Switching Between Left and Right Continuity in Network Calculus

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Abstract

The Network Calculus theory has been designed to compute upper bounds on delay and backlog in data networks. A lot of results have been developed to address different aspects. However, they are not all compatible with each other since they make different assumptions on the continuity of a core aspect of the model (the cumulative curves). However, real systems may mix several mechanisms. When modeling such a system, one has to choose one continuity hypothesis and limit the analysis to a subset of existing results. This paper addresses the continuity problem and argues formally that continuity issues are mathematical details that can be solved as long as the min-plus properties are used (minimal and maximal service, shaping). Conversely, it gives a counter-example for properties based on strict service, requiring a generalisation of the backlogged interval notion.

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1 Introduction

The Network Calculus theory, introduced in [7], has been instrumental in analysing the real-time performance of data networks [2, 12], and continues to be actively employed for enhancing existing results and analysing new networks [1, 14].

Central to this theory are functions representing the total amount of data sent by a source up to a given time, known as *cumulative curves* (see Section 6.1). Many results in this domain assume these cumulative curves to be left continuous functions (cf. [2, Def. 1.1] for details), as this assumption confers useful properties, which will be detailed in Section 10. We refer to the set of results developed with left continuous cumulative curves as the *left world*.

However, certain concepts, such as the decomposition of flows into packets, are more challenging to address in the left world. Hence, it can be interesting to consider right continuous cumulative curves to facilitate the decomposition of flows into packets [3]. Additionally, establishing connections between Network Calculus and Compositional Performance Analysis (CPA, [16]) necessitates right continuous cumulative curves (see [6] and [5, § 4.4] for discussion on continuity). The set of results developed with right continuous cumulative curves is termed the *right world*.

While some results have been proven under either assumption [4], utilising all results within a single model remains elusive.

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This paper suggests a more generic approach, demonstrating that if a system is modelled with one assumption, its arrival and service curves can be projected to conform to the other assumption.

To achieve this, we consider the sets of left continuous functions ($\mathcal{L}$) and right continuous functions ($\mathcal{R}$), along with two projection operators: the left and right projections denoted by $(.)_l$ and $(.)_r$ respectively, as depicted in Figure 1.

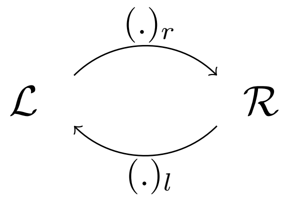

Figure 1 Graph with the set of left continuous functions ($\mathcal{L}$) and the set of right continuous functions ($\mathcal{R}$) and the operators to pass from one set to another: the left ($(.)_l$) and the right ($(.)_r$) projection.

The positive contributions of this paper encompass the arrival curve, the min-plus minimal service curve β, the maximal service curves β^M, and the shaping curves σ of a server (formal definitions provided in Section 7.1).

Theorems 24, 26, and 28 demonstrate that if a server S_l, the "left model" of a system element $\mathcal{N}$, offers a min-plus minimal service β, a maximal service β^M, and a shaping curve σ, then the "right-model" S_r offers the same service curves. Conversely, if a server S_r, the "right model" of a system element $\mathcal{N}$, offers a min-plus minimal service β, a maximal service β^M, and a shaping curve σ, then the "left model" S_l offers a min-plus minimal service $(\beta)_l$, a maximal service β^M, and a shaping curve σ.

Figure 2 summarises the contributions of this paper.

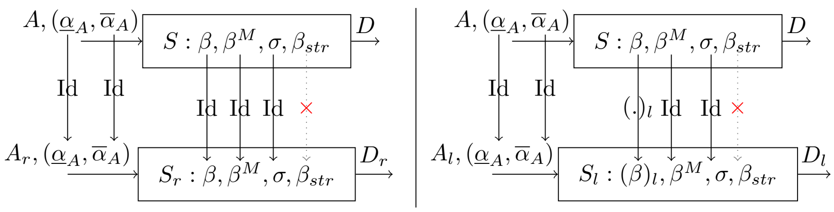

Figure 2 Summarise of the contribution: Passing from one world to another does not change the services and the arrival curves except the min-plus minimal service from the right world to the left one and the strict service which cannot be transferred.

The negative contributions reveal that due to variations in backlogged periods when cumulative arrival and departure functions are projected into the other world, the notion of strict service cannot be transferred.

The paper is organised as follows: we begin by introducing our motivation and the notations pertinent to the subsequent sections. As the Network Calculus theory is based on the $(\min, +)$ dioid, itself grounded on piece-wise continuous functions from non-negative reals to reals ($\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$), we proceed with four steps:

1. We highlight properties on sets of functions preserved by projecting the set in one world (Section 4).
2. We discuss how the min-plus convolution supports the left/right function sets (Section 5).

3. We review Network Calculus properties preserved (or slightly modified) when switching from one world to another: the cumulative curves (Section 6.1), the arrival curves (Section 6.2), and the service curves (including the simple, maximal, shaping, and strict service curves – Section 7).
4. We demonstrate, in Section 8, that the delay and backlog bounds still hold whatever the continuity of cumulative curves (in the literature, the theorems were only assuming left continuous cumulative curves).

Section 9 illustrates the use of these new results with a simple example. Section 10 presents related work, and Section 11 concludes.

2 Motivations

A cyber-physical system comprises numerous computational units, actuators, and communication networks. Our primary objective is to determine timing upper bounds within such systems. Various theories are available for timing analysis, each excelling in different scenarios. To mitigate overly pessimistic bounds, we propose integrating diverse theories and transferring information between them. Among these theories, Network Calculus (NC) stands out for its efficacy in analyzing networks, particularly in Time Sensitive Networks (TSN). However, NC may lack sufficient results for analyzing computational units or tend towards pessimism. To address this, we explore leveraging the connection with Compositional Performance Analysis (CPA) formalism as introduced in [6] and [10], aiming to enhance system analysis.

The connection between NC and CPA is intriguing as both are founded on analogous curves: NC employs cumulative curves representing the total data generated by a flow up to time t (denoted as $A(t)$), while CPA utilizes event curves representing the total packets generated by a flow up to time t (denoted as $E(t)$). The connection is established through a function P where $P(a)$ denotes the number of packets received for the amount of data a. By combining these curves under continuity assumptions, the relation $E = P(A)$ is derived (refer to [6] for detailed exposition).

However, when applying these theories to analyze an entire real-time system, several challenges emerge. For instance, modeling the interface between computational units and networks poses difficulties since tasks from a computational unit typically do not directly transmit to the network but may wait for other tasks, potentially introducing scaling issues. This aspect lies beyond the scope of our current work.

In this paper, we focus on addressing the theoretical connection between NC and CPA. The relation $E = P(A)$ necessitates right continuous curves, whereas Network Calculus predominantly employs left continuous curves. This misalignment prevents seamless integration of NC theorems/properties with those of CPA. Essentially, the choice between left or right continuity assumptions for modeling the system is exclusive, without a mechanism for transitioning between them, as depicted in Figure 3. To enhance this connection, we identify two potential approaches: either re-prove existing theorems/properties with the alternative continuity assumption or develop a gateway mechanism to project theorems/properties from one assumption to the other.

Inspired by a statement from [12, §1.1.1]: "It would be nice to stick to either left or right continuous functions. However, depending on the model, there is no best choice.", we aim to demonstrate that if a system is modeled with one continuity assumption in the Network Calculus formalism, its arrival and service curves can be projected to conform to the other assumption. Throughout this paper, we focus on a network modeled using the Network Calculus formalism and showcase the preservation of most properties related to the fundamentals of Network Calculus theory when switching between continuity assumptions.

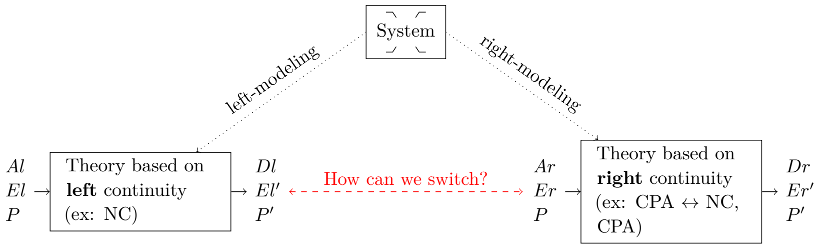

Figure 3 A system with two models: the "left model" with left continuous curves (Ar, Er, P), (Dr, Er', P'), and the "right model" with right continuous curves (Al, El, P), (Dl, El', P').

We shall commence by laying out the mathematical foundation of this paper.

3 Mathematical background

3.1 Notation

First, let $\mathbb{R}$, and $\mathbb{R}^+$, denote, respectively, the sets of reals and non-negative reals. $\mathbb{Z}$ denotes the set of integers.

Let $\lfloor . \rfloor$, respectively $\lceil . \rceil$, denote the floor and ceiling functions such that $\forall x \in \mathbb{R}$, $\lfloor x \rfloor \in \mathbb{Z}, \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ and $\lceil x \rceil \in \mathbb{Z}, \lceil x \rceil - 1 < x \leq \lceil x \rceil$.

Also, we denote by $\wedge$, respectively $\vee$, the minimum and the maximum operator *i.e.*, $a \wedge b \overset{def}{=} \min(a, b)$ and $a \vee b \overset{def}{=} \max(a, b)$. $[.]^+$ represents the non-negative closure: $\forall f$ a function, $[f]^+ \overset{def}{=} f \vee 0$. $[.]_\uparrow$ represents the non-decreasing closure: $\forall f$ a function, $[f]_\uparrow \overset{def}{=} \sup_{0 \leq s \leq t} \{f(s)\}$.

Finally, we will use the limit of functions and denote with $f(t+) \overset{def}{=} \lim_{\varepsilon \to 0, \varepsilon > 0} f(t + \varepsilon)$ the right limit at t of the function f if it exists and $f(t-) \overset{def}{=} \lim_{\varepsilon \to 0, \varepsilon > 0} f(t - \varepsilon)$ the left limit at t of the function f if it exists.

