Two-Dimensional Kripke Semantics I: Presheaves

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Abstract

The study of modal logic has witnessed tremendous development following the introduction of Kripke semantics. However, recent developments in programming languages and type theory have led to a second way of studying modalities, namely through their categorical semantics. We show how the two correspond.

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1 Introduction

The development of modal logic has undergone many phases [23, 15, 46, 98]. It is widely accepted that one of the most important developments was the relational semantics of Kripke [68, 69, 70] [15, §1] [46, §4.8]. Kripke semantics has proven time and again that it is intuitive and technically malleable, thereby exerting sustained influence over Computer Science.

However, over the last 30 years another way of studying modalities has evolved: looking at modal logic through the prism of the *Curry-Howard-Lambek correspondence* [72, 93, 99] yields new computational intuitions, often with surprising applications in both programming languages and formal proof. The tools of the trade here are type theory and category theory.

Up to now these two ways of looking at modalities have been discussed in isolation. The purpose of this paper is to establish a connection: I will show that the Kripke and categorical semantics of modal logic are part of a *duality*. It is well-known that dualities between Kripke and algebraic semantics exist: the *Jónsson-Tarski duality* is one of the cornerstones of classical modal logic [15, §5]. The main contribution of this paper is to show that such dualities can be elevated to the level of *proofs*. The punchline is that a *profunctor* $R : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$, considered as a *proof-relevant relation* on a category $\mathcal{C}$, uniquely corresponds to a categorical model of modal logic on the category of presheaves on $\mathcal{C}$.

There are two obstacles to overcome to get to that result. The first is that we must work over an *intuitionistic* substrate: most research on types and categories is forced to do so, for unavoidable reasons [72, §8]. We must therefore first develop a duality for *intuitionistic modal logic*. However, there is no consensus on what intuitionistic modal logic is! The problem is particularly acute in the presence of $\Diamond$ [27]. I will avoid this problem by making canonical choices at each step. First, I will formulate a Kripke semantics based on *bimodules*, i.e.

relations that are canonically compatible with a poset. Then, I will show how *Kan extension* uniquely determines two adjoint modalities, ⧫ and □, from any bimodule. The fact these arise automatically is evidence that they are the canonical choice of intuitionistic modalities.

The second obstacle stems from considering proofs. The jump from algebraic to categorical semantics involves adding an extra "dimension" of proofs. Consequently, in order to re-establish a duality, an additional dimension must be added to Kripke semantics as well. I call the result a *two-dimensional Kripke semantics*. Category theorists will find it anticlimactic: it amounts to the folklore observation that a proof-relevant Kripke semantics is essentially a semantics in a presheaf category.

Indeed, a sizeable proportion of this paper consists of folklore results that are well-known to experts. However, many of them are drawn from related but distinct areas: logic, order theory, category theory, and topos theory. As a result, it does not appear that all of them are well-known by a *single* expert. Thus, the synthesis presented here appears to be new.

The results I present in this paper show that there are deep connections between modal logic and presheaf categories. This is important, as the latter are ubiquitous in logic and related fields: presheaf models are used in fields as disparate as categorical homotopy theory [87, 24], type theory [57], concurrency [65, 21, 22], memory allocation [81, 82], synthetic guarded domain theory [14], second-order syntax and algebraic theories [34, 53, 35, 36, 37], higher-order abstract syntax [58], and so on. As a result, the connections presented here may enable synthetic reasoning *via* modalities in a variety of logical settings.

In Section 2 I recall the Kripke and algebraic semantics of intuitionistic logic, and outline the duality between Kripke semantics and certain complete Heyting algebras, the *prime algebraic lattices*. Then I extend this duality to intuitionistic modal logic in Section 3 by showing how a relation that is compatible with the intuitionistic order – a bimodule – gives rise to modalities through Kan extension. In Section 4 I add proofs to intuitionistic logic, and elevate the duality to one between "two-dimensional frames" and presheaf categories. I then repeat this exercise for intuitionistic modal logic in Section 5 by promoting bimodules to profunctors on the relational side, and adding an adjunction on the categorical side.

For general background in orders please refer to the book by Davey and Priestley [28]. Given a poset $(D, \sqsubseteq_D)$ let the *opposite* poset D^{op} be given by reversing the partial order; that is, $x \sqsubseteq_{D^{\mathsf{op}}} y$ iff $y \sqsubseteq_D x$. A *lattice* has all finite meets and joins. A *complete lattice* has arbitrary ones. A complete lattice is *infinitely distributive* just if the law $a \wedge \bigvee_i b_i = \bigvee_i a \wedge b_i$ holds. Such lattices are variously called *frames*, *locales*, or *complete Heyting algebras* [61, 76, 84].

2 Intuitionistic Logic I

There are many types of semantics for intuitionistic logic, including Kripke, Beth, topological, and algebraic semantics. Bezhanishvili and Holliday [11] argue that these form a strict hierarchy, with Kripke being the least general, and algebraic the most general. I will briefly review the elements of these extreme points of the spectrum.

The Kripke semantics of intuitionistic logic are given by *Kripke frames*, i.e. partially-ordered sets $(W, \sqsubseteq)$ [23, §2.2]. W is referred to as the set of *possible worlds*, and $\sqsubseteq$ as the *information order*. A world $w \in W$ is a "state of knowledge", and $w \sqsubseteq v$ means that moving from w to v potentially entails an increase in the amount of information.

Let $\mathsf{Up}(W)$ be the set of *upper sets* of W, i.e. the subsets $S \subseteq W$ such that $w \in S$ and $w \sqsubseteq v$ implies $v \in S$. A *Kripke model* $\mathfrak{M} = (W, \sqsubseteq, V)$ consists of a Kripke frame $(W, \sqsubseteq)$ as well as a function $V : \mathsf{Var} \to \mathsf{Up}(W)$. The *valuation* V assigns to each propositional variable $p \in \mathsf{Var}$ an upper set $V(p) \subseteq W$, which is the set of worlds in which p is true. The idea is that, once a proposition becomes true, it must remain true as information increases.

We are now able to inductively define a relation $\mathfrak{M}, w \vDash \varphi$ with the meaning that φ is true in world w of model $\mathfrak{M}$. The only interesting clause is that for implication:

$$\mathfrak{M}, w \vDash \varphi \to \psi \stackrel{\text{def}}{\equiv} \forall w \sqsubseteq v.\ \mathfrak{M}, v \vDash \varphi \text{ implies } \mathfrak{M}, v \vDash \psi$$

This definition is famously monotonic: if $\mathfrak{M}, w \vDash \varphi$ and $w \sqsubseteq v$ then $\mathfrak{M}, v \vDash \varphi$. Kripke semantics is sound and complete for intuitionistic logic [23, 15].

The algebraic semantics of intuitionistic logic consist of *Heyting algebras*. These are lattices such that every map $- \wedge x : L \to L$ has a right adjoint, i.e. for $x, y \in L$ there is an element $x \Rightarrow y \in L$ such that $c \wedge x \sqsubseteq y$ iff $c \sqsubseteq x \Rightarrow y$. Such lattices are always distributive. Assuming that one has an interpretation $[\![p]\!] \in L$ of each proposition p, each formula φ of intuitionistic logic is inductively mapped to an element $[\![\varphi]\!] \in L$ using the corresponding algebraic structure. I will not expound further on Heyting algebras; see [23, §7.3] [16, §1.1] [76, §I.8]. But I note that they are sound and complete for intuitionistic logic.

2.1 Prime algebraic lattices

Let $(W, \sqsubseteq)$ be any Kripke frame, and let $\mathbb{2} \stackrel{\text{def}}{=} \{0 \sqsubseteq 1\}$. Consider the poset $[W, \mathbb{2}]$ of monotonic functions from W to $\mathbb{2}$, ordered pointwise. This poset has a number of curious properties.

First, the monotonicity of $p : W \to \mathbb{2}$ means that if $p(w) = 1$ and $w \sqsubseteq v$, then $p(v) = 1$. Hence, the subset $U \stackrel{\text{def}}{=} p^{-1}(1)$ of W is an upper set. Conversely, every upper set $U \subseteq W$ gives rise to a monotonic $p_U : W \to \mathbb{2}$ by setting $p_U(w) = 1$ if $w \in U$, and 0 otherwise. Consequently, there is an *order bijection*

$$\mathsf{Up}(W) \cong [W, \mathbb{2}]$$

with the order on $\mathsf{Up}(W)$ being inclusion. I will liberally treat upper sets and elements of $[W, \mathbb{2}]$ as the same, but prefer the latter notation for reasons that will become clear later.

