



Exponential Lower Bounds via Exponential Sums

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Abstract

Valiant’s famous VP vs. VNP conjecture states that the symbolic permanent polynomial does not have polynomial-size algebraic circuits. However, the best upper bound on the size of the circuits computing the permanent is exponential. Informally, VNP is an exponential sum of VP-circuits. In this paper we study whether, in general, exponential sums (of algebraic circuits) *require* exponential-size algebraic circuits. We show that the famous Shub-Smale τ -conjecture indeed implies such an exponential lower bound for an exponential sum. Our main tools come from parameterized complexity. Along the way, we also prove an exponential fpt (fixed-parameter tractable) lower bound for the parameterized algebraic complexity class $\text{VW}_{\text{nb}}^0[\mathbb{P}]$, assuming the same conjecture. $\text{VW}_{\text{nb}}^0[\mathbb{P}]$ can be thought of as the weighted sums of (unbounded-degree) circuits, where only ± 1 constants are *cost-free*. To the best of our knowledge, this is the *first* time the Shub-Smale τ -conjecture has been applied to prove explicit exponential lower bounds.

Furthermore, we prove that when this class is fpt, then a variant of the counting hierarchy, namely the *linear counting hierarchy* collapses. Moreover, if a certain type of parameterized exponential sums is fpt, then integers, as well as polynomials with coefficients being *definable* in the linear counting hierarchy have subpolynomial τ -complexity.

Finally, we characterize a related class $\text{VW}[\mathbb{F}]$, in terms of permanents, where we consider an exponential sum of algebraic formulas instead of circuits. We show that when we sum over cycle covers that have one long cycle and all other cycles have constant length, then the resulting family of polynomials is *complete* for $\text{VW}[\mathbb{F}]$ on certain types of graphs.

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1 Introduction

Valiant [23] proposed an algebraic version of the P versus NP question and defined the class VP, the algebraic analogue of P, which contains polynomial families computable by polynomial sized algebraic circuits. An *algebraic circuit* (or, arithmetic circuit) C is a directed acyclic graph such that (1) every node has either in-degree (*fan-in*) 0 (the *input gates*) or 2 (the *computational gates*), (2) every input gate is labeled by elements from a field \mathbb{K} or variables from $\mathbf{X} = \{X_1, \dots, X_n\}$, (3) every computational gate is labeled by either +



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(addition gate) or \times (multiplication gate), with the obvious syntactic meaning, and (4) there is a unique gate of out-degree 0, the *output gate*. Clearly, every gate in a circuit computes a polynomial in $\mathbb{K}[\mathbf{X}]$. We say that the circuit C computes $P(\mathbf{X}) \in \mathbb{K}[\mathbf{X}]$ if the output gate of C computes $P(\mathbf{X})$. The *size* of C , denoted by $\text{size}(C)$, is the number of nodes in the circuit. An algebraic circuit is an *algebraic formula* if every gate in the circuit has out-degree 1 except for the output gate. The class VNP , the algebraic analogue of NP , is definable by taking *exponential sums* of the form

$$f(\mathbf{X}) = \sum_{e \in \{0,1\}^\ell} g(\mathbf{X}, e), \quad (1)$$

where g is computable by a polynomial-size circuit and ℓ is polynomial in the number of variables. It is known that one can also replace algebraic circuits by algebraic formulas, and still get the same class VNP [23, 18]. Valiant further proved that the permanent family is complete for VNP (over fields of characteristic not two). Recall that the permanent of a matrix $(X_{i,j})$ is defined as

$$\text{per } \mathbf{X} = \sum_{\pi \in S_n} X_{1,\pi(1)} \cdots X_{n,\pi(n)}. \quad (2)$$

The famous Valiant’s conjecture $\text{VP} \neq \text{VNP}$ is equivalent to the fact that the permanent does not have polynomial-size circuits. The representation of the permanent in (2), although it looks very natural, is not *optimal*. Ryser’s formula [19] yields an algebraic formula of size $O(2^n n^2)$. A formula of similar size was later found by Glynn [11]. Ryser’s formula is now over sixty years old and has not been improved since. This gives rise to the interesting question whether there is a formula or circuit of subexponential-size (in n) for the permanent? More generally, we can now ask the following question.

► **Question 1.** *Is an exponential sum f (as in Eq. (1)) always computable by an algebraic circuit or formulas of size subexponential in ℓ , that is, size $2^{o(\ell)}$? Or are there instances for which exponential-size is necessary?*

Note that exponential-size being necessary is a much *stronger* claim than $\text{VP} \neq \text{VNP}$. It could well be that $\text{VP} \neq \text{VNP}$ but still exponential sums like in (1) have subexponential size circuits! In this paper, we shed some light on the question what happens if exponential sums would always have *subexponential* size circuits.

Question 1 works as driving force between the famous Shub-Smale τ -conjecture [20] and *exponential* lower bounds on exponential sums. The τ -complexity $\tau(f)$ of an integer polynomial is the size of a smallest division-free circuit that computes f starting from the constants ± 1 . The τ -conjecture states the the number of integer zeroes of f is polynomially bounded in $\tau(f)$, see [20]. [20] shows that the τ -conjecture implies $\text{P}_{\mathbb{C}} \neq \text{NP}_{\mathbb{C}}$, in the Blum–Shub–Smale (BSS) model of computation over the complex numbers [5, 4].

Super-polynomial lower bounds assuming the τ -conjecture. Bürgisser [6] connected the τ -complexity of the permanent to various other conjectures. He showed that the τ -conjecture implies a *superpolynomial* lower bound on $\tau(\text{per}_n)$, implying the constant-free version of $\text{VP} \neq \text{VNP}$, namely $\text{VP}^0 \neq \text{VNP}^0$; for definitions, see Section 2.1. The proof strategy of [6] is as follows: assume $\tau(\text{per}_n) = \text{poly}(n)$, and conclude a complexity-theoretic “collapse” that the counting hierarchy CH (for a definition, see Section 3) is in P/poly . Consider the Pochhammer–Wilkinson polynomial $f_n(x) := \prod_{i=1}^n (x - i)$, and construct a unique $O(\log n)$ -variate multilinear polynomial B_n such that under a “suitable” substitution, one gets back f_n .

The coefficients of f_n as well as B_n , are efficiently computable (since $\text{CH} \subseteq \text{P/poly}$), implying $B_n \in \text{VNP}^0$. An inspection of Valiant's completeness result reveals that if $B_n \in \text{VNP}^0$, then there is a polynomially bounded sequence $p(n)$ such that $\tau(2^{p(n)}B_n) = \text{poly}(\log n)$, which implies $\tau(2^{p(n)}f_n) = \text{poly}(\log n)$, contradicting the τ -conjecture.

In [6], the superpolynomial lower bound on $\tau(\text{per}_n)$ was also implied by any of the quantities $\tau(n!)$, $\tau(\sum_{k=0}^n \frac{1}{k!} T^k)$, or $\tau(\sum_{k=0}^n k^r T^k)$ (for any fixed negative integer r) not being poly-logarithmically bounded as a function of n . Here, we remark that the separation proof of VP^0 and VNP^0 , even assuming *strong* bounds on the τ -conjecture, is merely *superpolynomial*: we *do not* get the (possibly) desirable exponential separation between VP^0 and VNP^0 . This leads to the following question.

► **Question 2.** *Does the τ -conjecture imply exponential algebraic lower bounds?*

Here, we mention that there are variants of the τ -conjecture, e.g., the *real τ -conjecture* [15, 21] or *SOS- τ -conjecture* [8], which also give strong algebraic lower bounds. There is also super polynomial lower bound known from a proof complexity theoretic view due to [1] from the original Shub-Smale τ -conjecture. However, the Shub-Smale τ -conjecture is *not known* to give an exponential lower bound for the permanent.

1.1 Our results

The results of our paper revolve around answering both Question 1-2 positively. The main result is the following.

► **Theorem 1 (Informal).** *The τ -conjecture implies an exponential lower bound for some explicit exponential sum.*

Remarks.

- (1) Although the existence of *some* polynomial requiring exponential circuits is clear from dimension/counting, the existence of an (even non-explicit) exponential sum polynomial requiring exponential-size circuits is *unclear*. Explicit here means that the family is in VNP .
- (2) One can also think of an exponential sum f in Equation (1), as $f = \sum_{e \in \{0,1\}^{\ell(n)}} U(\mathbf{X}, y, e)$, where $U(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is a *universal circuit* of size $\text{size}(U) = \text{poly}(\text{size}(g))$ with $\mathbf{Y} = (Y_1, \dots, Y_r)$ and $\mathbf{Z} = (Z_1, \dots, Z_{\ell(n)})$ and $y \in \mathbb{F}^r$ is chosen such that $U(\mathbf{X}, y, e) = g(\mathbf{X}, e)$; and the number of variables $\ell(n)$ is linear in n .
- (3) Since there's a polynomial (non-linear) blowup in the reduction of the exponential sum on the universal circuit from the permanent, we will only get a subexponential lower bound on the permanent polynomial assuming the τ -conjecture. We leave it as an open question to achieve an exponential lower bound on the permanent assuming the τ -conjecture.

The proof of Theorem 1 is rather indirect, and goes via *exponential sums*, which is our main object of study (and bridge between many results and classes).

log-variate exponential sum polynomial. Let $g(\mathbf{X}, \mathbf{Y})$ be some polynomial in n -many \mathbf{X} -variables and $\ell(n)$ -many \mathbf{Y} -variables, where $\ell(n) = O(n)$. Assume that g is computed by a circuit of size m . Then we define

$$\text{p-log-Expsum}_{m,k}(g) := \sum_{y \in \{0,1\}^{\ell(n)}} g(\mathbf{X}, y),$$

where $k = n / \log m$. The size of the exponential is measured in the number $\ell(n)$ of \mathbf{Y} -variables. In the end, we want to measure in the input size, the number n of \mathbf{X} -variables. To talk about subexponential complexity, $\ell(n)$ should be linearly bounded. g will be typically computed by a circuit (of unbounded degree). We want to view $\mathbf{p}\text{-log-Expsum}_{m,k}$ as a parameterized problem, the parameter will be k . Our definition of $\mathbf{p}\text{-log-Expsum}$, as a polynomial-sum, is motivated by the *log-parameterizations* which are used in the definition of the so-called M -hierarchy in the Boolean setting, see [9, 10].

We show that $\mathbf{p}\text{-log-Expsum}$ is most likely *not* fixed-parameter tractable (fpt). A polynomial family $p_{n,k}$ is fpt if both its size and degree are fpt bounded, i.e., of the form $f(k)q(n)$, for $q \leq \text{poly}(n)$, and $f : \mathbb{N} \rightarrow \mathbb{N}$ being *any* computable function. We connect $\mathbf{p}\text{-log-Expsum}$ with – (1) a linear variant of the counting hierarchy (we denote it by CH_{lin}), where the size of the oracle calls are bounded linearly in the size of the input; for definition see Section 3; and (2) integers definable in CH_{lin} , similar to Bürgisser [6]. Informally, an integer is definable in CH_{lin} , if its sign and bits are computable in the same class.