3.2 Sets and operators

After introducing the notations, we will define some specific sets of functions according to the min-plus theory (as said in the introduction) and to work with left and right worlds, some operators to pass from one to another: the left and right projections.

Sets

First, let us define piece-wise continuous function.

▶ **Definition 1** (Piece-wise continuous functions)**.** *A function f is said to be piece-wise continuous if it has a finite number of discontinuities on any finite interval.*

Then, we can introduce the useful sets of functions.

▶ **Definition 2** (Function sets: $\mathcal{F}$, $\mathcal{R}$, $\mathcal{L}$ and subsets)**.** *$\mathcal{F}$ is the set of piece-wise continuous functions from $\mathbb{R}^+$ to $\mathbb{R}$.*

Also, $\mathcal{R}$ (respectively $\mathcal{L}$) denotes the set of right continuous functions (respectively left continuous functions).

Let X be a subset of $\mathcal{F}$ (either $\mathcal{F}$, $\mathcal{R}$ or $\mathcal{L}$ in this paper). Then

- $X^\uparrow \subseteq X$ *is the subset of the non-decreasing functions of X,*
- $X_0^\uparrow = \{f \in X^\uparrow \mid f(0) = 0\}$,
- $\dot{X} = \left\{f \in X_0^\uparrow \mid f(0+) = 0\right\}$.

In this paper, our focus primarily revolves around non-decreasing functions. Consequently, we introduce an alternative formulation for the limits applicable to such functions. This alternative expression is intended to facilitate subsequent proofs.

▶ **Property 3** (Alternative expression for the right/left limit)**.** *Let $f \in \mathcal{F}^{\uparrow}$. Then*

$$f(t+) = \inf_{\varepsilon>0} f(t+\varepsilon) \qquad \text{and} \qquad \forall t > 0, f(t-) = \sup_{t\geq\varepsilon>0} f(t-\varepsilon).$$

The proof is given in Appendix A.

Operators

Now, we can introduce operators to switch from one world to another.

▶ **Definition 4** (Right and Left projections)**.** $(.)_r$ *defines the right projection operator such that*

$$\begin{aligned} (.)_r &: \mathcal{F} \to \mathcal{R} \\ &: f \mapsto (f_r : t \mapsto f(t+)). \end{aligned} \tag{1}$$

Correspondingly, $(.)_l$ *defines the left projection operator such that*

$$\begin{aligned} (.)_l &: \mathcal{F} \to \mathcal{L} \\ &: f \mapsto \left(f_l : t \mapsto \begin{cases} f(t-) & \text{if } t > 0 \\ f(0) & \text{if } t = 0 \end{cases} \right). \end{aligned} \tag{2}$$

Also, we can apply these projections to a set $X \subset \mathcal{F}$ as

$$(X)_l \overset{def}{=} \{(f)_l \mid f \in X\}, \qquad (X)_r \overset{def}{=} \{(f)_r \mid f \in X\}.$$

▶ **Remark 5.** *When there is no ambiguity, the parentheses will be removed in the projections,* i.e., $(f)_l$ *and* f_l *represent the same: the left projection of f. In the same way,* $(f)_r$ *and* f_r *represent the right projection of f.*

▶ **Remark 6.** *Network Calculus papers have considered either functions f from $\mathbb{R}$ to $\mathbb{R}$, or from $\mathbb{R}$ to $\mathbb{R}^+$ or from $\mathbb{R}^+$ to $\mathbb{R}^+$ or even from $\mathbb{R}$ to $\mathbb{R}^+$ with $\forall x \leq 0, f(x) = 0$. Most modern Network Calculus papers use functions from $\mathbb{R}^+$ to $\mathbb{R}^+$. However, this restriction of the function domain creates mathematical issues when we are evaluating the limit at zero. In this paper, we assume that, at zero, the left projection of a function is equal to the value of the function,* i.e., $\forall f \in \mathcal{F}, f_l(0) = f(0)$ *(as if the function were extended as $\forall x \leq 0, f(x) = f(0)$).*

A useful property is the monotony of the projections as shown in the following property.

▶ **Property 7** (Monotony of the projections)**.** *Let $f, g \in \mathcal{F}^{\uparrow}$ such that $f \leq g$. Then, $f_r \leq g_r$ and $f_l \leq g_l$.*

Let $f \in \mathcal{F}^{\uparrow}$. Then, $f_l \leq f \leq f_r$.

Proof. Let $f, g \in \mathcal{F}^{\uparrow}$ such that $f \leq g$. Then, according to Property 3, $\forall t \in \mathbb{R}^+$,

$$\inf_{\epsilon>0} f(t+\epsilon) \leq \inf_{\epsilon>0} g(t+\epsilon).$$

Also, $f_l(0) = f(0) \leq g(0) = g_l(0)$ and $\forall t > 0$,

$$\sup_{\epsilon>0} f(t-\epsilon) \leq \sup_{\epsilon>0} g(t-\epsilon).$$

Let $t \in \mathbb{R}^+$. If $t = 0$, $f_l(0) = f(0) \leq f(0)$ and $\forall \varepsilon > 0$, as f is non-decreasing, $f(0) \leq f(\varepsilon)$. Passing to the limit, $f(0) \leq f_r(0)$.
Otherwise, let $\varepsilon \in \mathbb{R}^+$ such that $0 < \varepsilon < t$. As f is non-decreasing, $f(t - \varepsilon) \leq f \leq f(t + \varepsilon)$ and passing to the limit, we have $f_l(t) \leq f \leq f_r(t)$. ◀

The background section concludes with the introduction of notations and operators. Subsequently, we will examine the effects of switching from one world to another on these defined sets.

4 Stability results on function sets

Now that we have introduced the relevant notations, sets, and operators, we will demonstrate which properties of the subsets of $\mathcal{F}$ remain unchanged when switching between worlds.

First, we will establish that the non-decreasing property remains preserved.

▶ **Property 8** (Stability of non-decreasing subsets of $\mathcal{F}$). *Let $\mathcal{L}^{\uparrow}, \mathcal{R}^{\uparrow}$ be defined as in Definition 2 and $(.)_r, (.)_l$ be the left and right projection as defined in Definition 4. Then*

$$(\mathcal{F}^{\uparrow})_l \subset \mathcal{L}^{\uparrow}, \qquad (\mathcal{F}^{\uparrow})_r \subset \mathcal{R}^{\uparrow}.$$

Proof. $\forall f \in \mathcal{F}^{\uparrow}$, $\forall x, y \in \mathbb{R}^+, x < y$, $f(x) \leq f(y)$

$$\implies \inf_{\epsilon>0} f(x + \epsilon) \leq \inf_{\epsilon>0} f(y + \epsilon) \Leftrightarrow f_l(x) \leq f_l(y).$$

Also, if $x = 0$, $f_l(0) = f(0) \leq f(y + \epsilon), \forall \epsilon > 0$.

Otherwise, $\sup_{\epsilon>0} f(x-\epsilon) \leq \sup_{\epsilon>0} f(y-\epsilon) \Leftrightarrow f_r(x) \leq f_r(y)$. Consequently, $\inf_{\epsilon>0} f(y+\epsilon) \Leftrightarrow f_l(0) \leq f_l(y)$ Thus, f_r and f_l are non-decreasing. ◀

Then, we can work on the stability of the subsets of $\mathcal{F}$.

▶ **Property 9** (Stability of the subsets of $\mathcal{F}$). *Let $\mathcal{R}_0^{\uparrow}, \dot{\mathcal{R}}, \mathcal{L}_0^{\uparrow}, \dot{\mathcal{L}}$ be defined as in Definition 2 and $(.)_r, (.)_l$ be the left and right projection as defined in Definition 4. Then*

$$(\dot{\mathcal{F}})_r \subset \dot{\mathcal{R}}, \quad (\dot{\mathcal{F}})_l \subset \dot{\mathcal{L}}, \tag{3}$$

$$(\mathcal{F}_0^{\uparrow})_l \subset \mathcal{L}_0^{\uparrow}, \qquad (\mathcal{F}_0^{\uparrow})_r \not\subset \mathcal{R}_0^{\uparrow}. \tag{4}$$

Proof. According to Property 8, the non-decreasing aspect of the subsets is preserved. We only need to look at value and continuity at zero.

- Proof of Equation (3): We need to see if the property such that the limit of the function when $t \to 0+$ is zero is preserved.
  Let $f \in \dot{\mathcal{F}}$ be a function. At $t = 0$, it is continuous so $f(0) = f_l(0) = 0$ and there exists a neighbourhood around 0: $[0; \epsilon[$ such that $\forall t \in [0; \epsilon[, f(t) = 0$ and it is continuous then $\forall t \in [0; \epsilon[, f_l(t) = f(0) = 0$. Then, $f_l \in \dot{\mathcal{L}}$ and $(\dot{\mathcal{F}})_l \subset \dot{\mathcal{L}}$.
  Similarly, we have that $(\dot{\mathcal{F}})_r \subset \dot{\mathcal{R}}$.
- Proof of Equation (4): First, we can easily note that it is not preserved for the right projection of the set $\mathcal{F}_0^{\uparrow}$ because $\exists f \in \mathcal{F}_0^{\uparrow} \mid f_r \notin \mathcal{R}_0^{\uparrow}$ with, for instance, the function:

$$f(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } t > 0. \end{cases}$$

  Indeed, $f_r(t) = 1, \forall t \in \mathbb{R}^+$ and $f_r \notin \mathcal{R}_0^{\uparrow}$. The property $f(0) = 0$ is not preserved by the right projection operator. Consequently, $(\mathcal{F}_0^{\uparrow})_r \not\subset \mathcal{R}_0^{\uparrow}$.
  However, $(\mathcal{F}_0^{\uparrow})_l \subset \mathcal{L}_0^{\uparrow}$. Indeed, Property 8 shows the non-decreasing stability and by definition, $f_l(0) = f(0) = 0$. ◀

In this section, we illustrate that the majority of properties related to the sets of functions are preserved during switching between the two worlds.

