $\alpha(x, y), \tau_1, \dots, \tau_k$ reduces to one of four possibilities on G – we have that $\psi(x, y)$ is equivalent one of the following: $\alpha(x, y), \neg\alpha(x, y), \top$ or \perp . We can encode these four options with two bits of information, and so we mark all vertices of G with two unary predicates (all vertices in the same way) accordingly. Finally, we define formula $\psi'(x, y)$ to be the formula which first checks the unary predicates on vertex x to determine which of the four possibilities $\varphi(x, y)$ is equivalent to on G , and based on this “outputs” the value of $\alpha(x, y), \neg\alpha(x, y), \top$ or \perp . To finish the argument, we argue that evaluating $\alpha(x, y)$ on $\widehat{G}[S]$ gives the same answer as evaluating $\alpha(x, y)$ on G for any u, v, S such that $N_r^G(u) \cup N_r^G(v) \subseteq S$. This follows from the properties of α (which hold for any graph, including $\widehat{G}[S]$) which guarantee that $\widehat{G}[S] \models \alpha(u, v) \iff G[N_r^G(u) \cup N_r^G(v)] \models \alpha(u, v) \iff G \models \alpha(u, v)$.

3 Vertex rankings in sparse graphs

In this section we introduce a way to assign, for given parameters r and m , to every vertex of a graph its (r, m) -rank. Intuitively, the (r, m) -rank of a vertex measures how complicated the r -neighborhood of v is. Our definition is algorithmic and is given in Section 3.1 together with the proof that the (r, m) -rank can be computed in FPT runtime with respect to the parameters r and m . Then we prove Theorems 3 and 4 relating graph classes for which suitable rankings of vertices exist to graph classes of bounded tree rank (Section 3.2) and bounded expansion (Section 3.3).

3.1 The ranking algorithm

We now describe the ranking algorithm that (based on the parameters r and m) for every graph G assigns to every vertex of G either a positive integer or ∞ .

Let G be a graph and r, m be positive integers. The (r, m) -ranking algorithm works as follows. Initially, each vertex is assigned rank ∞ . Then the algorithm proceeds in rounds for $i = 1, 2, 3, \dots$ as follows. In the i -th round, the algorithm considers all vertices of rank ∞ (these are the vertices that have not received a finite rank in rounds $1, \dots, i - 1$), and for each such vertex v it checks (in parallel) the following condition: Does there exist a set $S \subseteq V(G) \setminus \{v\}$ of size at most m such that $N_r^{G-S}(v) \setminus \{v\}$ contains only vertices of finite

rank? If yes, then v receives rank i , otherwise it keeps rank ∞ . (Notice that in the first round the algorithm assigns rank 1 to vertices of degree at most m .) The algorithm stops when all vertices obtain a finite rank or when in some round no new vertex receives a rank. Thus, in any case, the algorithm stops after at most $|V(G)|$ rounds.

► **Definition 12.** *Let G be a graph. The (r, m) -rank of a vertex $v \in V(G)$ is the element of $\mathbb{N} \cup \{\infty\}$ assigned to v by the ranking algorithm. The (r, m) -rank of G is the maximum (r, m) -rank of any vertex of G .*

We remark that the definition of (r, m) -ranking of vertices in a graph G can be phrased without any explicit mention of the ranking algorithm: All vertices of degree at most m are assigned rank 1, and for every vertex v of degree more than m we define the (r, m) -rank of v to be the minimum number k such that there exists a set S of vertices so that $N_r^{G-S}(v) \setminus \{v\}$ contains only vertices of smaller rank; or ∞ if such k does not exist. This is easily seen to be equivalent to the algorithmic definition given above, but we prefer to stick with the algorithmic perspective in what follows.

3.1.1 Algorithmic considerations

Notice that the straightforward implementation of the ranking algorithm runs in time $\mathcal{O}(n^{m+4})$: we have at most n rounds, in each round we consider at most n vertices, and for each vertex we can in time $\mathcal{O}(n^{m+2})$ try removing all sets S of size m in time $\mathcal{O}(n^m)$ and check whether $G - S$ satisfies the desired condition in time $\mathcal{O}(n^2)$. Thus, we trivially get an XP algorithm. However, we can replace the subroutine which checks whether there exists a set S with $|S| \leq m$ with desired properties by a simple branch-and-bound procedure with FPT runtime.

► **Lemma 13.** *There is an algorithm with runtime $\mathcal{O}(r^m \cdot n^2)$ which correctly decides the following problem: Given G , $v \in V(G)$, $A \subseteq V(G) \setminus \{v\}$ and $r, m \in \mathbb{N}$ as input, decide whether there exist a set of set $S \subseteq V(G) \setminus \{v\}$ with $|S| \leq m$ such that $N_r^{G-S}(v) \cap A = \emptyset$.*

Proof. The algorithm checks whether there is a path of length at most r from v to any vertex in A . If no such path exists, the algorithm outputs YES. If such a path P exists and $m = 0$, the algorithm outputs NO. If $m > 0$ then at least one of the vertices on the path P has to be in the solution S , and so we can branch on all vertices $u \in V(P) \setminus \{v\}$ (there are at most r of them) and call the algorithm with input $G - \{u\}, v, A \setminus \{u\}, r, m - 1$. Then the algorithm returns YES if for at least one such u the recursive call returned YES, and returns NO otherwise.

We thus get an algorithm with branching bounded by r and depth bounded by m , and finding the path from v to A takes time $\mathcal{O}(n^2)$, so the claimed runtime follows. ◀

As an immediate consequence of the lemma we get the following.