► **Theorem 2 (Informal).** *If $\mathbf{p}\text{-log-Expsum}$ is fixed-parameter tractable, then the following results hold.*

1. *The linear counting hierarchy (CH_{lin}) collapses.*
2. *Any sequence $a(n)$ definable in the linear counting hierarchy, as well as univariate polynomials with coefficients being definable in the linear counting hierarchy, have subpolynomial τ -complexity.*

For formal statements, see Theorem 13 and 21.

Finally, many algebraic complexity classes can be defined in terms of permanents. Most prominently, the “regular” permanent family (per_n) is complete for VNP . The class $\text{VW}[1]$ is an important class in parameterized complexity. It is defined as a bounded sum over constant depth weft-1 circuits. Bounded sum means that we sum over $\{0, 1\}$ -vectors with k ones and k is the parameter. Bläser and Engels [3] prove that $\text{VW}[1]$ is described by so-called k -permanents with k being the parameter. In a k -permanent, we only sum over permutations with $n - k$ self-loops. The crucial parameterized class of this work is $\text{VW}[\text{P}]$: it is defined as a bounded exponential sum over polynomially-sized arithmetic circuits computing a polynomial of degree that is polynomially bounded. While we do not characterise $\text{VW}[\text{P}]$ in terms of permanents, we characterize the related class $\text{VW}[\text{F}]$: Here instead of summing over circuits, we sum over *formulas*.¹ The permutations that we sum over for defining our permanent family will have one cycle of length k and all other cycles bounded by 4. Again, k is the parameter. We call the corresponding polynomials $(k, 4)$ -restricted permanents. It turns out that we also need to restrict the graph classes. We call a graph $G = (V, E)$ $(4, b)$ -nice if we can partition the set $V = V_1 \cup V_2$ disjointly, such that in the induced graph $G[V_1]$, every cycle is either a self-loop or has length > 4 and in the induced graph $G[V_2]$ has tree-width bounded by b . While this looks artificial at a first glance, it turns out that there is a constant b such that $(k, 4)$ -restricted permanent on $(4, b)$ -nice graphs describes the natural class $\text{VW}[\text{F}]$. There is a family of $(4, b)$ -nice graphs such that the corresponding family of $(k, 4)$ -restricted permanents is $\text{VW}[\text{F}]$ -hard. On the other hand, the $(k, 4)$ -restricted permanent family is in $\text{VW}[\text{F}]$ for every family of $(4, b)$ -nice graphs. Together, this implies:

► **Theorem 3 (VW[F]-Completeness).** *$(k, 4)$ -restricted permanent family on $(4, b)$ -nice graphs is $\text{VW}[\text{F}]$ -complete.*

¹ Maybe an explanation of the naming convention is helpful: In $\text{VW}[\text{P}]$, we sum of polynomial-size circuits, which describe the class VP . In $\text{VW}[\text{F}]$, we sum over polynomial size formulas, which define the class VF , the modern name for VP_e .

We also prove strong separations of algebraic complexity classes and parameterized algebraic complexity classes (Theorem 30), and exponential lower bounds in the parameterized setting (Theorem 36).

For VNP it is known that it does not matter whether we sum over formulas or circuits, that is, $\text{VNP} = \text{VNP}_e$. Whether $\text{VW}[\mathbb{P}] = \text{VW}[\mathbb{F}]$ remains an open questions for future research.

1.2 Proof ideas

In this section, we briefly sketch the proof ideas. The omitted proofs of the paper can be found in the longer arxiv version of the paper. We first present the proofs of Theorem 2, because the techniques and lemmas involved in proving them are the backbone of Theorem 1.

Proof idea of Theorem 2. We prove them in two parts.

Proof of Part (1): We prove even a stronger statement for the subexponential version of the linear counting hierarchy. The proof goes via induction on the level of the counting hierarchy. The criteria for some language B being in the $(k+1)$ -th level is that there should be some language A in the k -th level such that $|\{y \in \{0, 1\}^n : \langle x, y \rangle \in A\}| > 2^{n-1}$. Essentially, for a language A in the k -th level, we express $|\{y \in \{0, 1\}^n : \langle x, y \rangle \in A\}| > 2^{n-1}$ as an exponential sum over an algebraic circuit $\chi_A(x, y)$, which captures the characteristic function of A . Furthermore, one can show that p-log-Expsum is fpt (in an unbounded constant-free setting) iff $\sum_y g(\mathbf{X}, y)$ has $2^{o(n)}\text{poly}(m)$ size circuits, where g has a circuit of size m ; see Theorem 15 and 16. Putting these together, one gets that the exponential sum has a subexponential-size constant-free circuit. Lastly, we want to get the information about the highest bit of the sum (which is equivalent to looking at it mod 2^n), which can be efficiently *arithmetized*. In every step there is polynomial blowup in the size, and hence the size remains subexponential, yielding the desired result. For details, see Theorem 13.

Proof of Part (2): This proof is an adaption of [6, 14] in our context. Take a sequence $(a_n)_n \in \text{CH}_{\text{lin}}\mathbb{P}$. We define a multilinear polynomial $A(\mathbf{Y})$ such that the coefficient of $\mathbf{Y}^{\mathbf{j}}$ is the j -th bit of $a(n)$, where \mathbf{j} is the binary representation of j . Furthermore, checking $a(n, j) = b$ can be done by a subexponential circuit $C(\mathbf{N}, \mathbf{J})$, where \mathbf{N} and \mathbf{J} have $\log n$ and $\text{bit}(n)$ -many variables capturing n and j respectively. Moreover, one can define $F(\mathbf{N}, \mathbf{Y}, \mathbf{J}) = C(\mathbf{N}, \mathbf{J}) \cdot \prod_i (J_i Y_i + 1 - J_i)$ and show that A can be expressed as an exponential sum over $F(j, \mathbf{N}, \mathbf{Y})!$ This is clearly a p-log-Expsum instance, which finally yields that the τ -complexity of $a(n)$ is subpolynomial. A similar proof strategy also holds for the polynomials with coefficients being definable in $\text{CH}_{\text{lin}}\mathbb{P}$. For details, see Section 6.

Proof idea of Theorem 1. Take the Pochhammer polynomial $p_n(X) = \prod_{i=1}^n (X + i)$. The coefficient of X^{n-k} in p_n will be $\sigma_k(1, \dots, n)$, where $\sigma_k(z_1, \dots, z_n)$ is the k -th elementary symmetric polynomial in variables z_1, \dots, z_n . It is not hard to show that $\text{CH}_{\text{lin}}\mathbb{P}$ is closed under polynomially-many additions and multiplications (Theorem 19). Therefore, $(\sigma_k(1, \dots, n))_{n \in \mathbb{N}, k \leq n}$ is definable in the linear counting hierarchy (see Corollary 20). And by Theorem 21, $(p_n)_{n \in \mathbb{N}}$ has $n^{o(1)}$ -sized constant-free circuits if p-log-Expsum is fixed-parameter tractable. But p_n has n distinct integer roots. Assuming the τ -conjecture, p-log-Expsum is not fpt . On the other hand, one can show that when exponential sums over circuits of size m have circuits have size $2^{o(n)}\text{poly}(m)$, then the p-log-Expsum is fpt , by Theorem 16; in other words, p-log-Expsum is not fpt implies an exponential lower bound on an exponential sum. This finishes the proof.

Proof idea of Theorem 3. The hardness proof is *gadget* based (Theorem 42). The details are however quite complicated since we have to cleverly keep track of the cycle lengths. For the upper bound, we work along a tree decomposition. While it is known that the permanent can be computed in fpt time on graphs of bounded treewidth, we cannot simply adapt these algorithms, since we have to produce a formula. This can be achieved using a *balanced tree decomposition*.

1.3 Previous results

To prove (conditional) exponential lower bounds, the standard assumptions that $\text{P} \neq \text{NP}$ or $\text{VP} \neq \text{VNP}$ are not enough. It is consistent with our current knowledge that for instance $\text{P} \neq \text{NP}$, but NP -hard problems can have subexponential time algorithms. What we need is a complexity assumption stating that certain problems can only be solved in exponential time. This is the exponential time hypothesis (ETH) in the Boolean setting. Dell et al. [7] studied the exponential time complexity of the permanent, they prove that when there is an algorithm for computing the permanent in time $2^{o(n)}$, then this violates the counting version of the exponential time hypothesis $\#ETH$. $\#ETH$ states that there is a constant c such that no deterministic algorithm can count the number of satisfying assignments of a formula in 3-CNF in time 2^{cn} . For connections between parameterized and subexponential complexity in the Boolean setting, we refer to [9, 10].

Bläser and Engels [3] transfer the important definitions and results from parameterized complexity in the Boolean world to define a theory of parameterized algebraic complexity classes. In particular, they define the VW-hierarchy and prove that the clique polynomial and the k -permanent are $\text{VW}[1]$ -complete (under so-called fpt -substitutions). They also claim the hardness of the restricted permanent for the class $\text{VW}[t]$ for every constant t and sketch a proof. Note that $\text{VW}[F]$ contains each $\text{VW}[t]$. So we strengthen the hardness proof in [3] and complement it with an upper bound.

The main tool used by Bürgisser [6] to prove the results above is the counting hierarchy. The polynomial counting hierarchy was introduced by Wagner [24] to classify the complexity of Boolean counting problems. The fact that small circuits for the permanent collapses the counting hierarchy is used by Bürgisser to prove the results mentioned above.

Finally, there have been quite a few works [6, 14, 16, 15], where we have conditional separations on the constant-free version of VP and VNP , namely VP^0 and VNP^0 , or their variants, depending on the strength of the conjecture. But this is the first time that we are separating algebraic classes and proving exponential lower bounds, assuming the τ -conjecture.

1.4 Structure of the paper

In Section 2, we defined the basics of constant-free Valiant's model and the unbounded and parameterized setting. In Section 3, we introduce the linear counting hierarchy (CH_{lin}) and its basic properties. Section 4 connects Valiant's model to the counting hierarchy. Here, we formally introduce exponential sums and investigate their relation to the parameterized classes. The main result is that the fixed-parameter tractability of exponential sums collapses the counting hierarchy. The proofs are quite similar to [6], however, we need to pay special attention to the fact the witness size is linear. Section 5 introduces the definability (computability) of integers in the linear counting hierarchy, and some closure properties of the same. Section 6 proves the exponential lower bound on exponential sum assuming τ -conjecture. Section 7 introduces the parameterized VW-classes and its basic properties. In Section 8 we prove some easy conditional collapse results of the VW-hierarchy in various circuit models.