5 Min-plus convolution and continuity

After results on the stability of function sets, we address the second layer, the min-plus convolution of functions and the impact of the left and right projection.

Let us, first, recall the min-plus convolution from [2, 12].

▶ **Definition 10** (min-plus convolution)**.** *Let $f, g \in \mathcal{F}$ be two functions. The min-plus convolution operator, denoted $*$ is defined as*

$$f * g = \inf_{0 \le s \le t} \{f(s) + g(t-s)\} \tag{5}$$

$$= \inf_{\substack{0 \le u,s \\ u+s=t}} \{f(s) + g(u)\}. \tag{6}$$

Hereafter, min-plus convolution or convolution means the same.

Following the methodology employed in Section 4, we will illustrate the preservation of properties within sets of functions. We begin by recalling the preservation of the non-decreasing property, established in [2, Lemma 2.3].

▶ **Property 11** (Non-decreasing stability of convolution)**.** *Let f, g be two non-decreasing functions then $f * g$ is also a non-decreasing function* i.e.,

$$\forall f, g \in \mathcal{F}^{\uparrow} : f * g \in \mathcal{F}^{\uparrow}.$$

▶ **Theorem 12** (Left stability of the convolution)**.** *Let f, g be two non-decreasing left continuous functions. Then $f * g$ is also a non-decreasing left continuous function,* i.e.,

$$\forall f, g \in \mathcal{L}^{\uparrow} : f * g \in \mathcal{L}^{\uparrow}.$$

Subsequently, our attention will shift towards the results of the projection of a convolution. The theorem presented below stands as a cornerstone of this paper, as all findings pertaining to Network Calculus services and continuity are contingent upon it.

▶ **Theorem 13** (Convolution and projections)**.** *Let $f, f', g \in \mathcal{F}^{\uparrow}$ be three functions with $f'(0+) = f'(0)$. Then*

$$(f * g)_l = f_l * g_l, \tag{7}$$

$$(f' * g)_r = f'_r * g. \tag{8}$$

Proof of Equation (7). Let $f, g \in \mathcal{F}^{\uparrow}$ be two functions.

The proof consists of bounding the expression $f_l * g_l$.

On the one hand, we will prove that $f_l * g_l \le f * g$. Firstly, $g_l \le g \implies f_l * g_l \le f_l * g$. Also, $f_l \le f \implies f_l * g \le f * g$. Then, $f_l * g_l \le f * g$. Using Property 7, we have $(f_l * g_l)_l \le (f * g)_l$. And, from Corollary 12, since $f_l * g_l \in \mathcal{L}$ then $(f_l * g_l)_l = f_l * g_l$. Consequently, $f_l * g_l \le (f * g)_l$.

On the other hand, we will prove that $(f * g)_l \le f_l * g_l$. We will use the expression of the convolution and the limit: First, if $t = 0$, $(f * g)_l(0) \overset{(2)}{=} (f * g)(0) \overset{(5)}{=} \inf_{0 \le s \le 0} \{f(s) + g(t-s)\} = f(0) + g(0) \overset{(2)}{=} f_l(0) + g_l(0) = \inf_{0 \le s \le 0} \{f_l(s) + g_l(t-s)\} \overset{(5)}{=} (f_l) * (g_l)(0)$.

Otherwise, let $t \in \mathbb{R}^+, t > 0$. First, $(f * g)_l(t) = \sup_{t>\epsilon>0} \{f * g(t - \epsilon)\}$. As it is a supremum, showing that $(f * g)_l(t) \leq (f_l * g_l)(t)$ is equivalent to proving that $\forall \epsilon \in]0, t]$, $f * g(t - \epsilon) \leq f_l * g_l(t)$ or, equivalently, replacing $t - \epsilon$ by x: $\forall x \in [0, t[$, $f * g(x) \leq (f_l * g_l)(t)$ with $x = t - \epsilon$.

The same way, $(f_l * g_l)(t) \overset{(6)}{=} \inf_{\substack{0 \leq u,v \\ u+v=t}} \{f_l(u) + g_l(v)\}$. As it is an infimum, showing that $\forall x \in]0, t]$, $(f * g)(x) \leq f_l * g_l(t)$ is equivalent to prove that $\forall x \in [0, t[$, $\forall u, v \in \mathbb{R}^+$ such that $u + v = t$, $(f * g)(x) \leq f_l(u) + g_l(v)$.

Now, we have three cases, depending on the value of u and v:

1. If $u = 0$, then $f_l(u) + g_l(v) = f(0) + g_l(t)$. As $x < t$ then $\exists \epsilon \in [t, 0[$ such that $x = t - \epsilon$ and $(f*g)(x) = \inf_{u'+v'=t-\epsilon} \{f(u') + g(v')\} \leq f(0) + g(t-\epsilon)$ with $u = 0$ and $v = t - \epsilon - u$. As a consequence, $(f * g)(x) \leq f(0) + g(t - \epsilon) \leq f(0) + \sup 0 < \epsilon \leq t \{g(t - \epsilon)\} = f(0) + g_l(t) = f_l(u) + g_l(v)$.
2. If $v = 0$, the same argument holds replacing u by v in the previous proof.
3. Otherwise, if $u \neq 0$ and $v \neq 0$, as the left projection is a supremum, $\exists \eta, \tau \in \mathbb{R}^+$ such that $f_l(u) = \sup_{0<\epsilon\leq u} \{f(u - \epsilon)\} \geq f(u - \eta)$ and $g_l(v) = \sup_{0<\epsilon\leq v} \{g(v - \epsilon)\} \geq g(v - \tau)$. If we manage to prove $\forall x \in [0, t[$, $\forall u, v \in \mathbb{R}^+$ such that $u + v = t$, $\exists \eta, \tau \in \mathbb{R}^+$ such that $(f * g)(x) \leq f(u - \eta) + g(v - \tau)$, then the proof is done.
   We finally have two cases:
   a. If $\tau \geq \eta$: As $x \in [0, t[$, it exists $\epsilon \in \mathbb{R}^+$ such that $x = t - \epsilon + \tau - \eta$. Then, $(f * g)(x) = \inf_{u'+v'=t-\epsilon+\tau-\eta} \{f(u') + g(v')\} \leq f(t - \epsilon - \eta - v) + g(v - \tau)$ with $v' = v - \tau$ and $f(t - \epsilon - \eta - v) + g(v - \tau) = f(u - \eta - \epsilon) + g(v - \tau) \leq f(u - \eta) + g(v - \tau)$.
   b. If $\eta \geq \tau$: As $x \in [0, t[$, it exists $\epsilon \in \mathbb{R}^+$ such that $x = t - \epsilon + \eta - \tau$. Then, $(f * g)(x) = \inf_{u'+v'=t-\epsilon+\eta-\tau} \{f(u') + g(v')\} \leq f(u - \eta) + g(t - \epsilon - \tau - u)$ with $u' = u - \eta$ and $f(u - \eta) + g(t - \epsilon - \tau - u) = f(u - \eta) + g(v - \epsilon - \tau) \leq f(u - \eta) + g(v - \tau)$.
   As a consequence, $\forall u, v \in \mathbb{R}^+$ such that $u + v = t$, $\exists \eta, \tau \in \mathbb{R}^+$ such that $(f * g)(x) \leq f(u - \eta) + g(v - \tau) \leq f_l(u) + g_l(v)$.

Finally, combining the inequality, we have $\forall t \in \mathbb{R}^+, (f_l * g_l)(t) = (f * g)_l(t)$. ◀

Proof of Equation (8). Let $f', g \in \mathcal{F}^{\uparrow}$ be two non-decreasing functions such that $f'(0+) = f'(0)$ and t be a real. Let us first develop the expression of $(f' * g)_r(t)$.

$$
\begin{aligned}
(f' * g)_r(t) &\overset{(1)}{=} \inf_{\epsilon>0} \{f' * g(t + \epsilon)\} \\
&\overset{(5)}{=} \inf_{\epsilon>0} \left\{ \inf_{0\leq s\leq t+\epsilon} \{g(s) + f'(t + \epsilon - s)\} \right\} \\
&= \inf_{\epsilon>0} \left\{ \min \begin{cases} \inf_{0\leq s\leq t} \{g(s) + f'(t + \epsilon - s)\} \\ \inf_{t\leq s\leq t+\epsilon} \{g(s) + f'(t + \epsilon - s)\} \end{cases} \right\} \\
&= \min \begin{cases} \inf_{\epsilon>0} \{\inf_{0\leq s\leq t} \{g(s) + f'(t + \epsilon - s)\}\} \\ \inf_{\epsilon>0} \{\inf_{t\leq s\leq t+\epsilon} \{g(s) + f'(t + \epsilon - s)\}\} \end{cases}
\end{aligned}
$$

Now, the proof consists of proving that the second term is always lower than the second.

First, note that, in the second term, $\forall \epsilon > 0, s \in [t, t+\epsilon]$ then, $g(s) \geq g(t)$ and $f'(t+\epsilon-s) \geq f'(0)$. As a consequence, $\inf_{\epsilon>0} \{\inf_{t\leq s\leq t+\epsilon} \{g(s) + f'(t + \epsilon - s)\}\} \geq \inf_{\epsilon>0} \{g(t) + f'(0)\} = g(t) + f'(0)$.

However, the first term is $\inf_{\epsilon>0} \{\inf_{0\leq s\leq t} \{g(s) + f'(t + \epsilon - s)\}\} \leq \inf_{\epsilon>0} \{g(t) + f'(\epsilon)\}$ with $s = t$ and $\inf_{\epsilon>0} \{g(t) + f'(\epsilon)\} = g(t) + \inf_{\epsilon>0} \{f'(\epsilon)\} = g(t) + f(0+) = g(t) + f(0)$ by hypothesis.

