Laplace Distributors and Laplace Transformations for Differential Categories

Marie Kerjean
CNRS, Université Sorbonne Paris Nord, France

Jean-Simon Pacaud Lemay
School of Mathematical and Physical Sciences, Macquarie University, Sydney, Australia

Abstract

In a differential category and in Differential Linear Logic, the exponential conjunction ! admits structural maps, characterizing quantitative operations and symmetric co-structural maps, characterizing differentiation. In this paper, we introduce the notion of a Laplace distributor, which is an extra structural map which distributes the linear negation operation $(\_)^*$ over ! and transforms the co-structural rules into the structural rules. Laplace distributors are directly inspired by the well-known Laplace transform, which is all-important in numerical analysis. In the star-autonomous setting, a Laplace distributor induces a natural transformation from ! to the exponential disjunction ?, which we then call a Laplace transformation. According to its semantics, we show that Laplace distributors correspond precisely to the notion of a generalized exponential function e^x on the monoidal unit. We also show that many well-known and important examples have a Laplace distributor/transformation, including (weighted) relations, finiteness spaces, Köthe spaces, and convenient vector spaces.

2012 ACM Subject Classification Theory of computation → Categorical semantics; Theory of computation → Denotational semantics; Theory of computation → Linear logic

Keywords and phrases Differential Categories, Differential Linear Logic, Laplace Distributor, Laplace Transformation, Exponential Function

Digital Object Identifier 10.4230/LIPIcs.FSCD.2024.9

Funding *Marie Kerjean*: Part of the ANR DIFFERENCE # ANR-20-CE48-0002 and the ANR NuSCAP # ANR-20-CE48-0014.
Jean-Simon Pacaud Lemay: ARC DECRA (DE230100303) & AFOSR (FA9550-24-1-0008)

Acknowledgements We are grateful to Yoann Dabrowski for enriching discussions on Laplace and Fourier transformations.

1 Introduction

Differential Linear Logic (DiLL) [11], introduced by Ehrhard and Regnier [12], introduces the concept of differentiation in Linear Logic (LL), as introduced by Girard [15], by symmetrizing three out of the four rules for the aptly called exponential connective !. So LL features four exponential structural rules which dictate the use of $!A$; they are: the weakening rule w, the contraction rule c, the dereliction rule d, and the promotion rule P[1].

$$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\,\mathsf{w} \quad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,\mathsf{c} \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,\mathsf{d} \quad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,\mathsf{P}$$

It is worth mentioning that the promotion rule can be equivalently replaced by two rules: the functorial promotion rule $!_{\mathsf{f}}$ and the digging rule p.

$$\frac{\Gamma \vdash A}{!\Gamma \vdash !A}\,!_{\mathsf{f}} \quad \frac{\Gamma, !!A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,\mathsf{p}$$

[1] These rules are presented with bilateral sequent for simplicity, but they could also be made monolateral by using the exponential disjunction ?, which is the dual of !.

© Marie Kerjean and Jean-Simon Pacaud Lemay;
licensed under Creative Commons License CC-BY 4.0

DiLL then adds the co-structural rules which are the co-weakening rule $\overline{\mathsf{w}}$, the co-contraction rule $\overline{\mathsf{c}}$, and the co-dereliction rule $\overline{\mathsf{d}}$.

$$\frac{\vdash}{\vdash !A}\ \overline{\mathsf{w}} \quad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\ \overline{\mathsf{c}} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \overline{\mathsf{d}}$$

The co-dereliction rule $\overline{\mathsf{d}}$ expresses differentiation, while the co-contraction rule $\overline{\mathsf{c}}$ and the co-weakening rule $\overline{\mathsf{w}}$ are necessary for cut-elimination purposes. This beautifully results in a symmetry between the structural rules and co-structural rules, that has however never been properly explained. In this paper, we explain this symmetry using the *Laplace transform*.