2 Preliminaries I

2.1 Constant-free and unbounded models

Constant-free Valiant's classes. We will say that an algebraic circuit is *constant-free*, if no field elements other than $\{-1, 0, 1\}$ are used for labeling in the circuit. Clearly, constant-free circuits can *only* compute polynomials in $\mathbb{Z}[\mathbf{X}]$. For $f(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$, $\tau(f)$ is the size of a minimum size constant-free circuit that computes f , while $L(f)$ denotes the minimum size circuit that computes f . It is noteworthy to observe that, *unlike* Valiant's classical models, computing integers in the constant-free model can be costly; e.g., $\tau(2^{2^n} X^n) = \Omega(n)$, while $L(2^{2^n} X^n) = \Theta(\log n)$. On the other hand, for any $f \in \mathbb{Z}[\mathbf{X}]$, $L(f) \leq \tau(f)$.

Before defining the constant-free Valiant classes, we formalize the notion of *formal degree* of a node, denoted $\text{formal-deg}(\cdot)$. It is defined recursively as follows: (1) the formal degree of an input gate is 1, (2) if $u = v + w$, then $\text{formal-deg}(u) = \max(\text{formal-deg}(v), \text{formal-deg}(w))$, and (3) if $u = v \times w$, then $\text{formal-deg}(u) = \text{formal-deg}(v) + \text{formal-deg}(w)$. The formal degree of a circuit is defined as the formal degree of its output node.

The class *constant-free Valiant's P*, denoted by VP^0 , contains all p -families (f) in $\mathbb{Z}[\mathbf{X}]$, such that $\text{formal-deg}(f)$ and $\tau(f)$ are both p -bounded. Analogously, VNP^0 contains all p -families (f_n) , such that there exists a p -bounded function $q(n)$ and $(g_n) \in \text{VP}^0$, where

$$f_n(\mathbf{X}) = \sum_{\bar{y} \in \{0,1\}^{q(n)}} g_n(\mathbf{X}, y_1, \dots, y_{q(n)}).$$

It is not clear whether showing $\text{VP}^0 \neq \text{VNP}^0$ implies $\text{VP} \neq \text{VNP}$, it is *not even clear* whether $\text{VP}^0 \neq \text{VNP}^0 \implies \tau(\text{per}_n) = n^{\omega(1)}$. The *subtlety* here is that in the algebraic completeness proof for the permanent, *divisions by two* occur! However, a partial implication is known due to [6, Theorem 2.10]: Showing $\tau(2^{p(n)} f_n) = n^{\omega(1)}$, for some $f_n \in \text{VNP}^0$ and all p -bounded $p(n)$ would imply that $\tau(\text{per}_n) = n^{\omega(1)}$.

Arithmetization is a well-known technique in complexity theory. To arithmetize a Boolean circuit C computing a Boolean function φ , we use the arithmetization technique wherein we map $\varphi(x_1, \dots, x_n)$ to a polynomial $p(x_1, \dots, x_n)$ such that for any assignment of Boolean values $v_i \in \{0, 1\}$ to the x_i , $\varphi(v_1, \dots, v_n) = p(v_1, \dots, v_n)$ holds.

We define the arithmetization map Γ for variables x_i , and clauses c_1, \dots, c_m , as follows:

1. $x_i \mapsto x_i$,
2. $\neg x_i \mapsto 1 - x_i$,
3. $c_1 \vee \dots \vee c_m \mapsto 1 - \prod_{i \in [m]} (1 - \Gamma(c_i))$,
4. $c_1 \wedge \dots \wedge c_m \mapsto \prod_{i \in [m]} \Gamma(c_i)$.

This map allows us to transform C into an arithmetic circuit for p . For a Boolean circuit C , we denote the arithmetized circuit by $\text{arithmetize}(C)$. Here, we remark that the degree of $\text{arithmetize}(C)$ can become *exponentially* large; this is because there is no known depth-reduction for Boolean circuits, and hence the degree may double at each step, owing to an exponential blowup in the degree.

Valiant's classes in the unbounded setting. It is well-known that an algebraic circuit of size s , can compute polynomials of degree $\exp(s)$; e.g., $f(x) = x^{2^s}$, and $L(f) = O(s)$. This brings us to the next definition, the class VP_{nb} , originally defined in [17]. A sequence of polynomials $(f) = (f_n)_n \in \text{VP}_{\text{nb}}$, if the number of variables in f_n and $L(f_n)$ are both p -bounded (the degree *may be* exponentially large). The subscript “nb” signifies the “*not bounded*” phenomenon on the degree of the polynomial, in contrast to the original class VP . Similarly, a sequence of polynomials $(f) = (f_n)_n \in \text{VNP}_{\text{nb}}$, if there exists a p -bounded function $q(n)$ and $g_n(\mathbf{X}, Y_1, \dots, Y_{q(n)}) \in \text{VP}_{\text{nb}}$ where

$$f_n(\mathbf{X}) = \sum_{\bar{y} \in \{0,1\}^{q(n)}} g_n(\mathbf{X}, y_1, \dots, y_{q(n)}).$$

One can analogously define VP_{nb}^0 and VNP_{nb}^0 , in the constant-free setting. It is obvious that $\text{VP}_{\text{nb}} = \text{VNP}_{\text{nb}}$ implies $\text{VP} = \text{VNP}$, but the converse is *unclear*. However, [17] showed that over a ring of positive characteristic, the converse holds, i.e., $\text{VP} = \text{VNP}$ implies $\text{VP}_{\text{nb}} = \text{VNP}_{\text{nb}}$! On the other hand, [16] showed that $\text{VP}^0 = \text{VNP}^0$ implies that $\text{VP}_{\text{nb}}^0 = \text{VNP}_{\text{nb}}^0$, and the converse is unclear because it seems difficult to rule out the possibility that some polynomial family in VNP^0 does not lie in VP^0 , but still in VP (i.e., computable by polynomial-size algebraic circuits using *exponentially large-bit* integers).

2.2 Parameterized Valiant’s classes

Parameterized Valiant’s classes were introduced in [3]. We will briefly review the definitions and results there and extend them to the constant-free and unbounded setting. We first start with the fixed-parameter tractable classes. The W -hierarchies will be introduced later since we only need them in the second part of this work.

Our families of polynomials will now have two indices. They will be of the form $(p_{n,k})$. Here, n is the index of the family and k is the parameter. We will say a polynomial family $(p_{n,k})$ is a *parameterized p -family* if the number of variables is p -bounded in n and the degree is p -bounded in n, k . If there is no bound on the degree, we say it is *parameterized family*.

The most natural parameterization is by the degree: Let (p_n) be any p -family then we get a parameterized family $(p_{n,k})$ by setting $p_{n,k} :=$ the homogeneous part of degree k of p_n . For more details, we will refer the reader to [3].

We now define fixed-parameter variants of Valiant’s classes with the constant-free version.

► **Definition 4** (Algebraic FPT classes).

1. A parameterized p -family $(p_{n,k})$ is in VFPT iff $L(p_{n,k})$ is upper bounded by $f(k)q(n)$ for some p -bounded function q and some function $f : \mathbb{N} \rightarrow \mathbb{N}$ (such bound will be called an *fpt bound*). If one removes the requirement of p -family on $p_{n,k}$, and imposes only that the number of variables is p -bounded, one gets the class VFPT_{nb} .
2. A parameterized p -family $p_{n,k}$ is in VFPT^0 iff $\tau(p_{n,k})$ is upper bounded by $f(k)q(n)$ for some p -bounded function q and some function $f : \mathbb{N} \rightarrow \mathbb{N}$. Similarly, one gets $\text{VFPT}_{\text{nb}}^0$, if one removes the requirement of p -family, and imposes only that the number of variables is p -bounded.

We remark that in the above, f need not be computable as Valiant’s model is non-uniform.

► **Definition 5** (Fpt-projection). A parameterized family $f = (f_{n,k})$ is an *fpt-projection* of another parameterized family $g = (g_{n,k})$ if there are functions $r, s, t : \mathbb{N} \rightarrow \mathbb{N}$ such that r is p -bounded, s, t are functions and $f_{n,k}$ is a projection of $g_{r(n)s(k),k'}$ for some $k' \leq t(k)$ ². We write $f \leq_p^{\text{fpt}} g$.

However p -projection in Valiant’s world seems to be *weaker* compared to parsimonious poly-time reduction in the Boolean world; therefore we need a stronger notion of reduction for defining algebraic models of the Boolean $\#W$ -classes, see [3]. That’s why we are defining substitutions. We will analogously define it for constant-free model as well.

² k' might depend on n , but its size is bounded by a function in k . There are examples in the Boolean world, where this dependence on n is used.

► **Definition 6** (Fpt-substitution).

1. A parameterized family $f = (f_{n,k})$ is an fpt-substitution of another parameterized family $g = (g_{n,k})$ if there are functions $r, s, t, u : \mathbb{N} \rightarrow \mathbb{N}$ and polynomials $h_1, \dots, h_{u(r(n)s(k))} \in \mathbb{K}[\mathbf{X}]$ with both $L(h_i)$ and $\deg(h_i)$ fpt-bounded such that r, u are p -bounded, s, t are functions, and $f_{n,k}(\mathbf{X}) = g_{r(n)s(k),k'}(h_1, \dots, h_{u(r(n)s(k))})$ for some $k' \leq t(k)$. We write $f \leq_s^{\text{fpt}} g$. When we allow **unbounded** degree substitution of h_i (i.e. only $L(h_i)$ is fpt-bounded), we say that f is an fpt_{nb} -substitution of g . We denote this as $f \leq_s^{\text{fpt}_{\text{nb}}} g$.
2. A parameterized family $f = (f_{n,k})$ is a constant-free fpt-substitution of another parameterized family $g = (g_{n,k})$ if there are functions $r, s, t, u : \mathbb{N} \rightarrow \mathbb{N}$ and polynomials $h_1, \dots, h_{u(r(n)s(k))} \in \mathbb{K}[\mathbf{X}]$ with both $\tau(h_i)$ and $\deg(h_i)$ are fpt-bounded such that r, u are p -bounded, s, t are functions and $f_{n,k}(\mathbf{X}) = g_{r(n)s(k),k'}(h_1, \dots, h_{u(r(n)s(k))})$ for some $k' \leq t(k)$. We write $f \leq_s^{\tau\text{-fpt}} g$. If we remove the degree condition, we get fpt_{nb} -substitutions, denoted as $f \leq_s^{\tau\text{-fpt}_{\text{nb}}} g$.

One can define constant-free fpt-projections analogously. The following lemma should be immediate from the definitions, see [3] for a proof in the case of VFPT.

► **Lemma 7.** VFPT, VFPT_{nb} and their constant-free versions (VFPT^0 , $\text{VFPT}_{\text{nb}}^0$) are closed under fpt-projections and fpt-substitutions (constant-free fpt-projections and constant-free fpt-substitutions, respectively).

