Univalent Enriched Categories and the Enriched Rezk Completion

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Abstract

Enriched categories are categories whose sets of morphisms are enriched with extra structure. Such categories play a prominent role in the study of higher categories, homotopy theory, and the semantics of programming languages. In this paper, we study univalent enriched categories. We prove that all essentially surjective and fully faithful functors between univalent enriched categories are equivalences, and we show that every enriched category admits a Rezk completion. Finally, we use the Rezk completion for enriched categories to construct univalent enriched Kleisli categories.

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1 Introduction

Over the years, category theory [16] has established itself as a powerful mathematical framework with a wide variety of applications. The applications of category theory range from pure mathematics [9, 35] to computer science [21, 23, 24, 25]. This study resulted in the development of various notions of categories.

One of these notions is given by *enriched categories*. Enriched categories are categories whose morphisms are equipped with additional structure. Examples of such categories are plentiful. For instance, in the study of the semantics of effectful programming languages, one uses categories enriched over *directed complete partial orders* (DCPOs) [23, 24, 25]. The type of morphisms in categories enriched over DCPOs is given by a DCPO, and thus fixpoint equations of morphisms can be solved in such categories [34]. For similar purposes, categories enriched over partial orders have been used [19]. Other applications of enriched categories appear in homological algebra [35] where one is interested in categories enriched over abelian groups, abstract homotopy theory [9] where one looks at categories enriched over simplicial sets, and higher category theory [14] where one consider categories enriched over categories.

Univalent Foundations. Throughout this paper, we work in univalent foundations [27, 31]. Univalent foundations is an extension of dependent Martin-Löf Type Theory [17] with the *univalence axiom*. This axiom says that the identity of types is the same as equivalences between them. More specifically, we have a map that sends identities $A = B$ to equivalences

$A \simeq B$, and the univalence axiom states that this map is an equivalence. Concretely, this means that properties of types are invariant under equivalence and that two types share the same properties whenever we have an equivalence between them.

Univalent foundations is especially interesting for the study of category theory. In category theory, objects are viewed up to isomorphism: whenever there is an isomorphism between two objects, they share the same categorical properties. This is known as the *principle of equivalence*, and this principle is made precise using *univalent categories*.

Univalent Categories. In univalent foundations, the "correct" notion of category is given by univalent categories. Given two objects x and y in a category, we have a map sending identities $x = y$ to isomorphisms $x \cong y$. In a univalent category, this mapping is required to be an equivalence: identities between objects are thus the same as isomorphisms. Hence, whenever two objects are isomorphic, they satisfy the same properties. Semantically, this is the correct notion of category, because in the simplicial set model, univalent categories correspond to set-theoretic categories [11].

There are several consequences of univalence for categories. For instance, we have a *structure identity principle* for univalent categories. This principle says that the identity type of two categories is equivalent to the type of adjoint equivalences between them [2]. As a consequence, one gets that whenever two categories are equivalent, then they have the same properties. Another consequence is that every essentially surjective fully faithful functor is an adjoint equivalence as well. Usually, one uses the axiom of choice to prove this principle, but if the domain is univalent, then one can constructively prove this fact. Finally, every category is weakly equivalent to a univalent one, which is called its *Rezk completion* [2].

While most categories that one encounters in practice are univalent (e.g., Eilenberg-Moore categories and functor categories), some are not. An example is given by the Kleisli category. Usually, the Kleisli category $\mathsf{K}(\mathsf{T})$ of a monad T on a category C is defined to be the category whose objects are objects of C and whose morphisms from x to y are morphisms $x \to \mathsf{T}\, y$ in C. However, this does not give rise to a univalent category in general. One can give an alternative presentation of the Kleisli category as a full subcategory $\mathsf{Kleisli}(\mathsf{T})$ of the Eilenberg-Moore category to obtain a univalent category [4]. To prove the desired theorems about $\mathsf{Kleisli}(\mathsf{T})$, one uses that it is the Rezk completion of $\mathsf{K}(\mathsf{T})$ [36].

Univalent Enriched Categories. In this paper, we develop enriched category theory in univalent foundations. More specifically, we define univalent enriched categories, and we prove analogous theorems for univalent enriched categories as for univalent categories. We show that univalent enriched categories satisfy a structure identity principle, that every essentially surjective fully faithful functor is an adjoint equivalence, and that every enriched category admits a Rezk completion. We also use these theorems to construct univalent enriched Kleisli categories.

Related work. While there are numerous libraries that contain a formalization of categories, enriched categories have gotten less attention. Several libraries, such as Agda categories [10] using Agda [22], mathlib [18] using Lean, and the category-theory library [38] in `Coq` [29], contain a couple of basic definitions. In `UniMath`, Satoshi Kura also formalized several basic concepts of enriched categories. However, none of the aforementioned formalizations consider much of the theory of enriched categories, and they do not consider univalent enriched categories. In addition, we use enrichments (Definition 2.2), while the other formalizations use the definition as given by Kelly [12]. Enrichments have been used in the setting of skew-enriched categories [5], and, with a slightly different definition, in the study of strong monads [20, Definition 5.1].

Formalization. The results in this paper are formalized in the `Coq` [29] proof assistant using the `UniMath` library [33]. We use the `UniMath` library in this work, because we frequently use notions from bicategory theory that have only been formalized in UniMath up to now. Definitions and theorems are accompanied with links to their corresponding identifier in the formalization, and these links are underlined. The tool `coqwc` reports the following number of lines of code in the formalization.

```
   spec    proof comments
  17855     9242      520 total
```

Beside what is discussed in this paper, the formalization also contains (weighted) limits and colimits in enriched categories and models of the enriched effect calculus [7].

Contributions and Overview. The contributions of this paper are as follows.

- A construction of the bicategory of univalent enriched categories (Definition 2.6) and a proof that this bicategory is univalent (Theorem 2.7);
- a construction of the image factorization system of enriched categories (Construction 4.7);
- a proof that all fully faithful and essentially surjective enriched functors are adjoint equivalences (Theorem 4.8);
- a construction of the Rezk completion for enriched categories (Construction 5.4) and a proof of its universal property (Theorem 5.5);
- a construction of Kleisli objects (Construction 6.9) in the bicategory of univalent enriched categories.

In Section 3, we discuss numerous examples of enriched categories.

2 The Bicategory of Enriched Categories

In the remainder of this paper, we study univalent enriched categories, and in this section, we discuss a structure identity principle for univalent enriched categories, which says that identity of univalent enriched categories is the same as equivalence. Before we do so, we briefly recall *monoidal categories* to fix the notation for the remainder of the paper [3, 16].

▶ **Definition 2.1.** *A* ***monoidal category*** *consists of a category* V *together with*

- *an object* $\mathbb{1}_{\mathsf{V}} : \mathsf{V}$*;*
- *a bifunctor* $- \otimes - : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$*;*
- *isomorphisms* $\mathsf{l}_x : \mathbb{1}_{\mathsf{V}} \otimes x \to x$*,* $\mathsf{r}_x : x \otimes \mathbb{1}_V \to x$*, and* $\mathsf{a}_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$*;*

such that l_{-}*,* r_{-}*, and* $\mathsf{a}_{-,-,-}$ *are natural and such that the triangle and pentagon laws hold.*

A ***symmetric monoidal category*** *is a monoidal category* V *together with morphisms* $\mathsf{s}_{x,y} : x \otimes y \to y \otimes x$*, such that* $\mathsf{s}_{x,y}$ *and* $\mathsf{s}_{y,x}$ *are inverses,* $\mathsf{s}_{-,-}$ *is natural, and satisfies the hexagon law.*

A ***symmetric monoidal closed category*** *is a symmetric monoidal category* V *such that for every* $x : \mathsf{V}$ *the functor* $x \otimes -$ *has a right adjoint.*

If we have a symmetric monoidal closed category, then we have internal homs $x \multimap y$. We also have evaluation morphisms $\epsilon_{x,y} : (x \multimap y) \otimes x \to y$, and an internal lambda abstraction operation $\lambda(f) : x \to y \multimap z$ for every $f : x \otimes y \to z$.

Usually, a V-enriched category consists of a collection objects together with a hom-object $\mathcal{E}(x,y)$ in V for all x and y, such that we have appropriate identity morphisms and a composition operation [12]. Every enriched category $\mathcal{E}$ has an underlying category, which has the same collection of objects and whose morphisms from x to y are the same as morphisms $\mathbb{1} \to \mathcal{E}(x,y)$ in V.

However, we take a slightly different approach: we use *enrichments*. An enrichment for a category C consists of a hom-object $\mathcal{E}(x, y)$ in V for all x and y, such that we have the appropriate identity and composition morphisms and such that C is equivalent to the underlying category of the corresponding enriched category. As such, we view an enriched category as a category together with extra structure. This viewpoint also determines our notion of univalence for enriched categories: a univalent enriched category is an enriched category such that its underlying category is univalent. Note that our notion of univalence is similar to completeness for enriched ∞-categories [8]. Using enrichments, we can equivalently phrase univalent enriched categories as a univalent category together with an enrichment.

▶ **Definition 2.2.** *Let* V *be a monoidal category and let* C *be a category. A* V***-enrichment*** $\mathcal{E}$ *for* C *consists of*

- *for all objects* $x, y : \mathsf{C}$ *an object* $\mathcal{E}(x, y) : \mathsf{V}$*;*
- *for every object* $x : \mathsf{C}$ *a morphism* $\mathsf{id}^{\mathsf{e}}_{\mathcal{E}}(x) : \mathbb{1}_V \to \mathcal{E}(x, x)$*;*
- *for all objects* $x, y, z : \mathsf{C}$ *a morphism* $\mathsf{comp}_{\mathcal{E}}(x, y, z) : \mathcal{E}(y, z) \otimes \mathcal{E}(x, y) \to \mathcal{E}(x, z)$*;*
- *for all morphisms* $f : x \to y$ *in* C *a morphism* $\overrightarrow{f} : \mathbb{1}_V \to \mathcal{E}(x, y)$*;*
- *for every morphism* $f : \mathbb{1}_V \to \mathcal{E}(x, y)$ *a morphism* $\overleftarrow{f} : x \to y$ *in* C*.*

In addition, we require that $\overleftarrow{\overrightarrow{f}} = f$ *and* $\overrightarrow{\overleftarrow{f}} = f$ *and that the following diagrams commute.*

$$\begin{tikzcd}
\mathbb{1} \otimes \mathcal{E}(x,y) \arrow[r, "\mathsf{id}^{\mathsf{e}}(y) \otimes \mathrm{id}"] \arrow[dr, "\mathsf{l}_{\mathcal{E}(x,y)}"'] & \mathcal{E}(y,y) \otimes \mathcal{E}(x,y) \arrow[d, "\mathsf{comp}(x,y,y)"] \\
& \mathcal{E}(x,y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}(x,y) \otimes \mathbb{1} \arrow[r, "\mathrm{id} \otimes \mathsf{id}^{\mathsf{e}}(x)"] \arrow[dr, "\mathsf{r}_{\mathcal{E}(x,y)}"'] & \mathcal{E}(x,y) \otimes \mathcal{E}(x,x) \arrow[d, "\mathsf{comp}(x,x,y)"] \\
& \mathcal{E}(x,y)
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathcal{E}(y,z) \otimes \mathcal{E}(x,y)) \otimes \mathcal{E}(w,x) \arrow[r, "\mathsf{a}"] \arrow[dd, "\mathsf{comp}(x,y,z) \otimes \mathrm{id}"'] & \mathcal{E}(y,z) \otimes (\mathcal{E}(x,y) \otimes \mathcal{E}(w,x)) \arrow[d, "\mathrm{id} \otimes \mathsf{comp}(w,x,y)"] \\
& \mathcal{E}(y,z) \otimes \mathcal{E}(w,y) \arrow[d, "\mathsf{comp}(w,y,z)"] \\
\mathcal{E}(x,z) \otimes \mathcal{E}(w,x) \arrow[r, "\mathsf{comp}(w,x,z)"'] & \mathcal{E}(w,z)
\end{tikzcd}$$

When it is clear from the context, we leave the arguments of comp and id^{e} implicit. In addition, note that the morphism id^{e} is redundant in Definition 2.2, since we have that $\mathsf{id}^{\mathsf{e}}_{\mathcal{E}}(x) = \overrightarrow{\mathrm{id}}$. However, we decided to keep id^{e} in the definition, because then it is slightly more convenient to prove Proposition 2.3.