Second, given any $w \in W$, consider its *principal upper set* $\uparrow w \stackrel{\text{def}}{=} \{v \in W \mid w \sqsubseteq v\} \in [W, \mathbb{2}]$. This set consists of worlds with potentially more information than that found in world w. A simple argument shows that $w \sqsubseteq v$ iff $\uparrow v \subseteq \uparrow w$.[1] Thus, this gives an *order embedding*

$$\uparrow : W^{\mathsf{op}} \to [W, \mathbb{2}]$$

which can be shown to preserve meets and exponentials.

Third, the poset $[W, \mathbb{2}]$ is a *complete lattice*: arbitrary joins and meets are given pointwise. Viewing the elements of $[W, \mathbb{2}]$ as upper sets, these joins and meets correspond to arbitrary unions and intersections of upper sets, which are also upper. Moreover, this lattice satisfies the infinite distributive law, so it is a *complete Heyting algebra* – synonymously a *frame* or *locale* [61, 84]. Given two upper sets $X, Y \subseteq W$ their exponential is given by [29, §1.9]

$$X \Rightarrow Y \stackrel{\text{def}}{=} \{w \in W \mid \forall w \sqsubseteq v.\ v \in X \text{ implies } v \in Y\}$$

Fourth, the principal upper sets $\uparrow w$ are special, in that they are *prime*.[2] An element d of a complete lattice L is *prime* just if

$$d \sqsubseteq \bigsqcup X \quad \text{implies} \quad \exists x \in X.\ d \sqsubseteq x$$

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[1] This is an order-theoretic consequence of the Yoneda lemma.

[2] Such elements are variously called *completely join-irreducible* [86], *supercompact* [8] [84, §VII.8], *completely (join-)prime* [100], or simply *join-prime* [41, §1.3].

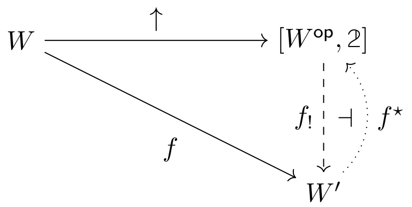
This says that d contains a tiny, indivisible fragment of information: as soon as it approximates a supremum, it must approximate something in the set that is being upper-bounded. The prime elements of $[W, \mathbb{2}]$ are exactly the principal upper sets $\uparrow w$ for some $w \in W$.

Fifth, the complete lattice $[W, \mathbb{2}]$ is *prime algebraic*. This means that all its elements can be reconstructed by "multiplying" or "sticking together" prime elements. In symbols, a complete lattice L is prime algebraic whenever for every element $d \in L$ we have

$$d = \bigsqcup \{p \in L \mid p \sqsubseteq d, p \text{ prime } \}$$

Such lattices are variously called *completely distributive, algebraic lattices* [28, §10.29] or *superalgebraic lattices* [84, §VII.8]. In fact, it can be shown that any such lattice is essentially of the form $[W, \mathbb{2}]$, i.e. a lattice of upper sets; this was shown by Raney in the 1950s [86], and independently by Nielsen, Plotkin and Winskel in the 1980s [80]. See the paper by Winskel for the use of prime algebraic lattices in semantics [100].

Finally, the fact every element can be reconstructed as a supremum of primes means that it is possible to canonically extend any monotonic $f : W \to W'$ to a monotonic $[W^{\mathsf{op}}, \mathbb{2}] \to W'$, as long as W' is a complete lattice. Diagrammatically, in the situation



(1)

there exists a unique $f_!$ which preserves all joins and satisfies $f_!(\uparrow w) = f(w)$. It is given by

$$f_!(S) \stackrel{\text{def}}{=} \bigsqcup \{f(w) \mid w \in S\}$$

$f_!$ is called the *(left) Kan extension* of f along $\uparrow$. As $f_!$ preserves all joins and $[W, \mathbb{2}]$ is complete it has a right adjoint $f^\star$, by the adjoint functor theorem [28, §7.34] [61, §I.4.2]. For any complete lattice W' this situation amounts to a bijection

$$\mathrm{Hom}_{\mathbf{Pos}}(W, W') \cong \mathrm{Hom}_{\mathbf{CSLatt}}([W^{\mathsf{op}}, \mathbb{2}], W')$$

where **CSLatt** is the category of complete lattices and join-preserving maps.

Suppose then that we have a Kripke model $(W, \sqsubseteq, V)$. The construction given above induces a Heyting algebra $[W, \mathbb{2}]$. Defining $[\![p]\!] \stackrel{\text{def}}{=} V(p)$ we obtain an algebraic model of intuitionistic logic, which interprets every formula φ as an upper set $[\![\varphi]\!] \in [W, \mathbb{2}]$. This is the upper set of worlds in which a formula is true [23, Theorem 7.20]:

▶ **Theorem 1.** *$w \vDash \varphi$ if and only if $w \in [\![\varphi]\!]$.*

Thus, every Kripke semantics corresponds to a prime algebraic lattice.

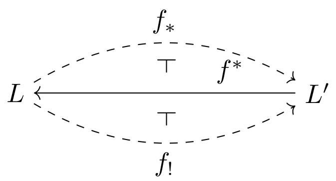
▶ Remark 2. This shows that a Kripke semantics is a particular kind of algebraic semantics. Thus, we can deduce the completeness of the latter from the completeness of the former: if a formula is valid in all Heyting algebras, it must be valid in all prime algebraic lattices, and hence valid in all Kripke semantics. If the Kripke semantics is complete, then the formula must be provable. Therefore, the algebraic semantics is then complete as well.

The opposite direction – viz. proving the completeness of Kripke semantics from completeness of the algebraic semantics – cannot be shown constructively. The reason is that it requires the construction of *prime filters*, which is a weak form of choice. I will investigate the details of this mismatch in a sequel paper.

2.2 Morphisms

The simplest kind of morphism between Kripke frames is a *monotonic* map $f : W \to W'$. Frames and monotonic maps form the category **Pos** of posets. Given a monotonic $f : W \to W'$ we can define a monotonic $f^* : [W', \mathbb{2}] \to [W, \mathbb{2}]$ by taking $p : W' \to \mathbb{2}$ to $p \circ f : W \to \mathbb{2}$. Viewing the elements of $[W', \mathbb{2}]$ as upper sets, f^* maps the upper set $S \subseteq W'$ to the set $\{v \in W' \mid f(v) \in S\} \subseteq W$, which is upper by the monotonicity of f. f^* preserves arbitrary joins and meets. It is thus the morphism part of a functor $[-, \mathbb{2}] : \mathbf{Pos}^{\mathsf{op}} \longrightarrow \mathbf{PrAlgLatt}$ to the category **PrAlgLatt** of prime algebraic lattices and complete lattice homomorphisms.

Moreover, the functor $[-, \mathbb{2}]$ is an equivalence! By the adjoint functor theorem any complete lattice homomorphism $f^* : L' \to L$ has a left and right adjoint:



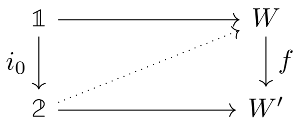
(2)

Given a prime algebraic lattice L, let $\mathsf{Prm}(L) \subseteq L$ be the sub-poset of prime elements. It can be shown that the left adjoint $f_!$ maps primes to primes [41, Lemma 1.23]. We can thus restrict it to a function $\mathsf{Prm}(L) \to \mathsf{Prm}(L')$. This defines a functor $\mathsf{Prm}(-) : \mathbf{PrAlgLatt} \longrightarrow \mathbf{Pos}^{\mathsf{op}}$ with the property that $\mathsf{Prm}([W, \mathbb{2}]) \cong W$. All in all, this amounts to a *duality*

$$\mathbf{Pos}^{\mathsf{op}} \simeq \mathbf{PrAlgLatt} \tag{3}$$

However, monotonic maps are not particularly well-behaved from the perspective of logic, as they do not preserve nor reflect "local" truth. This is the privilege of *open maps.*

▶ **Definition 3.** *Let $i_0 : \mathbb{1} \to \mathbb{2}$ map the unique point of $\mathbb{1} \stackrel{\text{def}}{=} \{*\}$ to $0 \in \mathbb{2}$. A monotonic map $f : W \to W'$ of Kripke frames is* open *just when it has the right lifting property with respect to $i_0 : \mathbb{1} \to \mathbb{2}$, i.e. when every commuting diagram of the form*