3 Linear counting hierarchy

In this section, we define the linear counting hierarchy, a variant of the counting hierarchy, which will allow us to talk about subexponential complexity. The original counting hierarchy was defined by Wagner [24]. We here restrict the witness length to be linear, which is important when dealing with exponential complexity. Allender et al. [2] also define a linear counting hierarchy. Their definition is not comparable to ours. We use an operator-based definition: The base class is deterministic polynomial time and the witness length is linearly bounded. Allender et al. use an oracle TM definition: The oracle Turing machine is probabilistic and linear time bounded, which automatically bounds the query lengths.

► **Definition 8.** Given a complexity class K , we define $\mathbf{C}.K$ to be the class of all languages A such that there is some $B \in K$ and a function $p : \mathbb{N} \rightarrow \mathbb{N}$, $p(n) = O(n^c)$ for some constant c , and some polynomial time computable function $f : \{0, 1\}^* \rightarrow \mathbb{N}$ such that,

$$x \in A \iff |\{y \in \{0, 1\}^{p(|x|)} : \langle x, y \rangle \in B\}| > f(x).$$

We start from $\mathbf{C}_0\mathbf{P} := \mathbf{P}$ and for all $k \in \mathbb{N}$, $\mathbf{C}_{k+1}\mathbf{P} := \mathbf{C}.\mathbf{C}_k\mathbf{P}$. Then the *counting hierarchy* is defined as $\text{CH} := \bigcup_{k \geq 0} \mathbf{C}_k\mathbf{P}$. We now define our linear counting hierarchy:

► **Definition 9.** Given a complexity class K , we define $\mathbf{C}_{\text{lin}}.K$ to be the class of all languages A such that there is some $B \in K$ and a function $\ell : \mathbb{N} \rightarrow \mathbb{N}$, $\ell(n) = O(n)$, and some polynomial time computable function $f : \{0, 1\}^* \rightarrow \mathbb{N}$ such that,

$$x \in A \iff |\{y \in \{0, 1\}^{\ell(|x|)} : \langle x, y \rangle \in B\}| > f(x).$$

We define $\mathbf{C}\text{-lin}_0\mathbf{P} := \mathbf{P}$ and for all $k \in \mathbb{N}$, $\mathbf{C}\text{-lin}_{k+1}\mathbf{P} := \mathbf{C}_{\text{lin}}.\mathbf{C}\text{-lin}_k\mathbf{P}$. The *linear counting hierarchy* is $\text{CH}_{\text{lin}} := \bigcup_{k \geq 0} \mathbf{C}\text{-lin}_k\mathbf{P}$.

Now, we slightly modify the above definition to get $\exists_{\text{lin}}.K$ and $\forall_{\text{lin}}.K$ in the following way: $x \in A \iff \exists y \in \{0, 1\}^{\ell(|x|)} : \langle x, y \rangle \in B$ and $x \in A \iff \forall y \in \{0, 1\}^{\ell(|x|)} : \langle x, y \rangle \in B$, respectively. Clearly, it can be said that $K \subseteq \exists_{\text{lin}}.K \subseteq \mathbf{C}_{\text{lin}}.K$ and $K \subseteq \forall_{\text{lin}}.K \subseteq \mathbf{C}_{\text{lin}}.K$.

We can define the linear counting hierarchy in a slightly easier manner.

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► **Definition 10.** Given a complexity class K , we define $\mathbf{C}'_{\text{lin}}.K$ to be the class of all languages A such that there is some $B \in K$ and a function $\ell : \mathbb{N} \rightarrow \mathbb{N}$, $\ell(n) = O(n)$, such that

$$x \in A \iff |\{y \in \{0,1\}^{\ell(|x|)} : \langle x, y \rangle \in B\}| > 2^{\ell(|x|)-1}.$$

It is clear that $\mathbf{C}'_{\text{lin}}.K \subseteq \mathbf{C}_{\text{lin}}.K$ for any class K . Moreover, by an easy adaption of the proof of [22, Lemma 3.3], for any language $K \in \text{CH}$, $\mathbf{C}_{\text{lin}}.K \subseteq \mathbf{C}'_{\text{lin}}.K$. Also, from the definition, we can say that $\text{CH}_{\text{lin}}\text{P} \subseteq \text{CH}$. Therefore, the following holds.

► **Fact 11.** $\text{C-lin}_{k+1}\text{P} = \mathbf{C}'_{\text{lin}}.\text{C-lin}_k\text{P}$.

We also need a subexponential version of the counting hierarchy. Let $\text{SUBEXP} = \text{DTime}(2^{o(n)})$. Then we set $\text{C-lin}_0\text{SUBEXP} = \text{SUBEXP}$ and for all $k \in \mathbb{N}$, $\text{C-lin}_{k+1}\text{SUBEXP} := \mathbf{C}_{\text{lin}}.\text{C-lin}_k\text{SUBEXP}$. Moreover, $\text{CH}_{\text{lin}}\text{SUBEXP} = \bigcup_{k \geq 0} \text{C-lin}_k\text{SUBEXP}$.

Here we define a few more terms that we shall use later in Section 5. We set $\text{NP}_{\text{lin}} = \exists_{\text{lin}}.\text{P}$, NP with linear witness size. In the same way, we can define the levels of the linear polynomial time hierarchy, Σ_i^{lin} and Π_i^{lin} , by applying the operators \exists_{lin} and \forall_{lin} in an alternating fashion to P . The linear polynomial hierarchy PH_{lin} is the union over all Σ_i^{lin} .

From the above definitions, we get the following conclusion.

► **Fact 12.** $\text{NP}_{\text{lin}} \subseteq \text{PH}_{\text{lin}} \subseteq \text{CH}_{\text{lin}}$.

4 Connecting Valiant's model to the counting hierarchy

In this section, we aim to prove that subexponential upper bounds for exponential sums imply a collapse of the linear counting hierarchy (for a definition, see Section 3). To show this, we will define a polynomial family p-log-Expsum and show that $\text{p-log-Expsum} \in \text{VFPT}_{\text{nb}}^0$ is equivalent to exponential sums having subexponential circuits (Corollary 34). $\text{p-log-Expsum} \in \text{VFPT}_{\text{nb}}^0$ will imply a collapse of the linear counting hierarchy (Theorem 13).

4.1 log-variate exponential sum polynomial family

In this section, we will define a parameterized log-variate exponential sum polynomial family,

$$\text{p-log-Expsum}_{m,k}(g) := \sum_{y \in \{0,1\}^{\ell(n)}} g_n(\mathbf{X}, y),$$

where \mathbf{X} has n variables, $\ell(n) = O(n)$, and g_n has circuits of size m ($n = \Omega(\log m)$), and the parameter is $k = \frac{n}{\log m}$. m and k are functions of n . Note that the running parameter of the family is m . When we write $\text{p-log-Expsum} \in \text{VFPT}$, we mean that $\{\text{p-log-Expsum}_{m,k}(g)\}_{m,k} \in \text{VFPT}$ for all families g . We are allowing g to have *unbounded* degree, i.e., g may not necessarily be a p -family. We will also be using constant-free circuits computing g in the constant-free context.

4.2 Collapsing of $\text{CH}_{\text{lin}}\text{SUBEXP}$

The main theorem of the section is the following:

► **Theorem 13.** If $\text{p-log-Expsum} \in \text{VFPT}_{\text{nb}}^0$, then for every language L in $\text{CH}_{\text{lin}}\text{SUBEXP}$, we have a constant-free algebraic circuit χ_L so that $x \in L \implies \chi_L(x) = 1$, $x \notin L \implies \chi_L(x) = 0$ and χ_L has size $2^{o(n)}$.

Proof. We prove the above statement by induction on the level of $\text{CH}_{\text{lin}}\text{SUBEXP}$. By definition, $\text{CH}_{\text{lin}}\text{SUBEXP} = \bigcup_{k \geq 0} \text{C-lin}_k\text{SUBEXP}$. For $k = 0$, $\text{C-lin}_k\text{SUBEXP} = \text{SUBEXP}$. Now by standard arithmetization, we can get a $2^{o(n)}$ size, unbounded degree constant-free circuit for each $L \in \text{SUBEXP}$, so that the above-mentioned condition holds.

Now, by induction hypothesis say, it is true up to k -th level of the hierarchy. We will prove that it is true for the $(k+1)$ -th level. Take any $B \in \text{C-lin}_{k+1}\text{SUBEXP}$. By Fact 11 and Definition 10, there exists $A \in \text{C-lin}_k\text{SUBEXP}$ such that

$$x \in B \iff |\{y \in \{0,1\}^{\ell(|x|)} : \langle x, y \rangle \in A\}| > 2^{\ell(|x|)-1},$$

where ℓ is some linear polynomial. By slight abuse of notation, let χ_A denote an algebraic circuit capturing the characteristic function for A , i.e.,

$$\chi_A(x, y) = 1 \iff \langle x, y \rangle \in A.$$

By the induction hypothesis, we can assume that χ_A has size $2^{o(|x|)}$. Now, one can equivalently write the following:

$$x \in B \iff \sum_{y \in \{0,1\}^{\ell(|x|)}} \chi_A(x, y) > 2^{\ell(|x|)-1}.$$

In this way, we get an instance of p-log-Expsum , $\sum_{y \in \{0,1\}^{\ell(|x|)}} \chi_A(x, y)$, where the size of χ_A is $m = 2^{o(|x|)}$ and it computes a polynomial of *unbounded degree* (there is no depth-reduction known for Boolean circuits and thus, it cannot be reduced).

As $\text{p-log-Expsum} \in \text{VFPT}_{\text{nb}}^0$, there is an algebraic circuit C such that $C(x) := \sum_{y \in \{0,1\}^{\ell(|x|)}} \chi_A(x, y)$ and C has subexponential-size by Theorem 15.

Trivially, $\tau(2^{\ell(|x|)-1}) \leq \text{poly}(|x|)$. So, we can make C first constant-free and then Boolean by the standard procedure of computing on the binary representation modulo $2^{\ell(n)}$. Let \tilde{C} is the Boolean circuit that computes the highest bit. We just arithmetize \tilde{C} and take $\chi_B = \text{arithmetize}(\tilde{C})$. Each time we convert the arithmetic circuit to a Boolean one and arithmetize the Boolean circuit, we incur only a small polynomial blow-up in size. Therefore, χ_B has subexponential-size, as desired. \blacktriangleleft

► **Remark 14.** Clearly, $\text{CH}_{\text{lin}}\text{P} \subseteq \text{CH}_{\text{lin}}\text{SUBEXP}$ and hence, $\text{p-log-Expsum} \in \text{VFPT}_{\text{nb}}^0$ implies that every language in $\text{CH}_{\text{lin}}\text{P}$ has subexponential-size constant-free algebraic circuits.