► **Theorem 14.** *The (r, m) -ranking of any graph G can be computed in time $\mathcal{O}(r^m \cdot n^4)$.*

3.2 Vertex rankings and bounded tree rank

In this section we prove Theorem 3. The proof is split into Lemmas 15 and 16, which correspond to the two directions of the theorem.

► **Lemma 15.** *Let G be a graph that contains $T_{d, m+1}$ as an r -shallow topological minor. Then G has (r, m) -rank more than d .*

Proof. Let T be a subgraph of G that corresponds to an $\leq r$ -subdivision of $T_{d,m+1}$. We prove that the principal vertices of T of height i (where the height is measured in $T_{d,m+1}$, and we think of leaves as having height 0) have (r, m) -rank at least $i + 1$ in G . For $i = 0$, the leaves of T clearly have rank at least 1. For $i > 1$, let v be a vertex of T corresponding to a vertex of height i in $T_{d,m+1}$. Then there are $m + 1$ internally disjoint paths connecting v to the corresponding $m + 1$ children that have rank at least i . Therefore we cannot disconnect v from all of these children by deleting m vertices other than v . Thus v has (r, m) -rank at least $i + 1$, which completes the proof. ◀

► **Lemma 16.** *Fix any $d, r \in \mathbb{N}$. For every m there exists $m' = m'(d, r, m)$ such that every graph G with (r, m') -rank more than d contains $T_{d,m}$ as an r -shallow topological minor.*

Proof. We will prove the lemma by induction on d . Actually, we will prove a slightly stronger statement – that for a suitably defined m' we have that every vertex of (r, m') -rank more than d is the root of an $\leq r$ -subdivision of $T_{d,m}$ in G .

For $d = 1$ we can set $m' = m - 1$. Then if there exists a vertex of (r, m') -rank more than 1 in G , this vertex has at least m neighbors, and thus is the root of a $T_{1,m}$ subgraph in G . For $d > 1$ we proceed as follows. For a given m , we wish to define a suitable m' . First, let $W_{d,m,r}$ denote the number of vertices of the graph obtained from $T_{d-1,m}$ by subdividing each edge r times. For convenience, set $M = m \cdot W_{d,m,r} + r \cdot m + m$. Let m'' be the number obtained by the inductive assumption applied to $d - 1, r$, and M . Then we define $m' := \max\{m'', r \cdot m\}$. Let v be a vertex of (r, m') -rank more than d in G . Then for every set $S \subseteq V(G) \setminus \{v\}$ of size at most m' , the set $N_r^{G-S}(v) \setminus \{v\}$ contains at least one vertex of (r, m') -rank at least d .

We claim that there exist paths P_1, P_2, \dots, P_m of length at most r that are disjoint other than at v , and so that each path P_i joins v to a vertex $u_i \neq v$ that has (r, m') -rank at least d . We find the paths greedily by adding one path at a time. Suppose that so far we have found the paths P_1, \dots, P_j for some $j < m$. Let S be the set of all vertices $u \neq v$ that are in any of the paths P_1, \dots, P_j . This set S contains at most r vertices from each of the paths, and thus S has size at most $r(m - 1) \leq m'$. Thus the set $N_r^{G-S}(v) \setminus \{v\}$ contains at least one vertex of (r, m') -rank at least d , and we can add another path P_{j+1} to our collection. This proves the claim. We note that this argument amounts to a Menger-type result about short paths, and this type of result has previously appeared in [16].

Since $m' \geq m''$, the (r, m'') -rank of each vertex of G is at least its (r, m') -rank. So by the inductive assumption applied to $d - 1, r$, and M , each vertex u_i is the root of an $\leq r$ -subdivision of $T_{d-1,M}$. Let us call this $\leq r$ -subdivision rooted at u_i by T_i .

We now greedily build up the desired r -shallow topological minor of $T_{d,m}$ rooted at v as follows. Suppose that for some $j < m$, we have found that T_1, \dots, T_j contain, respectively, subgraphs T'_1, \dots, T'_j so that

- for every $i \in \{1, \dots, j\}$, the subgraph T'_i of T_i is an $\leq r$ -subdivision of $T_{d-1,m}$ that is rooted at u_i ,
- the subgraphs T'_1, \dots, T'_j are pairwise vertex-disjoint, and
- for every $i \in \{1, \dots, j\}$, the vertex u_i is the only vertex that is in both T'_i and in any of the paths P_1, \dots, P_m .

We begin this greedy process with $j = 0$. Now suppose that it holds for some $j < m$. We will prove it for $j + 1$. This will complete the proof of Lemma 16.

So, we need to find a subgraph T'_{j+1} of T_{j+1} so that T'_{j+1} is an $\leq r$ -subdivision of $T_{d-1,m}$ that is rooted at u_{j+1} , and the only vertex in common between T'_{j+1} and any of T'_1, \dots, T'_j or P_1, \dots, P_m is u_{j+1} . Notice that each of the trees T_1, \dots, T_j has at most $W_{d,m,r}$ vertices by the definition of $W_{d,m,r}$. Moreover, each of the paths P_1, \dots, P_m contributes at most r

additional vertices. So, since $j < m$, the number of vertices in any of T'_1, \dots, T'_j or P_1, \dots, P_m is at most $m \cdot W_{d,m,r} + r \cdot m$. Recall from the definition that M is this quantity plus m . So we can just delete the branches of T_{j+1} in which any of the vertices of T'_1, \dots, T'_j or P_1, \dots, P_m , other than u_{j+1} , occur. In this manner we arrive at the desired subgraph T'_{j+1} . ◀

3.3 Vertex rankings and bounded expansion

In this section we prove Theorem 4, characterizing classes of bounded expansion using vertex rankings. The theorem follows from the two lemmas below combined with Theorems 9 and 10.