1.1 Differentiation on proofs

Before diving into the Laplace transform and its interpretation in categorical models of DiLL, let us give more intuitions on the co-structural rules of DiLL. The core intuition of LL is that a proof of a sequent $A \vdash B$ will be a *linear* proof, making use of A exactly once and not allowing contraction nor weakening on A. This is opposed to a proof of $!A \vdash B$, which can make a non-linear usage of A by using contraction or weakening. The basic rule of LL is that you can forget about linearity. Hence, the dereliction rule d transforms a linear proof into a non-linear proof, which intuitively is done so by just forgetting about the linearity property. DiLL takes the reverse path by introducing a co-dereliction rule $\overline{\mathsf{d}}$, which, after a cut, allows the transformation of a non-linear proof $!A \vdash B$ into a linear proof $A \vdash B$. From a semantical point of view, linearizing a non-linear function (which interprets a proof) is done so via differentiation. This analogy is made precise by introducing new cut-elimination rules between $\overline{\mathsf{d}}$ and structural rules. The cut-elimination between d and $\overline{\mathsf{d}}$ results in a cut between their premises, and this represents the fact that differentiating at 0 a linear function returns the same linear function. The cut-elimination between promotion p and $\overline{\mathsf{d}}$ is more intricate and uses $\overline{\mathsf{c}}$ and $\overline{\mathsf{w}}$: it represents the chain rule, which is the formula expressing how to differentiate a composition of functions.

Rules of DiLL can also be understood through the notions of functions and *distributions*. Naively, distributions are linear scalar maps which are computed on smooth functions. Let us for now suggestively denote $\mathcal{C}^\infty(A, B) := \mathcal{L}(!A, B)$ the set of smooth maps from A to B, and $A \multimap B := \mathcal{L}(A, B)$ the set of linear maps from A to B. Now, in Classical DiLL, elements of $!A$ can be interpreted as distributions, so we may suggestively write $!A \subseteq \mathcal{C}^\infty(A, I) \multimap I$. In most models, I is often interpreted as the field of real or complex numbers. Now, for each element x of A, the dereliction rule gives us the Dirac distribution at x, which is the distribution $\delta_x \in !A$ which evaluates a smooth function at x, so $\delta_x(f) = f(x)$. For finite-dimensional vector spaces, or in the model of convenient vector spaces [1] (which we discuss in Ex 14), it is sufficient to define what a non-linear map does on Dirac distributions. So, on Dirac distributions, the structural maps, which correspond to the structural rules of LL and the co-structural rules of DiLL, are given as follows:

$$\begin{gathered} \mathsf{p}_A(\delta_x) = \delta_{\delta_x} \qquad \mathsf{d}_A(\delta_x) = x \qquad \mathsf{c}_A(\delta_x) = \delta_x \otimes \delta_x \qquad \mathsf{w}_A(\delta_x) = 1 \\ \overline{\mathsf{d}}_A(x) = \mathsf{D}_0(\_)(x) \qquad \overline{\mathsf{c}}_A(\delta_x \otimes \delta_y) = \delta_{x+y} \qquad \overline{\mathsf{w}}_A(1) = \delta_0 \end{gathered} \tag{1}$$

where for the co-dereliction $\overline{\mathsf{d}}$, the D is the differential operator, that is, for a smooth function f, $\mathsf{D}_x(f)(y)$ is the derivative of f at point x along the vector y. We highlight that on the whole space $!A$, the co-contraction $\overline{\mathsf{c}} : !A \otimes !A \to !A$ is interpreted as the *convolution of distributions*:

$$\overline{\mathsf{c}}_A(\phi \otimes \psi) = \phi * \psi := f \mapsto \phi(x \mapsto \psi(y \mapsto f(x + y))).$$