In the remainder, we use the following operations for V-enrichments $\mathcal{E}$ for a category C. Given an object w and a morphism $f : x \to y$, we define f^{pre} as the following composition.

$$\mathcal{E}(w,x) \xrightarrow{\mathsf{l}^{-1}} \mathbb{1} \otimes \mathcal{E}(w,x) \xrightarrow{\overrightarrow{f} \otimes \mathrm{id}} \mathcal{E}(x,y) \otimes \mathcal{E}(w,x) \xrightarrow{\mathsf{comp}} \mathcal{E}(w,y)$$

For objects z and morphisms $f : x \to y$, we define f^{post} as the following composition.

$$\mathcal{E}(y,z) \xrightarrow{\mathsf{r}^{-1}} \mathcal{E}(y,z) \otimes \mathbb{1} \xrightarrow{\mathrm{id} \otimes \overrightarrow{f}} \mathcal{E}(y,z) \otimes \mathcal{E}(x,y) \xrightarrow{\mathsf{comp}} \mathcal{E}(x,z)$$

Note that a category together with a V-enrichment is the same as an enriched category as defined by Kelly [12].

▶ **Proposition 2.3.** *For a monoidal category* V*, the type of categories together with a* V*-enrichment is equivalent to the type of* V*-enriched categories.*

The reason why we use enrichments over the usual definition, is because it simplifies the proof of the structure identity principle for enriched categories. A structure identity principle is already present for univalent categories [2, Theorem 6.17], which can be reused directly if we define enriched categories as pairs of univalent categories together with an enrichment. However, if we would use the definition in [12] instead, then reusing this principle would be more cumbersome and thus the desired proof would be more involved.

To phrase the structure identity principle for enriched categories, we use *univalent bicategories*. More specifically, this principle for enriched categories is expressed by saying that the bicategory of enriched categories is univalent. We define the bicategory of enriched categories using *displayed bicategories* [1]. A displayed bicategory over a bicategory B represents structures and properties to be added to objects, 1-cells, and 2-cells in B. In our case, we define a displayed bicategory $\mathsf{dEnrichCat_V}$ over the bicategory $\mathsf{UnivCat}$ of univalent categories, and then $\mathsf{EnrichCat_V}$ is total bicategory of $\mathsf{dEnrichCat_V}$. The displayed objects over a univalent category C are V-enrichments for C, the displayed 1-cells over a functor are enrichments for functors, and the displayed 2-cells over a natural transformation are proofs that this transformation is enriched.

Note that from the machinery of displayed bicategories, we get a pseudofunctor $\mathsf{Und_V} : \mathsf{EnrichCat_V} \to \mathsf{UnivCat}$, which sends every enriched category to its underlying category. Using this pseudofunctor, we can understand an enrichment for C to be an object in the fiber of C along $\mathsf{Und_V}$.

▶ **Definition 2.4.** *Suppose that we have V-enrichments $\mathcal{E}_1$ and $\mathcal{E}_2$ for C_1 and C_2 respectively. A* V***-enrichment*** *$\mathcal{F}$ for a functor $\mathsf{F} : \mathsf{C}_1 \to \mathsf{C}_2$ from $\mathcal{E}_1$ to $\mathcal{E}_2$ is a family of morphisms $\mathcal{F}(x, y) : \mathcal{E}_1(x, y) \to \mathcal{E}_2(\mathsf{F}\,x, \mathsf{F}\,y)$ such that the following diagrams commute*

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\mathsf{id^e}"] \arrow[dr, "\mathsf{id^e}"'] & \mathcal{E}_1(x,x) \arrow[d, "\mathcal{F}(x,x)"] \\ & \mathcal{E}_2(\mathsf{F}\,x, \mathsf{F}\,x) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}_1(y,z) \otimes \mathcal{E}_1(x,y) \arrow[r, "\mathsf{comp}"] \arrow[d, "\mathcal{F}(y,z)\otimes\mathcal{F}(x,y)"'] & \mathcal{E}_1(x,z) \arrow[d, "\mathcal{F}(x,z)"] \\ \mathcal{E}_2(\mathsf{F}\,y, \mathsf{F}\,z) \otimes \mathcal{E}_2(\mathsf{F}\,x, \mathsf{F}\,y) \arrow[r, "\mathsf{comp}"'] & \mathcal{E}_2(\mathsf{F}\,x, \mathsf{F}\,z) \end{tikzcd}$$

In addition, we require that $\overrightarrow{\mathsf{F}\,f} = \overrightarrow{f} \cdot \mathcal{F}(x, y)$.

▶ **Definition 2.5.** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be V-enrichments for functors $\mathsf{F}_1, \mathsf{F}_2 : \mathsf{C}_1 \to \mathsf{C}_2$ from $\mathcal{E}_1$ to $\mathcal{E}_2$. A natural transformation $\tau : F_1 \Rightarrow F_2$ is called* V***-enriched*** *whenever the following diagram commutes*

$$\begin{tikzcd} & \mathcal{E}_1(x,y) \otimes \mathbb{1} \arrow[r, "\mathcal{F}_2 \otimes \overrightarrow{\tau\,x}"] & \mathcal{E}_2(\mathsf{F}_2\,x, \mathsf{F}_2\,y) \otimes \mathcal{E}_2(\mathsf{F}_1\,x, \mathsf{F}_2\,x) \arrow[dr, "\mathsf{comp}"] & \\ \mathcal{E}_1(x,y) \arrow[ur, "\mathsf{r}^{-1}"] \arrow[dr, "\mathsf{l}^{-1}"'] & & & \mathcal{E}_2(\mathsf{F}_1\,x, \mathsf{F}_2\,y) \\ & \mathbb{1} \otimes \mathcal{E}_1(x,y) \arrow[r, "\overrightarrow{\tau\,y} \otimes \mathcal{F}_1"'] & \mathcal{E}_2(\mathsf{F}_1\,y, \mathsf{F}_2\,y) \otimes \mathcal{E}_2(\mathsf{F}_1\,x, \mathsf{F}_1\,y) \arrow[ur, "\mathsf{comp}"'] & \end{tikzcd}$$

Note that the condition for V-enriched natural transformations can equivalently formulated by saying that the following diagram commutes.

$$\begin{tikzcd} \mathcal{E}_1(x,y) \arrow[r, "\mathcal{F}_1"] \arrow[d, "\mathcal{F}_2"'] & \mathcal{E}_2(\mathsf{F}_1\,x, \mathsf{F}_1\,y) \arrow[d, "(\tau\,y)^{\mathsf{post}}"] \\ \mathcal{E}_2(\mathsf{F}_2\,x, \mathsf{F}_2\,y) \arrow[r, "(\tau\,x)^{\mathsf{pre}}"'] & \mathcal{E}_2(\mathsf{F}_1\,x, \mathsf{F}_2\,y) \end{tikzcd} \tag{1}$$

Now we have everything in place to define the bicategory of enriched categories.

▶ **Definition 2.6.** *Let* V *be a monoidal category. We define the* ***displayed bicategory*** $\mathsf{dEnrichCat}_{\mathsf{V}}$ ***of enrichments*** *over* $\mathsf{UnivCat}$ *as follows.*

- *The displayed objects over a category* C *are* V*-enrichments for* C*;*
- *the displayed 1-cells over a functor* $\mathsf{F} : \mathsf{C}_1 \to \mathsf{C}_2$ *from* $\mathcal{E}_1$ *to* $\mathcal{E}_2$ *are* V*-enrichments for* F *from* C_1 *to* C_2*;*
- *the displayed 2-cells over a natural transformation* $\tau : \mathsf{F}_1 \Rightarrow \mathsf{F}_2$ *from* $\mathcal{F}_1$ *to* $\mathcal{F}_2$ *are proofs that* τ *is* V*-enriched.*

The ***bicategory of enriched categories*** *is defined to be the total bicategory of* $\mathsf{dEnrichCat}_{\mathsf{V}}$*, and we denote it by* $\mathsf{EnrichCat}_{\mathsf{V}}$*. Its objects are* ***univalent*** V***-enriched categories****, and we call the 1-cells and 2-cells of* $\mathsf{EnrichCat}_{\mathsf{V}}$ ***enriched functors*** *and* ***enriched transformations*** *respectively.*

Note that by construction a univalent V-enriched category is the same as a univalent category together with a V-enrichment. In addition, our univalence condition for V-enriched categories has no local variant in contrast to univalence for bicategories [1], since we only look at enrichments over monoidal 1-categories.

To show that Definition 2.6 actually gives rise to a displayed bicategory, one also needs to construct enrichments for the identity and composition, and one needs to prove that the identity transformation is enriched and that enriched transformations are preserved under composition and whiskering. Details on this construction are left to the formalization.

▶ **Theorem 2.7.** *If* V *is a univalent monoidal category, then the bicategory* $\mathsf{EnrichCat}_{\mathsf{V}}$ *is univalent.*

▶ **Proposition 2.8.** *Let* $\tau : \mathcal{F}_1 \Rightarrow \mathcal{F}_2$ *be a 2-cell in* $\mathsf{EnrichCat}_{\mathsf{V}}$*. Then* τ *is invertible if the underlying natural transformation of* τ *is a natural isomorphism.*

3 Examples of Enriched Categories

Before we continue our study of univalent enriched categories, we first look at numerous examples of enrichments that we use in the remainder of this paper. In Section 3.2, we characterize enrichments over a large class of structures.