► **Lemma 17.** *Let G be a graph with $\text{scol}_r(G) = m$. Then G has finite $(r, m - 1)$ -rank.*

We stress that in Lemma 17 we are not bounding the maximum value of $(r, m - 1)$ -rank of any vertex of G , but merely claiming that the $(r, m - 1)$ -ranking algorithm will assign to each vertex of G a finite value. This value, however, can be arbitrarily large (and in particular it is not bounded in terms of r and m). This can be seen already on the class of trees – the class of all trees has bounded strong coloring numbers (one can set $m := r + 1$), but on the complete m -ary tree of depth d the $(r, m - 1)$ -ranking algorithm will need d rounds to finish.

Proof of Lemma 17. Assume for contradiction that there exists a vertex of G of $(r, m - 1)$ -rank ∞ . Fix an order \leq on $V(G)$ that certifies that $\text{scol}_r(G) = m$. Let v be the smallest vertex of G with respect to \leq that has $(r, m - 1)$ -rank ∞ . Let A be the set of all vertices below v in \leq , and let i be the maximum $(r, m - 1)$ -rank of any vertex in A . We claim that in round $(i + 1)$ of the ranking algorithm vertex v received rank (meaning it has rank $i + 1$), which is a contradiction with our assumption on v . To see this, let S be the set of all vertices that are strongly r -reachable from v in G . Note that every path of length at most r from v to a vertex $w \geq v$ must contain a vertex in S . It follows that $N_r^{G-S}(v) \setminus \{v\}$ contains only vertices of finite rank, a contradiction. ◀

Lemma 17 guarantees that the ranking algorithm succeeds on graphs with bounded strong coloring numbers. The next lemma tells us that the ranking algorithm computes a good admissibility ordering for an input graph G .

► **Lemma 18.** *Let $r, m \in \mathbb{N}$ with $r \geq 1$. Let G be a graph with finite (r, m) -rank. Then $\text{adm}_r(G) \leq m$.*

Proof. Let \leq be an ordering of $V(G)$ so that if $v \leq w$, then the (r, m) -rank of v is at most the (r, m) -rank of w . Thus we are ordering vertices of G by their rank, breaking ties arbitrarily. We claim that this ordering certifies that $\text{adm}_r(G) \leq m$. Let v be any vertex of G , and let i be its (r, m) -rank. Let S be the set of vertices certifying that the (r, m) -rank of v is i . That is, S is a subset of $V(G) \setminus \{v\}$ of size at most m so that $N_r^{G-S}(v) \setminus \{v\}$ contains only vertices of (r, m) -rank strictly less than i .

Since any vertex $w \geq v$ has (r, m) -rank at least i , any path from v that ends in a vertex larger than v with respect to the ordering must contain a vertex of S . Therefore, if we consider a collection P_1, \dots, P_ℓ of paths that maximizes the r -backconnectivity of v , then each P_i has to intersect S . Since all of these paths are disjoint except for their common end v , we get that $\text{adm}_r(G) \leq m$, as desired. ◀

4 A lemma about edge-stable graphs

In this section we state and prove a lemma (Lemma 23) about the behaviour of the so-called k -near-twin relation on graphs that do not contain large half-graphs. This lemma will be crucial in the next sections and may be of independent interest in the theory of (monadically) stable graphs [20, 21] (see also [4] for the latest important developments in this area).

First we need a few definitions. A crucial role will be played by the notion of k -near-twins.

► **Definition 19.** *Let G be a graph, and let $k \in \mathbb{N}$. We say that two vertices are k -near-twins if $|N^G(v) \Delta N^G(u)| \leq k$.*

► **Definition 20.** *Let G be a graph, and let $k \in \mathbb{N}$. The k -near-twin graph of G , denoted by $NT_k(G)$, is the graph with vertex set $V(G)$ in which there is an edge between two vertices u and v if they are k -near-twins in G .*

► **Definition 21.** *A half-graph of order t is a graph with vertex set $\{u_1, \dots, u_t, w_1, \dots, w_t\}$ such that there is an edge between u_i and w_j if and only if $i \leq j$.*

► **Definition 22.** *We say that a graph G contains a half-graph of order t as a semi-induced subgraph if there are distinct vertices $u_1, \dots, u_t, w_1, \dots, w_t$ in G such that there is an edge between u_i and w_j if and only if $i \leq j$.*

Note that in the definition we do not say anything about edges between the vertices within the set $\{w_1, \dots, w_t\}$ or edges between the vertices within the set $\{u_1, \dots, u_t\}$; the edges within these sets can be arbitrary.

Our main lemma in this section is the following.

► **Lemma 23** (*). *There is a function $h : \mathbb{N}^2 \rightarrow \mathbb{N}$ so that for any $k, t \in \mathbb{N}$, if G is a graph with no half-graph of order t as a semi-induced subgraph, and u and v are vertices in the same component of the k -near-twin graph of G , then u and v are $h(k, t)$ -near-twins in G .*

The technical proof of the lemma can be found in the full version of the paper.

5 Interpretations of graph classes of tree rank 2

In this section we prove Theorems 5 and 6. First we will establish some properties of classes of tree rank at most 2 (Section 5.1). After this, in Section 5.2, we give a characterization of graph classes interpretable in graph classes of tree rank at most 2 by interpretations of *bounded range*. This section contains the technical core of our results – the key result is Theorem 31, proof of which is split into Lemmas 32 and 33. The case of general interpretations (Theorem 5) is presented in Section 5.3; it will easily follow from results in Section 5.2. Finally, we prove our main algorithmic result, Theorem 6, in Section 5.4.