*in **Pos** has a diagonal filler (dashed) that makes it commute.*

In other words, f is open if whenever $f(w) \sqsubseteq v'$ there exists a $w' \in W$ with $w \sqsubseteq w'$ and $f(w') = v'$.[3] Open maps send upper sets to upper sets [23, Prop. 2.13]. Thus

▶ **Lemma 4.** *Let $\mathfrak{M} = (W, \sqsubseteq, V)$ and $\mathfrak{N} = (W', \sqsubseteq, V')$ be Kripke models, and $f : W \to W'$ be open. Suppose $V = f^{-1} \circ V'$, i.e. $w \in V(p)$ iff $f(w) \in V'(p)$. Then $\mathfrak{M}, w \vDash \varphi$ iff $\mathfrak{N}, f(w) \vDash \varphi$.*

Write $W \vDash \varphi$ to mean that $(W, \sqsubseteq, V), w \vDash \varphi$ for any valuation V and $w \in W$. Then

▶ **Lemma 5.** *If $f : W \to W'$ is open and surjective, then $W \vDash \varphi$ implies $W' \vDash \varphi$.*

[3] Such morphisms are often called *p-morphisms* [23, §2.3] or *bounded morphisms* [15, §2.1]. According to Goldblatt [46] open maps were introduced by de Jongh and Troelstra [29] in intuitionistic logic, and by Segerberg [91] in modal logic. More rarely they are called *functional simulations*, and led us to bisimulations [90, §3.2]. The name is chosen because such maps are open with respect to the *Alexandrov topology* on a poset, whose open sets are the upper sets [61, §1.8].

Recall now the induced map $f^* : [W', 2] \to [W, 2]$ for a monotonic $f : W \to W'$. The following lemma allows us to characterise the openness and surjectivity of f in terms of f^*.

▶ **Lemma 6.**

1. $f : W \to W'$ *is open iff* $f^* : [W', 2] \to [W, 2]$ *preserves exponentials.*
2. $f : W \to W'$ *is surjective iff* $f^* : [W', 2] \to [W, 2]$ *is injective.*

Consequently, the duality (3) may be restricted to two wide subcategories:

$$\mathbf{Pos}^{\mathsf{op}}_{\mathrm{open}} \simeq \mathbf{PrAlgLatt}_{\Rightarrow} \qquad \mathbf{Pos}^{\mathsf{op}}_{\mathrm{open,surj}} \simeq \mathbf{PrAlgLatt}_{\Rightarrow,\mathrm{inj}} \tag{4}$$

The morphisms of the categories to the left of $\simeq$ are open (resp. and surjective) maps, and the morphisms of the categories to its right are *complete Heyting homomorphisms*, i.e. complete lattice homomorphisms that preserve exponentials (resp. and are injective).

Finally, let us consider the classical case – as a sanity-check. This amounts to restricting **Pos** to its subcategory of discrete orders, i.e. **Set**. In this case every map is open. The corresponding restriction on the other side is to the category **CABA** of *complete atomic Boolean algebras*, yielding the usual *Tarski duality* $\mathbf{Set}^{\mathsf{op}} \simeq \mathbf{CABA}$ [67].

2.3 Related work

The origins of the construction of a Heyting algebra from a Kripke frame seems to be lost in the mists of time. The earliest occurrence I have located is in the book by Fitting [39, §1.6], where it is attributed to an exercise in the book by Beth [10].

The duality (3) appears to be folklore – folklore enough to be included as an exercise in new textbooks [41, Ex. 1.3.10]; see also Erné [31]. However, I have not been able to find any mention of the dualities of (4) in the literature.

Both the dualities (3) and (4) involve just prime algebraic lattices, which is a far cry from encompassing all Heyting algebras. It is possible to do so, by enlarging the category **Pos** to a class of ordered topological spaces called *descriptive frames* [23, §8.4]. The resulting duality is called *Esakia duality* [32] [41, §4.6] [12, §2.3].

A survey on dualities for classical modal logic is given by Kishida [67].

3 Modal Logic I

The task now is to extend the results of Section 2 to *intuitionistic modal logic.*

There is disagreement on what a minimal intuitionistic modal logic is. This arises no matter the methodology we choose – be it relational, algebraic, or proof-theoretic. The situation becomes even more complex if we include a diamond modality ($\Diamond$): see Das and Marin [27] and Wolter and Zakharyaschev [103] for a discussion.

In this paper I will adopt the *intuitionistic propositional logic with Galois connections* of Dzik, Järvinen, and Kondo [30], for reasons that will become clear in a moment. This extends intuitionistic logic with modalities $\blacklozenge$ and $\Box$, and the two inference rules

$$\frac{\blacklozenge\varphi \to \psi}{\varphi \to \Box\psi} \qquad \text{and} \qquad \frac{\varphi \to \Box\psi}{\blacklozenge\varphi \to \psi}$$

These rules correspond to a *Galois connection* [28, §7.23], i.e. an adjunction $\blacklozenge \dashv \Box$ between posets. They imply the derivability of the following rules, amongst others [30, Prop. 2.1].

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi} \qquad \frac{\varphi}{\Box\varphi} \qquad \frac{}{\Box\top} \qquad \frac{\blacklozenge\bot}{\bot} \qquad \frac{\varphi \to \psi}{\blacklozenge\varphi \to \blacklozenge\psi} \qquad \frac{}{\blacklozenge(\varphi \vee \psi) \leftrightarrow \blacklozenge\varphi \vee \blacklozenge\psi}$$

$$\frac{}{\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi}$$

The notation of the "black diamond" modality $\blacklozenge$ may appear unusual. However, I will argue that this logic is, in a way, the canonical intuitionistic modal logic.

The Kripke semantics of classical modal logic is given by a *modal frame* (W, R), which consists of a set W and an arbitrary *accessibility relation* $R \subseteq W \times W$ [15, §1]. If the same set of worlds W is already part of an intuitionistic Kripke frame $(W, \sqsubseteq)$ we must take care to ensure that $\sqsubseteq$ and R are *compatible*. There are many compatibility conditions that one can consider [85] [92, §3.3]. However, I will take a hint from the category theory literature, and seek a canonical definition of what it means for a relation to be compatible with a poset.

Recall that relations can be presented as functions $R : W \times W \to 2$ which map a pair of worlds (w, v) to 1 iff $w \mathrel{R} v$. I will ask that R is such function, but with a twist:

▶ **Definition 7.** *A* bimodule $R : W_1 \twoheadrightarrow W_2$ *is a monotonic map* $R : W_1^{\mathsf{op}} \times W_2 \to 2$.

Thus, a relation $R \subseteq W_1 \times W_2$ is a bimodule just if $w' \sqsubseteq w \mathrel{R} v \sqsubseteq v'$ implies $w' \mathrel{R} v'$. This means that R can absorb changes in information on either side: contravariantly on the first component, and covariantly on the second. This is a standard, minimal way to define what it means to be "a relation in **Pos**". It is strongly reminiscent of bimodules in abstract algebra.

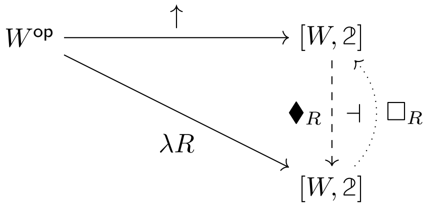
We can then define a *modal Kripke frame* $(W, \sqsubseteq, R)$ to be a Kripke frame $(W, \sqsubseteq)$ equipped with a bimodule $R : W \twoheadrightarrow W$. A *modal Kripke model* $\mathfrak{M} = (W, \sqsubseteq, R, V)$ adds to this a function $V : \mathsf{Var} \to \mathsf{Up}(W)$. We extend $\mathfrak{M}, w \vDash \varphi$ to modal formulae:

$$\mathfrak{M}, w \vDash \blacklozenge\varphi \overset{\text{def}}{\equiv} \exists v.\ v \mathrel{R} w \text{ and } \mathfrak{M}, v \vDash \varphi$$
$$\mathfrak{M}, w \vDash \Box\varphi \overset{\text{def}}{\equiv} \forall v.\ w \mathrel{R} v \text{ implies } \mathfrak{M}, v \vDash \varphi$$

There are a number of things to note about this definition. First, there is a deep duality between the clauses: not only do we exchange $\forall$ for $\exists$, but we also flip the variance of R. As a result, $\blacklozenge$ uses the relation in the *opposite variance* to the more traditional $\Diamond$ modality (hence the change in notation). Second, the clause for the $\Box$ modality is the traditional one; some streams of work on intuitionistic modal logic adopt a slightly different one [85, 92], which is equivalent to this in the presence of the bimodule condition. Finally, this definition is monotonic: the bimodule conditions on R suffice to show that if $\mathfrak{M}, w \vDash \varphi$ and $w \sqsubseteq v$ then $\mathfrak{M}, v \vDash \varphi$. Dzik et al. [30, §5] prove that this semantics is sound and complete.