Moreover, the structural rules of LL can also be naturally expressed on functions. Indeed, in Classical DiLL, we have an involutive duality $*$ where A^* is the linear dual of A, that is, $A^* = A \multimap I$. Using the linear dual, one also introduces the connector $?A = (!A^*)^*$, which is interpreted as a space of smooth functions, $?A \subseteq \mathcal{C}^\infty(A^*, I)$. We also get the multiplicative disjunction $A \parr B = (A^* \otimes B^*)^*$, which we may think of as a completed tensor product. Then the contraction $\mathsf{c}^?_A : ?A \parr ?A \to ?A$ is interpreted by the pointwise multiplication of scalar functions, the weakening $\mathsf{w}^?_A : \mathbb{K} \to ?A$ maps scalars r to constant functions $cst_r : x \mapsto r$, and the dereliction $\mathsf{d}^?_A : A \to ?A$ maps elements of A to their evaluation at a point x:

$$\mathsf{c}^?_A(f \otimes g) = f \cdot g \qquad \mathsf{w}^?_A(r) = \mathrm{cst}_r \qquad \mathsf{d}^?_A(x) = (\ell \in A^* \mapsto \ell(x)) \tag{2}$$

All these intuitions can be made formal in specific models of DiLL; see Section 6.

1.2 A higher-order Laplace transform

The categorification of functional analysis and differential geometry entertains close links with the semantics of the sequent calculus for LL and DiLL. Differential categories were introduced by Blute, Cocket, and Seely [3], and originated from the semantics of DiLL. Since their introduction, differential categories now have a rich mathematical literature and have been quite successful in categorifying various important concepts from differential calculus and differential geometry, as well as various other aspects of differentiation throughout mathematics and computer science. This paper follows this line of research. Following the categorification of the exponential functions in a differential category by the second named author in [20], and the completion of DiLL by the addition of a co-digging rule by the authors in [18], here we give a categorical interpretation of the Laplace transform and study its properties. We explain why it is the reason behind the symmetry in DiLL rules, which we exploit categorically.

The Laplace transform is a central component of calculus and engineering, as it changes differential equations into polynomial equations. As such, the Laplace transform is a very useful tool for solving differential equations. In its first-order version, the Laplace transform takes a function $f : \mathbb{R} \to \mathbb{R}$ to a function $\mathcal{L}(f) : \mathbb{C} \to \mathbb{R}$, defined as:

$$\mathcal{L}(f)(s) = \int_0^\infty f(t) e^{-st}\, dt$$

However, this first-order version does not necessarily fit well with the categorical semantics of DiLL. Instead of using integration to make functions act on functions, one can use distributions by following the general idea of interpreting distributions as generalized functions. Consider a distribution ϕ with compact support, that is, $\phi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})'$ is a linear form on the space of smooth functions (where $F' := \mathcal{L}(F, \mathbb{R})$ is the space of linear scalar functions on a vector space F). Then we may write:

$$\mathcal{L}(\phi)(s) = \phi(t \mapsto e^{-st})$$

So for a higher-order distribution $\phi \in \mathcal{C}^\infty(E, \mathbb{R})'$, where E stands for a possibly infinite-dimensional vector space, we get:

$$\mathcal{L} : \begin{cases} \mathcal{C}^\infty(E, \mathbb{R})' \longrightarrow \mathcal{C}^\infty(E', \mathbb{R}) \\ \qquad\qquad \phi \longmapsto \left(x^* \mapsto \left(\phi\left(t \mapsto e^{x^*(t)}\right)\right)\right) \end{cases} \tag{3}$$

Following the intuitions developed above, this gives us a new understanding of the Laplace transformation in terms of connectives of LL, resulting in a natural transformation of type $\mathcal{L}_A : !A \to ?A$. This idea was only recently noticed in the context of DiLL, thanks to the higher-order presentation of the Laplace transform in a specific polarized model of DiLL discussed in [18, Prop V.8].