► **Theorem 15.** *If $\text{p-log-Expsum} \in \text{VFPT}_{\text{nb}}^0$, then $\sum_{y \in \{0,1\}^{\ell(n)}} g(\mathbf{X}, y)$ has circuits of size $2^{o(n)} \text{poly}(m)$.*

Proof. Assume that p-log-Expsum has circuits of size $f(n/\log m) \text{poly}(m)$. We can assume that f is an increasing function. Let $i(n) = \max(\{1\} \cup \{j \mid f(j) \leq n\})$. $i(n)$ is nondecreasing and unbounded. Moreover, $f(i(n)) \leq n$ for all but finitely many n .

We will prove that $\sum_{y \in \{0,1\}^{\ell(n)}} g(\mathbf{X}, y)$ has circuits of size $2^{n/i(n)} \text{poly}(m)$. If $m \geq 2^{n/i(n)}$, then $f(n/\log m) \leq f(i(n)) \leq n$, thus there are circuits of size $n \cdot \text{poly}(m) = \text{poly}(m)$. If $m < 2^{n/i(n)}$, then let $\hat{m} = 2^{n/i(n)}$. We can take a circuit C for g and pad it to a circuit \hat{C} of size s with $\hat{m} \leq s \leq O(\hat{m})$, such that \hat{C} has the same variables as C . Then let $\hat{k} = n/\log \hat{m}$. Thus, $\sum_{y \in \{0,1\}^{\ell(n)}} g(\mathbf{X}, y)$ has circuits of size $f(\hat{k}) \text{poly}(\hat{m}) = n \cdot \text{poly}(2^{n/i(n)})$. \blacktriangleleft

We will need the unbounded version as stated above, but a similar proof also works for the bounded case. The same is true of the non-constant-free version. We will also need the following converse direction:

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► **Theorem 16.** Let $\sum_{y \in \{0,1\}^{\ell(n)}} g(\mathbf{X}, y)$ have circuits of size $2^{o(n)} \text{poly}(m)$ for each g of size m . Then $\text{p-log-Expsum} \in \text{VFPT}_{\text{nb}}^0$.

Proof. Let C_n be a circuit for $\sum_{y \in \{0,1\}^{\ell(n)}} g(\mathbf{X}, y)$ of size $2^{O(n/i(n))} \text{poly}(m)$ for some non-decreasing and unbounded function i . Let f be a nondecreasing function such that $f(i(n)) \geq 2^n$. We claim that p-log-Expsum has circuits of size $f(k) \text{poly}(m)$ with $k = n/\log m$. If $m \geq 2^{n/i(n)}$, then C_n has size $\text{poly}(m) \leq f(k) \text{poly}(m)$. Otherwise, $k = n/\log m \geq i(n)$ and therefore $f(k) \geq 2^n$. Thus, the trivial circuit for $\sum_{y \in \{0,1\}^{\ell(n)}} g(\mathbf{X}, y)$ has size $f(k) \text{poly}(m)$. ◀

5 Integers definable in $\text{CH}_{\text{lin}}\text{P}$

In [6, Section 3], integers are studied that are definable in the counting hierarchy. We adapt this notation to the *linear* counting hierarchy. Formally, we are given a sequence of integers $(a(n, k))_{n \in \mathbb{N}, k \leq q(n)}$ for some p -bounded function $q : \mathbb{N} \rightarrow \mathbb{N}$. We can assume that $|a(n, k)| \leq 2^{n^c}$ for some constant c . In other words, the bit-size of $a(n, k)$ is at most *exponential*, as we think n, k has been represented in binary by $O(\log n)$ bits. Now consider two languages,

$$\begin{aligned} \text{sgn}(a) &:= \{(n, k) : a(n, k) \geq 0\} \quad \text{and} \\ \text{Bit}(|a|) &:= \{(n, k, j, b) : j\text{th bit of } |a(n, k)| \text{ is } b\}. \end{aligned}$$

Here in both of these two languages, n, k, j are given in binary representation.

► **Definition 17.** We say an integer sequence $(a(n, k))_{n \in \mathbb{N}, k \leq q(n)}$ for some p -bounded function q is definable in $\text{CH}_{\text{lin}}\text{P}$ whenever both of $\text{sgn}(a)$ and $\text{Bit}(|a|)$ are in $\text{CH}_{\text{lin}}\text{P}$.

Chinese remainder language. Now, we define another language and make a connection to the definition of an integer sequence to be definable in $\text{CH}_{\text{lin}}\text{P}$, via the *Chinese remainder representation*. Given that the bit-size of $a(n, k)$ is at most n^c , we consider the set of all primes $p < n^{2c}$. The product of all such primes is $> 2^{n^c}$. Therefore, from $a(n, k) \bmod p$, for all primes $p < n^{2c}$, we can recover $a(n, k)$. Consider

$$\text{CR}(a) := \{(n, k, p, j, b) : p \text{ prime, } p < n^{2c}, j\text{-th bit of } (a(n, k) \bmod p) \text{ is } b\}.$$

Now we show an essential criterion for a sequence to be in $\text{CH}_{\text{lin}}\text{P}$. It is an adaption with some additional modifications and observations from [12], which were further implemented in [6, Theorem 3.5].

► **Theorem 18.** Let $(a(n, k))_{n \in \mathbb{N}, k \leq q(n)}$ be a integer sequence of exponential bit-size ($|a(n, k)| < 2^{n^c}$). Then, $(a(n, k))$ is definable in $\text{CH}_{\text{lin}}\text{P}$ iff both $\text{sgn}(a)$ and $\text{CR}(a)$ are in $\text{CH}_{\text{lin}}\text{P}$.

Now, we can prove an important *closure* property of non-negative integers definable in $\text{CH}_{\text{lin}}\text{P}$, which we shall use later.

► **Theorem 19 (Closure properties).** Let $(a(n, k))_{n \in \mathbb{N}, k \leq q(n)}$ be a non-negative integer sequence for some p -bounded function $q : \mathbb{N} \rightarrow \mathbb{N}$ with $a(n, k)$ having bit-size $< n^c$ and it is definable in $\text{CH}_{\text{lin}}\text{P}$. Consider the sum and product of $a(n, k)$ defined as follows:

$$b(n) := \sum_{k=0}^{q(n)} a(n, k) \quad \text{and} \quad c(n) := \prod_{k=0}^{q(n)} a(n, k).$$

Then, both of $(b(n))_{n \in \mathbb{N}}$ and $(c(n))_{n \in \mathbb{N}}$ are definable in $\text{CH}_{\text{lin}}\text{P}$.

► **Corollary 20.** Take $a(n, k) := \sigma_{n,k}(1, \dots, n)$, $k \leq n$, where $\sigma_{n,k}(z_1, \dots, z_n)$ is the k -th elementary symmetric polynomial on variables z_1, \dots, z_n . Then, $(a(n, k))_{n \in \mathbb{N}, k \leq n}$ is definable in $\text{CH}_{\text{lin}}\text{P}$.

6 Connecting the counting hierarchy to the τ -conjecture

In this section, we connect the τ -conjecture to the counting hierarchy. Specifically, we show that the collapse of $\text{CH}_{\text{lin}}\text{P}$ implies that some explicit polynomial, whose coefficients are definable in $\text{CH}_{\text{lin}}\text{P}$, is “easy”. Formally, we prove the following theorem:

► **Theorem 21.** Say, $(a(n))_{n \in \mathbb{N}}$ and $(b(n, k))_{k \leq q(n), n \in \mathbb{N}}$ are both definable in $\text{CH}_{\text{lin}}\text{P}$. Here q is some p -bounded function. If $\text{p-log-Expsum} \in \text{VFPT}_{\text{nb}}^0$ then the following holds:

1. $\tau(a(n)) = n^{o(1)}$,
2. If $f_n(X) := \sum_{k=1}^{q(n)} b(n, k)X^k$ then $\tau(f_n) = n^{o(1)}$.

Proof. We can assume that if $a(n)$ is definable in $\text{CH}_{\text{lin}}\text{P}$, $|a(n)| \leq 2^{n^c}$, that is, the bit-size of any integer definable in $\text{CH}_{\text{lin}}\text{P}$ is polynomially bounded. Furthermore, if $\text{p-log-Expsum} \in \text{VFPT}_{\text{nb}}^0$, then every language in $\text{CH}_{\text{lin}}\text{P}$ has subexponential-size circuits by Theorem 13. We will use both facts below.

Proof of part (1). Let $a(n) = \sum_{j=1}^{p(n)} a(n, j)2^j$ be the binary decomposition of $a(n)$ and $p(n) = O(n^c)$. Define a new polynomial:

$$A_{\lceil \log n \rceil}(Y_1, \dots, Y_{\text{bit}(n)}) := \sum_{j=0}^{p(n)} a(n, j)Y_1^{j_1} \dots Y_{\text{bit}(n)}^{j_{\text{bit}(n)}},$$

where $\text{bit}(n) := \lceil \log(p(n)) \rceil$. By our assumption, we can decide if $a(n, j) = b$ by a subexponential-size circuit, given input n and j in binary. Say, $C_r(\mathbf{N}, \mathbf{J})$ is the corresponding circuit, where $r = \lceil \log n \rceil$. We have $C_r(n_1, \dots, n_{\lceil \log n \rceil+1}, j_1, \dots, j_{\text{bit}(n)}) = a(n, j)$, where the n_i 's and the j_i 's are the bits of n and j , respectively. Consider the polynomial

$$F_r(J_1, \dots, J_{cr+1}, N_1, \dots, N_{r+1}, Y_1, \dots, Y_{cr+1}) := C_r(\mathbf{N}, \mathbf{J}) \cdot \prod_{i=1}^{cr+1} (J_i Y_i + 1 - J_i).$$

Now, by our assumption and Theorem 13, we can say that F_r has $2^{o(r)}$ size constant-free algebraic circuits (of unbounded degree). Consider the exponential sum

$$\tilde{F}_r(\mathbf{N}, \mathbf{Y}) := \sum_{j \in \{0,1\}^{cr+1}} F_r(j, \mathbf{N}, \mathbf{Y}).$$

It is an instance of p-log-Expsum with $\tau(F_r) = 2^{o(r)}$. By assumption, this implies that $\tau(\tilde{F}_r) = 2^{o(r)}$. Finally, note that $A_{\lceil \log n \rceil}(\mathbf{Y}) = \tilde{F}_r(n_1, \dots, n_{r+1}, \mathbf{Y})$, and $a(n) = A_{\lceil \log n \rceil}(2^{2^0}, \dots, 2^{2^{\text{bit}(n)-1}})$. Therefore,

$$\tau(a(n)) \leq \tau(\tilde{F}_r) + \tau(2^{2^{\text{bit}(n)-1}}) \leq n^{o(1)},$$

as desired.