The algebraic semantics of this logic is given by a Heyting algebra H equipped with two monotonic maps $\blacklozenge, \Box : H \to H$ which form an adjunction $\blacklozenge \dashv \Box$, i.e. a Galois connection. Dzik et al. [30, §4] prove that this semantics is also sound and complete.

We are now in a position to relate the Kripke and algebraic semantics of this intuitionistic modal logic. Let $(W, \sqsubseteq, R)$ be a modal Kripke frame, and consider the map $\lambda R : W^{\mathsf{op}} \to [W, 2]$ obtained by cartesian closure of **Pos**. This map takes $w \in W$ to the upper set $\{v \in W \mid w \mathrel{R} v\}$ of worlds accessible from w. Putting λR in (1) we obtain through Kan extension the diagram



(5)

where we write $\blacklozenge_R$ for $\lambda R_!$ and $\Box_R$ for $\lambda R^\star$. It can be shown that these maps are given by

$$\blacklozenge_R(S) \overset{\text{def}}{=} \{w \in W \mid \exists v.\ v\, R\, w \text{ and } v \in S\}$$
$$\Box_R(S) \overset{\text{def}}{=} \{w \in W \mid \forall v.\ w\, R\, v \text{ implies } v \in S\}$$

Thus, any bimodule R defines an adjunction $\blacklozenge_R \dashv \Box_R$ on $[W, 2]$. Correspondingly, any adjunction $\blacklozenge \dashv \Box$ on $[W, 2]$ yields a monotonic map $\blacklozenge \circ {\uparrow}(-) : W^{\mathsf{op}} \to [W, 2]$, which uniquely corresponds to a bimodule $W^{\mathsf{op}} \times W \to 2$ by the cartesian closure of **Pos**.

Thus, starting from a bimodule, i.e. a relation that is compatible with the information order, we have canonically obtained a model of intuitionistic modal logic on $[W, 2]$ through Kan extension: $[W, 2]$ is a complete Heyting algebra, and we define $[\![\blacklozenge\varphi]\!] = \blacklozenge_R[\![\varphi]\!]$ and $[\![\Box\varphi]\!] = \Box_R[\![\varphi]\!]$. We immediately obtain a modal analogue to Theorem 1:

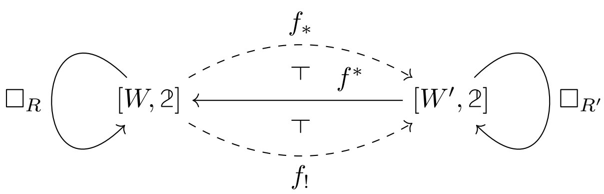
▶ **Theorem 8.** *For any modal formula φ, $w \vDash \varphi$ if and only if $w \in [\![\varphi]\!]$.*

3.1 Morphisms

We define a category **Bimod** with bimodules $R : W_1 \;-\!\!\!\!\!\!\rightarrow\!\!\!\!\!\rightarrow\!\!\!\!\!\!\rightarrow W_2$ as objects. A *bimodule morphism* from $R : W_1 \rightarrow W_2$ to $R' : W_1' \rightarrow W_2'$ is a pair (f, g) of monotonic maps $f : W_1 \to W_1'$ and $g : W_2 \to W_2'$ such that $R(w, v) \sqsubseteq R'(f(w), g(v))$. Stated in terms of relations, it must be that $w\, R\, v$ implies $f(w)\; R'\; g(v)$.

We define the subcategory **EBimod** to consist of (endo)bimodules $R : W \rightarrow W$ and pairs of maps (f, f). Thus, objects are bimodules on a single poset W, and morphisms are monotonic maps $f : W \to W'$ that preserve the relation, i.e. $w\, R\, v$ implies $f(w)\; R\; f(v)$. In other words, the objects of **EBimod** are modal Kripke frames, and the morphisms are monotonic, relation-preserving maps.

Recall the adjunctions and modalities induced by a monotonic $f : W \to W'$:



(6)

▶ **Lemma 9.** *$f : W \to W'$ is a morphism of bimodules $f : R \to R'$ iff $f^*\Box_{R'} \subseteq \Box_R f^*$.*

This constitutes a duality

$$\mathbf{EBimod}^{\mathsf{op}} \simeq \mathbf{PrAlgLattO} \tag{7}$$

where **PrAlgLattO** is the category with objects $(L, \Box_L)$, where L is a prime algebraic lattice and $\Box_L : L \to L$ is an operator that preserves all meets. By the adjoint functor theorem, such operators always have a left adjoint $\blacklozenge_L : L \to L$. Thus, this category contains algebraic models of intuitionistic modal logic (but not all of them). By the preceding section each such adjunction corresponds uniquely to a bimodule. The morphisms of **PrAlgLattO** are complete lattice homomorphisms $h : L \to L'$ such that $h\Box_L \sqsubseteq \Box_{L'} h$. By the preceding lemma they correspond precisely to morphisms of bimodules.

However, as with monotone maps, morphisms of bimodules do not preserve local truth; for that we need a notion of *modally open* maps.

▶ **Definition 10.** *Let* $(W, \sqsubseteq, R)$ *and* $(W', \sqsubseteq, R')$ *be modal Kripke frames. A bimodule morphism* $f : R \to R'$ *is* modally open *just if whenever* $f(w)\, R'\, v$ *then there exists a* $w' \in W$ *with* $w\, R\, w'$ *and* $f(w') \sqsubseteq v$.

This is similar to Definition 3, but ever so slightly weaker: instead of requiring $f(w') = v'$, it requires that the information in $f(w')$ can be increased to v'. Like Definition 3, it can also be written homotopy-theoretically, but that requires some ideas from double categories that are beyond the scope of this paper. We have the analogous result about preservation of truth:

▶ **Lemma 11.** *Let* $\mathfrak{M} = (W, \sqsubseteq, R, V)$ *and* $\mathfrak{N} = (W', \sqsubseteq, R', V')$ *be modal Kripke models,* $f : W \to W'$ *be open and modally open, and* $V = f^{-1} \circ V'$. *Then* $\mathfrak{M}, w \vDash \varphi$ *iff* $\mathfrak{N}, f(w) \vDash \varphi$.

▶ **Lemma 12.** *Let* $f : W \to W'$ *be open, modally open, and surjective. If* $W \vDash \varphi$ *then* $W' \vDash \varphi$.

The following result relates the modal openness of f to f^*.

▶ **Lemma 13.** $f : R \to R'$ *is modally open iff* $\Box_R f^* = f^* \Box_{R'}$ *iff* $f_! \blacklozenge_R = \blacklozenge_{R'} f_!$.

Thus, the duality (7) may be restricted to dualities between wide subcategories:

$$\mathbf{EBimod}^{\mathrm{op}}_{\mathrm{moo}} \simeq \mathbf{PrAlgLattO}_{\Rightarrow o} \qquad \mathbf{EBimod}^{\mathrm{op}}_{\mathrm{moo,\ surj}} \simeq \mathbf{PrAlgLattO}_{\Rightarrow o,\mathrm{inj}} \tag{8}$$

The morphisms to the left of $\simeq$ are open and modally open (resp. and surjective); and the to the right of it preserve exponentials and commute with operators (resp. and are injective).

Let us consider the restriction of this duality to the classical setting – as a sanity check. A bimodule on a discrete poset is just a relation on a set. The corresponding restriction on the right is to CABAs with operators, and complete homomorphisms which commute with operators. We thus obtain the *Thomason duality* $\mathbf{MFrm}^{\mathrm{op}}_{\mathrm{open}} \simeq \mathbf{CABAO}$ between Kripke frames and modally open maps on the left, and CABAs with operators to the right [96, 67].

3.2 Related work

Many works have presented a Kripke semantics for intuitionistic modal logic. All such semantics assume two accessibility relations: a preorder for the intuitionistic dimension, and a second relation for the modal dimension. What varies is their *compatibility conditions*.

The first work to present such a semantics appears to be that of Fischer Servi [38]. One of the required compatibility conditions is $(\sqsubseteq) \circ R \subseteq R \circ (\sqsubseteq)$. This is weaker than having a bimodule, but sufficient to prove soundness.