5.1 Classes of tree rank at most 2

We start with a characterization of graph classes of tree rank 2.

► **Definition 24.** *A class of graphs \mathcal{C} has locally almost⁴ bounded degree if there exist functions $f, d : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $r \in \mathbb{N}$, every $G \in \mathcal{C}$, and every $v \in V(G)$, the set $N_r^G(v)$ contains at most $f(r)$ vertices with degree larger than $d(r)$ in G .*

⁴ We remark that the adjective “almost” is sometimes used differently in context of structural graph parameters. For example, for a graph class \mathcal{C} , having almost bounded flipwidth (see [22]) means that for every ϵ there exists c such that every $G \in \mathcal{C}$ has flipwidth at most $c\epsilon$. Our usage of the adjective “almost” is different – it refers to having a bounded number of exceptions.

137:14 On Classes of Bounded Tree Rank, Their Interpretations, and Efficient Sparsification

► **Lemma 25.** *A class \mathcal{C} of graphs has tree rank at most 2 if and only if has locally almost bounded degree.*

In the proof of the lemma we will use the following simple result from [13, Lemma 13].

► **Lemma 26.** *Let $r, t \in \mathbb{N}$, let H be a graph of radius at most r , and let $S \subseteq V(H)$. If $|S| \geq t^r + 1$, then there exists $u \in V(H)$ such that there are t vertices in S that are distinct from u and can be reached from u by internally vertex-disjoint paths of length at most r .*

Proof of Lemma 25. By Theorem 3, it suffices to prove that a class \mathcal{C} is of locally almost bounded degree if and only if for every $r \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that every graph in \mathcal{C} has (r, m) -rank at most 2.

First, assume that \mathcal{C} is of locally almost bounded degree with respect to functions f and d . Let $r \in \mathbb{N}$, and set $m := \max\{f(r), d(r)\}$. We claim that the (r, m) -ranking algorithm will assign the number 1 or 2 to each $v \in V(G)$. In the first round, all vertices of degree at most $d(r) \leq m$ will get rank 1. In the second round, we know that each $N_r^G(v)$ contains at most $f(r)$ vertices of degree larger than $d(r)$. Thus we can delete a set $S \subseteq V(G) \setminus \{v\}$ of size at most $f(r)$ so that $N_r^{G-S}(v) \setminus \{v\}$ contains only vertices of rank 1. The result follows.

For the other direction, suppose that for every $r \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that every graph in \mathcal{C} has (r, m) -rank at most 2. Let $r \in \mathbb{N}$, and set $f(r) := (m+1)^r$ and $d(r) := m$. Let $v \in V(G)$ be arbitrary, and let S be the set of vertices with (r, m) -rank exactly 2 in $N_r^G(v)$ (these are the vertices of degree more than m). Going for a contradiction, we may assume that $|S| > f(r)$, since otherwise we are done. Thus, by Lemma 26 applied to $r, t := m+1$ and S in the subgraph H of G induced on $N_r^G(v)$, there exists a vertex $u \in N_r^G(v)$ such that there are $m+1$ vertices in S that are distinct from u and can be reached from u by internally vertex-disjoint paths of length at most r . As all of the vertices in S have (r, m) -rank 2, this shows that the (r, m) -rank of u is more than 2, which is a contradiction. Thus \mathcal{C} is of locally almost bounded degree with functions f and d , as desired. ◀

Algorithmic considerations. For our algorithms we will need to assume that the functions which are used in the definitions of our graph classes are efficiently computable. In particular, we will need to be able to efficiently test whether a given graph G comes from a fixed graph class of locally almost bounded degree given by functions d and f . This can be done under fairly relaxed conditions on functions d and f , which we now describe. The key observation is that for any graph G we need to check the conditions imposed by functions f and d from Definition 24 only for values of r with $r \leq n$ (where $n = |V(G)|$), and that checking the conditions is trivial whenever $f(r) > n$ or $d(r) > n$.

With this in mind, we say that a function $h : \mathbb{N} \rightarrow \mathbb{N}$ is *nice* if there exists an algorithm which inputs two numbers r and n represented in binary with $r \leq n$ and in time $\text{poly}(n)$ either correctly answers that $h(r) > n$ or outputs $h(r)$ (which is upper bounded by n).

Note that a function which is very fast growing and complicated to compute (with respect to its input which has length $\lceil \log(r) \rceil$) can still be nice. This is because it may be easy to check that $h(r) > n$, in which case no further computation is required, and if $h(r) \leq n$, we know that n is much larger than $\lceil \log(r) \rceil$, and then we have $\text{poly}(n)$ time at our disposal to compute $h(r)$. It is easy to verify that functions such as 2^r , $\text{tow}_\ell(r)$ (tower of twos of height ℓ with r on top), $\text{tow}_r(2)$ (tower of twos of height r) and also r^r are nice functions.

Coming back to graph classes of tree rank 2, we will use the following definition to specify graph classes for which we can prove Theorem 6.

► **Definition 27.** We say that a class \mathcal{C} of graphs of tree rank 2 is efficiently bounded if there exists a nice function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $r \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that no $G \in \mathcal{C}$ contains $T_{2,m}$ as an r -shallow topological minor.

By inspecting the proof of Lemma 25 one easily checks that any efficiently bounded class \mathcal{C} of tree rank 2 is a class of locally almost bounded degree for which there exist functions f and d which certify this. If we moreover assume that \mathcal{C} is maximum class of locally bounded degree with respect to f and g (meaning it contains every G that satisfies conditions given by f and d for all $r \leq |V(G)|$), then we can efficiently test for any G whether $G \in \mathcal{C}$. This is easily achieved by testing the conditions from Definition 24 for all values r up to $|V(G)|$. Since the functions f and d are nice, these values are either too large (if $f(r) > n$ or $d(r) > n$ which can be efficiently checked), or efficiently computable. This yields the following lemma.