As a consequence, we have that
$\inf_{\epsilon>0} \{\inf_{0\leq s\leq t} \{g(s) + f'(t+\epsilon-s)\}\} \quad \leq \quad \inf_{\epsilon>0} \{\inf_{t\leq s\leq t+\epsilon} \{g(s) + f'(t+\epsilon-s)\}\}$.
And,

$$
\begin{aligned}
(f' * g)_r(t) &= \min \begin{cases} \inf_{\epsilon>0} \{\inf_{0\leq s\leq t} \{g(s) + f'(t+\epsilon-s)\}\} \\ \inf_{\epsilon>0} \{\inf_{t\leq s\leq t+\epsilon} \{g(s) + f'(t+\epsilon-s)\}\} \end{cases} \\
&= \inf_{\epsilon>0} \left\{ \inf_{0\leq s\leq t} \{g(s) + f'(t+\epsilon-s)\} \right\} \\
&= \inf_{0\leq s\leq t} \left\{ \inf_{\epsilon>0} \{g(s) + f'(t+\epsilon-s)\} \right\} \\
&= \inf_{0\leq s\leq t} \left\{ g(s) + \inf_{\epsilon>0} \{f'(t+\epsilon-s)\} \right\} \\
&= \inf_{0\leq s\leq t} \{g(s) + f'_r(t-s)\} \\
&= (f'_r * g)(t).
\end{aligned}
$$
◀

According to Theorem 13, it can be inferred that right continuity exhibits a greater degree of robustness compared to left continuity with respect to convolution. Specifically, achieving the right continuity necessitates only one function to exhibit this property within the convolution. Conversely, attaining left continuity requires both functions involved in the convolution to possess left continuity.

6 Continuity and flows

Now, the aim is to see if the Network Calculus properties are preserved passing from one world to another. First, we will see the impact on cumulative curves. Then, we will show that the notion of arrival curves is preserved.

6.1 Cumulative curves

In Network Calculus, flows are modelled by cumulative functions $A \in \mathcal{F}_0^{\uparrow}$ such that $A(t)$ counts the total amount of data generated by the flow up to time t. Since a flow is a cumulative amount of data, it must be a non-decreasing function. The condition on a finite number of discontinuities is related to the discrete aspect of computer behaviour and simplifies the mathematical part. The condition of zero value at 0 is related to the fact that all results in Network Calculus are based on differences: a bound on $A(t)$ has to be understood as a bound on $A(t) - A(0)$.

However, according to Section 4 and to find an equivalence between left and right worlds, we will enforce the right continuity at the origin, that is to say, consider cumulative curves in the set $\dot{\mathcal{F}}$ instead of $\mathcal{F}_0^{\uparrow}$. Indeed, we first have that this set is stable using the right/left projection operators and secondly, we have results concerning the min-plus convolution that tilts the balance in favour of the sets $\dot{\mathcal{R}}/\dot{\mathcal{L}}$.

By requiring that a cumulative curve must be in $\dot{\mathcal{R}}/\dot{\mathcal{L}}$ instead of $\mathcal{R}^{\uparrow}/\mathcal{L}^{\uparrow}$, we forbid an instantaneous burst of data just at time origin (but an instantaneous burst may happen at any other instant). A justification should be that Network Calculus requires that the network elements must be ready just before the applications: the time origin of the network is strictly smaller than the time origin of the applications.

6.2 Arrival curves

The notion of arrival curve is used to bound the amount of data sent by a flow on any interval of time. There are two notions: the maximal arrival curve and the minimum arrival curve. By definition $(\overline{\alpha}, \underline{\alpha}) \in \mathcal{F}^{\uparrow}$ are maximal and minimal arrival curves of a cumulative curve A if

$$\forall t, d \in \mathbb{R}^{+}, \qquad \underline{\alpha}(d) \leq A(t+d) - A(t) \leq \overline{\alpha}(d).$$

► **Property 14.** *Let $\overline{\alpha} \in \mathcal{F}^{\uparrow}$ be a maximal arrival curve of a cumulative curve $A \in \dot{\mathcal{F}}$. Then, $\overline{\alpha}$ is a maximal arrival curve of A_r and A_l.*

Proof. Let $\overline{\alpha} \in \mathcal{F}^{\uparrow}$ be a maximal arrival curve of a cumulative curve $A \in \dot{\mathcal{F}}$. Let $t, d \in \mathbb{R}^{+}$.
We have different cases:

1. If $t > 0$: According to the definition of maximal arrival curves, $\forall \varepsilon \in [0, t]$, $A(t - \varepsilon + d) - A(t - \varepsilon) \leq \overline{\alpha}(d)$;
2. If $t = 0$ and $d > 0$: According to the definition of maximal arrival curves, $\forall \varepsilon \in [0, d]$, $A(d - \varepsilon) - A(0) = A(d - \varepsilon) - A_l(0) \leq \overline{\alpha}(d)$;
3. If $t = d = 0$: $A_l(t + d) - A_l(t) = A_l(0) - A_l(0) = A(0) - A(0) \leq \overline{\alpha}(0)$.

Passing to the limit when $\varepsilon \to 0$ in the two first cases, we have $A_l(t + d) - A_l(t) \leq \overline{\alpha}(d)$.
Then, $\overline{\alpha}$ is a maximal arrival curve of A_l.

Also, $\forall \varepsilon > 0$, $A(t + \varepsilon + d) - A(t + \varepsilon) \leq \overline{\alpha}(d)$.
Passing to the limit when $\varepsilon \to 0$, $A_r(t + d) - A_r(t) \leq \overline{\alpha}(d)$.
Then, $\overline{\alpha}$ is a maximal arrival curve of A_r. ◄

► **Property 15.** *Let $\underline{\alpha} \in \mathcal{F}^{\uparrow}$ be a minimal arrival curve of a cumulative curve $A \in \dot{\mathcal{F}}$. Then, $\underline{\alpha}$ is a minimal arrival curve of A_r and A_l.*

The proof is similar to the previous one.

Consequently, the notions of arrival curves are preserved passing from one world to another.

7 Continuity and servers

Now, we will work on the servers and prove that their properties are preserved when passing from one world to another. This section contains the main contributions of the paper.

7.1 Defining servers in both worlds

A server S describes relationships between input and output flows, $S \subseteq \mathcal{F}_0^{\uparrow} \times \mathcal{F}_0^{\uparrow}$. Then $(A, D) \in S$, denoted as $A \xrightarrow{S} D$, means that a server S receives an input (arrival) flow A, and delivers the output (departure) D. A server S might be, for example, a single buffer served at a constant rate, a complex communication node, or even a complete network.

An important assumption made with servers is that $D \leq A$, meaning that data only exits after having entered. Basic Network Calculus assumes also that there is no loss, data creation, compression or deflating, even if some projections have been defined in [8].

As previously said, most results in the literature consider only non-decreasing left continuous cumulative curves such that they are zero at $t = 0$ (i.e. $S \subseteq \mathcal{L}_0^{\uparrow} \times \mathcal{L}_0^{\uparrow}$). In the following, we want to consider $S \subseteq X \times X$ for different subsets $X \subseteq \mathcal{F}_0^{\uparrow}$. To do that, we need to rewrite the common definitions related to the servers parameterised by X.

▶ **Definition 16** (Server). *Let $X \subseteq \mathcal{F}_0^{\uparrow}$. A server $S^X \subseteq X \times X$ is a right-total relation between flow cumulative functions ($\forall A \in X, \exists D, (A, D) \in S^X$) that satisfies $(A, D) \in S^X \Rightarrow A \geq D$.*

We denote $A \overset{S^X}{\to} D$ for $(A, D) \in S^X$.

Then, for any $(A, D) \in S^X$ ($X \subseteq \mathcal{F}_0^{\uparrow}$), we can compute the backlog at any instant t as $A(t) - D(t)$. It represents the quantity of data that is waiting to be served. An important notion is the backlogged period, the interval during which there is a backlog.

▶ **Definition 17** (Backlogged period (BP)). *Let $X \subseteq \mathcal{F}_0^{\uparrow}$. Let S^X be a server and $(A, D) \in S^X$. An interval I is a backlogged period (also noted BP) for (A, D) if*

$$\forall t \in I, A(t) - D(t) > 0.$$

▶ **Definition 18** (Start of backlogged period). *Let $X \subseteq \mathcal{F}_0^{\uparrow}$. Let S^X be a server and $(A, D) \in S^X$. The start of a backlogged period of $t \in \mathbb{R}^+$ is defined by*

$$Start_{A,D}(t) \overset{def}{=} \sup\{u \leq t \mid D(u) = A(u)\}.$$

▶ **Remark 19.** *Note that with $X \subseteq \mathcal{L}_0^{\uparrow}$, $\forall t \in I$, I a backlogged period for $(A, D) \in S^X$, then we have equality between the arrival and departure cumulative curves at the start of the backlogged period:*

$$A(Start_{A,D}(t)) = D(Start_{A,D}(t)).$$

This property is used extensively in Network Calculus proofs, and is lost if the cumulative curves are not in the left world.

Four different properties related to service can be defined.