3.1 General Examples

▶ **Example 3.1.** *Let* V *be a symmetric monoidal closed category. We define a* V*-enrichment for* V*, which we call the* ***self-enrichment*** *and denote by* $\mathsf{self}(\mathsf{V})$*, as follows.*

- *We define* $\mathsf{self}(\mathsf{V})(x, y)$ *to be* $x \multimap y$*.*
- *The enriched identity* $\mathsf{id}^{\mathsf{e}}(x) : \mathbb{1} \to x \multimap x$ *is defined to be* $\lambda(\mathsf{l}_x)$*.*
- *The composition* $\mathsf{comp}(x, y, z) : y \multimap z \otimes x \multimap y \to x \multimap z$ *is the exponential transpose of the following composition of morphisms.*

$$((y \multimap z) \otimes (x \multimap y)) \otimes x \xrightarrow{\mathsf{a}} (y \multimap z) \otimes ((x \multimap y) \otimes x) \xrightarrow{\mathrm{id} \otimes \epsilon} (y \multimap z) \otimes y \xrightarrow{\epsilon} z$$

- *Given* $f : x \to y$*, we define* $\overrightarrow{f} : \mathbb{1} \to x \multimap y$ *to be* $\lambda(\mathsf{l}_x \cdot f)$*.*
- *For* $f : \mathbb{1} \to x \multimap y$*, we define* $\overleftarrow{f}$ *to be the following composition of morphisms.*

$$x \xrightarrow{\mathsf{l}_x^{-1}} \mathbb{1} \otimes x \xrightarrow{f \otimes \mathrm{id}_x} (x \multimap y) \otimes x \xrightarrow{\epsilon} y$$

If we assume that V *is univalent, then* $\mathsf{self}(\mathsf{V})$ *is a univalent enriched category.*

▶ **Example 3.2.** *Let* C *be a category together with a* V*-enrichment* $\mathcal{E}$*, and let* P *be a predicate on the objects of* C*. From all of this, we obtain a* V***-enrichment*** $\mathsf{FSub_e}(\mathsf{P})$ ***for the full subcategory*** $\mathsf{FSub}(\mathsf{P})$*, such that* $\mathsf{FSub_e}(\mathsf{P})(x,y) := \mathcal{E}(x,y)$*. If* C *is univalent, then the full subcategory of* C *is also univalent, and in that case, this construction gives rise to a univalent enriched category.*

▶ **Example 3.3.** *Suppose that* V *is a symmetric monoidal category, and let* C *be a category together with a* V*-enrichment* $\mathcal{E}$*. We define the* V*-enrichment* $\mathcal{E}^{\mathsf{op}}$*, called the* ***opposite enrichment****, for* C^{op} *as follows.*

- $\mathcal{E}^{\mathsf{op}}(x,y) := \mathcal{E}(y,x)$;
- $\mathsf{id}^{\mathsf{e}}_{\mathcal{E}^{\mathsf{op}}}(x) := \mathsf{id}^{\mathsf{e}}_{\mathcal{E}}(x)$;
- $\mathsf{comp}_{\mathcal{E}^{\mathsf{op}}}(x,y,z) := \mathcal{E}(z,y)\otimes\mathcal{E}(y,x) \xrightarrow{\mathsf{s}} \mathcal{E}(y,x)\otimes\mathcal{E}(z,y) \xrightarrow{\mathsf{comp}} \mathcal{E}(z,x)$

The operations $\overrightarrow{f}$ *and* $\overleftarrow{f}$ *in* $\mathcal{E}^{\mathsf{op}}$ *are inherited from* $\mathcal{E}$*. In addition,* $\mathcal{E}^{\mathsf{op}}$ *gives rise to a univalent enriched category if* C *is univalent.*

In fact, using Example 3.3 one can construct a duality involution on $\mathsf{EnrichCat}_{\mathsf{V}}$.

▶ **Example 3.4.** *Suppose that* V *is a symmetric monoidal category that has equalizers, and suppose that we have two enriched functors* $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{E}_1 \to \mathcal{E}_2$*. We have the category* $\mathsf{Dialg}(\mathcal{F}_1,\mathcal{F}_2)$ *of dialgebras whose objects are pairs* (x,f) *consisting of an object* $x : \mathcal{E}_1$ *together with a morphism* $f : \mathcal{F}_1\,x \to \mathcal{F}_2\,x$*. Morphisms from* (x,f) *to* (y,g) *are morphisms* $h : x \to y$ *such that the following diagram commutes.*

$$\begin{tikzcd} \mathcal{F}_1\,x \arrow[r, "\mathcal{F}_1\,h"] \arrow[d, "f"'] & \mathcal{F}_1\,y \arrow[d, "g"] \\ \mathcal{F}_2\,x \arrow[r, "\mathcal{F}_2\,h"'] & \mathcal{F}_2\,y \end{tikzcd}$$

We define a V***-enrichment*** $\mathsf{Dialg_e}(\mathcal{F}_1,\mathcal{F}_2)$ ***for*** $\mathsf{Dialg}(\mathcal{F}_1,\mathcal{F}_2)$*. Suppose that we have objects* (x,f) *and* (y,g) *in* $\mathsf{Dialg}(\mathcal{F}_1,\mathcal{F}_2)$*. We define the object* $\mathsf{Dialg_e}(\mathcal{F}_1,\mathcal{F}_2)((x,f),(y,g))$ *as the equalizer of the following diagram.*

$$\begin{tikzcd} & & \mathcal{E}_2(\mathcal{F}_1\,x, \mathcal{F}_1\,y) \arrow[dr, "g^{\mathsf{post}}"] & \\ \mathsf{Dialg_e}(\mathcal{F}_1,\mathcal{F}_2)((x,f),(y,g)) \arrow[r, hook] & \mathcal{E}_1(x,y) \arrow[ur, "\mathcal{F}_1"] \arrow[dr, "\mathcal{F}_2"'] & & \mathcal{E}_2(\mathcal{F}_1\,x, \mathcal{F}_2\,y) \\ & & \mathcal{E}_2(\mathcal{F}_2\,x, \mathcal{F}_2\,y) \arrow[ur, "f^{\mathsf{pre}}"'] & \end{tikzcd}$$

To define the enriched identity and composition morphisms, one uses the universal property of equalizers. If C *is univalent, then so is the category of dialgebras, and in that case,* $\mathsf{Dialg_e}(\mathcal{F}_1,\mathcal{F}_2)$ *is a univalent enriched category.*

Using Example 3.4, one can show that the bicategory $\mathsf{EnrichCat}_{\mathsf{V}}$ has inserters.

▶ **Example 3.5.** *Let* V *be a complete symmetric monoidal category, and suppose that we have enriched categories* $\mathcal{E}_1$ *and* $\mathcal{E}_2$*. Note that we have a category* $[\mathcal{E}_1,\mathcal{E}_2]$ *whose objects are given by enriched functors from* $\mathcal{E}_1$ *to* $\mathcal{E}_2$*, and whose morphisms are given by enriched natural transformations. We define a* V***-enrichment*** $\mathsf{EFunctor}(\mathcal{E}_1,\mathcal{E}_2)$ ***for*** $[\mathcal{E}_1,\mathcal{E}_2]$ *as the equalizer of the morphisms displayed below.*

$$\prod_{x:\mathcal{E}_1} \mathcal{E}_2(\mathcal{F}_1\,x, \mathcal{F}_2\,x) \underset{g}{\overset{f}{\rightrightarrows}} \prod_{x,y:\mathcal{E}_1} \mathcal{E}_1(x,y) \multimap \mathcal{E}_2(\mathcal{F}_1\,x, \mathcal{F}_2\,y)$$

Here f is defined to be the following composition of morphisms

$$\prod_{x:\mathcal{E}_1} \mathcal{E}_2(\mathcal{F}_1\,x, \mathcal{F}_2\,x) \xrightarrow{\pi_y} \mathcal{E}_2(\mathcal{F}_1\,y, \mathcal{F}_2\,y) \xrightarrow{\varphi} \mathcal{E}_1(x,y) \multimap \mathcal{E}_2(\mathcal{F}_1\,x, \mathcal{F}_2\,y)$$

where φ is the exponential transpose of

$$\mathcal{E}_2(\mathcal{F}_1\,y, \mathcal{F}_2\,y) \otimes \mathcal{E}_1(x,y) \xrightarrow{\mathrm{id} \otimes \mathcal{F}_1} \mathcal{E}_2(\mathcal{F}_1\,y, \mathcal{F}_2\,y) \otimes \mathcal{E}_2(\mathcal{F}_1\,x, \mathcal{F}_1\,y) \xrightarrow{\mathsf{comp}} \mathcal{E}_2(\mathcal{F}_1\,x, \mathcal{F}_2\,y).$$

We define g analogously. The fact that $[\mathcal{E}_1, \mathcal{E}_2]$ is univalent, follows from the fact that $\mathsf{EnrichCat}_{\mathsf{V}}$ *is univalent (Theorem 2.7).*

Inspired by Example 3.5, we can refine Example 3.1. More specifically, given a small category C, we define a V-enrichment for the functor category from C to V. The construction is analogous to Example 3.5, and details can be found in the formalization.

Finally, we look at the change of base operation for enriched categories, and for this operation, a subtlety arises. Given a lax monoidal functor $\mathsf{F} : \mathsf{V}_1 \to \mathsf{V}_2$, our goal is to define a pseudofunctor $\mathsf{EnrichCat}_{\mathsf{V}_1} \to \mathsf{EnrichCat}_{\mathsf{V}_2}$. On objects, this operation acts as follows: given a univalent category C together with an enrichment $\mathcal{E}$, then we get an enriched category $\mathsf{F}^*(\mathcal{E})$ whose objects are objects in C and such that $\mathsf{F}^*(\mathcal{E})(x,y) = \mathsf{F}\,(\mathcal{E}(x,y))$. However, the underlying category of this enriched category is **not** necessarily univalent. For instance, if we take F to be the unique monoidal functor from Set to the terminal category, then the underlying category of $\mathsf{F}^*(\mathcal{E})$ would have sets as objects, and inhabitants of the unit type as the morphisms. For this reason, we add a restriction to F in order to define the change of base of enriched categories.

▶ **Definition 3.6.** *Let $\mathsf{F} : \mathsf{V}_1 \to \mathsf{V}_2$ by a lax monoidal functor. We say that F* ***preserves underlying categories*** *if for all $x : \mathsf{V}_1$ the function that sends morphisms $f : \mathbb{1}_{\mathsf{V}_1} \to x$ to $\mathbb{1}_{\mathsf{V}_2} \xrightarrow{\epsilon_{\mathsf{F}}} \mathsf{F}\,\mathbb{1}_{\mathsf{V}_1} \xrightarrow{\mathsf{F}\,f} \mathsf{F}\,x$ is an equivalence of types. If we have $f : \mathbb{1}_{\mathsf{V}_2} \to \mathsf{F}\,x$, then we denote the action of the inverse by $\zeta_{\mathsf{F}}(f)$.*

The requirement in Definition 3.6 says that the underlying category is preseerved by change of base along F. With this additional assumption, we define the change of base of enriched categories.