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Proof of part (2). Again we can assume that $|b(n, k)|$ has polynomially many bits. Let $b(n, k) = \sum_{j=1}^{p(n)} b(n, k, j)2^j$ be the binary decomposition with $p(n) = O(n^{c'})$ and $q(n) = O(n^c)$. Define

$$B_{\lceil \log n \rceil}(Y_1, \dots, Y_{\mu(n)}, Z_1, \dots, Z_{\lambda(n)}) := \sum_{k=0}^{q(n)} \sum_{j=0}^{p(n)} b(n, k, j) Y_1^{j_1} \dots Y_{\mu(n)}^{j_{\mu(n)}} Z_1^{k_1} \dots Z_{\lambda(n)}^{k_{\lambda(n)}}.$$

Here $\mu(n) := \lceil \log(p(n)) \rceil$ and $\lambda(n) := \lceil \log(q(n)) \rceil$. Let the variable sets be $\mathbf{J} = (J_1, \dots, J_{c'r+1})$, $\mathbf{N} = (N_1, \dots, N_{r+1})$, $\mathbf{K} = (K_1, \dots, K_{cr+1})$, $\mathbf{Y} = (Y_1, \dots, Y_{c'r+1})$, $\mathbf{Z} = (Z_1, \dots, Z_{cr+1})$, where again $r = \lceil \log n \rceil$. Define a new polynomial F_r as follows:

$$F_r(\mathbf{J}, \mathbf{K}, \mathbf{N}, \mathbf{Y}, \mathbf{Z}) := D_r(\mathbf{N}, \mathbf{J}, \mathbf{K}) \cdot \prod_{m=1}^{c'r+1} (J_m Y_m + 1 - J_m) \prod_{s=1}^{cr+1} (K_s Z_s + 1 - Z_s).$$

Like in the previous part of the proof, $(D_r(\mathbf{N}, \mathbf{J}, \mathbf{K}))_r$ is the circuit family for computing $(b(n, k, j))$. In particular,

$$D_r(n_1, \dots, n_{r+1}, j_1, \dots, j_{\mu(n)}, k_1, \dots, k_{\lambda(n)}) = b(n, k, j).$$

By our assumption, D_r has $2^{o(r)}$ size constant-free algebraic circuits (of unbounded degree). Consider,

$$\tilde{F}_r(\mathbf{N}, \mathbf{Y}, \mathbf{Z}) = \sum_{j \in \{0,1\}^{c'r+1}} \sum_{k \in \{0,1\}^{cr+1}} F_r(j, k, \mathbf{N}, \mathbf{Y}, \mathbf{Z}).$$

It is an instance of $\mathbf{p}\text{-log-Expsum}$ with $\tau(F_r)$ is $2^{o(r)}$. Since $\mathbf{p}\text{-log-Expsum} \in \text{VFPT}_{\text{nb}}^0 \implies \tau(\tilde{F}_r) = 2^{o(r)}$. Now, $B_{\lceil \log n \rceil}(\mathbf{Y}, \mathbf{Z}) = F_r(n_1, \dots, n_{r+1}, \mathbf{Y}, \mathbf{Z})$ and

$$f_n(X) = B_{\lceil \log n \rceil}(2^{2^0}, \dots, 2^{2^{\mu(n)-1}}, X^{2^0}, \dots, X^{2^{\lambda(n)-1}}).$$

Therefore, $\tau(f_n) \leq \tau(B_{\lceil \log n \rceil}) + \tau(2^{2^{\mu(n)}}) + \tau(X^{2^{\lambda(n)}}) \leq n^{o(1)}$, as desired. \blacktriangleleft

► **Theorem 22.** *If the τ -conjecture is true, then $\mathbf{p}\text{-log-Expsum} \notin \text{VFPT}_{\text{nb}}^0$.*

Proof. Take the Pochhammer polynomial $p_n(X) = \prod_{i=1}^n (X + i)$. The coefficient of X^{n-k} in p_n will be $\sigma_k(1, \dots, n)$, where $\sigma_k(z_1, \dots, z_n)$ is the k -th elementary symmetric polynomial in variables z_1, \dots, z_n . And $(\sigma_k(1, \dots, n))_{n \in \mathbb{N}, k \leq n}$ is definable in linear counting hierarchy by Corollary 20. By Theorem 21, $(p_n)_{n \in \mathbb{N}}$ has $n^{o(1)}$ size constant-free circuit if $\mathbf{p}\text{-log-Expsum}$ is fixed-parameter tractable. But p_n has distinct n many integer roots. So, assuming the τ -conjecture, $\mathbf{p}\text{-log-Expsum}$ is not *fpt*. \blacktriangleleft

► **Remark 23.** Instead of taking the Pochhammer polynomial, there are many other possible choices for some explicit polynomial, see [6].

Finally, we prove the exponential lower bound for an exponential sum, proving Theorem 1.

► **Theorem 24 (Exponential algebraic lower bound).** *If the τ -conjecture is true, then there exists an n -variate polynomial family $\sum_{y \in \{0,1\}^n} g_n(X, y)$, which requires $2^{\Omega(n)}$ -size circuits.*

Proof. If the τ -conjecture is true, then Theorem 22 shows that $\mathbf{p}\text{-log-Expsum} \notin \text{VFPT}_{\text{nb}}^0$. By the contrapositive statement of Theorem 16, the existence of such a hard exponential sum follows. \blacktriangleleft

► **Remark 25.** The family g_n simply is a universal circuit of size polynomial in n , where the polynomial is large enough to simulate the computation of the Turing machine that shows that the n -th Pochhammer polynomial is definable in $\text{CH}_{\text{lin}}\text{P}$.

7 Preliminaries II: The VW-hierarchy

In this section, we define different variants of the VW-hierarchy, which will be analogous to $\#W$ -hierarchy, see [3]. We will consider circuits that can have unbounded fanin gates.

► **Definition 26** (Weft). *For an algebraic circuit C , the weft of C is the maximum number of unbounded fan-in gates on any path from a leaf to the root.*

For $n \geq k \in \mathbb{N}$, let $\langle \binom{n}{k} \rangle$ be the set of all vectors in $\{0, 1\}^n$ which have exactly k many 1s.

► **Definition 27.**

1. A parameterized p -family $f_{n,k}(\mathbf{X})$ is in $\text{VW}[F]$ iff there exists a p -bounded function $q(n)$ and p -family $g_n(\mathbf{X}, y_1, \dots, y_{q(n)})$ such that $f_{n,k} \leq_s^{\text{fpt}} \sum_{\bar{y} \in \langle \binom{q(n)}{k} \rangle} g_n(\mathbf{X}, y_1, \dots, y_{q(n)})$ and g_n can be computed by a polynomial-size formula.
2. A parameterized family $f_{n,k}(\mathbf{X})$ is in $\text{VW}_{\text{nb}}[F]$ iff there exists a p -bounded function $q(n)$ and family $g_n(\mathbf{X}, y_1, \dots, y_{q(n)})$ such that $f_{n,k} \leq_s^{\text{fpt}_{\text{nb}}} \sum_{\bar{y} \in \langle \binom{q(n)}{k} \rangle} g_n(\mathbf{X}, y_1, \dots, y_{q(n)})$ and g_n can be computed by a polynomial-size formula.
3. A parameterized p -family $f_{n,k}(\mathbf{X})$ is in $\text{VW}^0[F]$ iff there exists a p -bounded function $q(n)$ and p -family $g_n(\mathbf{X}, y_1, \dots, y_{q(n)})$ such that $f_{n,k} \leq_s^{\tau\text{-fpt}} \sum_{\bar{y} \in \langle \binom{q(n)}{k} \rangle} g_n(\mathbf{X}, y_1, \dots, y_{q(n)})$ and g_n can be computed by a constant-free, polynomial-size formula.
4. A parameterized family $f_{n,k}(\mathbf{X})$ is in $\text{VW}_{\text{nb}}^0[F]$ iff there exists a p -bounded function $q(n)$ and family $g_n(\mathbf{X}, y_1, \dots, y_{q(n)})$ such that $f_{n,k} \leq_s^{\tau\text{-fpt}_{\text{nb}}} \sum_{\bar{y} \in \langle \binom{q(n)}{k} \rangle} g_n(\mathbf{X}, y_1, \dots, y_{q(n)})$ and g_n can be computed by a constant-free, polynomial-size formula.

In some sense, $\text{VW}[F]$ is a substitution of a *weighted sum* of formulas. We will define $\text{VW}[P]$ as a weighted sum as above, but summing over an arbitrary circuit of polynomial-size. Similarly, we can define $\text{VW}^0[P]$, and its counterpart in the unbounded setting, i.e. $\text{VW}_{\text{nb}}[P]$, and $\text{VW}_{\text{nb}}^0[P]$.

Finally, we will define the completeness notion:

► **Definition 28.** *We will say a parameterized p -family $f_{n,k}$ is $\text{VW}[F]$ -hard if every $g_{n,k} \in \text{VW}[F]$, $g_{n,k} \leq_s^{\text{fpt}} f_{n,q}$. Similarly, we can define completeness for $\text{VW}[P]$.*

We can also define completeness and hardness in the constant-free and unbounded models.

8 Conditional collapsing of VW-hierarchy and applications

Let us recall the definition of k -degree n -variate ($n \geq k$) *elementary symmetric polynomial* $\sigma_{n,k}(\mathbf{X}) := \sum_{y \in \langle \binom{n}{k} \rangle} X_1^{y_1} X_2^{y_2} \dots X_n^{y_n}$. It is known that $(\sigma_{n,k})_n \in \text{VP}^0$, with a simple dynamic programming algorithm; see [13, Section 4]. Let us define a new polynomial family $B_{n,k}(\mathbf{X})$, which will be important in the latter part of the section: $B_{n,k}(\mathbf{X}) := \sum_{t=0}^{n-k} (-1)^t \binom{k+t}{k} \cdot \sigma_{n,k+t}(\mathbf{X})$. The following claim is crucial:

▷ **Claim 29.** For $y \in \{0, 1\}^n$, $B_{n,k}(y) = \begin{cases} 1, & \text{if } y \in \langle \binom{n}{k} \rangle, \\ 0, & \text{otherwise.} \end{cases}$

Proof. For a string $y \in \{0, 1\}^n$, we will call the *weight* of y , denoted $\text{wt}(y)$, the number of 1's present in y . Note that if $\text{wt}(y) < k$, then $\sigma_{n,k}(y) = 0$ implying $B_{n,k}(y) = 0$. Similarly if $\text{wt}(y) = k$, then $B_{n,k}(y) = \sigma_{n,k}(y)$, which will be exactly equal to 1. Now if $\text{wt}(y) = k + r$ where $r > 0$, then

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$$\begin{aligned}
 B_{n,k}(y) &= \sum_{t=0}^{n-k} (-1)^t \binom{k+t}{k} \cdot \sigma_{n,k+t}(y) = \sum_{t=0}^r (-1)^t \binom{k+t}{k} \cdot \sigma_{n,k+t}(y) \\
 &= \sum_{t=0}^r (-1)^t \binom{k+t}{k} \cdot \binom{k+r}{k+t} \\
 &= \sum_{t=0}^r (-1)^t \frac{(k+r)!}{k!t!(r-t)!}.
 \end{aligned}$$

Let us further define the tri-variate polynomial $Q(x, y, z) := (x + y - z)^{k+r} \in \mathbb{Z}[x, y, z]$. Note that the coefficient of x^k in $Q(x, y, z)$ is

$$\sum_{t=0}^r y^{r-t} z^t (-1)^t \cdot \frac{(k+r)!}{k!t!(r-t)!}.$$

Now putting $y = z = 1$, we get the coefficient exactly equal to $B_{n,k}(y)$; since $r \neq 0$, we can say that the coefficient of x^k in $Q(x, 1, 1)$ is 0, which finally implies that $B_{n,k}(y) = 0$. \triangleleft

Now we are ready to prove the following transfer theorem from the parameterized Valiant's classes to Valiant's algebraic models.