The first work to recognise the importance of bimodules was Sotirov's 1979 thesis. His results are summarised in a conference abstract [94, §4]: they include the completeness of a minimal intuitionistic modal logic with a $\Box$, the K axiom, and the necessitation rule. Božić and Došen [19] repeat the study for the same logic, but for a semantics based on the Fischer Servi compatibility conditions. However, they note that their completeness proof actually constructs half a bimodule (a "condensed" relation). They also point out that bimodules, which they call "strictly condensed" relations, are sound and complete for their logic. Wolter and Zakharyaschev [101, §2] argue that bimodule and Fischer Servi semantics are equi-expressive.

Plotkin and Stirling [85] attempt to systematise the Kripke semantics of intuitionistic modal logic. Their frame conditions allow "transporting a modal relation upwards" along any potential increases of information on either side. This paper and all its descendants – notably the thesis of Simpson [92, §3.3] – adopt a different satisfaction clause for $\Box$ which uses both $\sqsubseteq$ and R. In the presence of the bimodule conditions this satisfaction clause is equivalent to the classical one, which I use here.

The bimodule condition and the *complex algebra* construction (or fragments thereof) have made scattered appearances in the literature: in the early work of Sotirov [94] and Božić and Došen [19]; in the work of Wolter and Zakharyaschev [102, 101, 103], Hasimoto [54, §4], and Orłowska and Rewitzky [83]; and of course in Dzik et al. [30, §7]. With the exception of the last one, none of these references discuss the $\blacklozenge$ modality. Moreover, in none of these references are the categorical aspects of this construction discussed.

As mentioned before, dualities between frames and algebras have played a significant role in modal logic. Thomason [96] and Goldblatt [45] also considered morphisms of frames, respectively obtaining *Thomason duality* and (categorical) *Jónsson-Tarski duality* between descriptive frames and Boolean Algebras with Operators (BAOs) [46, §6.5]. Kishida [67] surveys a number of dualities for classical modal logic.

The duality (7) is stated by Gehrke [40, Thm. 2.5] who attributes it to Jónnson [64], even though no such theorem appears in that paper.

The dualities of (8) are the direct intuitionistic analogues to that of Thomason. I have not been able to find them anywhere in the literature.

According to the extensive survey of Menni and Smith [78], the idea that the commonly-used modalities $\Box$ and $\Diamond$ are often part of adjunctions $\blacklozenge \dashv \Box$ and $\Diamond \dashv \blacksquare$ is implicitly present throughout the development of modal logic. However, these were not made explicit in a logic until the 2010s, when they appeared in the work of Dzik et al. [30] and Sadrzadeh and Dyckhoff [89]. The same perspective plays a central rôle in the exposition of Kishida [67].

The $\blacklozenge$ modality has appeared before in *tense logics* as a "past" modality [33, 47].

4 Intuitionistic Logic II

In the rest of this paper we will *categorify* [7] the notion of Kripke semantics. The main idea is to replace posets by categories, so that the order $w \sqsubseteq v$ is replaced by a morphism $w \to v$. As there might be multiple morphisms $w \to v$, this allows the recording of not just the fact v may signify more information than w, but also the *manner* in which it does so. The reflexivity and transitivity of the poset are then replaced by the identity and composition laws of the category. This adds a dimension of *proof-relevance* to Kripke semantics.

A corresponding change in our algebraic viewpoint will be that of replacing the set 2 of truth values with the category **Set**. This is a classic Lawverean move [73]. Notice that this is lopsided, as is usual in intuitionistic logic: while the falsity 0 is only represented by one value, viz. the empty set, the truth 1 can be represented by any non-empty set X. The elements of X can be thought of as a *proofs* of a true statement.

Let us then trade the frame $(W, \sqsubseteq)$ for an arbitrary category $\mathcal{C}$. It remains to define what it means to have a *proof* that the formula φ holds at a world $w \in \mathcal{C}$. We denote the set of all such proofs by $[\![\varphi]\!]_w$. Assuming we are given a set $[\![p]\!]_w$ for each proposition p and world w, here is a first attempt:

$$[\![\bot]\!]_w \stackrel{\text{def}}{=} \emptyset \qquad [\![\top]\!]_w \stackrel{\text{def}}{=} \{*\} \qquad [\![\varphi \wedge \psi]\!]_w \stackrel{\text{def}}{=} [\![\varphi]\!]_w \times [\![\psi]\!]_w \qquad [\![\varphi \vee \psi]\!]_w \stackrel{\text{def}}{=} [\![\varphi]\!]_w + [\![\psi]\!]_w$$

$$[\![\varphi \to \psi]\!]_w \stackrel{\text{def}}{=} (v : \mathcal{C}) \to \mathrm{Hom}_{\mathcal{C}}(w, v) \to [\![\varphi]\!]_v \to [\![\psi]\!]_v$$

where for a family of sets $(B_a)_{a\in A}$ we let

$$(a : A) \to B_a \stackrel{\text{def}}{=} \left\{ f : A \to \bigcup_{a\in A} B_a \;\middle|\; \forall a \in A. f(a) \in B_a \right\}$$

This closely follows the usual Kripke semantics, but adds proofs. For example, a proof in $[\![\varphi_1 \wedge \varphi_2]\!]_w$ is a pair (x, y) of a proof $x \in [\![\varphi_1]\!]_w$ and a proof $y \in [\![\varphi_2]\!]_w$. Similarly, a proof $F \in [\![\varphi \to \psi]\!]_w$ is a function which maps a proof of "increase in information" $f : w \to v$ to a function $F(v)(f) : [\![\varphi]\!]_v \to [\![\psi]\!]_v$. In turn, this function maps proofs in $[\![\varphi]\!]_v$ to proofs in $[\![\psi]\!]_v$.

To show that this definition is monotonic we have to demonstrate it on proofs: given a proof $x \in [\![\varphi]\!]_w$ and a morphism $f : w \to v$ we have to define a proof $f \cdot x \in [\![\varphi]\!]_v$. Assuming that we are given this operation for propositions, we can extend it by induction; e.g.

$$\begin{aligned} f \cdot (x, y) &\stackrel{\text{def}}{=} (f \cdot x, f \cdot y) && \in [\![\varphi \wedge \psi]\!]_v \\ f \cdot F &\stackrel{\text{def}}{=} (z : \mathcal{C}) \mapsto (g : \mathrm{Hom}_{\mathcal{C}}(v, z)) \mapsto (x : [\![\varphi]\!]_z) \mapsto F(z)(g \circ f)(x) && \in [\![\varphi \to \psi]\!]_v \end{aligned}$$

Moreover, this definition is compatible with $\mathcal{C}$, in the sense that $g \cdot (f \cdot x) = (g \circ f) \cdot x$ and $\mathsf{id}_w \cdot x = x$. We thus obtain a (covariant) *presheaf* $[\![\varphi]\!] : \mathcal{C} \longrightarrow \mathbf{Set}$ for each formula φ.

It is well-known that the proofs of intuitionistic logic form a *bicartesian closed category* (biCCC), i.e. a category with finite (co)products and exponentials [71]. A biCCC can be seen as a categorification of a Heyting algebra: formulae are objects of the category, and proofs are morphisms. We will not expound on this further; see [72, 26, 4].

It should therefore be the case that the semantics described above form a biCCC. Indeed, it is a well-known fact of topos theory that the *category of presheaves* $[\mathcal{C}, \mathbf{Set}]$ is a biCCC. In fact, the construction of exponentials [76, §I.6] reveals that our definition above is deficient: we should restrict $[\![\varphi \to \psi]\!]_w$ to contain only those functions F that satisfy a *naturality condition*, i.e. those which for any $f : w \to v_1$, $g : v_1 \to v_2$, and $x \in [\![\varphi]\!]_{v_1}$ satisfy

$$g \cdot F(v_1)(f)(x) = F(v_2)(g \circ f)(g \cdot x)$$

From this point onwards I will identify two-dimensional Kripke semantics with categorical semantics in a category of presheaves $[\mathcal{C}, \mathbf{Set}]$.

4.1 Presheaf categories

The category $[\mathcal{C}, \mathbf{Set}]$ of covariant presheaves is eerily similar to prime algebraic lattices. In a sense they are just the same; but, having traded $\mathbb{2}$ for $\mathbf{Set}$, they have become proof-relevant.

First, letting $P \in [\mathcal{C}, \mathbf{Set}]$, an element $x \in P(w)$ is a proof that P holds at a "world" $w \in \mathcal{C}$. A morphism $f : w \to v$ of $\mathcal{C}$ then leads to a proof $f \cdot x \stackrel{\text{def}}{=} P(f)(x) \in P(v)$ that P holds at v. Thus, the presheaf P is very much like an upper set.