▶ **Example 3.7.** *Let $\mathsf{F} : \mathsf{V}_1 \to \mathsf{V}_2$ by a lax monoidal functor that preserves underlying categories, and let C be a category together with a V-enrichment $\mathcal{E}$. We define the* ***change-of-base enrichment*** *$\mathsf{F}^*(\mathcal{E})$ for C as follows. The hom-object $\mathsf{F}^*(\mathcal{E})(x,y)$ is defined to be $\mathsf{F}\,(\mathcal{E}(x,y))$, and the enriched identity $\mathsf{id}^{\mathsf{e}}(x)$ is defined as the composition*

$$\mathbb{1}_{\mathsf{V}_2} \xrightarrow{\epsilon_{\mathsf{F}}} \mathsf{F}\,\mathbb{1}_{\mathsf{V}_1} \xrightarrow{\mathsf{F}\,(\mathsf{id}^{\mathsf{e}}(x))} \mathsf{F}\,(\mathcal{E}(x,x))$$

Composition is defined similarly.

$$\mathsf{F}\,(\mathcal{E}(y,z)) \otimes \mathsf{F}\,(\mathcal{E}(x,y)) \xrightarrow{\mu_{\mathsf{F}}} \mathsf{F}\,(\mathcal{E}(y,z) \otimes \mathcal{E}(x,y)) \xrightarrow{\mathsf{F}\,\mathsf{comp}} \mathsf{F}\,(\mathcal{E}(x,y))$$

If we have a morphism $f : x \to y$, then we define $\overrightarrow{f}$ to be

$$\mathbb{1}_{\mathsf{V}_2} \xrightarrow{\epsilon_{\mathsf{F}}} \mathsf{F}\,\mathbb{1}_{\mathsf{V}_1} \xrightarrow{\mathsf{F}\,(\overrightarrow{f})} \mathsf{F}\,(\mathcal{E}(x,x)).$$

Finally, for a morphism $f : \mathbb{1}_{\mathsf{V}_2} \to \mathsf{F}\,(\mathcal{E}(x,x))$, we define $\overleftarrow{f}$ to be $\overleftarrow{\zeta_{\mathsf{F}}(f)}$. Note that here we use that F preserves underlying categories. In addition, if we assume that C is univalent, then we get a univalent enriched category $\mathsf{F}^*(\mathcal{E})$.

3.2 Enrichments over Structures

Next we characterize two classes of enrichments. First, we characterize enrichments for the category Set of sets equipped with its cartesian monoidal structure.

▶ **Proposition 3.8.** *Let C be a category. The type of Set-enrichments for C is contractible.*

From Proposition 3.8, we can conclude that the type of categories is equivalent to the type of Set-enriched categories. Second, we characterize enrichments for structured sets with a cartesian monoidal structure. To do so, we first define a general notion of structured sets.

▶ **Definition 3.9.** *A **cartesian notion of structure** $\mathcal{S}$ consists of*
- *a set $\mathsf{P}_{\mathcal{S}}\,\mathsf{X}$ of structures on X for every set X;*
- *a proposition $\mathsf{H}_{(\mathsf{p_X},\mathsf{p_Y})}(f)$ which represents that f is a structure preserving map from $\mathsf{p_X}$ to $\mathsf{p_Y}$, for all functions $f : \mathsf{X} \to \mathsf{X}$ and structures $\mathsf{p_X} : \mathsf{P}_{\mathcal{S}}\,\mathsf{X}$ and $\mathsf{p_Y} : \mathsf{P}_{\mathcal{S}}\,\mathsf{Y}$;*
- *an inhabitant $\mathsf{p_{unit}} : \mathsf{P}_{\mathcal{S}}\,\mathsf{unit}$;*
- *a structure $\mathsf{p_X} \times \mathsf{p_Y} : \mathsf{P}_{\mathcal{S}}\,(\mathsf{X} \times \mathsf{Y})$ for all $\mathsf{p_X} : \mathsf{P}_{\mathcal{S}}\,\mathsf{X}$ and $\mathsf{p_Y} : \mathsf{P}_{\mathcal{S}}\,\mathsf{Y}$,.*

This data is required to satisfy the following axioms.
- *For every set X and structure $\mathsf{p_X} : \mathsf{P}_{\mathcal{S}}\,\mathsf{X}$, we have $\mathsf{H}_{(\mathsf{p_X},\mathsf{p_X})}(\mathrm{id}_{\mathsf{X}})$;*
- *for all functions $f : \mathsf{X} \to \mathsf{Y}$ and $g : \mathsf{Y} \to \mathsf{Z}$ such that $\mathsf{H}_{(\mathsf{P}_{\mathcal{S}}\,\mathsf{X},\mathsf{P}_{\mathcal{S}}\,\mathsf{Y})}(f)$ and $\mathsf{H}_{(\mathsf{P}_{\mathcal{S}}\,\mathsf{Y},\mathsf{P}_{\mathcal{S}}\,\mathsf{Z})}(g)$, we have $\mathsf{H}_{(\mathsf{P}_{\mathcal{S}}\,\mathsf{X},\mathsf{P}_{\mathcal{S}}\,\mathsf{Z})}(g \circ f)$;*
- *given structures $\mathsf{p_X}, \mathsf{p'_X} : \mathsf{P}_{\mathcal{S}}\,\mathsf{X}$ such that $\mathsf{H}_{(\mathsf{p_X},\mathsf{p_{X'}})}(\mathrm{id}_{\mathsf{X}})$ and $\mathsf{H}_{(\mathsf{p_{X'}},\mathsf{p_X})}(\mathrm{id}_{\mathsf{X}})$, we have $\mathsf{p_X} = \mathsf{p'_X}$;*
- *given a structure $\mathsf{p_X} : \mathsf{P}_{\mathcal{S}}\,\mathsf{X}$ on a set X, we have $\mathsf{H}_{(\mathsf{p_X},\mathsf{p_{unit}})}(\lambda(x : \mathsf{X}).\mathsf{tt})$ where tt is the unique element of unit;*
- *given structures $\mathsf{p_X} : \mathsf{P}_{\mathcal{S}}\,\mathsf{X}$ and $\mathsf{p_Y} : \mathsf{P}_{\mathcal{S}}\,\mathsf{Y}$ on sets X and Y respectively, we have $\mathsf{H}_{(\mathsf{p_X}\times\mathsf{p_Y},\mathsf{p_X})}(\pi_1)$ and $\mathsf{H}_{(\mathsf{p_X}\times\mathsf{p_Y},\mathsf{p_Y})}(\pi_2)$;*
- *for all functions $f : \mathsf{X} \to \mathsf{Y}$ and $g : \mathsf{X} \to \mathsf{Z}$ such that $\mathsf{H}_{(\mathsf{P}_{\mathcal{S}}\,\mathsf{X},\mathsf{P}_{\mathcal{S}}\,\mathsf{Y})}(f)$ and $\mathsf{H}_{(\mathsf{P}_{\mathcal{S}}\,\mathsf{X},\mathsf{P}_{\mathcal{S}}\,\mathsf{Z})}(g)$, we have $\mathsf{H}_{(\mathsf{P}_{\mathcal{S}}\,\mathsf{X},\mathsf{p_Y}\times\mathsf{p_Z})}(\lambda(x : \mathsf{X}).(f\,x, g\,x))$.*

Note that Definition 3.9 is extension of standard notions of structures defined in [31, Definition 9.8.1]: the added data and axioms guarantee that the resulting category has binary products and a terminal object.

▶ **Problem 3.10.** *Given a cartesian notion of structure $\mathcal{S}$, to construct a univalent cartesian category $\mathsf{Str}(\mathcal{S})$.*

▶ **Construction 3.11** (for Problem 3.10)**.** In [31, Section 9.8], it is shown how every standard notion of structure gives rise to a univalent category. The terminal object is given by $(\mathsf{unit}, \mathsf{p_{unit}})$, and the product of $(\mathsf{X}, \mathsf{p_X})$ and $(\mathsf{Y}, \mathsf{p_Y})$ is given by $(\mathsf{X} \times \mathsf{Y}, \mathsf{p_X} \times \mathsf{p_Y})$. ⌟

▶ **Proposition 3.12.** *Let C be a category and let $\mathcal{S}$ be a cartesian notion of structure. Then the type of $\mathsf{Str}(\mathcal{S})$-enrichments for C is equivalent to a structure $\mathsf{hom}_{(x,y)} : \mathsf{P}_{\mathcal{S}}\,(x \to y)$ for all objects $x, y : \mathsf{C}$ such that for all $x, y, z : \mathsf{C}$ we have $\mathsf{H}_{(\mathsf{hom}_{(y,z)}\times\mathsf{hom}_{(x,y)},\mathsf{hom}_{(x,z)})}(\lambda f.\pi_2\,f \cdot \pi_1\,f)$.*

As such, to give a $\mathsf{Str}(\mathcal{S})$-enrichment for C one needs to endow every hom-set of C with an $\mathcal{S}$-structure such that the composition operation is a structure preserving map.

▶ **Example 3.13.** *We have a cartesian notion of structure DCPO of **directed complete partial orders structures** (DCPOs) such that $\mathsf{P_{DCPO}}\,\mathsf{X}$ is the set of DCPOs on X and such that $\mathsf{H}_{(\mathsf{p_X},\mathsf{p_Y})}(f)$ expresses that f is a Scott continuous map. As such, a DCPO-enriched category is given by a category whose hom-sets are directed complete partial orders, and whose composition operation is a Scott-continuous map.*

We also have a cartesian notion of structure DCPPO *of* ***pointed directed complete partial orders structures*** *(DCPPOs) such that* $\mathsf{P}_{\mathsf{DCPPO}}\,\mathsf{X}$ *is the set of DCPPOs on* X *and such that* $\mathsf{H}_{(\mathsf{p_X},\mathsf{p_Y})}(f)$ *expresses that* f *is a Scott continuous map. Hence,* DCPPO*-enriched categories are categories whose hom-sets are pointed directed complete partial orders, and whose composition operation is a Scott-continuous map.*

▶ **Remark 3.14.** *In Example 3.13, we defined a cartesian notion of structure by* pointed *DCPOs and Scott continuous maps without requiring these maps to be strict. For pointed DCPOs and strict Scott continuous maps, one can also define such a structure. However, in applications, one is often interested in a different monoidal structure for pointed DCPOs with strict maps, namely the one given by the smash product. Note that one can construct this symmetric monoidal category constructively [30, Theorem 2.9.1].*

The formalization contains a further extension of Definition 3.9, called a ***structure supporting smash products****, and a proof that every such structure gives rise to a symmetric monoidal closed category. For such structures, the smash product is constructed a quotient of types, and one can instantiate this notion using pointed sets and pointed partial orders.*

4 Image Factorization

We continue our study of univalent enriched categories by proving that every essentially surjective and fully faithful (enriched) functor is an adjoint equivalence. Classically, one would use the axiom of choice to prove this fact: to define the inverse, one needs to pick preimages and those are only guaranteed to be unique up to isomorphism. One can give a constructive proof of this fact if one assumes that the domain of the functor in question is univalent.

The way we approach this result, is via *orthogonal factorization systems* in bicategories. More specifically, we show that the essentially surjective and the fully faithful enriched functors form an orthogonal factorization system [15, Lemma 4.3.5]. From this fact, one directly obtains that every essentially surjective and fully faithful functor is an adjoint equivalence. The proof is similar to how in orthogonal factorization systems in categories the intersection of the left and right class of maps are precisely the isomorphisms.

We start by defining *orthogonal* maps in bicategories.