1.3 Laplace transformation from co-structural to structural rules

Since we have a categorical understanding of higher-order distribution theory, we, therefore, have all the ingredients in hand to axiomatize $\mathcal{L}$ categorically in a suitable differential category. Differential categories are reviewed in Section 2. For now, we simply state that we categorify the Laplace transform as a natural transformation of type $\mathcal{L} : !A \to ?A$, which we call a **Laplace transformation** (Def 8). Semantically, the axioms say that $\mathcal{L}$ transforms the interpretation of co-structural rules into the interpretation of structural rules:

$$\mathcal{L} : !A \mapsto ?A; \overline{\mathsf{w}} \mapsto \mathsf{w}; \overline{\mathsf{c}} \mapsto \mathsf{c}; \overline{\mathsf{d}} \mapsto \mathsf{d}.$$

These are all analogues of very well-known facts in calculus. For example, the Laplace transform converts convolution into multiplication, which is recaptured by the fact that our Laplace transformation $\mathcal{L}$ turns $\overline{\mathsf{c}}$ into $\mathsf{c}^?$. It may be useful to redo these well-known computations, which will help clearly show how $\mathcal{L}$, as given in (3), computes on co-structural morphisms, as given in (1). Here x^* is an element of $A^* = \mathcal{L}(A, \mathbb{R})$, ans as such acts on elements t of A.

$$\mathcal{L}(\overline{\mathsf{w}}_A(r)) = x^* \mapsto r \cdot \delta_0(t \mapsto e^{x^*(t)}) = x^* \mapsto r \cdot e^0 = x^* \mapsto r = \mathsf{w}^?_A(r)$$

$$\mathcal{L}(\overline{\mathsf{d}}_A(y)) = x^* \mapsto D_0(t \mapsto e^{x^*(t)})(y) = x^* \mapsto x^*(y) = \mathsf{d}^?_A(y)$$

$$\begin{aligned}
\mathcal{L}(\overline{\mathsf{c}}_A(\phi \otimes \psi)) &= x^* \mapsto (\phi * \psi)(t \mapsto e^{x^*(t)}) = x^* \mapsto \phi\left(s \mapsto \psi(t \mapsto e^{x^*(t+s)})\right) \\
&= x^* \mapsto \phi\left(s \mapsto \psi(t \mapsto e^{x^*(t)+x^*(s)})\right) && (x^* \text{ is linear}) \\
&= x^* \mapsto \phi\left(s \mapsto \psi(t \mapsto e^{x^*(t)}e^{x^*(s)})\right) \\
&= x^* \mapsto \phi(s \mapsto e^{x^*(s)} \cdot \psi(t \mapsto e^{x^*(t)})) && (\psi \text{ is linear}) \\
&= x^* \mapsto \phi(s \mapsto e^{x^*(s)}) \cdot \psi(t \mapsto e^{x^*(t)}) && (\phi \text{ is linear}) \\
&= \mathsf{c}^?_{A^*}(\mathcal{L}(\phi) \otimes \mathcal{L}(\psi))
\end{aligned}$$

Observe how all these equations are intrinsically linked with the basic properties of the exponential function e^x. We will make this precise in Section 4. Indeed, generalizations of the exponential function in a differential category were defined by the second named author in [20], and are axiomatized by analogues of three fundamental properties of the exponential function: that $e^{x+y} = e^x e^y$ and $e^0 = 1$, and also that e^x is its own derivative. We will explain how the notion of a Laplace transformation is fundamentally linked to that of a generalized exponential function on the monoidal unit I (Def 3).

Moreover, since we are in the monoidal closed, we may uncurry the Laplace transformation to get an extranatural transformation $\partial_A : !A^* \otimes !A \to I$, which we call a **Laplace evaluator** (Def 2), or take the dual to get a natural transformation $\ell_A : !A^* \to (!A)^*$, which we call a **Laplace distributor** (Def 1).