► **Lemma 28.** Let \mathcal{C} be a class of graphs with locally almost bounded degree given by nice functions $f, d : \mathbb{N} \rightarrow \mathbb{N}$. Assume that \mathcal{C} is maximum such class with respect to f and d . Then there is a polynomial time algorithm that determines whether $G \in \mathcal{C}$.

In subsequent sections, it would be desirable to show at various places that some function is nice (or at least can be upper bounded by some nice function). We will not spell out these arguments explicitly; we just note here that one can often combine two nice functions to obtain another nice function. In particular, one easily checks that if f and g are functions such that for all r we have $f(r) \geq r$ and $g(r) \geq r$, then also $h := g \circ f$ is nice with $h(r) \geq r$. This because for given r, n with $r \leq n$ we can check whether $f(r) > n$, and if yes then we know that $h(r) = g(f(r)) > n$. On the other hand if $f(r) \leq n$, then the input condition for function g is satisfied, and we can just check whether $g(f(r)) > n$ or compute the value of $g(f(r))$ efficiently.

5.2 The case of interpretations of bounded range

We now proceed by giving a characterization of graph classes that can be obtained from graph classes of tree rank at most 2 by an interpretation of bounded range (Theorem 31).

In what follows we will use the notions of k -near-twins and near-twin graphs introduced in Section 4.

► **Definition 29.** We say that a graph G is (k, m) -near-covered if for every set $S \subseteq V(G)$ such that the elements of S are mutually not k -near-twins we have $|S| \leq m$.

We remark that the definition of (k, m) -near-covered graphs introduced in [12] was slightly different – there it was required that there exists a set of at most m vertices such that every vertex v of G is a k -near-twin of some $w \in S$. One can easily check that the two definitions are functionally equivalent (i.e. up to changing k and m in one definition to k' and m' in the other definition). We use Definition 29 because it will be more convenient in our arguments.

► **Definition 30.** A graph class \mathcal{C} is locally almost near-covered if there exist functions $k, m : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $r \in \mathbb{N}$, every $G \in \mathcal{C}$, and every $v \in V(G)$, the subgraph of G induced on $N_r^G(v)$ is $(k(r), m(r))$ -near-covered.

► **Theorem 31.** A class \mathcal{C} is interpretable from a class of locally almost bounded degree by a bounded range interpretation if and only if \mathcal{C} is locally almost near-covered.

The forward direction of Theorem 31 is handled by the following lemma, which can be proven by combining the locality-based Lemma 11 with the characterization of graph classes interpretable in graph classes of bounded degree in terms of near-covered graphs given in [12].

► **Lemma 32** (*). *Let I be a bounded range interpretation, and let \mathcal{C} be a class of locally almost bounded degree. Then $I(\mathcal{C})$ is locally almost near-covered.*

The rest of this section is dedicated to proving the backward direction of Theorem 31

► **Lemma 33**. *Let \mathcal{C} be a locally almost near-covered graph class. Then \mathcal{C} is interpretable in a class \mathcal{D} of locally almost bounded degree by an interpretation I of bounded range.*

Moreover, there is a polynomial algorithm that to every graph G computes a graph $\mathcal{S}(G)$ so that if $G \in \mathcal{C}$, then $\mathcal{S}(G) \in \mathcal{D}$ and $G = I(\mathcal{S}(G))$.

We will need the following simple lemma showing that locally almost near-covered graph classes do not contain arbitrarily large half-graphs.

► **Lemma 34**. *Let \mathcal{C} be a locally almost near-covered graph class with respect to functions k and m , and set $t := m(2) \cdot k(2) + m(2) + 1$. Then no graph in \mathcal{C} contains a half-graph of order t as a semi-induced subgraph.*

Proof. Assume for contradiction that such a graph $G \in \mathcal{C}$ exists. Let $u_1, \dots, u_t, w_1, \dots, w_t$ be the vertices of a half-graph of order t that G contains as a semi-induced subgraph. Then all of these vertices $u_1, \dots, u_t, w_1, \dots, w_t$ are in $N_2^G(u_1)$. Also, for any $i, j \in \{1, 2, \dots, t\}$ with $i - j > k(2)$, we know that u_i and u_j are not $k(2)$ -near twins, since their adjacency differs in vertices $w_j, w_{j+1}, \dots, w_{i-1}$. Therefore, the vertices $u_1, u_{k(2)+2}, u_{2k(2)+3}, \dots, u_{m(2) \cdot k(2) + m(2) + 1}$ form a set of $m(2) + 1$ pairwise not $k(2)$ -near-twins, a contradiction with \mathcal{C} being locally almost near-covered. ◀

In the proof of Lemma 33 we will also use the following lemma taken from [12, Corollary 5.3]. We note that in that paper, an extra assumption was made that amounts to saying that no vertex in A is a k -near-twin of any vertex in B . However, that assumption was not used in the proof given in [12], and so same proof works to prove the following lemma.

► **Lemma 35**. *Let G be a graph and let A and B be two subsets of $V(G)$ that are either disjoint or have $A = B$ and such that $|A| \geq 5k + 1$, $|B| \geq 5k + 1$, all vertices in A are pairwise k -near-twins, and all vertices in B are pairwise k -near-twins. Then either*

1. *every vertex of A is adjacent to at most $2k$ vertices of B and every vertex of B is adjacent to at most $2k$ vertices of A , or*
2. *every vertex of A is adjacent to all but at most $2k$ vertices of B and every vertex of B is adjacent to all but at most $2k$ vertices of A*

We will also use the following immediate corollary of Lemma 35.