▶ **Definition 20** (kind of services). *Let $X \subseteq \mathcal{F}_0^{\uparrow}$. Let S^X be a server.*

1. *S^X offers a min-plus minimal service of curve $\beta \in \mathcal{F}^{\uparrow}$ if*

$$\forall A, \forall D : A \overset{S^X}{\to} D \implies D \geq A * \beta. \tag{9}$$

2. *S^X offers a maximal service of curve $\beta^M \in \mathcal{F}^{\uparrow}$ if*

$$(A, D) \in S^X \implies D \leq A * \beta^M. \tag{10}$$

3. *S^X is a shaper of shaping curve $\sigma \in \mathcal{F}^{\uparrow}$ if*

$$(A, D) \in S^X \implies D \leq D * \sigma. \tag{11}$$

4. *S^X offers a strict service of curve $\beta_{st} \in \mathcal{F}^{\uparrow}$ if*

$$\forall (A, D) \in S^X, \forall (s, t] \text{ a } BP, D(t) - D(s) \geq \beta(t - s). \tag{12}$$

In the previous definitions, the set X is, in most results of the Network Calculus, the set of left continuous non-decreasing functions such that they are zero at $t = 0$ (*i.e.*, $X = \mathcal{L}_0^{\uparrow}$).

Before looking at which properties are preserved between the left and right worlds, we need to define the left/right projections of a server.

▶ **Definition 21** (Right/Left projection of servers). *Let $X \subseteq \mathcal{F}_0^{\uparrow}$ and S^X be a server. The right and left projections of a server are respectively defined by*

$$(S^X)_r = \left\{(A_r, D_r) \mid \exists (A, D) \in S^X\right\},$$
$$(S^X)_l = \left\{(A_l, D_l) \mid \exists (A, D) \in S^X\right\}.$$

▶ **Property 22.** *The right or left projection of a server is a server.*

Proof. Let $X \in \mathcal{F}_0^{\uparrow}$ and S^X be a server. We need to prove that $\forall (A', D') \in (S^X)_r, A' \geq D'$.

Let $(A', D') \in (S^X)_r$ then $(A', D') = (A_r, D_r) \mid \exists (A, D) \in S^X$. But $A \geq D$ and $(A, D) \in S^X$ so $(A, D) \in \mathcal{F}_0^{\uparrow}$, according to Property 7, $A_r \geq D_r$. Then, $(S^X)_r$ is a server.

In the same way, $(S^X)_l$ is also a server. ◀

Now, we want to see if the related notion (the min-plus minimal service, the maximal service, the shaping and the strict minimal service) are stable under the projections, *i.e.*, if the properties associated to them are preserved passing from one world to another.

7.2 Quasi equivalence of min-plus minimal services

Let us begin with the minimal min-plus services. Theorem 24 shows that if a min-plus minimal service with left continuous cumulative curves offers a service curve then this service curve is also the one of the right projection of the server. Reciprocally, if a min-plus minimal service with right continuous arrival curves offers a service curve then, the *left projection of* this service curve is also the one of the left projection of the server. Before proving this theorem, we will demonstrate that more generally, the inequality of the min-plus minimal service is (almost) preserved from any functions in $\dot{\mathcal{F}}$ to one of the two worlds.

▶ **Lemma 23** (Projections of min-plus inequality)**.** *Let $A, D \in \dot{\mathcal{F}}$ and $\beta \in \mathcal{F}^{\uparrow}$ such that $D \geq A * \beta$. Then, $D_r \geq A_r * \beta$ and $D_l \geq A_l * \beta_l$.*

Proof. Let $(A, D) \in \dot{\mathcal{F}}$ and $\beta \in \mathcal{F}^{\uparrow}$ such that $D \geq A * \beta$. The proof is split into two parts, one for each inequality:

1. As $A, D \in \dot{\mathcal{F}}$, using Property 7, we have $D \geq A * \beta \implies D_r \geq (A * \beta)_r$.
   According to Theorem 13, we have that $(A * \beta)_r = A_r * \beta$. Consequently, $D \geq A * \beta \implies D_r \geq A_r * \beta$.
2. As $A, D \in \dot{\mathcal{F}}$, using Property 7, we have $D \geq A * \beta \implies D_l \geq (A * \beta)_l$.
   According to Theorem 13, $(A * \beta)_l = A_l * \beta_l$. Consequently, $D_l \geq A_l * \beta_l$. ◀

This lemma allows us to conclude regarding the switch of the min-plus minimal service from one world to another.

▶ **Theorem 24** (Quasi equivalence of min-plus minimal service)**.** *Let $S^{\dot{\mathcal{R}}}$ be a server offering a min-plus minimal service curve $\beta \in \mathcal{F}^{\uparrow}$. Then $(S^{\dot{\mathcal{R}}})_l$ offers a min-plus minimal service curve β_l.*

Let $S^{\dot{\mathcal{L}}}$ be a server offering a min-plus minimal service curve $\beta \in \mathcal{F}^{\uparrow}$. Then β is a min-plus minimal service curve for $(S^{\dot{\mathcal{R}}})_r$.

The proof consists of applying Lemma 23 on the min-plus minimal service inequality.

The theorem, termed quasi-equivalence, arises because the min-plus minimal service curve is not perfectly conserved. It mandates a left projection switch from the right world to the left one. This stems from two key observations: firstly, left continuity is not inherently absorbing, meaning it demands both functions involved in the convolution to be left continuous; secondly, we cannot lower bound the left projection since it represents the largest left continuous function below the given function.

This lack of preservation of the min-plus minimal service results in non-equivalence. That is, utilising the two results successively to traverse from one world to another does not yield the same outcome. Put differently, some information is lost during the round trip. To illustrate this, we introduce a counter-example. Let $\beta : t \mapsto \begin{cases} 0 & \text{if } t < 3 \\ t & \text{otherwise} \end{cases}$, and let $S^{\dot{\mathcal{L}}}$ denote the server associating any input cumulative curve $A \in \dot{\mathcal{L}}$ with any output cumulative curve $D \in \dot{\mathcal{L}}$ such that $A \geq D \geq A * \beta$. Utilising Theorem 24, a server $\overline{S}^{\dot{\mathcal{R}}}$ exists offering a min-plus minimal service curve β, such that (A_r, D_r) constitutes a valid input/output pair for $\overline{S}^{\dot{\mathcal{R}}}$. Subsequently, by employing Theorem 24 once more, a server $\overline{\overline{S}}^{\dot{\mathcal{L}}}$ exists offering a min-plus minimal service curve β_l, such that $((A_r)_l, (D_r)_l) \overset{(23)}{=} (A, D)$ forms a valid input/output pair for $\overline{\overline{S}}^{\dot{\mathcal{L}}}$. However, $S^{\dot{\mathcal{L}}} \neq \overline{\overline{S}}^{\dot{\mathcal{L}}}$. Specifically, the pair $(A : t \mapsto \lfloor t \rfloor, D = A * \beta_l)$ belongs to $\overline{\overline{S}}^{\dot{\mathcal{L}}}$ but not to $S^{\dot{\mathcal{L}}}$ because $A * \beta_l(3) < A * \beta(3)$, as illustrated in Figure 4.

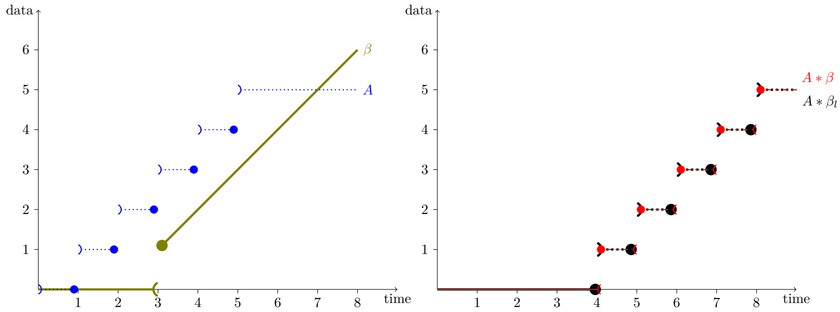

Figure 4 Illustration of a counter-example showing $S^{\dot{\mathcal{L}}} \neq \overline{\overline{S}}^{\dot{\mathcal{L}}}$, *i.e.*utilising the two results successively to traverse from one world to another does not yield the same outcome.

However, it should be noted that $\beta_l \leq \beta$, indicating that β_l still is a min-plus minimal service (potentially with greater pessimism), for which the equivalence between the two worlds holds.

To sum up, we proved that if β is a min-plus minimal service curve for a server $S^{\dot{\mathcal{L}}}$ then it is also for its right projection: $S_r^{\dot{\mathcal{L}}}$. Reciprocally, if β is a min-plus minimal service curve for a server $S^{\dot{\mathcal{R}}}$, then $(\beta)_l$ is a min-plus minimal service curve for its right projection: $S_l^{\dot{\mathcal{R}}}$. That is to say, the equivalence holds only if the min-plus minimal service curve is left continuous.

7.3 Equivalence of maximal services

Following the approach outlined in Section 7.2, we will establish the equivalence between the right and left worlds concerning maximal services. In other words, our objective is to prove that a maximal service curve for the left servers remains a maximal service curve for the right projection server, and vice versa. Prior to demonstrating this equivalence, we will establish, in a broader context, that the inequality of the maximal service curve switching from any functions in $\dot{\mathcal{F}}$ to either of the two worlds remains preserved.

▶ **Lemma 25** (Projections of maximal inequality). *Let $A, D \in \dot{\mathcal{F}}$ and $\beta^M \in \mathcal{F}^{\uparrow}$ such that $D \leq A * \beta^M$. Then, $D_r \leq A_r * \beta^M$ and $D_l \leq A_l * \beta^M$.*

Proof. Let $A, D \in \dot{\mathcal{F}}$ and $\beta^M \in \mathcal{F}^{\uparrow}$ such that $D \leq A * \beta^M$. The proof is split into two parts, one for each inequality:

1. As $A, D \in \dot{\mathcal{F}}$, using Property 7, we have $D \leq A * \beta^M \implies D_r \leq (A * \beta^M)_r$. According to Theorem 13, we have that $(A * \beta^M)_r = A_r * \beta^M$. Consequently, $D \leq A * \beta^M \implies D_r \leq A_r * \beta^M$.
2. As $A, D \in \dot{\mathcal{F}}$, using Property 7, we have $D \leq A * \beta^M \implies D_l \leq (A * \beta^M)_l \leq A_l * \beta^M$. ◀

This lemma allows us to conclude regarding the switch of the maximal service from one world to another.