▶ **Definition 4.1.** *Let* B *be a bicategory and let* $f : x_1 \to x_2$ *and* $g : y_1 \to y_2$ *be 1-cells. Then we say that* f *is* ***orthogonal*** *to* g*, written* $f \perp g$*, if the following diagram of categories is a weak pullback in the bicategory of categories.*

$$\begin{tikzcd} \underline{\mathsf{B}}(x_2, y_1) \arrow[r, "f^{\mathsf{pre}}"] \arrow[d, "g^{\mathsf{post}}"'] & \underline{\mathsf{B}}(x_1, y_1) \arrow[d, "g^{\mathsf{post}}"] \\ \underline{\mathsf{B}}(x_2, y_2) \arrow[r, "f^{\mathsf{pre}}"'] & \underline{\mathsf{B}}(x_1, y_2) \end{tikzcd}$$

where the functors f^{pre} *and* g^{post} *are given by precomposition with* f *and postcomposition with* g *respectively.*

Let us reflect on Definition 4.1. Weak pullbacks of categories are given by iso-comma categories. The objects in the iso-comma category $\mathsf{F}/_{\cong}\mathsf{G}$ of functors $\mathsf{F} : \mathsf{C}_1 \to \mathsf{C}_3$ and $\mathsf{G} : \mathsf{C}_2 \to \mathsf{C}_3$ are given by triples (x, y, f) of objects $x : \mathsf{C}_1$ and $y : \mathsf{C}_2$ together with an isomorphism $f : \mathsf{F}\,x \cong \mathsf{G}\,y$. Note that we have a functor $\mathsf{O}_{(f,g)} : \underline{\mathsf{B}}(x_2, y_1) \to f^{\mathsf{pre}}/_{\cong}g^{\mathsf{post}}$. The functor $\mathsf{O}_{(f,g)}$ maps 1-cells $h : x_2 \to y_1$ to the triple $(h \cdot g, f \cdot h, \alpha_{f,h,g})$ where α is the associator of B. Orthogonality can equivalently be phrased by saying that the functor $\mathsf{O}_{(f,g)}$ is an adjoint equivalence. Essential surjectivity of $\mathsf{O}_{(f,g)}$ says that every square has a diagonal filler as follows.

$$\begin{tikzcd} x_1 \arrow[r, "h_1"] \arrow[d, "f"'] & y_1 \arrow[d, "g"] \\ x_2 \arrow[r, "h_2"'] \arrow[ur, dashed, "l"] & y_2 \end{tikzcd}$$

More concretely, given the diagram above, there is a lift $l : x_2 \to y_1$ making the two triangles commute up to invertible 2-cell. Fully faithfulness of $\mathsf{O}_{(f,g)}$ says that whenever we have two lifts $l_1, l_2 : x_2 \to y_1$ together with 2-cells $\tau_1 : l_1 \cdot g \Rightarrow l_2 \cdot g$ and $\tau_2 : f \cdot l_1 \Rightarrow f \cdot l_2$, we have a unique 2-cell $\zeta : l_1 \Rightarrow l_2$ such that $\zeta \triangleright g = \tau_1$ and $f \triangleleft \zeta = \tau_2$.

▶ **Definition 4.2.** *Let B be a bicategory. An **orthogonal factorization system** on B consists of two classes of maps, which we denote by $\mathcal{L}$ and $\mathcal{R}$, such that*

- *$\mathcal{L}$ and $\mathcal{R}$ are closed under invertible 2-cells;*
- *for all 1-cells f and g such that $\mathcal{L}\, f$ and $\mathcal{R}\, g$, we have $f \perp g$;*
- *for every 1-cell f, we have a factorization $f \cong l \cdot r$ such that $\mathcal{L}\, l$ and $\mathcal{R}\, r$.*

In this section, we are interested in a particular factorization system on $\mathsf{EnrichCat}_{\mathsf{V}}$, which is given by the fully faithful and the essentially surjective enriched functors.

▶ **Definition 4.3.** *Let $\mathcal{F} : \mathcal{E}_1 \to \mathcal{E}_2$ be an enriched functor.*

- *We say that $\mathcal{F}$ is **fully faithful** if for all objects $x, y : \mathcal{E}_1$ the morphism $\mathcal{F}(x, y)$ is an isomorphism.*
- *We say that $\mathcal{F}$ is **essentially surjective** if its underlying functor is essentially surjective. That is to say, for all $y : \mathcal{E}_2$ we have an inhabitant of $\|\sum(x : \mathcal{E}_1), \mathcal{F}\, x \cong y\|$.*
- *We say that $\mathcal{F}$ is a **weak equivalence** if $\mathcal{F}$ is both fully faithful and essentially surjective.*

Every enriched functor can be factorized as an essentially surjective functor followed by a fully faithful functor by taking the *full image*.

▶ **Example 4.4.** *Let $\mathcal{F} : \mathcal{E}_1 \to \mathcal{E}_2$ be an enriched functor. We define a predicate P on the objects of $\mathcal{E}_2$ such that $\mathsf{P}\, y :\equiv \|\sum(x : \mathcal{E}_1), \mathcal{F}\, x \cong y\|$. The **full image** $\mathsf{Im}(\mathcal{F})$ of $\mathcal{F}$ is defined to be the full subcategory of $\mathcal{E}_2$ with respect to P.*

▶ **Proposition 4.5.** *Suppose that we have univalent enriched categories $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, and $\mathcal{E}_4$. If we have enriched functors $\mathcal{F} : \mathcal{E}_1 \to \mathcal{E}_2$ and $\mathcal{G} : \mathcal{E}_3 \to \mathcal{E}_4$ such that $\mathcal{F}$ is essentially surjective and $\mathcal{G}$ is fully faithful, then $\mathcal{F} \perp \mathcal{G}$.*

▶ **Problem 4.6.** *To construct an orthogonal factorization system on $\mathsf{EnrichCat}_{\mathsf{V}}$.*

▶ **Construction 4.7** (for Problem 4.6)**.** The classes $\mathcal{L}$ and $\mathcal{R}$ are given by the essentially surjective and the fully faithful enriched functors respectively. The desired factorization is given by Example 4.4, and the proof of orthogonality is given in Proposition 4.5. It remains to show that essentially surjective and fully faithful enriched functors are closed under enriched natural isomorphisms, and the details for that is given in the formalization. ⌟

From this factorization system, we directly obtain that weak equivalence are actually adjoint equivalences.

▶ **Theorem 4.8.** *Every fully faithful and essentially surjective enriched functor $\mathcal{F} : \mathcal{E}_1 \to \mathcal{E}_2$ is an adjoint equivalence.*

Proof. Suppose that $\mathcal{F} : \mathcal{E}_1 \to \mathcal{E}_2$ is fully faithful and essentially surjective. Consider the following diagram

$$\begin{tikzcd} \mathcal{E}_1 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathcal{F}"'] & \mathcal{E}_1 \arrow[d, "\mathcal{F}"] \\ \mathcal{E}_2 \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "l"] & \mathcal{E}_2 \end{tikzcd}$$

Due to the orthogonality of fully faithful and essentially surjective morphisms, this diagram has a lift l such that both triangles commute up to invertible 2-cell. From this, we get that f is an equivalence, and since equivalences can be refined to adjoint equivalences, f is an adjoint equivalence. ◀

▶ **Remark 4.9.** *For the proof of Theorem 4.8, we require both the domain and codomain of $\mathcal{F}$ to be univalent. This restriction is a consequence of the bicategorical machinery, because we phrase everything in the bicategory* $\mathsf{EnrichCat}_{\mathsf{V}}$ *whose objects are univalent enriched categories. In the case that only the domain of $\mathcal{F}$ is univalent, one could still use the same construction as in Construction 4.7.*

5 The Rezk Completion

The next aspect in our study of univalent enriched categories, is the *enriched Rezk completion*. There are two features to a suitable Rezk completion for enriched categories. First of all, one needs to show that every enriched category is weakly equivalent to a univalent one (Construction 5.4). This construction is similar to the Rezk completion of categories [2]: in both cases, we construct the enriched Rezk completion as the image of the Yoneda embedding. Note that for Construction 5.4 we assume that V is a complete symmetric monoidal closed category to guarantee that we have an enrichment for the desired presheaf category. Second of all, one needs to prove a universal property (Theorem 5.5). This property says that every enriched functor from an enriched category $\mathcal{E}$ to some univalent enriched category can be extended to the Rezk completion of $\mathcal{E}$.

Our construction of the enriched Rezk completion makes use of the *Yoneda lemma* [12]. As such, we first define representable presheaves and the Yoneda embedding, and we prove the Yoneda lemma.

▶ **Definition 5.1.** *Let V be a complete symmetric monoidal closed category, and let $\mathcal{E}$ be an V-enriched category. Given an object $y : \mathcal{E}$, we define the* ***representable functor*** $\mathsf{r}_0(y) : [\mathcal{E}^{\mathsf{op}}, \mathsf{self}(\mathsf{V})]$ *as follows.*

- *For objects $x : \mathcal{E}$, we define $\mathsf{r}_0(y)\, x := \mathcal{E}(x, y)$;*
- *for morphisms $f : x_1 \to x_2$, we define $\mathsf{r}_0(y)\, f$ to be $f^{\mathsf{pre}} : \mathcal{E}(x_2, y) \to \mathcal{E}(x_1, y)$.*

Given a morphism $f : y_1 \to y_2$ in $\mathcal{E}$, we define the ***representable natural transformation*** $\mathsf{r}_1(f) : \mathsf{r}_0(y_1) \to \mathsf{r}_0(y_2)$ *to be $f^{\mathsf{post}} : \mathcal{E}(x, y_1) \to \mathcal{E}(x, y_2)$ for every $x : \mathcal{E}$.*

Finally, the ***enriched Yoneda embedding*** $\mathsf{y}_{\mathcal{E}} : \mathcal{E} \to [\mathcal{E}^{\mathsf{op}}, \mathsf{self}(\mathsf{V})]$ *is defined to be $\mathsf{r}_0(y)$ on objects $y : \mathcal{E}$ and $\mathsf{r}_1(f)$ on morphisms $f : y_1 \to y_2$.*

Note that in Definition 5.1, one also needs to construct V-enrichments for $\mathsf{r}_0(y)$ and $\mathsf{y}_{\mathcal{E}}$, and prove that $\mathsf{r}_1(f)$ is V-enriched. The details for that can be found in [12] and the formalization.

▶ **Proposition 5.2.** *The enriched Yoneda embedding is fully faithful.*

▶ **Problem 5.3.** *Given a category $\mathcal{E}$ enriched over a univalent complete symmetric monoidal category V, to construct a univalent enriched category $\mathcal{R}(\mathcal{E})$ and a weak equivalence $\mathsf{P} : \mathcal{E} \to \mathcal{R}(\mathcal{E})$.*

▶ **Construction 5.4** (for Problem 5.3)**.** We define $\mathcal{R}(\mathcal{E})$ to be the image of the Yoneda embedding $\mathsf{y}_{\mathcal{E}}$. The enriched functor $\mathsf{P} : \mathcal{E} \to \mathcal{R}(\mathcal{E})$ is essentially surjective by construction (Example 4.4). By the Yoneda lemma (Proposition 5.2), P is fully faithful as well. ⌟

Note that Construction 5.4 might increase the universe level. Let us assume that the type of objects of $\mathcal{E}$ and V live in $\mathcal{U}$ and $\mathcal{V}$ respectively. Since objects of $\mathcal{R}(\mathcal{E})$ are enriched presheaves from $\mathcal{E}$ to $\mathsf{self}(\mathsf{V})$ that are in the image of $\mathsf{y}_{\mathcal{E}}$, the type of objects of $\mathcal{R}(\mathcal{E})$ lives in $\mathcal{U} \sqcup \mathcal{V}$. In many examples, $\mathcal{V}$ is a larger universe that $\mathcal{U}$, because we require the category V to have all products indexed by the objects in $\mathcal{E}$.