1.4 Content and Outline

This paper starts in Section 2 with a review of differential categories, the categorical semantics of DiLL, in which we will categorify $\mathcal{L}$. In Section 3, we introduce the concept of a Laplace distributor, which we axiomatize as a natural transformation operating in differential linear closed categories which transforms co-structural rules into structural rules. In Section 4, we show that the presence of a Laplace distributor in a differential linear closed category is equivalent to the presence of a generalized exponential function $\mathsf{e} : !I \multimap I$ on the monoidal unit. Laplace distributors in the context of isomix star-autonomous categories are studied in Section 5, where we show that we obtain our desired Laplace transformation $\mathcal{L}$ in a differential linear isomix category. In Section 6, we give examples of Laplace distributors/evaluators/transformations in well-known differential categories. We then conclude in Section 7 with a discussion of future work.

2 Background: Differential Categories

In this section, we quickly review differential categories, mostly to set terminology and notation. We will follow the same terminology and notation used in [18]. For a more in-depth introduction to the basics of monoidal categories and the overall categorical semantics of linear logic, we refer the reader to see [22], and for an in-depth introduction to differential categories and examples, we refer them to see [2, 11].

The underlying categorical structure of a differential category is that of an *additive symmetric monoidal category*. Recall that a **symmetric monoidal category** [22, Sec 4.4] interprets the multiplicative fragment of LL. So for an arbitrary symmetric monoidal category, we denote the underlying category as $\mathcal{C}$, the monoidal product as $\otimes$, the monoidal unit as I, the natural associativity isomorphism as $\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, the natural right unital isomorphism as $\rho_A : A \otimes I \to A$, and the natural symmetry isomorphism as $\sigma_{A,B} : A \otimes B \to B \otimes A$. So then an **additive symmetric monoidal category** [2, Def 3] is a symmetric monoidal category $\mathcal{C}$ which is enriched over the category of commutative monoids, that is, each homset $\mathcal{C}(A, B)$ is a commutative monoid, with addition operation $+$ and zero $0 : A \to B$, and such that composition and the monoidal product $\otimes$ are compatible with the additive structure. This extra structure of additive enrichment for a differential category is necessary to express the famous Leibniz rule from differential calculus.

The categorical interpretation of the exponential fragment of LL is given by a *monoidal coalgebra modality*. So for a symmetric monoidal category $\mathcal{C}$, a **coalgebra modality** [2, Def 1] is a comonad $! : \mathcal{C} \to \mathcal{C}$ with comultiplication $\mathsf{p}_A : !A \to !!A$ called the **digging** and counit $\mathsf{d}_A : !A \to A$ called the **dereliction**, which comes equipped with two other natural transformations: $\mathsf{c}_A : !A \to !A \otimes !A$ called the **contraction** and $\mathsf{w}_A : !A \to I$ called the **weakening**, making each $!A$ a cocommutative comonoid and the digging a comonoid morphism. Then a **monoidal coalgbera modality** [2, Def 2] is a coalgebra modality $!$ which furthermore comes equipped with a natural transformation $\mu_{A,B} : !A \otimes !B \to !(A \otimes B)$ and a map $\mu_I : I \to !I$ which makes $!$ into a lax monoidal functor; p, d, c, and w into monoidal transformations; and c, and w into !-coalgebra morphisms.

For an additive symmetric monoidal category, a monoidal coalgebra modality can equivalently be described in terms of an **additive bialgebra modality** [2, Def 5]. So in particular, for a monoidal coalgebra modality $!$ on an additive symmetric monoidal category, we can build natural transformations $\overline{\mathsf{c}}_A : !A \otimes !A \to !A$ called the **co-contraction** and $\overline{\mathsf{w}}_A : I \to !A$ called the **co-weakening**, which in particular makes every $!A$ a commutative monoid, and in fact a bimonoid [2, Prop 1]. Then a **monoidal differential modality** is a monoidal coalgebra modality (equiv. additive bialgebra modality) $!$ on an additive symmetric monoidal category which comes equipped with a natural transformation $\overline{\mathsf{d}}_A : A \to !A$ called a **co-dereliction** [2, Def 9], whose axioms are analogues of the fundamental rules of differential calculus such as the Leibniz rule and chain rule. Then a **differential linear category** [2, Sec 6] is an additive symmetric monoidal category equipped with a monoidal differential modality. One could also consider differential linear categories with finite products $\times$, which are called *differential storage categories*. In particular, in such a setting, we have the all-important *Seely isomorphisms* $!(A \times B) \cong !A \otimes !B$ [2, Def 10]. However, since products don't necessarily play a role in the story of this paper, we will not review them here and invite the reader to see [2, Sec 7] for details. Then a categorical model of (Classical) DiLL is a differential storage category that is also monoidal closed (resp. star-autonomous), which we discuss in Section 3 (resp. Section 5).