▶ **Theorem 26** (Equivalence of maximal service). *Let $S^{\dot{\mathcal{R}}}$ be a server offering a maximal service curve $\beta^M \in \mathcal{F}^{\uparrow}$. Then $(S^{\dot{\mathcal{R}}})_l$ offers a maximal service curve β^M.*

Let $S^{\dot{\mathcal{L}}}$ be a server offering a min-plus minimal service curve $\beta^M \in \mathcal{F}^{\uparrow}$. Then $(S^{\dot{\mathcal{L}}})_r$ offers a maximal service curve β^M.

The proof consists of applying Lemma 25.

To sum up, we just proved that if β^M is a maximal service curve for a server $S^{\dot{\mathcal{L}}}$ then it is also for its right projection: $S^{\dot{\mathcal{L}}}_r$. Reciprocally, if β^M is a maximal service curve for a server $S^{\dot{\mathcal{R}}}$, then β^M is a min-plus minimal service curve for its right projection: $S^{\dot{\mathcal{R}}}_l$.

7.4 Equivalences of shapers

Now, let's examine the (null) effect of changing the continuity properties for shapers. Our objective is to illustrate that a shaping curve for the left servers remains a shaping curve for the right projection server, and vice versa. Prior to proving this equivalence, we will demonstrate, in a broader context, that the inequality of the shaping curve switching from any functions in $\dot{\mathcal{F}}$ to either of the two worlds remains preserved.

▶ **Lemma 27** (Projections of shaping inequality). *Let $A, D \in \dot{\mathcal{F}}$ and $\sigma \in \mathcal{F}^{\uparrow}$ such that $D \leq D * \sigma$. Then, $D_r \leq D_r * \sigma$ and $D_l \leq D_l * \sigma$.*

Proof. Let $A, D \in \dot{\mathcal{F}}$ and $\sigma \in \mathcal{F}^{\uparrow}$ such that $D \leq D * \sigma$. The proof is split into two parts, one for each inequality:

1. As $A, D \in \dot{\mathcal{F}}$, using Property 7, we have $D \leq D * \sigma \implies D_r \leq (D * \sigma)_r$.
   According to Theorem 13, we have that $(D * \sigma)_r = D_r * \sigma$. Consequently, $D \leq D * \sigma \implies D_r \leq D_r * \sigma$.
2. As $A, D \in \dot{\mathcal{F}}$, using Property 7, we have $D \leq D * \sigma \implies D_l \leq (D * \sigma)_l \leq D_l * \sigma$. ◀

This lemma allows us to conclude regarding the switch of the shaper from one world to another.

▶ **Theorem 28.** *Let $S^{\dot{\mathcal{R}}}$ be a server offering a shaping curve $\sigma \in \mathcal{F}^{\uparrow}$. Then $(S^{\dot{\mathcal{R}}})_l$ offers a shaping curve σ.*

Let $S^{\dot{\mathcal{L}}}$ be a server offering a shaping curve $\sigma \in \mathcal{F}^{\uparrow}$. Then σ is a shaping curve for $(S^{\dot{\mathcal{R}}})_r$.

The proof consists of applying Lemma 27.

To sum up, we just proved that if σ is a shaping curve for a server $S^{\dot{\mathcal{L}}}$ then it is also for its right projection: $S^{\dot{\mathcal{L}}}_r$. Reciprocally, if σ is a shaping curve for a server $S^{\dot{\mathcal{R}}}$, then σ is a min-plus minimal service curve for its right projection: $S^{\dot{\mathcal{R}}}_l$.

7.5 No direct equivalence of strict minimal services

We may expect a similar result for the strict min-plus service.

As in the definition of the strict minimal service, the curve is a strict minimal service curve for any $(s, t]$ backlogged period (Equation 12), we first need to study these backlogged periods. First, let us start with this latter notion: the backlogged periods and show that it is not preserved passing from one world to another with an illustration. Thus, it is not possible to have an equivalence for the strict minimal services and it is shown with a counter-example.

Let us take an example to illustrate that the strict minimal service is not preserved. Consider a system $A \xrightarrow{S} B$ and $B \xrightarrow{S'} C$ and its right projection $A_r \xrightarrow{S_r} B_r$ and $B_r \xrightarrow{S'_r} C_r$ with

$$A(t) = A_r(t) = t \wedge 3, \tag{13}$$

$$B(t) = ((\lceil t \rceil - 1) \vee 0) \wedge 3, \tag{14}$$

$$C(t) = C_r(t) = [(t-1) \wedge 3]^+. \tag{15}$$

The top of Fig. 5 represents these systems and the bottom of Fig. 5 shows the cumulative curves A, B, B_r and C (the left side shows the left continuous ones (A, B and C) and the right side shows the right continuous ones (A, B_r and C)).

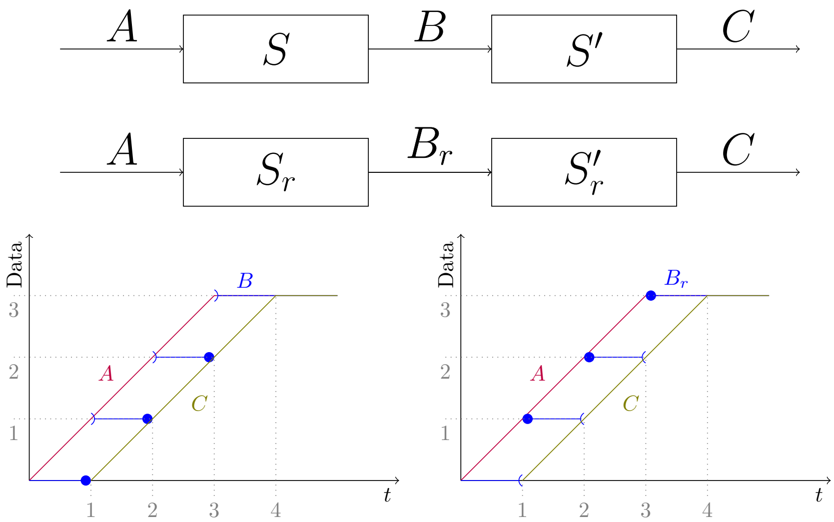

Figure 5 Sequence of two servers.

If we study the backlogged periods of (A, B) and its right projection (A, B_r) then, with the current definition of the backlogged period (Definition 17), we see that $[1\ ;\ 4]$ is a backlogged period for the left continuous functions (A, B). However, $[1\ ;\ 4]$ is not one for (A, B_r), the right projection of (A, B) because *e.g.* $A(2) = B(2)$. Consequently, the backlogged period for the left continuous functions is not in general one for the right projections.

Related to this analysis, we show that, with the current definition of the backlogged period (Definition 17), there is no equivalence between left and right worlds. Consequently, we cannot have equivalence between the strict min-plus service curve, illustrated by a counter-example: let $\beta(t) = 0 \vee (2(t-1))$. Indeed, $\forall (s,t]$ backlogged period of (B,C), $C(t) - C(s) \geq \beta(t-s)$ because $\forall (s,t]$ backlogged period, $t - s < 1 \implies \beta(t-s) = 0$. However, $(1, 3.9]$ is a backlogged period for (B_r, C) and $C(3.9) - C(1) = 2.9 \leq 3.8 = \beta(3.9-1)$. Consequently, β is a strict minimal service curve for the server in the left world but not for its right projection. Fig. 6 illustrates this service curve β.

Then, this example shows that, with the current definition of the backlogged period, we cannot have any equivalence of the strict minimal service between the left and the right world.

Now going from right to left we can also construct a similar example showing that the equivalence is not possible according to the current definition of the backlogged period.

As a result, the preservation of the strict minimal service is not guaranteed when switching from one world to another. Nevertheless, it's worth noting, as indicated in [4, Theorem 17 and 19], that a strict minimal service remains a min-plus minimal service. Consequently, while switching between worlds is feasible, the notion of "strict" is lost in the process.

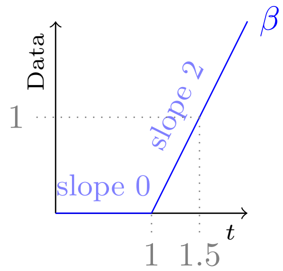

Figure 6 Graph of a curve $\beta(t) = 2.[t-1]^+$.

Some preliminary works on a new definition of strict service stable by change of continuity can be found in [9, § 4.5.2].

7.6 Summary

In conclusion of this section, we demonstrate that both the maximal service curve and the shaping curve remain preserved when switching between the two worlds. Regarding the min-plus minimal service curves, we establish that a left service curve is preserved, but in the general case, an additional left projection is required to switch from the right world to the left one (a min-plus minimal service curve from the left world remain a min-plus minimal service curve for the right one). These results are summarised in Figure 2.

Finally, we ascertain that there is no equivalence for the strict minimal service curve due to the absence of preservation of the notion of backlogged period.

Now, our attention shifts to the primary aspect of the analysis: the bounds on delay and backlog. Our objective is to determine whether these bounds remain valid when switching between the two worlds.

8 Continuity and delay/backlog bounds

Network Calculus is designed to compute upper bounds on delays and backlog. Let us recall the definition of the horizontal and vertical deviations which are the operators utilised to calculate these bounds.

▶ **Definition 29** (Horizontal and vertical deviation). *Let $f, g \in \mathcal{F}_0^{\uparrow}$, the horizontal and vertical deviation between f and g are respectively defined as*

$$\mathrm{hDev}(f,g) \overset{def}{=} \sup_{t\in\mathbb{R}^+} \left\{ \inf \left\{ d \in \mathbb{R}^+ \mid f(t) \le g(t+d) \right\} \right\}, \tag{16}$$

$$\mathrm{vDev}(f,g) \overset{def}{=} \sup_{t\in\mathbb{R}^+} \left\{ f(t) - g(t) \right\}. \tag{17}$$

Based on this definition, we can introduce the worst-case delay and backlog.