► **Theorem 30.** $\text{VW}^0[\text{P}] \neq \text{VFPT}^0 \implies \text{VP}^0 \neq \text{VNP}^0$. Similarly, $\text{VW}[\text{P}] \neq \text{VFPT} \implies \text{VP} \neq \text{VNP}$.

Proof. We will prove the contraposition. Assume that $\text{VP}^0 = \text{VNP}^0$. As mentioned before, we know that $(\sigma_{n,k})_n \in \text{VP}^0$. Further, since $k \in [n]$, for $t \leq n - k$, it is trivial to see that $\tau\left(\binom{k+t}{k}\right) = n^{O(1)}$. Therefore, for each $0 \leq t \leq n - k$, $(-1)^t \binom{k+t}{k} \cdot \sigma_{n,k+t}(\mathbf{X})$ has a VP^0 -circuit. Since VP^0 is closed under polynomially many additions, it follows that $(B_{n,k})_n \in \text{VP}^0$.

Let $q_{n,k} \in \text{VW}^0[\text{P}]$. By definition, there is a polynomial family $p_{n,k}$ of the above form $p_{n,k}(\mathbf{X}) := \sum_{y \in \binom{[n]}{k}} g_n(\mathbf{X}, y)$, where $g_n(\mathbf{X}, \mathbf{Y})$ is in VP^0 , such that $q_{n,k} \leq_s^{fpt} p_{n,k}$. By Claim 29, it follows that

$$p_{n,k} = \sum_{y \in \{0,1\}^n} g_n(\mathbf{X}, y) \cdot B_{n,k}(y).$$

We have already proved above that $B_{n,k}$ has $\text{poly}(n)$ sized constant-free circuits. Hence, $g_n(\mathbf{X}, y)B_{n,k}(y)$ has constant-free $\text{poly}(n)$ -size circuit. Therefore, by definition and our primary assumption, it follows that $p_{n,k} \in \text{VNP}^0 = \text{VP}^0 \subseteq \text{VFPT}^0$. Since, VFPT^0 is closed under constant-free fpt-substitution (Lemma 7), it follows that $q_{n,k} \in \text{VFPT}^0$, implying $\text{VW}^0[\text{P}] \subseteq \text{VFPT}^0$.

The proof in the usual (not constant-free) model also follows essentially along the same line as above. \triangleleft

► **Remark 31.** The above theorem holds in the unbounded regime as well, i.e., $\text{VW}_{\text{nb}}^0[\text{P}] \neq \text{VFPT}_{\text{nb}}^0 \implies \text{VP}_{\text{nb}}^0 \neq \text{VNP}_{\text{nb}}^0$ (which further implies $\text{VP}^0 \neq \text{VNP}^0$, see [16]). Similarly, $\text{VW}_{\text{nb}}[\text{P}] \neq \text{VFPT}_{\text{nb}} \implies \text{VP}_{\text{nb}} \neq \text{VNP}_{\text{nb}}$.

We now aim to prove a *conditional separation* of $\text{VW}_{\text{nb}}^0[\text{P}]$ and $\text{VFPT}_{\text{nb}}^0$, by showing that $\text{VW}_{\text{nb}}^0[\text{P}] = \text{VFPT}_{\text{nb}}^0$ implies a collapse of the linear counting hierarchy. To show this, we will show that $\text{VW}_{\text{nb}}^0[\text{P}] = \text{VFPT}_{\text{nb}}^0 \implies \text{p-log-Expsum} \in \text{VFPT}_{\text{nb}}^0$ (Corollary 34) from which the collapse of the linear counting hierarchy follows.

► **Theorem 32.** Let $f(\mathbf{X}) = \sum_{y \in \{0,1\}^{\ell(n)}} g(\mathbf{X}, y)$, where $\ell(\cdot)$ is a linear function and g is computed by an arithmetic circuit of size $m = 2^{O(n^c)}$ for some constant c . Then, $f(\mathbf{X})$ can be written as

$$f(\mathbf{X}) = \sum_{e \in \binom{[b(m)]}{k}} G(\mathbf{X}, e),$$

for some p -bounded function b and $k = \ell(n)/\log m$ and G has $\text{poly}(m)$ size circuits.

Proof. Let $f(\mathbf{X})$ be an instance of p -log-Expsum, i.e., $f(\mathbf{X}) = \sum_{y \in \{0,1\}^n} g(\mathbf{X}, y)$, where $g(\mathbf{X}, \mathbf{Y})$ has size m constant-free circuit. Here we mention that, although we just take sum over n variables here for the ease of presentation, the same proof also works if we sum over $\ell(n)$ many variables for some linear function ℓ .

Let us partition the variable set $\mathbf{Y} = \{Y_1, \dots, Y_n\} = E_1 \sqcup \dots \sqcup E_k$. Here $k = n/\log m$, and for all i , $|E_i| = \log m$. For each $S \subseteq E_i$, we take a *new variable* Z_i^S and we do this for all i . Define $\overline{Z}_i := \{Z_i^S : S \subseteq E_i\}$ and $\mathbf{Z} = \bigcup_i \overline{Z}_i$. The number of \mathbf{Z} -variables is $2^{\log m} \cdot k$, which is polynomial in m .

Let us call an assignment of \mathbf{Z} variables a *good assignment*, if *exactly* one variable in each set \overline{Z}_i is set to be 1. Below we show that there is a one-to-one correspondence between $\{0,1\}$ assignments to the \mathbf{Y} variables and *good assignments* to the \mathbf{Z} variables.

Let φ be a homomorphism from $R[\mathbf{Y}] \rightarrow R[\mathbf{Z}]$, where $R := \mathbb{F}[\mathbf{X}]$, such that $\varphi : Y_i \mapsto \prod_{S \subseteq E_i, Y_j \notin S} (1 - Z_i^S)$. Let us define $\tilde{g}(\mathbf{X}, \mathbf{Z}) := \varphi(g)$. Now let us fix an assignment $y \in \{0,1\}^n$ to the \mathbf{Y} variables. We construct a corresponding good assignment of \mathbf{Z} . For each E_i of \mathbf{Y} , we have some $S_i \subseteq E_i$ such that *each* variable of E_i , which is in S_i , gets value 1. The remaining variables in $E_i \setminus S_i$ get value 0 (so that it corresponds to y). Pick this particular $S_i \subseteq E_i$. Note that this S_i is *unique* (it can be the empty set). Now set $Z_i^{S_i} = 1$ and $Z_i^S = 0$, if $S \neq S_i$, for all $i \in [k]$.

Each variable in $\bigcup_i S_i$ gets the value 1 and variables in $\bigcup_i (E_i \setminus S_i)$ are assigned 0. Under the map φ , any $Y_j \in E_1 \setminus S_1$ is replaced by $\prod_{S \subseteq E_1, Y_j \notin S} (1 - Z_1^S)$. Since, $S_1 \subseteq E_1$ and $Y_j \notin S_1$, $(1 - Z_1^{S_1})$ occurs in the product. And, hence the product becomes 0. Now, let $Y_\ell \in S_1$ and $\varphi(Y_\ell) = \prod_{S \subseteq E_1, Y_\ell \notin S} (1 - Z_1^S)$. As $Y_\ell \in S_1$, $(1 - Z_1^{S_1})$ does not contribute to the product. Thus, under the assignment defined before, $\varphi(Y_\ell)$ becomes 1. This argument holds for any E_i . Therefore, one can conclude that

$$f = \sum_{e: e \text{ is a good assignment}} \tilde{g}(\mathbf{X}, e).$$

Note that the weft of the circuit for \tilde{g} has increased by 1 (from that of g), and the size has also increased by a polynomial (in m) factor. To capture a k -weight good assignment exactly, define a new polynomial $p(\mathbf{Z}) \in \mathbb{F}[\mathbf{Z}]$ as follows:

$$p(\mathbf{Z}) := \prod_{i=1}^k \left(\sum_{S \subseteq E_i} Z_i^S \right).$$

Clearly, p has a weft-2 circuit of size $\text{poly}(m)$. Further, it is simple to see that for any k -weight $\{0,1\}$ assignment e to the \mathbf{Z} variables, $p(e) = 1$ iff e is a *good assignment* because from each of the product terms, only one variable will survive. Therefore,

$$f = \sum_{e \in \binom{[b(m)]}{k}} p(e) \cdot \tilde{g}(\mathbf{X}, e), \quad \text{where } b(m) = |\mathbf{Z}|.$$

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We set $G(\mathbf{X}, \mathbf{Z}) := p(\mathbf{Z})\tilde{g}(\mathbf{X}, \mathbf{Z})$. By the construction, \tilde{g} has weft $\leq t + 1$, p has weft ≤ 2 , and \tilde{g}, p have $\text{poly}(m)$ size circuits. So, this ends our proof. \blacktriangleleft

► **Remark 33.** The construction above increases the weft by one.

► **Corollary 34.** $\text{VW}_{\text{nb}}^0[\mathbb{P}] = \text{VFPT}_{\text{nb}}^0 \implies \text{p-log-Expsum} \in \text{VFPT}_{\text{nb}}^0$.

Proof. In Theorem 32 we have reduced an instance of p-log-Expsum to an instance of $\text{VW}_{\text{nb}}^0[\mathbb{P}]$ with parameter $k = \ell(n)/\log m$. By our assumption $\text{VW}_{\text{nb}}^0[\mathbb{P}] = \text{VFPT}_{\text{nb}}^0$ and thus we can say that $\text{p-log-Expsum} \in \text{VFPT}_{\text{nb}}^0$. \blacktriangleleft

► **Remark 35.** If one restricts p-log-Expsum to exponential sums over g , where g is a p -family (i.e., it has polynomial degree and size), denoted $\text{p-log-Expsum}_{\text{bd}}$ (bd for bounded-degree), then the above proof similarly implies that $\text{VW}^0[\mathbb{P}] = \text{VFPT}^0 \implies \text{p-log-Expsum}_{\text{bd}} \in \text{VFPT}^0$.