Second, the *representable presheaves* $\mathbf{y}(w) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(w, -) : \mathcal{C} \to \mathbf{Set}$ are the proof-relevant analogues of the principal upper set. By the Yoneda lemma they constitute an *embedding*

$$\mathbf{y} : \mathcal{C}^{\mathsf{op}} \longrightarrow [\mathcal{C}, \mathbf{Set}]$$

which moreover preserves limits and exponentials [4].

Third, the category $[\mathcal{C}, \mathbf{Set}]$ is both complete and cocomplete, with limits and colimits computed pointwise [76, §I]. It is also "distributive" in an appropriate sense [3, §3.3], which makes it into a *Grothendieck topos*. It is thus a cartesian closed category, with exponential

$$(P \Rightarrow Q)(w) \stackrel{\text{def}}{=} \mathrm{Hom}(P \times \mathbf{y}(w), Q)$$

which is essentially the two-dimensional semantics of implication I gave above.

Fourth, the representables $\mathbf{y}(w)$ are special, in that they are *tiny* [104].

▶ **Definition 14.** *An object* $d \in \mathcal{D}$ *is* tiny *just if* $\mathrm{Hom}(d,-) : \mathcal{D} \to \mathbf{Set}$ *preserves colimits.*[4]

Tininess is a proof-relevant version of primality: it implies that for any $f : w \to \varinjlim_i v_i$ there exists an i such that f is equal to the composition of a morphism $w \to v_i$ with the injection $v_i \to \varinjlim_i v_i$. By the Yoneda lemma it follows that all representables $\mathbf{y}(w)$ are tiny, as they satisfy the above definition for $\mathcal{D} \stackrel{\text{def}}{=} [\mathcal{C}, \mathbf{Set}]$ and $d \stackrel{\text{def}}{=} \mathbf{y}(w)$.

Fifth, the so-called *co-Yoneda lemma* [75, §III.7] shows that every $P \in [\mathcal{C}, \mathbf{Set}]$ is a colimit of representables. This means that it can be reconstructed by sticking together tiny elements:

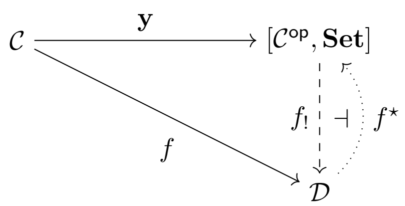
$$P \cong \varinjlim_{(w,x)\in \mathsf{el}\,P} \mathbf{y}(w)$$

Like with prime algebraic lattices, there is a converse to this result: every category which is generated by sticking together tiny elements is in fact a presheaf category:

▶ **Theorem 15** (Bunge [20])**.** *A category which is cocomplete and strongly generated by a small set of tiny objects is equivalent to* $[\mathcal{C}, \mathbf{Set}]$ *for some small category* $\mathcal{C}$*.*

A textbook presentation of this result can be found in the book by Kelly [66, §5.5].

Finally, the fact every element can be reconstructed as a colimit of representables means that it is possible to uniquely extend any functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ to a cocontinuous functor $[\mathcal{C}^{\mathsf{op}}, 2] \longrightarrow \mathcal{D}$, as long as $\mathcal{D}$ is cocomplete. Diagrammatically, in the situation



(9)

there exists an essentially unique cocontinuous $f_!$ with $f_!(\mathbf{y}(w)) = f(w)$. It is given by

$$f_!\left(\varinjlim_{(w,x)\in \mathsf{el}\,P} \mathbf{y}(w)\right) \stackrel{\text{def}}{=} \varinjlim_{(w,x)\in \mathsf{el}\,P} f(w)$$

$f_!$ is called the *left Kan extension* of f along $\mathbf{y}$. It has a right adjoint $f^\star$ which is explicitly given by $f^\star(d) \stackrel{\text{def}}{=} \mathrm{Hom}(f(-), d)$. This amounts to an isomorphism

$$\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) \cong \mathrm{Hom}_{\mathbf{Cocont}}([\mathcal{C}^{\mathsf{op}}, \mathbf{Set}], \mathcal{D})$$

where **Cat** is the category of categories, and **Cocont** is the category of cocomplete categories and cocontinuous functors: see [4, Prop. 9.16] [88, Cor. 6.2.6, Rem. 6.5.9] and [75, § X.3, Cor. 2] [66, Th. 4.51].

All in all, presheaf categories are the categorification of prime algebraic lattices.

4.2 Cauchy-complete and spacelike categories

Replacing posets with categories does not come for free: the extra dimension of morphisms leads to situations that have no analogues in poset. Some of these are problematic when thinking of $\mathcal{C}$ as a two-dimensional Kripke frame. Perhaps the most bizarre is the presence of *idempotents*, i.e. morphisms $e : w \to w$ with the property that $e \circ e = e$. Such morphisms represent a non-trivial increase in information which confusingly leaves us in the same world.

[4] In the literature this property is often referred to as *external tininess* (cf. internal tininess).

Table 1 Categorification of Kripke semantics.

poset	category
monotonic map	functor
upper sets	presheaves
principal upper set	representable presheaf
prime element	tiny object
prime algebraic lattice	presheaf category
bimodule	profunctor

The presence of idempotents causes issues. For example, recall that, in prime algebraic lattices, primes and principal upper sets coincide. The astute reader will have noticed we did *not* claim the analogous result in presheaf categories: tiny objects are not necessarily representable in $[\mathcal{C}, \mathbf{Set}]$. For that, we need $\mathcal{C}$ to be *Cauchy-complete* [18, 17].

▶ **Definition 16.** *A category is* Cauchy-complete *just if every idempotent splits, i.e. if every idempotent is equal to $s \circ r$ for a section-retraction pair s and r.*

Note that every complete category is Cauchy-complete, including $\mathbf{Set}$ and $[\mathcal{C}, \mathbf{Set}]$.

This leads us to another troublesome situation, namely that of having section-retraction pairs, i.e. $s : w \to v$ and $r : v \to w$ with $r \circ s = \mathsf{id}_w$. In this case w and v contain no more information than each other, but are not isomorphic. We may ask that this does not arise.

▶ **Definition 17.** *A category satisfies the* Hemelaer condition *[55, Prop. 5.8] just if every section-retraction pair is an isomorphism.*

Combining these two conditions is equivalent to the following definition.

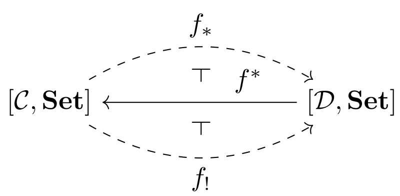
▶ **Definition 18.** *A category is* spacelike *if every idempotent is an identity.*

Lawvere has identified this condition as having particular importance in recognising petit toposes [77]. We will not use it much, as it restricts the dualities we wish to develop.

In the rest of this paper we will assume that our base categories $\mathcal{C}$ are Cauchy-complete, so that tiny objects coincide with representables.

4.3 Morphisms

The simplest kind of morphism between categories is a functor. Given a $f : \mathcal{C} \longrightarrow \mathcal{D}$ we can define a functor $f^* : [\mathcal{D}, \mathbf{Set}] \longrightarrow [\mathcal{C}, \mathbf{Set}]$ that takes $P : \mathcal{D} \longrightarrow \mathbf{Set}$ to $P \circ f : \mathcal{C} \longrightarrow \mathbf{Set}$. This functor has left and right adjoints, which are given by Kan extension [62, A4.1.4]:



(10)

Therefore f^* preserves all limits and colimits, i.e. it is (co)continuous. In short, the presheaf construction gives a functor $[-, \mathbf{Set}] : \mathbf{Cat}_{cc}^{op} \longrightarrow \mathbf{PshCat}$, where $\mathbf{Cat}_{cc}$ is the category of small Cauchy-complete categories and functors; and $\mathbf{PshCat}$ is the category of presheaf categories (over Cauchy-complete base categories) and (co)continuous functors.

Moreover, this functor is an equivalence. Given a presheaf category we can obtain its base as the subcategory of tiny objects [62, A1.1.10]. But how can we extract $f : \mathcal{C} \longrightarrow \mathcal{D}$ from any (co)continuous functor $f^* : [\mathcal{D}, \mathbf{Set}] \longrightarrow [\mathcal{C}, \mathbf{Set}]$? First, as presheaf categories are locally presentable, the adjoint functor theorem implies that f^* has left and right adjoints, as in (10) [1, §1.66]. This gives what topos theorists call an *essential geometric morphism*. Johnstone [62, §A4.1.5] shows that every such morphism is induced by a $f : \mathcal{C} \longrightarrow \mathcal{D}$, as $f_!$ preserves representables (when $\mathcal{D}$ is Cauchy-complete). We thus obtain a duality

$$\mathbf{Cat}_{\mathrm{cc}}^{\mathrm{op}} \simeq \mathbf{PshCat} \tag{11}$$

As with posets, functors here fail to preserve truth; for that we need a notion of openness.