3 Laplace Distributor

In this section, we introduce the notion of a *Laplace distributor*, which is an extra structural natural transformation in a differential linear category that is also *closed*. To properly define a Laplace distributor, we will first have to set up some notation in the closed setting.

So for a **symmetric monoidal closed category** [22, Sec 4.7], we denote the internal homs by $A \multimap B$ and the evaluation map by $\epsilon_{A,B} : (A \multimap B) \otimes A \to B$. Explicitly, recall that closed means that for every map $f : C \otimes A \to B$, there exists a unique map $\lambda(f) : C \to A \multimap B$, called the **curry** of f, such that[2]:

$$(\lambda(f) \otimes 1_A); \epsilon_{A,B} = f \tag{4}$$

Now for every map $f : X \to A$ and $g : B \to Y$, we denote by $f \multimap g : A \multimap B \to X \multimap Y$ to be the unique map such that:

$$((f \multimap g) \otimes 1_X)\,; \epsilon_{X,Y} = (1_{A \multimap B} \otimes f); \epsilon_{A,B}; g \tag{5}$$

We note that for a monoidal coalgebra modality $!$ on a symmetric monoidal category, we have canonical maps $\xi_{A,B} : !(A \multimap B) \to !A \multimap !B$ defined as the unique map such that:

$$(\xi_{A,B} \otimes 1_{!A})\,; \epsilon_{!A,!B} = \mu_{A \multimap B, A}; !\epsilon_{A,B} \tag{6}$$

Then by a **differential linear closed category** we mean a differential linear category whose underlying symmetric monoidal category is closed.

Now, a Laplace distributor is a natural transformation that transforms the co-structural rules of the modality into its structural rule. This is expressed in terms of dual objects. In a symmetric monoidal closed category, the **dual** of an object A is the object $A^* := A \multimap I$. It is important to note that in an arbitrary symmetric monoidal closed category, the dual operation is not necessarily involutive, that is, A^{**} is not necessarily always equal/isomorphic to A. This will be a situation we discuss later in Section 5 below. We do however have a canonical isomorphism $\upsilon_I : I \to I^*$ which is defined as the unique map such that:

$$(\upsilon_I \otimes 1_I); \epsilon_{I,I} = \rho_I \qquad\qquad \upsilon_I^{-1} = \rho_{I^*}^{-1}; \epsilon_{I,I} \tag{7}$$

[2] In a category, we write identity maps as $1_A : A \to A$, and we write composition diagrammatically, that is, the composition of maps $f : A \to B$ and $g : B \to C$ is denoted $f; g : A \to C$.

as well another canonical isomorphism $\nu_{A,B} : (A \otimes B)^* \to A \multimap B^*$ defined as the unique map such that:

$$\begin{aligned} &((\nu_{A,B} \otimes 1_A) \otimes 1_B)\,; (\epsilon_{A,B^*} \otimes 1_B); \epsilon_{B,I} = \alpha^{-1}_{(A\otimes B)^*,A,B}; \epsilon_{A\otimes B,I} \\ &\left(\nu^{-1}_{A,B} \otimes 1_{A\otimes B}\right); \epsilon_{A\otimes B,I} = \alpha_{A\multimap B^*,A,B}; (\epsilon_{A,B^*} \otimes 1_B); \epsilon_{B,I} \end{aligned} \tag{8}$$