► **Corollary 36**. *Let G be a graph, and let A and B be two subsets of $V(G)$ that satisfy the assumptions of Lemma 35. If there exists a vertex $v \in A$ with $|N^G(v) \cap B| > 2k$, then in the graph G' obtained from G by flipping the edges between A and B , every vertex $u \in A$ has $|N^{G'}(u) \cap B| \leq 2k$.*

Finally, we will use the following Ramsey-type result, which is routine to prove.

► **Lemma 37**. *There exists functions $R, D : \mathbb{N}^3 \rightarrow \mathbb{N}$ so that for all $k, m, s \in \mathbb{N}$, the following holds for every graph G with no subgraph isomorphic to the complete bipartite graph $K_{s,s}$. If $X \subseteq V(G)$ is such that $|X| \geq R(k, m, s)$ and every vertex in X has degree at least $D(k, m, s)$, then there exists $Y \subseteq X$ such that $|Y| \geq m$ and every vertex $y \in Y$ has at least k neighbors in $V(G) \setminus Y$ that are not adjacent to any other $y' \in Y \setminus \{y\}$.*

Proof of Lemma 33. Let \mathcal{C} be a locally almost near-covered graph class, and let k and m be the functions certifying this. We will describe a procedure \mathcal{S} that to every $G \in \mathcal{C}$ assigns a graph $\mathcal{S}(G)$ such that $G = I(\mathcal{S}(G))$ for a fixed interpretation I of bounded range. This procedure will also return some graph $\mathcal{S}(G)$ even when G is not in \mathcal{C} , however in this case we make no guarantees about the graph $\mathcal{S}(G)$. We will then show that the graph class $\mathcal{D} := \{\mathcal{S}(G) : G \in \mathcal{C}\}$ is of locally almost bounded degree. Since by construction we will have $\mathcal{C} \subseteq I(\mathcal{D})$, this will finish the proof.

Let $G \in \mathcal{C}$ be arbitrary. Let $h := h(k(3), t)$, where h is the function from Lemma 23 and $t := m(2) \cdot k(2) + m(2) + 1$ is the integer from Lemma 34. Thus every pair of vertices in the same component of the $k(3)$ -near-twin graph of G are h -near-twins in G . We define a partition \mathcal{F} of $V(G)$ by putting two vertices into the same part if they are in the same connected component of the $k(3)$ -near-twin graph of G (this creates a partition since being in the same connected component is an equivalence relation). We call parts $A, B \in \mathcal{F}$ (possibly with $A = B$) *mutually heavy* if $|A| \geq 5h + 1$, $|B| \geq 5h + 1$, and there exists a vertex $v \in B$ with $|N^G(v) \cap A| > 2h$. (Note that in this case, by Lemma 35, there also exists a vertex $u \in A$ with $|N^G(u) \cap B| > 2h$.) Similarly, we call a part $A \in \mathcal{F}$ *heavy* if it is mutually heavy with any part $B \in \mathcal{F}$ (possibly with $B = A$).

Creating the sparse graph $\mathcal{S}(G)$ from G . We can now describe the sparse graph $\mathcal{S}(G)$ associated to G . We start from G and proceed as follows:

1. For any heavy part $A \in \mathcal{F}$, we introduce a new vertex v_A , make it adjacent to all vertices in A , and mark it with a unary predicate R .
 2. For any mutually heavy parts $A, B \in \mathcal{F}$, we flip between A and B . If $A = B$, then we mark v_A with a unary predicate F . If $A \neq B$, then we put an edge between v_A and v_B .
- All other adjacencies that were not part of any flip remain as in G . Note that we can define this graph $\mathcal{S}(G)$ even when G is not in \mathcal{C} . Also note that by Corollary 36, for all mutually heavy parts $A, B \in \mathcal{F}$ and every vertex $v \in A$, we have that $|N^{\mathcal{S}(G)}(v) \cap B| \leq 2h$. We will use this important property later on.

Algorithmic considerations. It is easily seen that the construction of $\mathcal{S}(G)$ can be done in polynomial time from G .

Recovering G from $\mathcal{S}(G)$ by an interpretation. We now show that there is an interpretation I such that $G = I(\mathcal{S}(G))$. Let u and v be two vertices from $V(G)$. In the construction of $\mathcal{S}(G)$, the adjacency between two vertices only changed in the second part of the construction.

- Thus, the formula $\psi(x, y)$ will complement the adjacency between distinct vertices x and y if
- x and y have the same (unique) neighbor w marked with the predicate R and this w is also marked with predicate F , or
 - x and y each have a different neighbor marked with the predicate R and these neighbors are adjacent.

In all other cases we have $\psi(x, y) = E(x, y)$. The conditions above are easily expressed by a first-order formula. Note that $\psi(x, y)$ only keeps the existing edges or creates edges between vertices at distance at most 3 and therefore ψ is of range 3. To define the vertex set of G from the graph $\mathcal{S}(G)$, the formula $\delta(x)$ just keeps the vertices that are not marked with R (the original vertices of G).

Showing that the class $\mathcal{D} := \{\mathcal{S}(G) : G \in \mathcal{C}\}$ is of locally almost bounded degree. It remains to argue that \mathcal{D} is of locally almost bounded degree. Due to space restrictions this argument is omitted from the conference version and can be found in the full version of the paper. ◀

5.3 The case of full interpretations

In this section we finish the proof of our main result, Theorem 5, which states that a class \mathcal{C} of graphs is interpretable in a graph class of tree rank 2 if and only if \mathcal{C} is a perturbation of a locally almost near-covered graph class.