▶ **Definition 19.** *$f : \mathcal{C} \longrightarrow \mathcal{D}$ is* open *just if $f^* : [\mathcal{D}, \mathbf{Set}] \longrightarrow [\mathcal{C}, \mathbf{Set}]$ preserves exponentials.*

▶ **Lemma 20.** *If $f : \mathcal{C} \longrightarrow \mathcal{D}$ is open then there is a natural isomorphism $\theta_w : [\![\varphi]\!]_w \cong [\![\varphi]\!]_{f(w)}$.*

Definition 19 is somewhat underwhelming, as it does not give explicit conditions that one can check – unlike Definition 3. However, obtaining such a description appears difficult.

Some information may be gleaned by considering $(f^*, f_*) : [\mathcal{C}, \mathbf{Set}] \longrightarrow [\mathcal{D}, \mathbf{Set}]$ as a *geometric morphism*. Such a morphism is *open* [60] [63, C3.1] just if both the canonical maps $f^*(c \Rightarrow d) \to f^*(c) \Rightarrow f^*(d)$ and $f^*(\Omega) \to \Omega$ are monic. Johnstone [63, C3.1] proves that (f^*, f_*) is open iff for any $\beta : f(w) \to v'$ in $\mathcal{D}$ there exists an $\alpha : w \to w'$ in $\mathcal{C}$ and a section-retraction pair $s : v' \to f(w')$ and $r : f(w') \to v'$ with $s \circ \beta = f(\alpha)$. This superficially seems like a categorification of Definition 3. However, it only guarantees that f^* is *sub-cartesian-closed*, whereas we need an isomorphism for Lemma 20 to hold.

A stronger condition is to ask that (f^*, f_*) be *locally connected*, i.e. that f^* commute with dependent products [63, C3.3]. All such morphisms are open geometric morphisms. This is stronger than what we need, but sufficient conditions on f can be given [63, C3.3.8].

Finally, an even stronger condition is to ask that (f^*, f_*) be *atomic*, i.e. that f^* is a *logical* functor. This means it preserves exponentials and the subobject classifier [63, A2.1, C3.5]. All atomic geometric morphisms are locally connected. This is again stronger than what we need, and a characterisation in terms of f is elusive: see MathOverflow [95].

It is easier to characterise when (f^*, f_*) is a *surjective geometric morphism*, i.e. when f^* is faithful [62, A2.4.6]. This happens exactly when f is *retractionally surjective*, i.e. whenever every $d \in \mathcal{D}$ is the retract of $f(c)$ for some $c \in \mathcal{C}$ [62, A2.4.7]. If $\mathcal{D}$ satisfies the Hemelaer condition this reduces to f being essentially surjective.

Write $\mathcal{C} \vDash \varphi$ to mean that $[\![\varphi]\!]_w$ is non-empty for any $w \in \mathcal{C}$ and any interpretation of $[\![p]\!]$.

▶ **Lemma 21.** *Let $f : \mathcal{C} \to \mathcal{D}$ be open and retractionally surjective. If $\mathcal{C} \vDash \varphi$ then $\mathcal{D} \vDash \varphi$.*

We may thus restrict the duality (11) to dualities

$$\mathbf{Cat}_{\mathrm{cc,\ open}}^{\mathrm{op}} \simeq \mathbf{PshCat}_{\Rightarrow} \qquad\qquad \mathbf{Cat}_{\mathrm{cc,\ open,\ rs}}^{\mathrm{op}} \simeq \mathbf{PshCat}_{\Rightarrow,\mathrm{f}} \tag{12}$$

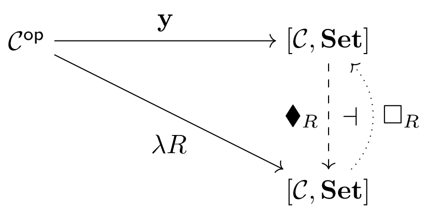
In the first instance the category to the left of $\simeq$ is that of small Cauchy-complete categories and open functors; and to the right of $\simeq$ it is presheaf categories and (co)complete, cartesian closed functors. In the second instance the category to the left of $\simeq$ is that of small Cauchy-complete categories and open, retractionally surjective functors; and to the right of $\simeq$ it is presheaf categories and (co)complete, faithful, cartesian closed functors.

5 Modal Logic II

To make a two-dimensional Kripke semantics for modal logic we have to categorify relations. We took the first step by considering bimodules, i.e. information-order-respecting relations. The second step can be taken by replacing $\mathbb{2}$ with **Set**; this leads us to the notion of a relation between categories, also known as a *profunctor* or *distributor* [9] [17, §7].

▶ **Definition 22.** *A* profunctor $R : \mathcal{C} \to\!\!\!\!\!\!\to \mathcal{D}$ *is a functor* $R : \mathcal{C}^{\mathsf{op}} \times \mathcal{D} \to \mathbf{Set}$*.*

To formulate a two-dimensional Kripke semantics for modal logic we replace modal Kripke frames with a small Cauchy-complete category $\mathcal{C}$ with an (endo)profunctor $R : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$. To obtain the modalities we can now play the same trick: putting $\lambda R : \mathcal{C}^{\mathsf{op}} \to [\mathcal{C}, \mathbf{Set}]$ into (9) we canonically obtain the following diagram by Kan extension:



(13)

Conversely, any adjunction $\blacklozenge \dashv \Box$ on $[\mathcal{C}, \mathbf{Set}]$ corresponds to the (endo)profunctor on $\mathcal{C}$ given by $(c_1, c_2) \mapsto \mathrm{Hom}_{\mathcal{C}}(c_1, \blacklozenge c_2)$.

We may then define $[\![\blacklozenge\varphi]\!] \stackrel{\mathrm{def}}{=} \blacklozenge_R [\![\varphi]\!] : \mathcal{C} \longrightarrow \mathbf{Set}$ and $[\![\Box\varphi]\!] \stackrel{\mathrm{def}}{=} \Box_R [\![\varphi]\!] : \mathcal{C} \longrightarrow \mathbf{Set}$. It is worth unfolding what a proof of $\Box\varphi$ at a world w is to obtain an explicit description:

$$[\![\Box\varphi]\!]_w = (\Box_R [\![\varphi]\!])(w) = \mathrm{Hom}_{[\mathcal{C},\mathbf{Set}]}(\lambda R(w), [\![\varphi]\!]) = \mathrm{Hom}_{[\mathcal{C},\mathbf{Set}]}(R(w, -), [\![\varphi]\!]) \tag{14}$$

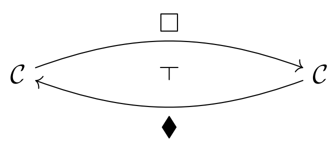
Thus, a proof that φ holds at w is a natural transformation $\alpha : R(w, -) \Rightarrow [\![\varphi]\!]$. This has the expected shape of Kripke semantics for $\Box$: for each $v \in \mathcal{C}$ and proof $x \in R(w, v)$ that v is accessible from w it gives us a proof $\alpha_v(x) \in [\![\varphi]\!]_v$ that φ holds at v.

It is a little harder to see what a proof of $\blacklozenge\varphi$ at a world w is. It becomes more perspicuous if we use the coend formula for the left Kan extension [74, §2.3]:

$$[\![\blacklozenge\varphi]\!] = \lambda R_! [\![\varphi]\!] \cong \int^{v \in \mathcal{C}} \mathrm{Hom}_{[\mathcal{C},\mathbf{Set}]}(\mathbf{y}(v), [\![\varphi]\!]) \times \lambda R(v) \cong \int^{v \in \mathcal{C}} [\![\varphi]\!]_v \times R(v, -) \tag{15}$$

Hence, a proof that $\blacklozenge\varphi$ holds at w consists of a world $v \in \mathcal{C}$, a proof that $R(v, w)$, and a proof that φ holds at v – which is exactly what one would expect. The difference is that the coend quotients some of these pairs according to the action of $\mathcal{C}$. See Mac Lane and Moerdijk [76, §VII.2] for a textbook exposition on why this is a tensor product of $[\![\varphi]\!]$ and λR.