Similarly, we also prove a lower bound for the class $\text{VW}_{\text{nb}}[\mathbb{P}]$, assuming an fpt lower bound on p-log-Expsum .

► **Theorem 36.** *Say that any family $F_{m,k}(\mathbf{X}) = \sum_{e \in \binom{[m]}{k}} G(\mathbf{X}, e) \in \text{VW}_{\text{nb}}^0[\mathbb{P}]$ has $2^{o(n)}\text{poly}(m)$ size constant-free circuits where $\tau(G) \leq m$, $n := k \log m/c$, for some constant c and b is some p -bounded function. Then, $\text{p-log-Expsum} \in \text{VFPT}_{\text{nb}}^0$.*

Proof. Take an instance of p-log-Expsum , $f(\mathbf{X}) = \sum_{y \in \{0,1\}^{\ell(n)}} g(\mathbf{X}, y)$, for some $\ell(n) = O(n)$. And g has a constant-free circuit of size m . By Theorem 32, we can make it an instance of $\text{VW}^0[\mathbb{P}]$ and say,

$$f = \sum_{e \in \binom{[m]}{k}} \tilde{g}(\mathbf{X}, e), \quad \text{where } b \text{ is } p\text{-bounded, } k = \ell(n)/\log m$$

By our assumption, f has a constant-free circuit of size $2^{o(n)}\text{poly}(m) = 2^{O(n/i(n))}\text{poly}(m)$ for some unbounded and non-decreasing function $i : \mathbb{N} \rightarrow \mathbb{N}$. Let h be a non-decreasing function, so that $h(i(n)) \geq 2^n$. We shall prove that f has $h(k)\text{poly}(m)$ size constant-free circuit. If $m \geq 2^{n/i(n)}$, clearly, f has $\text{poly}(m)$ size constant-free circuit. Otherwise, if $m < 2^{n/i(n)}$, this will imply $i(n) \leq n/\log m = k$. And hence, $h(k) \geq 2^n$. So, f has $h(k)\text{poly}(m)$ size constant-free circuit. \blacktriangleleft

9 Restricted permanent

A *cycle cover* of a directed graph is a collection of node-disjoint directed cycles such that each node is contained in exactly one cycle. Cycle covers of a directed graph stand in one-to-one relation with permutations of the nodes.

► **Definition 37.** *A cycle cover is (k, c) -restricted, if it contains one cycle of length k and all other cycles have length $\leq c$.*

Let $G = (V, E)$ be directed graph and $w : E \rightarrow R$ be a weight function. Here R is a ring and typically the ring of polynomials. The weight of a cycle cover C of G is the product of the weights of the edges in it, that is, $w(C) = \prod_{e \in C} w(e)$.

► **Definition 38.** *The (k, c) -restricted permanent of an edge-weighted directed graph G is*

$$\text{per}^{(k, \leq c)}(G) = \sum_C w(C),$$

where the sum is over all (k, c) -restricted cycle covers.

If $X = (X_{i,j})$ is a variable matrix, then $\text{per}_n(X)$ is the permanent of the complete directed graph with the edge weights $w(i, j) = X_{i,j}$. The (k, c) -restricted permanent family $\text{per}^{(k, \leq c)} = (\text{per}_n^{(k, \leq c)}(X_n))$, where X_n is an $n \times n$ -variables matrix. $\text{per}^{(k, \leq c)}$ is a parameterized family, n is the input size, k is the parameter, and c will be some constant to be determined later.

On general graphs, the restricted permanent is very powerful, even if we keep the parameter fixed.

► **Proposition 39.** *The $(2, 2)$ -restricted permanent family is VNP-complete.*

If we restrict the underlying graph appropriately, then the restricted permanent is complete for the class $\text{VW}[F]$. Recall that the girth of an undirected graph is the length of a shortest cycle in the graph. When we talk of the girth of a directed graph, we mean the girth of the graph when we disregard the direction of edges. Furthermore, when we talk about the treewidth of a directed graph, we mean the treewidth of the underlying undirected graph.

► **Definition 40.** *A directed graph $G = (V, E)$ is (c, b) -nice if we can partition the nodes $V = V_1 \cup V_2$ into two disjoint sets, such that*

1. *the graph induced by V_1 has girth $> c$ (not counting self-loops),*
2. *every node in V_1 has a self-loop, and*
3. *the graph induced by V_2 has tree-width bounded by b .*
4. *every cycle that contains vertices from V_1 and V_2 has length $> c$.*

Our main result is the following completeness result.

► **Theorem 41.** *Let c and b be constants. Let (G_n) be a family of (c, b) -nice graphs. Then the (k, c) -restricted permanent is in $\text{VW}[F]$.*

► **Theorem 42.** *Let the underlying field have characteristic 0. There is a constant b and a family of $(4, b)$ -nice graphs (H_n) such that the $(3k, 4)$ -restricted permanent of H_n forms a family of $\text{VW}[F]$ -hard polynomials.*

References

- 1 Yaroslav Alekseev, Dima Grigoriev, Edward A. Hirsch, and Iddo Tzameret. Semi-algebraic proofs, IPS lower bounds and the τ -conjecture: Can a natural number be negative? *CoRR*, abs/1911.06738, 2019. [arXiv:1911.06738](https://arxiv.org/abs/1911.06738).
- 2 Eric Allender, Michal Koucký, Detlef Ronneburger, Sambuddha Roy, and V. Vinay. Time-space tradeoffs in the counting hierarchy. In *Proceedings of the 16th Annual IEEE Conference on Computational Complexity, Chicago, Illinois, USA, June 18-21, 2001*, pages 295–302. IEEE Computer Society, 2001. doi:10.1109/CCC.2001.933896.
- 3 Markus Bläser and Christian Engels. Parameterized Valiant’s Classes. In Bart M. P. Jansen and Jan Arne Telle, editors, *14th International Symposium on Parameterized and Exact Computation (IPEC 2019)*, volume 148 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 3:1–3:14, Dagstuhl, Germany, 2019. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.IPEC.2019.3.
- 4 Lenore Blum, Felipe Cucker, Mike Shub, and Steve Smale. Algebraic settings for the problem “ $P \neq NP$?”. In *The Collected Papers of Stephen Smale: Volume 3*, pages 1540–1559. World Scientific, 2000. doi:10.1142/9789812792839_0025.
- 5 Lenore Blum, Mike Shub, and Steve Smale. On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines. *Bulletin (New Series) of the American Mathematical Society*, 21(1):1–46, 1989. URL: <https://www.ams.org/journals/bull/1989-21-01/S0273-0979-1989-15750-9/S0273-0979-1989-15750-9.pdf>.

- 6 Peter Bürgisser. On defining integers and proving arithmetic circuit lower bounds. *computational complexity*, 18:81–103, April 2009. doi:10.1007/s00037-009-0260-x.
- 7 Holger Dell, Thore Husfeldt, Dániel Marx, Nina Taslaman, and Martin Wahlen. Exponential time complexity of the permanent and the Tutte polynomial. *ACM Trans. Algorithms*, 10(4):21:1–21:32, 2014. doi:10.1145/2635812.
- 8 Pranjali Dutta. Real τ -conjecture for sum-of-squares: A unified approach to lower bound and derandomization. In *International Computer Science Symposium in Russia*, pages 78–101. Springer, 2021.
- 9 Jörg Flum and Martin Grohe. Parametrized complexity and subexponential time (column: Computational complexity). *Bull. EATCS*, 84:71–100, 2004.
- 10 Jörg Flum and Martin Grohe. *Parameterized Complexity Theory*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2006. doi:10.1007/3-540-29953-X.
- 11 David G. Glynn. The permanent of a square matrix. *European Journal of Combinatorics*, 31(7):1887–1891, 2010. doi:10.1016/j.ejc.2010.01.010.
- 12 William Hesse, Eric Allender, and D.A. Barrington. Uniform constant-depth threshold circuits for division and iterated multiplication. *Journal of Computer and System Sciences*, 65(4):695–716, 2002. doi:10.1016/S0022-0000(02)00025-9.
- 13 Stasys Jukna and Georg Schnitger. On the optimality of bellman–ford–moore shortest path algorithm. *Theoretical Computer Science*, 628:101–109, 2016.
- 14 Pascal Koiran. Valiant’s model and the cost of computing integers. *computational complexity*, 13:131–146, 2005.
- 15 Pascal Koiran. Shallow circuits with high-powered inputs. In *Innovations in Computer Science – ICS*, 2011. URL: <https://hal-ens-lyon.archives-ouvertes.fr/ensl-00477023v4/document>.
- 16 Pascal Koiran and Sylvain Perifel. Interpolation in Valiant’s theory. *Computational Complexity*, 20:1–20, 2011.
- 17 Guillaume Malod. The complexity of polynomials and their coefficient functions. In *Twenty-Second Annual IEEE Conference on Computational Complexity (CCC’07)*, pages 193–204. IEEE, 2007.
- 18 Guillaume Malod and Natacha Portier. Characterizing Valiant’s algebraic complexity classes. *J. Complexity*, 24(1):16–38, 2008. doi:10.1016/j.jco.2006.09.006.
- 19 Herbert John Ryser. *Combinatorial Mathematics*, volume 14 of *Carus Mathematical Monographs*. Mathematical Association of America, 1963.
- 20 Michael Shub and Steve Smale. On the intractability of Hilbert’s Nullstellensatz and an algebraic version of “ $NP \neq P$?”. *Duke Mathematical Journal*, 81(1):47–54, 1995. URL: <http://www.cityu.edu.hk/ma/doc/people/smales/pap97.pdf>.
- 21 Sébastien Tavenas. *Bornes inférieures et supérieures dans les circuits arithmétiques*. PhD thesis, Ecole Normale Supérieure de Lyon, 2014. URL: <https://tel.archives-ouvertes.fr/tel-01066752/document>.
- 22 Jacobo Torán. Complexity classes defined by counting quantifiers. *J. ACM*, 38:753–774, July 1991. doi:10.1145/116825.116858.
- 23 Leslie G. Valiant. Completeness classes in algebra. In Michael J. Fischer, Richard A. DeMillo, Nancy A. Lynch, Walter A. Burkhard, and Alfred V. Aho, editors, *Proceedings of the 11th Annual ACM Symposium on Theory of Computing, April 30 – May 2, 1979, Atlanta, Georgia, USA*, pages 249–261. ACM, 1979. doi:10.1145/800135.804419.
- 24 Klaus W. Wagner. The complexity of combinatorial problems with succinct input representation. *Acta Informatica*, 23(3):325–356, 1986. doi:10.1007/BF00289117.