In addition, we can extend Construction 5.4 to the case where V is not necessarily univalent. To do so, we first take the Rezk completion of monoidal categories [40] of V to obtain a weak equivalence $\mathsf{P} : \mathsf{V} \to \mathcal{R}(\mathsf{V})$. Since P is fully faithful, it preserves underlying categories. Hence, if we have a category $\mathcal{E}$ enriched over V, we obtain a category $\mathsf{P}^*(\mathcal{E})$ enriched over $\mathcal{R}(\mathsf{V})$ using Example 3.7. Then we use Construction 5.4 to obtain the Rezk completion.

Finally, we verify the universal property of the Rezk completion. This property is formulated using the precomposition functor.

▶ **Theorem 5.5.** *Suppose that we have a enriched functor $\mathcal{F} : \mathcal{E}_1 \to \mathcal{E}_2$ and a univalent enriched category $\mathcal{E}_3$. Then the precomposition functor $\mathcal{F}^{\mathsf{pre}} : [\mathcal{E}_2, \mathcal{E}_3] \to [\mathcal{E}_1, \mathcal{E}_3]$ is an adjoint equivalence of categories.*

To verify Theorem 5.5, we use that $\mathcal{E}_3$ is univalent. This implies that the categories $[\mathcal{E}_2, \mathcal{E}_3]$ and $[\mathcal{E}_1, \mathcal{E}_3]$ are both univalent, and thus it suffices to check that $\mathcal{F}^{\mathsf{pre}} : [\mathcal{E}_2, \mathcal{E}_3] \to [\mathcal{E}_1, \mathcal{E}_3]$ is essentially surjective and fully faithful. The proofs of Lemmata 5.6 and 5.7 have some overlap with the ordinary categorical case [2, Theorem 8.4]. However, here we must also check that the obtained functors and natural transformations actually are enriched.

▶ **Lemma 5.6.** *Suppose that we have a enriched functor $\mathcal{F} : \mathcal{E}_1 \to \mathcal{E}_2$ and a univalent enriched category $\mathcal{E}_3$. The functor $\mathcal{F}^{\mathsf{pre}} : [\mathcal{E}_2, \mathcal{E}_3] \to [\mathcal{E}_1, \mathcal{E}_3]$ is fully faithful.*

Proof. The proof that $\mathcal{F}^{\mathsf{pre}} : [\mathcal{E}_2, \mathcal{E}_3] \to [\mathcal{E}_1, \mathcal{E}_3]$ is faithful, is in essence the same as for ordinary categories [2, Lemma 8.1], so we only show that $\mathcal{F}^{\mathsf{pre}}$ is full. Let $\mathcal{G}_1, \mathcal{G}_2 : [\mathcal{E}_2, \mathcal{E}_3]$ be two enriched functors, and suppose that we have an enriched transformation $\tau : \mathcal{F} \cdot \mathcal{G}_1 \Rightarrow \mathcal{F} \cdot \mathcal{G}_2$. We show how to construct the desired enriched natural transformation $\theta : \mathcal{G}_1 \Rightarrow \mathcal{G}_2$.

For all objects $x : \mathcal{E}_2$ the following type is contractible.

$$\sum (f : \mathcal{G}_1\, x \to \mathcal{G}_2\, x), \prod (w : \mathcal{E}_1)(i : \mathcal{F}\, w \cong x), \tau\, w \cdot \mathcal{G}_2\, i = \mathcal{G}_1\, i \cdot f$$

The contractibility of this type follows from our assumption that $\mathcal{F}$ is essentially surjective. From this, we obtain the data of the the desired transformation θ. The fact that θ is V-enriched is shown by using Equation (1) and the fact that $\mathcal{F}$ is essentially surjective. ◀

▶ **Lemma 5.7.** *The functor $\mathcal{F}^{\mathsf{pre}} : [\mathcal{E}_2, \mathcal{E}_3] \to [\mathcal{E}_1, \mathcal{E}_3]$ is essentially surjective.*

Proof. Suppose that we have an enriched functor $\mathcal{G} : [\mathcal{E}_1, \mathcal{E}_3]$. We only demonstrate how to construct the desired enriched functor $\mathcal{H} : [\mathcal{E}_2, \mathcal{E}_3]$.

Suppose that we have $x : \mathcal{E}_2$. Then there is a unique object $y : \mathcal{E}_3$ and function $\varphi : \prod (w : \mathcal{E}_1)(i : \mathcal{F}\, x \cong y), \mathcal{G}\, w \cong y$ such that for all objects $w_1, w_2 : \mathcal{E}_1$, isomorphisms $i_1 : \mathcal{F}\, w_1 \cong x$ and $i_2 : \mathcal{F}\, w_2 \cong x$, and morphisms $k : w_1 \to w_2$ satisfying $\mathcal{F}\, k \cdot i_2 = i_1$, we have $\mathcal{G}\,k \cdot \varphi\,w_2\,i_2 = \varphi\,w_1\,i_1$. Uniqueness follows from the fact that $\mathcal{F}$ is fully faithful, and the desired element is constructed by using that $\mathcal{F}$ is essentially surjective. One can show that the obtained action on objects gives rise to a functor H from the underlying category of $\mathcal{E}_2$ to that of $\mathcal{E}_3$. We also have isomorphisms $\varphi\,w\,i : \mathcal{G}\,w \cong H\,x$ for all $w : \mathcal{E}_1$ and $i : \mathcal{F}\,x \cong y$.

Next we construct an enrichment for this functor. Suppose, that we have two objects $x, y : \mathcal{E}_2$. Then there is a unique morphism $f : \mathcal{E}_2(x, y) \to \mathcal{E}_3(H\,x, H\,y)$ in V such that for all objects $w_1, w_2 : \mathcal{E}_2$ and isomorphisms $i_1 : \mathcal{F}\,w_1 \cong x$ and $i_2 : \mathcal{F}\,w_2 \cong x$, f is equal to the following composition of morphisms

$$\begin{tikzcd}
\mathcal{E}_2(x,y) \arrow[r, "i_1^{\mathsf{pre}}"] & \mathcal{E}_2(\mathcal{F}\,w_1, y) \arrow[r, "(i_2^{-1})^{\mathsf{pre}}"] & \mathcal{E}_2(\mathcal{F}\,w_1, \mathcal{F}\,w_2) \arrow[r, "(\mathcal{F})^{-1}"] & \mathcal{E}_1(w_1, w_2) \arrow[dlll, "\mathcal{G}"] \\
\mathcal{E}_3(\mathcal{G}\,w_1, \mathcal{G}\,w_2) \arrow[r, "((\varphi\,w_1\,i_1)^{-1})^{\mathsf{pre}}"'] & \mathcal{E}_3(H\,x, \mathcal{G}\,w_2) \arrow[r, "\varphi\,w_2\,i_2^{\mathsf{post}}"'] & \mathcal{E}_3(H\,x, H\,y) &
\end{tikzcd}$$

This follows from the fact that $\mathcal{F}$ is a weak equivalence. As such, we get the desired enriched functor $\mathcal{H} : [\mathcal{E}_2, \mathcal{E}_3]$. ◀

6 Enriched Monads

We end our study of univalent enriched categories by looking at enriched monads. More specifically, we discuss Kleisli objects (Construction 6.9) in the bicategory of enriched categories. At first glance, it might not seem that univalence plays an interesting role, but upon closer look, this question is rather subtle.

Usually, the Kleisli category of a monad T on a category C is defined to be the category whose objects are objects of C and whose morphisms from x to y are morphisms $x \to \mathsf{T}\,y$ in C. We denote this category by $\mathsf{K}(\mathsf{T})$. In general, this category is not univalent (for example the constant monad on the unit set). This situation can be rectified by defining the Kleisli category in a slightly different way [4], namely as the image of the free algebra functor from C to the Eilenberg-Moore category $\mathsf{EM}(\mathsf{T})$ of T. The resulting univalent category is denoted by $\mathsf{Kleisli}(\mathsf{T})$. To derive the usual theorems about Kleisli categories, one can instantiate the formal theory of monads [13, 28, 36], meaning that it suffices to prove the universal property for Kleisli objects. Proving the desired universal property is a nice exercise using the universal property of the Rezk completion (Theorem 5.5).

The key notion of this section, enriched monads, can be defined concisely as monads internal to $\mathsf{EnrichCat}_{\mathsf{V}}$. Recall that monads in bicategories are defined as follows.

▶ **Definition 6.1.** *Let* B *be a bicategory. A* ***monad*** m ***in*** B *is given by*

- *an object* $\mathsf{ob}_m : \mathsf{B}$*;*
- *a 1-cell* $\mathsf{mor}_m : \mathsf{ob}_m \to \mathsf{ob}_m$*;*
- *a 2-cell* $\eta_m : \mathrm{id}_{\mathsf{ob}_m} \Rightarrow m$*;*
- *a 2-cell* $\mu_m : m \cdot m \Rightarrow m$.

such that the following diagrams commute.

$$\begin{tikzcd}
\mathsf{mor}_m \arrow[r, Rightarrow, "\rho^{-1}"] \arrow[rrd, Rightarrow, "\mathrm{id}"'] & \mathsf{mor}_m \cdot \mathrm{id}\,x \arrow[r, Rightarrow, "\mathsf{mor}_m \triangleleft \eta_m"] & \mathsf{mor}_m \cdot \mathsf{mor}_m \arrow[d, Rightarrow, "\mu_m"] & \mathrm{id} \cdot \mathsf{mor}_m \arrow[l, Rightarrow, "\eta_m \triangleright \mathsf{mor}_m"'] & \mathsf{mor}_m \arrow[l, Rightarrow, "\lambda^{-1}"'] \arrow[lld, Rightarrow, "\mathrm{id}"] \\
 & & \mathsf{mor}_m & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{mor}_m \cdot (\mathsf{mor}_m \cdot \mathsf{mor}_m) \arrow[rr, Rightarrow, "\mathsf{mor}_m \lhd \mu_m"] \arrow[d, Rightarrow, "\alpha"'] & & \mathsf{mor}_m \cdot \mathsf{mor}_m \arrow[d, Rightarrow, "\mu_m"] \\
(\mathsf{mor}_m \cdot \mathsf{mor}_m) \cdot \mathsf{mor}_m \arrow[r, Rightarrow, "\mu_m \rhd \mathsf{mor}_m"'] & \mathsf{mor}_m \cdot \mathsf{mor}_m \arrow[r, Rightarrow, "\mu_m"'] & \mathsf{mor}_m
\end{tikzcd}$$

Here λ and ρ are the left and right unitors of B*, and α is the associator of* B*.*

For enriched categories, one can further unfold this definition and phrase it in terms of enrichments. This results in the notion of *enrichments for monads.*

▶ **Definition 6.2.** *Suppose that we have a category* C*, a monad* T *on* C*, and a* V*-enrichment for* C*. Then a* V***-enrichment** for* T *consists of a* V*-enrichment for the endofunctor* T *such that the unit* η_T *and multiplication* μ_T *are* V*-enriched natural transformations.*

When we say *enriched monad*, we mean a monad together with an enrichment. In the remainder of this section, we are concerned with *Kleisli objects* in the bicategory of enriched categories. To define Kleisli objects, we first define their cocones. Note that for these definitions, we talk about arbitrary bicategories B and monads internal to B.