▶ **Definition 30** (Worst-case delay and backlog). *Let $X \subseteq \mathcal{F}_0^{\uparrow}$, a server $S^X \subseteq X \times X$ and $(A, D) \in S^X$. The worst-case delay is defined as* $\mathrm{hDev}(A, D)$ *and the worst-case backlog as* $\mathrm{vDev}(A, D)$.

These functions are in general defined only for $X = \dot{\mathcal{L}}$, except [4] which states that these bounds are equal if we exchange left/right continuity, *i.e.*, $\forall f, g \in \mathcal{F}^{\uparrow}$ such that $g \le f$,

$$\mathrm{hDev}(f_l, g_l) = \mathrm{hDev}(f_r, g_r), \qquad \mathrm{vDev}(f_l, g_l) = \mathrm{vDev}(f_r, g_r). \tag{18}$$

When looking at the proof details, the result that is proved is stronger: it shows that $\forall f, g \in \mathcal{F}^{\uparrow}$ such that $g \leq f$,

$$\text{hDev}(f, g) = \text{hDev}(f_l, g_l) = \text{hDev}(f_r, g_r). \tag{19}$$

Conversely, if f and g do not have the same continuity, the equality on the backlog is not anymore valid. Consider the example depicted in Figure 7.

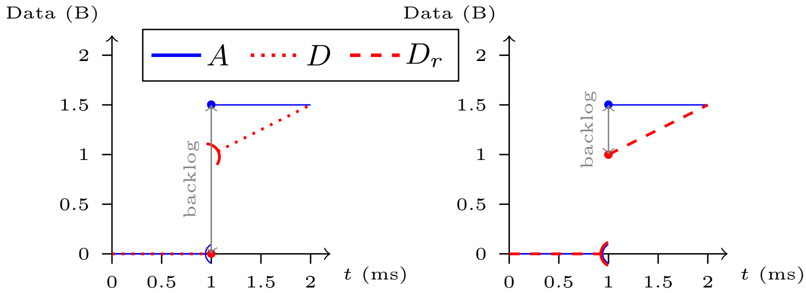

Figure 7 Illustration of an example of an input cumulative curve (A) and its output cumulative curve (D).

In Figure 7, we can see that, if A is right continuous and D is left continuous, the maximal backlog is equal to 1.5B whereas if the continuity of the two flows is right continuous, the maximal backlog is 0.5B.

The main result of Network Calculus is the ability to have upper bounds on delay and backlog on real behaviour (A, D) from contracts $(\overline{\alpha}, \beta)$, cf [2, Theorem 5.2] or [12, Theorem 1.4.2]. This is well known for $A, D \in \dot{\mathcal{L}}$, but [4] did not provide results for other continuities.

▶ **Theorem 31** (Delay and Backlog bounds). *Let $X \subseteq \dot{\mathcal{F}}$. Let S^X be a server, $\overline{\alpha} \in \mathcal{F}^{\uparrow}$ and $\beta \in \mathcal{F}_0^{\uparrow}$. If $(A, D) \in S^X$ such that A has maximal arrival curve $\overline{\alpha}$ and S^X offers a min-plus minimal service curve β, then the maximum delay and backlog can be respectively bounded with*

$$\text{hDev}(A, D) \leq \text{hDev}(\overline{\alpha}, \beta), \tag{20}$$

$$\text{vDev}(A, D) \leq \text{vDev}(\overline{\alpha}, \beta). \tag{21}$$

Then, this result is not a new theorem itself but a generalisation regarding the continuity of the cumulative curves. It allows us to have a bound on the maximum delay and backlog according to the service and the arrival curves for any cumulative curves, *i.e.*, regarding any continuity. Previously, the bounds can be computed only in the left world.

Proof. A check of proofs in [2, Theorem 5.2] shows that continuity is not involved in the proof. The proof in [12, Theorem 1.4.2] involves a continuity argument, even the proof starts with "The proof for the general [continuity] case is essentially the same but involves some ϵ cutting". ◀

Consequently, we are now able to compute bounds on the maximum delay and backlog for cumulative curves with any continuity.

Note that it has been already proved in [4] that $\text{hDev}(A_l, D_l) = \text{hDev}(A_r, D_r)$ and $\text{vDev}(A_l, D_l) = \text{vDev}(A_r, D_r)$, so the delays (not the bounds on delays) are the same in both worlds.

Moreover, considering $\beta \in \mathcal{L}^{\uparrow}$, according to Section 6.2 and Section 7, we can show that if a server S offers the min-plus minimal service β and $\overline{\alpha}$ is a maximal arrival of a flow A crossing the server S then, $\mathrm{hDev}(\overline{\alpha}, \beta)$ and $\mathrm{vDev}(\overline{\alpha}, \beta)$ are bounds on the maximum delay and backlog respectively regardless of the continuity of the cumulative curves. That is to say that $\mathrm{hDev}(\overline{\alpha}, \beta)$ and $\mathrm{vDev}(\overline{\alpha}, \beta)$ are valid bounds of maximum delay and backlog respectively, $\forall (A, D) \in S^X$, $X \subseteq \dot{\mathcal{F}}$ as soon as S offers a min-plus minimal service curve $\beta \in \mathcal{L}^{\uparrow}$ and A has maximal arrival curve $\overline{\alpha}$.

However, in the general case ($\beta \notin \mathcal{L}^{\uparrow}$), when switching from the left world to the right one, one may keep the same bounds (cf. Theorem 24), but switching from the right world to the left one, the bounds of the maximal delay and backlog can differ. Consider $S^{\dot{\mathcal{R}}}$ offering the min-plus minimal service β. For $(A, D) \in S^{\dot{\mathcal{R}}}$ and $\overline{\alpha}$ a maximal arrival curve for A, $\mathrm{hDev}(A_l, D_l) \leq \mathrm{hDev}(\overline{\alpha}, \beta_l)$ but $\mathrm{hDev}(\overline{\alpha}, \beta_l) \geq \mathrm{hDev}(\overline{\alpha}, \beta)$ and the same $\mathrm{vDev}(A_l, D_l) \leq \mathrm{vDev}(\overline{\alpha_l}, \beta_l)$ but $\mathrm{vDev}(\overline{\alpha}, \beta_l) \geq \mathrm{vDev}(\overline{\alpha}, \beta)$.

9 Usage of the results

In this section, we will exemplify how to pass from one world to another using the previous results. Consider a system S shared by two flows A, A', followed by a packetizer P, as in Figure 8.

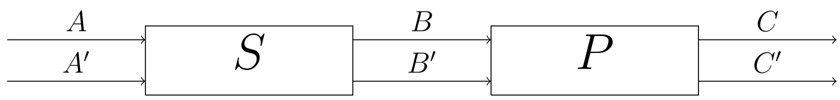

Figure 8 A system composed of a server (S) and a packetizer (P) shared by two data flows.

Assume that S uses a Static Priority Preemptive policy (SPP) and offers a strict minimal service curve β. Using the most common approach, we assume that the cumulative curves are left continuous and (A, A') are packetized. Also, A has higher priority than A'.

Let us do the two first steps in the left world. According to [2, Thm 7.6], $[\beta - \alpha_A]^+_{\uparrow}$ is a strict minimal service curve for A'.

And, according to [12, Prop. 1.3.5], a strict minimal service implies a min-plus minimal service.

Now, using Theorem 24, we can transfer this service from the left to the right world and state that $[\beta - \alpha_A]^+_{\uparrow}$ is a min-plus minimal service curve for $(A_r, B_r) \in S_r$.

Now, in the right world, according to [12, Thm 1.7.1], $\beta_{A'} = [[\beta - \alpha_A]^+_{\uparrow} - L'^M]^+$ (with L'^M the maximum packet length of A'_r) is a minimal service curve for the sequence $S; P$ that maps A'_r into C'_r. (notice that $[[\beta - \alpha_A]^+_{\uparrow} - L'^M]^+ = [\beta - \alpha_A - L'^M]^+_{\uparrow}$).

Finally, using Theorem 24, we can transfer this service from the right to the left world, i.e. $(\beta_{A'})_l$ is a min-plus service for $S; P$. as illustrated in Figure 9.

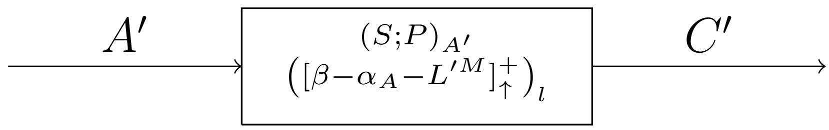

Figure 9 Illustration of the network $\mathcal{N}$ reduced according to the flow (A, C).

This is an example of the usage of the results presented in this paper. However, the aim is to use more results to improve analysis, for instance, the link with CPA theory as [6] suggests.

10 State of the art

As we previously said, most of the theory is based on the assumption of left continuous cumulative curves as, justified in [2, §1.3] and [12, §1.2.1]. In particular, this property implies that for any backlog instant t, there is a specific instant s, the start of the backlogged period such that $A(s) = D(s)$. This property is used in most proofs related to the derivation of the residual services. But, as noted in the previous sections, some notions are difficult in the left world. It was already noticed in [12, §1.1.1] that "It would be nice to stick to either left or right continuous functions. However, depending on the model, there is no best choice".

An example of a system easier to define and manipulate in the right world is the *packetizer*, the system which waits for the reception of all bits of a packet before transmitting it. A definition in the left world is quite complex (*e.g.* [2, Def. 8.2]) whereas it is simple in the right world (*e.g.* [12, Def. 1.7.3] and [6, Def. 10]), and this complexity can create some errors or difficulties as noticed in [2, §8.3].