For the forward direction, it is known (see for example [1, 3]) that every interpretation can be decomposed into two steps in the following sense.

► **Lemma 38.** *For every interpretation I there exists an interpretation I' of bounded range and $k \in \mathbb{N}$ such that for every G we have that $I(G)$ is a k -flip of $I'(G)$.*

The forward direction of Theorem 5 then follows immediately from Lemmas 32 and 38. The backward direction follows from Lemma 33: If \mathcal{C} is a perturbation of a locally almost near-covered graph class, then \mathcal{C} is a perturbation of $I(\mathcal{D})$ for some graph class \mathcal{D} of locally almost bounded degree. Since perturbations (k -flips) can be modelled by interpretations, add unary predicates to graphs from \mathcal{D} and adjust I to I' such that $\mathcal{C} \subseteq I'(\mathcal{D})$.

5.4 Proof of Theorem 6

In this section we prove Theorem 6 by showing that for every graph class \mathcal{C} interpretable in a graph class of tree rank 2, there is a polynomial time algorithm that to every $G \in \mathcal{C}$ computes $H \in \mathcal{D}$ such that $G = I(H)$, where I is a fixed interpretation and \mathcal{D} is a fixed class of graphs of locally almost bounded degree (which is the same as tree rank 2 by Lemma 25).

We will rely on the following lemma which was proven in [1].

► **Lemma 39.** *Let \mathcal{D} be an NIP class of graphs, and let I be an interpretation. Then there exist s, b , and a formula $\psi(x, y)$ of range b such that for every $G \in I(\mathcal{D})$ there exist $S \subseteq V(G)$ of size at most s , an S -flip G' of G , and a graph $H \in \mathcal{D}$ such that $G' = \psi(H)$.*

We do not define the notion of a graph class being NIP (see for example the relevant sections in [1]), but just note that this notion is very general and it is easily established that classes of tree rank 2 are NIP, so the lemma can be applied in our setting.

We now proceed with the proof of Theorem 6. Since \mathcal{C} is interpretable in a graph class of tree rank 2, there exist an interpretation I and a class \mathcal{D} of tree rank at most 2 such that $\mathcal{C} \subseteq I(\mathcal{D})$. By applying Lemma 39 to \mathcal{D} and I , we obtain numbers s and b and a formula ψ of range b . Since ψ is of bounded range, we know by Lemma 32 that the class $\psi(\mathcal{D})$ is locally almost near-covered. By Lemma 33 applied to $\psi(\mathcal{D})$, there exists a graph class \mathcal{D}' of locally almost bounded degree, an interpretation I' , and a polynomial time algorithm \mathcal{A} that to any $G' \in \psi(\mathcal{D})$ computes a graph $\mathcal{S}(G') \in \mathcal{D}'$ such that $G' = I'(\mathcal{S}(G'))$. Moreover, by inspecting the proofs of Lemma 32 and Lemma 33 one can check that the functions f and d which certify that \mathcal{D}' is a class of locally almost bounded degree are nice functions, and so we can use Lemma 28 to check membership in \mathcal{D}' .

We now consider the algorithm that for a graph $G \in \mathcal{C}$ does the following:

1. Go through all subsets S of $V(G)$ of size at most s .
2. For each such set S , go through all possible S -flips G' of G .
3. Apply the algorithm \mathcal{A} from Lemma 33 to G' to obtain a graph $\mathcal{S}(G')$.
4. Check whether $G' = I'(\mathcal{S}(G'))$ and use Lemma 28 to check whether $\mathcal{S}(G') \in \mathcal{D}'$, if yes, then output $\mathcal{S}(G')$.

We now argue the correctness of the algorithm. By Lemma 39, at least one G' will be such that $G' = \psi(H)$ for some $H \in \mathcal{D}$. Then, since $G' \in \psi(\mathcal{D})$, we have (by Lemma 33) that the algorithm \mathcal{A} returns a graph $\mathcal{S}(G')$ that is in \mathcal{D}' . Since for $\mathcal{S}(G')$ we have $G' = I'(\mathcal{S}(G'))$

and since from G' one can easily recover G by performing a bounded number of flips, we can adjust the interpretation I' to an interpretation I'' such that $G = I''(\mathcal{S}(G'))$, as desired. (We add unary predicates to $\mathcal{S}(G')$ in order to “mark” the sets that we flip between.)

The runtime of the algorithm is easily seen to be polynomial in $|V(G)|$: We have $|V(G)|^s$ possible sets of size s and consequently we have $|V(G)|^s$ iterations, where the number s depends only on the graph class \mathcal{C} . In each iteration we invoke the polynomial time algorithm \mathcal{A} from Lemma 33.

6 Conclusions

We conclude with two open ended questions that may deserve further attention.

1. Is there a way of defining vertex rankings for dense graphs analogous to our rankings? Ideally, such rankings should be computable in FPT time.
2. While the proof of Lemma 33 is technical, our construction of the graph $\mathcal{S}(G)$ from the graph G is simple: First we determine which components of the k -near-twin graph of G to flip between, and then we use this information to construct $\mathcal{S}(G)$. It may be interesting to see under which conditions on G we can claim that the graph $\mathcal{S}(G)$ comes from a class of sparse graphs. Also, using our construction recursively may lead to interesting results: If $\mathcal{S}(G)$ is not a sparse graph, one may consider $\mathcal{S}(\mathcal{S}(G))$, and so on.