▶ **Definition 6.3.** *Let* B *be a bicategory and let m be a monad in* B*. A **Kleisli cocone** k for m in* B *consists of an object* $\mathsf{ob}_k : \mathsf{B}$*, a 1-cell* $\mathsf{mor}_k : \mathsf{ob}_m \to \mathsf{ob}_k$*, and a 2-cell* $\mathsf{cell}_k : m \cdot \mathsf{mor}_k \Rightarrow \mathsf{mor}_k$ *such that the following diagrams commute.*

$$\begin{tikzcd}
\mathrm{id}\,\mathsf{ob}_m \cdot \mathsf{mor}_k \arrow[r, Rightarrow, "\eta_m \rhd \mathsf{mor}_k"] \arrow[dr, Rightarrow, "\lambda"'] & m \cdot \mathsf{mor}_k \arrow[d, Rightarrow, "\mathsf{cell}_k"] \\
& \mathsf{mor}_k
\end{tikzcd}$$

$$\begin{tikzcd}
(m \cdot m) \cdot \mathsf{mor}_k \arrow[r, Rightarrow, "\alpha^{-1}"] \arrow[d, Rightarrow, "\mu_m \rhd \mathsf{mor}_k"'] & m \cdot (m \cdot \mathsf{mor}_k) \arrow[r, Rightarrow, "m \lhd \mathsf{cell}_k"] & m \cdot \mathsf{mor}_k \arrow[d, Rightarrow, "\mathsf{cell}_k"] \\
m \cdot \mathsf{mor}_k \arrow[rr, Rightarrow, "\mathsf{cell}_k"'] & & \mathsf{mor}_k
\end{tikzcd}$$

▶ **Definition 6.4.** *A Kleisli cocone k is **universal** if the following conditions are satisfied.*

- *For every Kleisli cocone q there is a 1-cell* $\mathsf{Kl}_\mathsf{mor}(q) : \mathsf{ob}_k \to \mathsf{ob}_q$ *and an invertible 2-cell* $\mathsf{Kl}_\mathsf{com}(q) : \mathsf{mor}_k \cdot \mathsf{Kl}_\mathsf{mor}(q) \Rightarrow \mathsf{mor}_q$ *such that the following diagram commutes.*

$$\begin{tikzcd}
m \cdot (\mathsf{mor}_k \cdot \mathsf{Kl}_\mathsf{mor}(q)) \arrow[rr, Rightarrow, "m \lhd \mathsf{Kl}_\mathsf{com}(q)"] \arrow[d, Rightarrow, "\alpha"'] & & m \cdot \mathsf{mor}_q \arrow[d, Rightarrow, "\mathsf{cell}_q"] \\
(m \cdot \mathsf{mor}_k) \cdot \mathsf{Kl}_\mathsf{mor}(q) \arrow[r, Rightarrow, "\mathsf{cell}_k \rhd \mathsf{Kl}_\mathsf{mor}(q)"'] & \mathsf{mor}_k \cdot \mathsf{Kl}_\mathsf{mor}(q) \arrow[r, Rightarrow, "\mathsf{Kl}_\mathsf{com}(q)"'] & \mathsf{mor}_q
\end{tikzcd}$$

- *Suppose that we have an object x :* B*, two 1-cells* $g_1, g_2 : \mathsf{ob}_k \to x$*, and a 2-cell* $\tau : \mathsf{mor}_k \cdot g_1 \Rightarrow \mathsf{mor}_k \cdot g_2$ *such that the following diagram commutes.*

$$\begin{tikzcd}
m \cdot (\mathsf{mor}_k \cdot g_1) \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "m \lhd \tau"'] & (m \cdot \mathsf{mor}_k) \cdot g_1 \arrow[r, Rightarrow, "\mathsf{cell}_k \rhd g_1"] & \mathsf{mor}_k \cdot g_1 \arrow[d, Rightarrow, "\tau"] \\
m \cdot (\mathsf{mor}_k \cdot g_2) \arrow[r, Rightarrow, "\alpha"'] & (m \cdot \mathsf{mor}_k) \cdot g_2 \arrow[r, Rightarrow, "\mathsf{cell}_k \rhd g_2"'] & \mathsf{mor}_k \cdot g_2
\end{tikzcd}$$

Then there is a unique 2-cell $\mathsf{Kl}_\mathsf{cell}(\tau) : g_1 \Rightarrow g_2$ *such that* $\mathsf{cell}_k \lhd \mathsf{Kl}_\mathsf{cell}(\tau) = \tau$*.*

*We say that a bicategory **has Kleisli objects** if there is a universal Kleisli cocone for every monad m.*

As discussed before, there are multiple ways to define Kleisli categories. We first define an enrichment for $\mathsf{K}(\mathsf{T})$.

▶ **Example 6.5.** *Let* T *be an enriched monad on an enriched category* $\mathcal{E}$. *We define a* V***-enrichment*** $\mathsf{K}_\mathsf{e}(\mathsf{T})$ ***for*** $\mathsf{K}(\mathsf{T})$ *as follows.*

- *We define* $\mathsf{K}_\mathsf{e}(\mathsf{T})(x, y)$ *to be* $\mathcal{E}(x, \mathsf{T}\, y)$.
- *We define* $\mathsf{id}^\mathsf{e}(x)$ *to be* $\overrightarrow{\eta_\mathsf{T}\, x}$.
- *We define* $\mathsf{comp}(x, y, z)$ *as the following composition of morphisms.*

$$\mathcal{E}(y, \mathsf{T}\, z) \otimes \mathcal{E}(x, \mathsf{T}\, y) \xrightarrow{\mathsf{T} \otimes \mathrm{id}} \mathcal{E}(\mathsf{T}\, y, \mathsf{T}\,(\mathsf{T}\, z)) \otimes \mathcal{E}(x, \mathsf{T}\, y) \xrightarrow{\mathsf{comp}} \mathcal{E}(x, \mathsf{T}\,(\mathsf{T}\, z)) \xrightarrow{(\mu_\mathsf{T}\, z)^{\mathsf{post}}} \mathcal{E}(x, \mathsf{T}\, z)$$

The operations $\overrightarrow{f}$ *and* $\overleftarrow{f}$ *in* $\mathsf{K}_\mathsf{e}(\mathsf{T})$ *are inherited from* $\mathcal{E}$.

Next we define an enrichment for $\mathsf{Kleisli}(\mathsf{T})$. Since $\mathsf{Kleisli}(\mathsf{T})$ is defined as a full subcategory of the Eilenberg-Moore category $\mathsf{EM}(\mathsf{T})$, we define an enrichment for $\mathsf{EM}(\mathsf{T})$ first.

▶ **Example 6.6.** *Suppose that* V *has equalizers, and let* T *be an enriched monad on an enriched category* $\mathcal{E}$. *Note that we can define the Eilenberg-Moore category of* T *as a full subcategory of* $\mathsf{Dialg}(\mathsf{T}, \mathrm{id})$. *By Examples 3.2 and 3.4 we obtain the desired* V***-enrichment*** $\mathsf{EM}_\mathsf{e}(\mathsf{T})$ ***on*** $\mathsf{EM}(\mathsf{T})$.

Using Example 6.6 one can show that $\mathsf{EnrichCat}_\mathsf{V}$ has Eilenberg-Moore objects. In general, we have an enriched functor $\mathsf{FreeAlg}_\mathsf{T} : \mathcal{E} \to \mathsf{EM}_\mathsf{e}(\mathsf{T})$. This functor sends every object x to the free algebra $\mathsf{T}\, x$. Now we define an enrichment for $\mathsf{Kleisli}(\mathsf{T})$.

▶ **Example 6.7.** *Suppose that* V *is a monoidal category with equalizers, and let* T *be an enriched monad on an enriched category* $\mathcal{E}$. *Note that* $\mathsf{Kleisli}(\mathsf{T})$ *is constructed as a full subcategory of the Eilenberg-Moore category, and thus by Example 6.6 we obtain the* V***-enrichment*** $\mathsf{Kleisli}_\mathsf{e}(\mathsf{T})$ ***for*** $\mathsf{Kleisli}(\mathsf{T})$.

The category defined in Example 6.7 is univalent if we assume $\mathcal{E}$ to be univalent. This is because the Eilenberg-Moore category of a monad on a univalent category is always univalent and because univalence is preserved under full subcategories. In addition, note that in Example 6.7 we assume that V has equalizers, whereas in Example 6.5, we do not.

We finish this section by showing that $\mathsf{Kleisli}_\mathsf{e}(\mathsf{T})$ satisfies the required universal property. The main idea behind the proof is that we have a weak equivalence $\mathsf{incl}_\mathsf{T} : \mathsf{K}_\mathsf{e}(\mathsf{T}) \to \mathsf{Kleisli}_\mathsf{e}(\mathsf{T})$, and this weak equivalence allows use to instantiate Theorem 5.5.

▶ **Problem 6.8.** *Given a monoidal category* V *with equalizers, to construct Kleisli objects in the bicategory* $\mathsf{EnrichCat}_\mathsf{V}$.

▶ **Construction 6.9** (for Problem 6.8)**.** Given an enriched monad T on $\mathcal{E}$, the Kleisli object of T in $\mathsf{EnrichCat}_\mathsf{V}$ is given by $\mathsf{Kleisli}_\mathsf{e}(\mathsf{T})$. The main work lies in verifying the universal property. This check happens in three steps.

First, we define a weak equivalence $\mathsf{incl}_\mathsf{T} : \mathsf{K}_\mathsf{e}(\mathsf{T}) \to \mathsf{Kleisli}_\mathsf{e}(\mathsf{T})$. This enriched functor sends every object x to the free algebra on x. The action on morphisms is given by the following composition

$$\mathcal{E}(x, \mathsf{T}\, y) \xrightarrow{\mathsf{T}} \mathcal{E}(\mathsf{T}\, x, \mathsf{T}\,(\mathsf{T}\, y)) \xrightarrow{(\mu_y)^{\mathsf{post}}} \mathcal{E}(\mathsf{T}\, x, \mathsf{T}\, y)$$

Second, we check that $\mathsf{K}_\mathsf{e}(\mathsf{T})$ gives rise to Kleisli objects in the bicategory of (not necessarily univalent) enriched categories. For this, one can use the same proof as used, for example, by Street [28, Theorem 15].