How well does this fit the categorical semantics of modal logic? As with intuitionistic modal logic, there is also a number of proposals of what that might be. A fairly recent idea is to define it as the semantics of a *Fitch-style calculus*, as studied by Clouston [25]. This is exactly a bicartesian closed category $\mathcal{C}$ equipped with an adjunction:



(16)

The left adjoint $\blacklozenge$ is often written as lock. It does not commonly appear as a modality, but as an operator on contexts that corresponds to "opening a box" in Fitch-style natural deduction [59, §5.4]. The modality $\Box$ is a right adjoint, so that it automatically preserves all limits, including products. This idea has proven remarkably robust: variations on it have worked well for modal dependent type theories [13, 51, 52, 50, 49]. The fact that an adjunction on a presheaf category corresponds precisely to a two-dimensional Kripke semantics is further evidence that this is the correct notion of categorical model of modal logic.

Finally, note that (14) and (15) look suspiciously similar to the modal structure of Normalization-by-Evaluation models for modal type theories. This is explicitly visible in the paper by Valliappan et al. [97, §2], and also implicitly present in the paper by Gratzer [48].

5.1 Morphisms

Define the category **Prof** to have as objects profunctors. A morphism $(f, g, \alpha) : R \to S$ from $R : \mathcal{C} \twoheadrightarrow \mathcal{D}$ to $S : \mathcal{C}' \twoheadrightarrow \mathcal{D}'$ consists of functors $f : \mathcal{C} \longrightarrow \mathcal{C}'$ and $g : \mathcal{D} \longrightarrow \mathcal{D}'$, and a natural transformation $\alpha : R(-,-) \Rightarrow S(f(-), g(-))$. The subcategory **EProf** consists of endoprofunctors $R : \mathcal{C} \twoheadrightarrow \mathcal{C}$, and triples of the form (f, f, α). I will synecdochically refer to $\alpha : R(-,-) \Rightarrow S(f(-), f(-))$ as a morphism of **EProf**. Thus, objects are two-dimensional Kripke frames, and morphisms are functors that proof-relevantly preserve the relation.

▶ **Lemma 23.** *Morphisms of endoprofunctors $\alpha : R(-,-) \Rightarrow S(f(-), f(-))$ are in bijection with natural transformations $\gamma : f^*\Box_S \Rightarrow \Box_R f^*$.*

Proof. Unfolding the definitions, $\gamma : \mathrm{Hom}(S(f(-),-),-) \Rightarrow \mathrm{Hom}(R(-,-), f^*(-))$. As $f_! \dashv f^*$ this is exactly a transformation $\mathrm{Hom}(S(f(-),-),-) \Rightarrow \mathrm{Hom}\Big(f_! R(-,-), -\Big)$. By the Yoneda lemma, any such transformation arises by precomposition with a unique transformation $f_! R(-,-) \Rightarrow S(f(-),-)$. By $f_! \dashv f^*$ again, this uniquely corresponds to a transformation $\alpha : R(-,-) \Rightarrow f^* S(f(-),-) = S(f(-), f(-))$. ◀

We thus obtain a duality

$$\mathbf{EProf}_{\mathrm{cc}}^{\mathrm{op}} \simeq \mathbf{PshCatO} \tag{17}$$

where **PshCatO** is the category of presheaf categories $[\mathcal{C}, \mathbf{Set}]$ equipped with a continuous $\Box : [\mathcal{C}, \mathbf{Set}] \longrightarrow [\mathcal{C}, \mathbf{Set}]$. Note that, as presheaf categories are locally finitely presentable, $\Box$ always has a left adjoint $\blacklozenge$. Thus, the objects are categorical models of modal logic. Morphisms are pairs (f, γ) of a (co)continuous $f : \mathcal{C} \longrightarrow \mathcal{D}$ and a natural transformation $\gamma : f^*\Box \Rightarrow \Box f^*$.

As before, open functors do not preserve truth; for that we need a notion of modal openness. Let $\alpha : R(-,-) \Rightarrow S(f(-), f(-))$. As pointed out in the proof of Lemma 23 such an α uniquely corresponds to a transformation $t_\alpha : f_! R(-,-) \Rightarrow S(f(-),-)$. Its components

$$t_{\alpha,c,v} : \int^{w \in V} R(c,w) \times \mathrm{Hom}_{\mathcal{D}}(f(w), v) \to S(f(c), v)$$

map $x \in R(c,w)$ and $k : f(w) \to v$ to $S(\mathsf{id}_{f(c)}, k)(\alpha_{c,v}(x))$. We can then say that

▶ **Definition 24.** *$\alpha : R(-,-) \Rightarrow S(f(-), f(-))$ is* modally open *just if t_α is an isomorphism.*

This asks that for every proof $y \in S(f(c), v)$ we should be able to find an object $w \in \mathcal{C}$, a proof $x \in R(c,w)$, and a morphism $k : f(w) \to v$, so that $y = S(\mathsf{id}_{f(c)}, k)(\alpha_{c,v}(x))$. This is clearly a categorification of Definition 10, and leads to the following lemma:

▶ **Lemma 25.** *α is modally open iff the corresponding $f^*\Box_S \Rightarrow \Box_R f^*$ is an isomorphism.*

Proof. The proof of Lemma 23 precomposes with t_α to get γ. Thus γ is iso iff t_α is. ◀

Thus, the duality (17) may be restricted to dualities between the wide subcategories

$$\mathbf{EProf}^{\mathrm{op}}_{\mathrm{cc,\ moo}} \simeq \mathbf{PshCatO}_{\Rightarrow o} \qquad\qquad \mathbf{EProf}^{\mathrm{op}}_{\mathrm{cc,\ moo,\ rs}} \simeq \mathbf{PshCatO}_{\Rightarrow of} \tag{18}$$

The morphisms to the left of $\simeq$ are modally open, open maps (resp. and retractionally surjective); and the morphisms to the right of $\simeq$ are (f, γ) where f is cartesian closed (resp. and faithful) and $\gamma : f^*\Box \cong \Box f^*$ is a natural isomorphism.

6 Other related work

Perhaps the work most closely related to this paper is that on *Kripke-style lambda models* by Mitchell and Moggi [79]. These amount to elaborating the first-order definitions of applicative structure and λ-model in the internal language of a presheaf category, with the base category being a partial order. In practice this means that the interpretation of function types is only a subfunctor of the exponential of presheaves [79, §8]. However, Mitchell and Moggi prove that these models are sound and complete for the $(\times \to)$ fragment, even in the presence of empty types. They also use some general theorems about open geometric morphisms to prove that any cartesian closed category can be presented as such a model.

Another piece of work that bears kinship with the present one is Hermida's fibrational account of relational modalities [56]. Hermida shows that both the relational modalities $\Diamond$ and $\Box$ can be obtained canonically as extensions of predicate logic to relations, with the modalities arising as compositions of adjoints. The black diamond $\blacklozenge$ makes a brief cameo as the induced left adjoint to $\Box$, as does the dual black box [56, §3.3]. While the decompositions obtained by Hermida seem more refined than the results here, Kan extension does not make an explicit appearance. As such, the relationship to the present work is yet to be determined.

Awodey and Rabe [6] give a Kripke semantics for extensional Martin-Löf type theory (MLTT), in which contexts are posets and types are presheaves over them. They use topos-theoretic machinery to prove that every locally cartesian closed category can be embedded in a presheaf category over a poset; this result seems similar to one of Mitchell and Moggi, but the proof appears entirely different. As a consequence, they show that presheaf categories over posets form a complete class of models for extensional MLTT, in fact a subclass of locally cartesian closed categories.

Alechina et al. [2] present dualities between Kripke and algebraic semantics for constructive S4 and propositional lax logic. Their interpretation of $\Box$ follows that of Plotkin, Stirling and Simpson [85, 92].

Ghilardi and Meloni [42] explore a presheaf-like interpretation of (predicate) modal logic, which is similar to ours, albeit non-proof-relevant. They work over the identity profunctor $\mathrm{Hom}(-,-)$. They are hence forced to weaken the definition of presheaf. See also [43, 44].

Awodey, Kishida and Kotzsch [5] give a topos-theoretic semantics for a higher-order version of intuitionistic S4 modal logic. They also briefly survey much previous work on presheaf-based and topos-theoretic semantics for first-order modal logic. Their work is not proof-relevant.

Finally, there is clear methodological similarity between the results obtained here and the results of Winskel and collaborators on open maps and bisimulation [65, 22]. One central difference is that Winskel et al. are mainly concerned with open maps between presheaves themselves, whereas I only consider open maps between (two-dimensional) frames.

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