References

- 1 Édouard Bonnet, Jan Dreier, Jakub Gajarský, Stephan Kreutzer, Nikolas Mählmann, Pierre Simon, and Szymon Toruńczyk. Model checking on interpretations of classes of bounded local cliquewidth. In *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '22*, New York, NY, USA, 2022. Association for Computing Machinery. doi:10.1145/3531130.3533367.
- 2 Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, Pierre Simon, Stéphan Thomassé, and Szymon Toruńczyk. Twin-width iv: Ordered graphs and matrices. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2022*, pages 924–937, New York, NY, USA, 2022. Association for Computing Machinery. doi:10.1145/3519935.3520037.
- 3 Samuel Braulfeld, Jaroslav Nešetřil, Patrice Ossona de Mendez, and Sebastian Siebertz. Decomposition horizons: from graph sparsity to model-theoretic dividing lines. *CoRR*, abs/2209.11229, 2022. doi:10.48550/arXiv.2209.11229.
- 4 Jan Dreier, Ioannis Eleftheriadis, Nikolas Mählmann, Rose McCarty, Michał Pilipczuk, and Szymon Toruńczyk. First-order model checking on monadically stable graph classes, 2023. arXiv:2311.18740.
- 5 Jan Dreier, Nikolas Mählmann, and Sebastian Siebertz. First-order model checking on structurally sparse graph classes. In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing, STOC 2023*, pages 567–580, New York, NY, USA, 2023. Association for Computing Machinery. doi:10.1145/3564246.3585186.
- 6 Jan Dreier, Nikolas Mählmann, Sebastian Siebertz, and Szymon Toruńczyk. Indiscernibles and Flatness in Monadically Stable and Monadically NIP Classes. In Kousha Etessami, Uriel Feige, and Gabriele Puppis, editors, *50th International Colloquium on Automata, Languages, and Programming (ICALP 2023)*, volume 261 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 125:1–125:18, Dagstuhl, Germany, 2023. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ICALP.2023.125.
- 7 Zdeněk Dvořák. Constant-factor approximation of the domination number in sparse graphs. *European Journal of Combinatorics*, 34(5):833–840, 2013.

- 8 Kord Eickmeyer and Ken-ichi Kawarabayashi. FO model checking on map graphs. In Ralf Klasing and Marc Zeitoun, editors, *Fundamentals of Computation Theory – 21st International Symposium, FCT 2017, Bordeaux, France, September 11-13, 2017, Proceedings*, volume 10472 of *Lecture Notes in Computer Science*, pages 204–216. Springer, 2017. doi:10.1007/978-3-662-55751-8_17.
- 9 Markus Frick and Martin Grohe. The complexity of first-order and monadic second-order logic revisited. *Annals of Pure and Applied Logic*, 130(1-3):3–31, 2004.
- 10 Haim Gaifman. On local and non-local properties. *Studies in Logic and the Foundations of Mathematics*, 107:105–135, 1982.
- 11 Jakub Gajarský and Petr Hliněný. Kernelizing MSO properties of trees of fixed height, and some consequences. *Log. Methods Comput. Sci.*, 11(1), 2015. doi:10.2168/LMCS-11(1:19)2015.
- 12 Jakub Gajarský, Petr Hliněný, Jan Obdržálek, Daniel Lokshantov, and M. S. Ramanujan. A new perspective on FO model checking of dense graph classes. *ACM Trans. Comput. Log.*, 21(4):28:1–28:23, 2020. doi:10.1145/3383206.
- 13 Jakub Gajarský, Michał Pilipczuk, Marek Sokolowski, Giannos Stamoulis, and Szymon Toruńczyk. Elementary first-order model checking for sparse graphs, 2024. arXiv:2401.16230.
- 14 Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. Deciding first-order properties of nowhere dense graphs. *J. ACM*, 64(3):17:1–17:32, 2017. doi:10.1145/3051095.
- 15 Michael Lampis. First Order Logic on Pathwidth Revisited Again. In Kousha Etessami, Uriel Feige, and Gabriele Puppis, editors, *50th International Colloquium on Automata, Languages, and Programming (ICALP 2023)*, volume 261 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 132:1–132:17, Dagstuhl, Germany, 2023. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ICALP.2023.132.
- 16 L. Lovász, V. Neumann-Lara, and M. Plummer. Mengerian theorems for paths of bounded length. *Periodica Mathematica Hungarica*, 9(4):269–276, 1978.
- 17 Jaroslav Nešetřil and Patrice Ossona De Mendez. *Sparsity: graphs, structures, and algorithms*, volume 28. Springer Science & Business Media, 2012.
- 18 Jaroslav Nešetřil, Patrice Ossona de Mendez, Michał Pilipczuk, Roman Rabinovich, and Sebastian Siebertz. Rankwidth meets stability. In Dániel Marx, editor, *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms, SODA 2021, Virtual Conference, January 10–13, 2021*, pages 2014–2033. SIAM, 2021. doi:10.1137/1.9781611976465.120.
- 19 Jaroslav Nešetřil, Roman Rabinovich, Patrice Ossona de Mendez, and Sebastian Siebertz. Linear rankwidth meets stability. In *31st ACM-SIAM Symposium on Discrete Algorithms, SODA 2020*, pages 1180–1199. SIAM, 2020. doi:10.1137/1.9781611975994.72.
- 20 Saharon Shelah. Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory. *Annals of Mathematical Logic*, 3(3):271–362, 1971.
- 21 Saharon Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific Journal of Mathematics*, 41(1):247–261, 1972.
- 22 Szymon Toruńczyk. Flip-width: Cops and robber on dense graphs. *FOCS 2023, accepted*, arXiv abs/2302.00352, 2023.
- 23 Xuding Zhu. Colouring graphs with bounded generalized colouring number. *Discrete Mathematics*, 309(18):5562–5568, 2009.