Third, we conclude that the universal property also holds for $\mathsf{Kleisli_e}(\mathsf{T})$, and we only show how to construct 1-cells arising from the mapping property. Suppose, that we have some Kleisli cocone q in $\mathsf{EnrichCat_V}$. We get an enriched functor $\mathcal{F} : \mathsf{K_e}(\mathsf{T}) \to \mathsf{ob}_q$. From Theorem 5.5, we get the desired 1-cell $\tilde{\mathcal{F}} : \mathsf{Kleisli_e}(\mathsf{T}) \to \mathsf{ob}_q$. ⌟

Note the similarities between Construction 6.9 and the construction of Kleisli objects for univalent categories [36, Construction 6.10].

7 Conclusion

In this paper, we studied univalent enriched categories, and we discussed several aspects of their study. Our notion of univalent enriched category was based on enrichments, and we viewed enriched categories as a category together with an enrichment. First, we proved a structure identity principle for univalent enriched categories, which we formulated using univalent bicategories. The proof used displayed bicategories. Second, we showed that all weak equivalences between univalent enriched categories are adjoint equivalences. Here we made use of orthogonal factorization systems. Third, we discussed the Rezk completion of enriched categories, which we constructed using the Yoneda lemma. We also used the Rezk completion to construct Kleisli objects in the bicategory of univalent enriched categories.

Along the way, we saw a couple of interesting points where univalence interacted with enrichment. When we defined the change-of-base operation in Example 3.7, we restricted ourselves to lax functors that preserve underlying categories. This was to guarantee that the resulting category would remain univalent. In addition, we assumed that the monoidal category V has equalizers in the construction of the univalent Kleisli category (Example 6.7).

There are several ways to extend the results in this paper. A wide variety of notions in category theory can be defined internally to a bicategory. However, for enriched categories, these internal notions are not always the correct ones. For example, the notion of a fully faithful 1-cell can be defined internally to a bicategory using a representable definition, but the obtained notion does not correspond to the one given in Definition 4.3. To obtain the desired notions, one could use the theory of equipments [39], and one interesting extension of this work would be to develop the equipment of enriched categories. Such work would build forth upon recent work on univalent double (bi)categories [26, 37, 32]. Another interesting extension would be formalizing applications of enriched categories, such as models of the enriched effect calculus [7] or enriched profunctor optics [6].

References

1 Benedikt Ahrens, Dan Frumin, Marco Maggesi, Niccolò Veltri, and Niels van der Weide. Bicategories in univalent foundations. *Mathematical Structures in Computer Science*, pages 1–38, 2022. `doi:10.1017/S0960129522000032`.

2 Benedikt Ahrens, Krzysztof Kapulkin, and Michael Shulman. Univalent categories and the Rezk completion. *Mathematical Structures in Computer Science*, 25:1010–1039, 2015. `doi:10.1017/S0960129514000486`.

3 Benedikt Ahrens, Ralph Matthes, Niels van der Weide, and Kobe Wullaert. Displayed monoidal categories for the semantics of linear logic. In Amin Timany, Dmitriy Traytel, Brigitte Pientka, and Sandrine Blazy, editors, *Proceedings of the 13th ACM SIGPLAN International Conference on Certified Programs and Proofs, CPP 2024, London, UK, January 15-16, 2024*, pages 260–273. ACM, 2024. `doi:10.1145/3636501.3636956`.

4 Benedikt Ahrens, Paige Randall North, Michael Shulman, and Dimitris Tsementzis. The univalence principle. *CoRR*, abs/2102.06275, 2021. `arXiv:2102.06275`.

5 Alexander Campbell. Skew-enriched categories. *Appl. Categ. Structures*, 26(3):597–615, 2018. doi:10.1007/s10485-017-9504-0.

6 Bryce Clarke, Derek Elkins, Jeremy Gibbons, Fosco Loregian, Bartosz Milewski, Emily Pillmore, and Mario Román. Profunctor Optics, a Categorical Update. *Compositionality*, 6:1, February 2024. doi:10.32408/compositionality-6-1.

7 Jeff Egger, Rasmus Ejlers Møgelberg, and Alex Simpson. The enriched effect calculus: syntax and semantics. *J. Log. Comput.*, 24(3):615–654, 2014. doi:10.1093/logcom/exs025.

8 David Gepner and Rune Haugseng. Enriched ∞-categories via non-symmetric ∞-operads. *Adv. Math.*, 279:575–716, 2015. doi:10.1016/j.aim.2015.02.007.

9 Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition. doi:10.1007/978-3-0346-0189-4.

10 Jason Z. S. Hu and Jacques Carette. Formalizing category theory in Agda. In Catalin Hritcu and Andrei Popescu, editors, *CPP '21: 10th ACM SIGPLAN International Conference on Certified Programs and Proofs, Virtual Event, Denmark, January 17-19, 2021*, pages 327–342. ACM, 2021. doi:10.1145/3437992.3439922.

11 Krzysztof Kapulkin and Peter LeFanu Lumsdaine. The Simplicial Model of Univalent Foundations (after Voevodsky). *Journal of the European Mathematical Society*, 23(6):2071–2126, 2021. doi:10.4171/JEMS/1050.

12 Max Kelly. *Basic concepts of enriched category theory*, volume 64. CUP Archive, 1982.

13 Stephen Lack and Ross Street. The formal theory of monads. II. *J. Pure Appl. Algebra*, 175(1-3):243–265, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly. doi:10.1016/S0022-4049(02)00137-8.

14 Tom Leinster. A survey of definitions of n-category. *Theory Appl. Categ.*, 10:1–70, 2002.

15 Fosco Loregian and Emily Riehl. Categorical notions of fibration. *Expo. Math.*, 38(4):496–514, 2020. doi:10.1016/j.exmath.2019.02.004.

16 Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.

17 Per Martin-Löf. *Intuitionistic type theory*, volume 1 of *Studies in proof theory*. Bibliopolis, 1984.

18 The mathlib Community. The Lean mathematical library. In Jasmin Blanchette and Catalin Hritcu, editors, *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs, CPP 2020, New Orleans, LA, USA, January 20-21, 2020*, pages 367–381. ACM, 2020. doi:10.1145/3372885.3373824.

19 Dylan McDermott and Alan Mycroft. Galois connecting call-by-value and call-by-name. In Amy P. Felty, editor, *7th International Conference on Formal Structures for Computation and Deduction, FSCD 2022, August 2-5, 2022, Haifa, Israel*, volume 228 of *LIPIcs*, pages 32:1–32:19. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPICS.FSCD.2022.32.

20 Dylan McDermott and Tarmo Uustalu. What makes a strong monad? In Jeremy Gibbons and Max S. New, editors, *Proceedings Ninth Workshop on Mathematically Structured Functional Programming, MSFP@ETAPS 2022, Munich, Germany, 2nd April 2022*, volume 360 of *EPTCS*, pages 113–133, 2022. doi:10.4204/EPTCS.360.6.

21 Eugenio Moggi. Computational lambda-calculus and monads. In *Proceedings of the Fourth Annual Symposium on Logic in Computer Science (LICS '89), Pacific Grove, California, USA, June 5-8, 1989*, pages 14–23. IEEE Computer Society, 1989. doi:10.1109/LICS.1989.39155.

22 Ulf Norell. Dependently typed programming in Agda. In *Proceedings of the 4th international workshop on Types in language design and implementation*, pages 1–2, 2009.

23 Gordon Plotkin and John Power. Notions of computation determine monads. In *International Conference on Foundations of Software Science and Computation Structures*, pages 342–356. Springer, 2002.

24 Gordon Plotkin and John Power. Algebraic operations and generic effects. *Applied categorical structures*, 11(1):69–94, 2003.

25 John Power. Models for the computational lambda-calculus. In Ted Hurley, Mícheál Mac an Airchinnigh, Michel P. Schellekens, and Anthony Karel Seda, editors, *First Irish Conference on the Mathematical Foundations of Computer Science and Information Technology, MFCSIT 2000, Cork, Ireland, July 20-21, 2000*, volume 40 of *Electronic Notes in Theoretical Computer Science*, pages 288–301. Elsevier, 2000. doi:10.1016/S1571-0661(05)80056-8.

26 Nima Rasekh, Niels van der Weide, Benedikt Ahrens, and Paige Randall North. Insights From Univalent Foundations: A Case Study Using Double Categories. *CoRR*, abs/2402.05265, 2024. doi:10.48550/arXiv.2402.05265.

27 Egbert Rijke. Introduction to homotopy type theory. *arXiv preprint arXiv:2212.11082*, 2022.

28 Ross Street. The formal theory of monads. *J. Pure Appl. Algebra*, 2(2):149–168, 1972. doi:10.1016/0022-4049(72)90019-9.

29 The Coq Development Team. The Coq Proof Assistant, September 2022. doi:10.5281/zenodo.7313584.

30 Christopher Francis Townsend. *Preframe techniques in constructive locale theory*. PhD thesis, Imperial College London (University of London), 1996.

31 The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. https://homotopytypetheory.org/book, Institute for Advanced Study, 2013.

32 Dominic Verity. Enriched categories, internal categories and change of base. *Repr. Theory Appl. Categ.*, 20:1–266, 2011.

33 Vladimir Voevodsky, Benedikt Ahrens, Daniel Grayson, et al. Unimath — a computer-checked library of univalent mathematics. available at http://unimath.org. doi:10.5281/zenodo.10849216.

34 Mitchell Wand. Fixed-point constructions in order-enriched categories. *Theor. Comput. Sci.*, 8:13–30, 1979. doi:10.1016/0304-3975(79)90053-7.

35 Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994. doi:10.1017/CBO9781139644136.

36 Niels van der Weide. The formal theory of monads, univalently. In Marco Gaboardi and Femke van Raamsdonk, editors, *8th International Conference on Formal Structures for Computation and Deduction, FSCD 2023, July 3-6, 2023, Rome, Italy*, volume 260 of *LIPIcs*, pages 6:1–6:23. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2023. doi:10.4230/LIPICS.FSCD.2023.6.

37 Niels van der Weide, Nima Rasekh, Benedikt Ahrens, and Paige Randall North. Univalent double categories. In Amin Timany, Dmitriy Traytel, Brigitte Pientka, and Sandrine Blazy, editors, *Proceedings of the 13th ACM SIGPLAN International Conference on Certified Programs and Proofs, CPP 2024, London, UK, January 15-16, 2024*, pages 246–259. ACM, 2024. doi:10.1145/3636501.3636955.

38 John Wiegley. category-theory: Category Theory in Coq, 2023. URL: https://github.com/jwiegley/category-theory.

39 R. J. Wood. Abstract pro arrows I. *Cahiers de topologie et géométrie différentielle*, 23(3):279–290, 1982.

40 Kobe Wullaert, Ralph Matthes, and Benedikt Ahrens. Univalent monoidal categories. In Delia Kesner and Pierre-Marie Pédrot, editors, *28th International Conference on Types for Proofs and Programs, TYPES 2022, June 20-25, 2022, LS2N, University of Nantes, France*, volume 269 of *LIPIcs*, pages 15:1–15:21. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPICS.TYPES.2022.15.