A definition of Network Calculus in the right world can be found in [11, §4]. It introduces the basics of the Network Calculus theory on the cumulative curves. However, only the basics and some scheduling are developed.

Some works also consider the relations between both worlds. For instance, it is shown in [4] that the delay and the backlog are not influenced by the continuity of the cumulative curves. Moreover, the same hierarchy between server flavours holds in the right world (strict minimal service implies min-plus minimal services) and the expression of the FIFO and Static Priority services are almost identical ('almost' due to continuity) and lead to the same numerical results.

Focusing on the assumption of right continuous cumulative curves, [6] developed the link between the Network Calculus and the Compositional Performance Analysis (CPA) theory. The idea is to pass from the quantity of data of the Network Calculus (A) to the number of events in the CPA theory (E) using the packetization P, and the link is $E = P(A)$. But, this paper is developed using the right continuous assumption and this choice is explained in [5, §4.4].

Some results mixing the continuity of functions and the operators of the Network Calculus are also developed in [15].

All these previous works show a new point of view in the theory of Network Calculus and other results are developed changing some assumptions. To continue in this idea, [13] is another projection to the theory inverting the amount of data and the time that leads to the max-plus theory and other results.

11 Conclusion

Network Calculus represents flow behaviour with piece-wise continuous functions. For some results (especially the ones handling data amount), it is easier to consider left continuous functions, whereas for others (especially the ones handling packets), right continuous are more convenient. However, it is unsound to use, in the same analysis, several results developed from incompatible hypotheses. Then, to use results developed with different hypotheses, the only solution was, up to now, to choose one hypothesis and to redevelop the missing results (as done in [4]). But this is tedious and error-prone, whereas the engineer's intuition was that these continuity problems do not really matter.

To do so, this paper has developed some results related to the min-plus convolution, one of the fundamental operators of the Network Calculus, regarding either the right continuous or the left continuous functions.

Also, the main result is that most properties linked to the convolution are preserved. Thus, the equivalences between the left and the right world are proven for the arrival curves, the maximal service curves and the shaping curves. Concerning the min-plus minimal service curves, this paper demonstrates that, 1) if the min-plus minimal service curve is already left continuous, it holds in both world, 2) whatever its continuity is, it can be kept when going from the left world to the right world, and 3) going from the right world to the left one, it necessitates a left projection. A counter-example shows that this projection is mandatory in the general case.

Conversely, the strict minimal service is not preserved due to the backlogged periods. A counter-example showed that there is no equivalence with the current definition. As a consequence, any strict minimal service in the right world cannot be analysed with theorems developed in the left world. It can only be considered as a min-plus minimal service in the left world, or use the only (up to our knowledge) result developed for strict service: the blind multiplexing of [4]. For example, a system implementing a Deficit Round Robin scheduling policy in the right world can only be analysed as a blind server, leading to pessimistic bounds. To have transfer results, it seems to be necessary to improve the definition of the backlogged period which is currently not "stable" passing from one world to another. Then, a similar strict minimal service using this improved backlogged period may be considered [9].

In future works, we would like to use these results to improve the analysis results by combining them with CPA theory, as proposed in [6, 10].

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A Proof of Property 3

Proof. The proof comes from `https://encyclopediaofmath.org/wiki/Increasing_function` and is reported here for completeness. Let $f \in \mathcal{F}^{\uparrow}$. If x_0 is a right-sided (or left-sided) limit point of the set $\mathbb{R}^+$, if f is a non-decreasing function and if the set $A = \{y : y = f(x), x > x_0, x \in \mathbb{R}^+\}$ is bounded from below - or if $\{y : y = f(x), x < x_0, x \in \mathbb{R}^+\}$ is bounded from above - then, as $x \to x_0+$ (or, correspondingly, as $x \to x_0-$), $x \in \mathbb{R}^+$, the values $f(x)$ will have a finite limit; if the set is not bounded from below (or, correspondingly, from above), the values $f(x)$ have an infinite limit equal to $-\infty$ (or, correspondingly, to $+\infty$). If f is non-decreasing on $\mathbb{R}^+$ and $x_0 \in \mathbb{R}^+$, then the set A referred to above is automatically bounded from below by $f(x_0)$, unless it is empty. If, in addition, x_0 is a limit point of $\{x \in \mathbb{R}^+ : x > x_0\}$, then the right-hand limit of f at x_0 is simply the infimum of A: $\lim_{x \to x_0-} f(x) = \inf A$. A similar proof holds for left-hand limits. ◀

B Alternative expression of the min-plus convolution

Let us introduce a new expression for the convolution equivalent to the previous ones (Equation (5) and (6)). This expression can be useful to grasp the notion of "sliding" one function over the other.

▶ **Property 32** (Alternative expression of the convolution). *Let $f, g \in \mathcal{F}$ be two functions. Then*

$$\forall t \in \mathbb{R}^+ : f * g(t) = \inf_{0 \le s} \left\{ f(t \wedge s) + g([t-s]^+) \right\}. \tag{22}$$

Proof. Let $f, g \in \mathcal{F}$ be two functions.

$$\inf_{0 \le s} \left\{ f(t \wedge s) + g([t-s]^+) \right\} = \min \begin{cases} \inf_{0 \le s < t} \left\{ f(t \wedge s) + g([t-s]^+) \right\} \\ \inf_{s \ge t} \left\{ f(t \wedge s) + g([t-s]^+) \right\} \end{cases}$$

Note that $\begin{cases} 0 \le s < t \implies t \wedge s = s \text{ and } [t-s]^+ = t - s \\ s \le t \implies t \wedge s = t \text{ and } [t-s]^+ = 0 \end{cases}$, so

$$= \min \begin{cases} \inf_{0 \le s < t} \left\{ f(s) + g(t-s) \right\} \\ \inf_{s \ge t} \left\{ f(t) + g(0) \right\} \end{cases}$$

$$= \min \begin{cases} \inf_{0\leq s<t} \{f(s) + g(t-s)\} \\ (f(t) + g(0)) \end{cases}$$
$$= \inf_{0\leq s\geq t} \{f(s) + g(t-s)\} = f * g(t).$$ ◀

This property can be illustrated in Figure 10 adapted from [2, Fig. 2.1]. he convolution operation is typically depicted by imagining one function sliding over the other, as shown at the top of Figure 10. Property 32 encapsulates this concept precisely: for each s, consider $h_s : t \mapsto g(t \wedge s) + f([t-s]^+)$. Then, $f * g = \inf_{s\geq 0} h_s$. This relation is depicted at the bottom of Figure 10, where h_s is plotted for two different values of s.

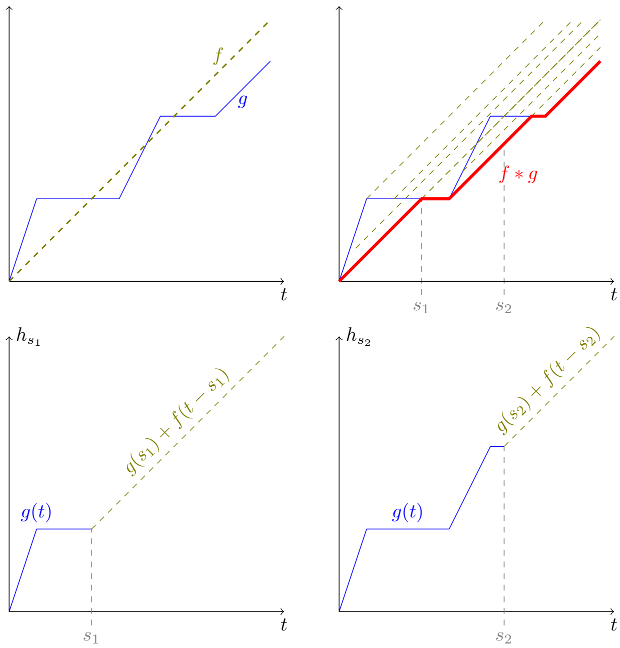

Figure 10 Top: Illustration of the convolution ($f * g$: thick) between f (dashed) and g (plain), from [2, Fig. 2.1].
Bottom: $h_s : t \mapsto g(t \wedge s) + f([t-s]^+)$ for two specific values of s.

C Composition of projections

The following properties allow us to traverse from one world to another the cumulative curves.

▶ **Property 33** (Composition of the projections). *Let $f \in \dot{\mathcal{F}}$. Then*

$$(f_r)_l = f_l, \qquad \text{and} \qquad (f_l)_r = f_r. \tag{23}$$

Proof. Let us start with $(f_r)_l = f_l$. Let $x \in \mathbb{R}^+$. For $x = 0$, as $f \in \dot{\mathcal{F}}$, f is continue at 0, then $(f_r)_l(0) \overset{(2)}{=} f_r(0) = f(0)$ Otherwise, let $(u_n)_{n\in\mathbb{N}}$ be a decreasing sequence converging to 0. Then, $(f_r)_l(x) = \lim_{n\to\infty} f_r(x - u_n)$.

Let $(v_m)_{m\in\mathbb{N}}$ be a decreasing sequence converging to 0. Then,
$(f_r)_l(x) = \lim_{n\to\infty}(\lim_{m\to\infty} f(x - u_n + v_m))$.

$\forall n \in \mathbb{N}$, since (v_m) is decreasing, $\exists m \in \mathbb{N}$ such that $u_n - v_m > 0$. Let us note $\phi(n)$ this number, and define $(w_k)_{k\in\mathbb{N}}$ by $w_k = u_k - v_{\phi(k)}$. We have $\lim_{k\to\infty} w_k = 0$, and $\forall k : w_k > 0$, so $\lim_{k\to\infty} f(x - w_k) = f_l(x)$. Since any sub-sequence converges to the same limit as the sequence itself (when it exists), $(f_r)_l(x) = f_l(x)$.

The same way, we also have $(f_l)_r = f_r$. ◀