Moreover, we also get canonical maps $\Theta_{A,B} : A^* \otimes B^* \to (A \otimes B)^*$ (which is not necessarily an isomorphism) defined as the unique map such that:

$$(\Theta_{A,B} \otimes 1_{A\otimes B})\,; \epsilon_{A\otimes B,I} = \tau_{A^*,B^*,A,B}; (\epsilon_{A,I} \otimes \epsilon_{B,I}); \rho_I \tag{9}$$

where $\tau_{A,B,C,D} : (A\otimes B)\otimes(C\otimes D) \to (A\otimes C)\otimes(B\otimes D)$ is the canonical natural isomorphism which swaps the middle two terms. The dual operation also induces a contravariant functor, where in particular for every map $f : A \to B$, we also have a map of dual type $f^* : B^* \to A^*$ which is defined as the unique map such that:

$$(f^* \otimes 1_A); \epsilon_{A,I} = (1_{B^*} \otimes f); \epsilon_{B,I} \tag{10}$$

Then a Laplace distributor is a natural transformation which distributes $*$ over !, hence the name, and associates the co-structural maps to the dual of their mirror structural map.

▶ **Definition 1.** *For a differential linear closed category, a **Laplace distributor** is a natural transformation $\ell_A : !A^* \to (!A)^*$ such that the following diagrams commute:*

$$\begin{tikzcd} !A^* \otimes !A^* \arrow[r, "\bar{\mathsf{c}}_{A^*}"] \arrow[d, "\ell_A\otimes\ell_A"'] & !A^* \arrow[dd, "\ell_A"] \\ (!A)^* \otimes (!A)^* \arrow[d, "\Theta_{!A,!A}"'] \arrow[r, phantom, "(\ell.\mathsf{c}.1)"] & \\ (!A \otimes !A)^* \arrow[r, "\mathsf{c}^*_A"'] & (!A)^* \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "\bar{\mathsf{w}}_{A^*}"] \arrow[d, "\upsilon_I"'] \arrow[dr, phantom, "(\ell.\mathsf{w}.1)"] & !A^* \arrow[d, "\ell_A"] \\ I^* \arrow[r, "\mathsf{w}^*_A"'] & (!A)^* \end{tikzcd} \qquad \begin{tikzcd} A^* \arrow[r, "\bar{\mathsf{d}}_{A^*}"] \arrow[dr, bend right, "\mathsf{d}^*_A"'] & !A^* \arrow[d, "\ell_A"] \arrow[dl, phantom, "(\ell.\mathsf{d}.1)"] \\ & (!A)^* \end{tikzcd}$$

$$\begin{tikzcd} !(A \otimes B)^* \arrow[rr, "\ell_{A\otimes B}"] \arrow[d, "!\nu_{A,B}"'] & & (!(A\otimes B))^* \arrow[dd, "\mu^*_{A,B}"] \\ !\,(A \multimap B^*) \arrow[d, "\xi_{A,B^*}"'] \arrow[r, phantom, "(\ell.\mu)"] & & \\ !A \multimap !B^* \arrow[r, "1_{!A}\multimap\ell_B"'] & !A \multimap (!B)^* \arrow[r, "\nu^{-1}_{!A,!B}"'] & (!A \otimes !B)^* \end{tikzcd}$$

Examples of Laplace distributors can be found in Section 6. Let us provide some intuition for a Laplace distributor using our distribution analogy. First, for every linear functional $x^* : A \multimap I$, we have the Dirac distribution $\delta_{x^*} \in !A^*$. So the Laplace distributor produces a linear functional $\ell_A(\delta_{x^*}) : !A^* \multimap I$, which through the call-by-name translation of Linear Logic in Intuitionistic Logic corresponds to a smooth function $A \Rightarrow I$. Then for $z \in A$ and $x^*, y^* \in A$, the first three axioms of a Laplace distributor say that:

$$\ell_A(\delta_{x^*+y^*})(\delta_z) = \ell_A(\delta_{y^*})(\delta_z) \cdot \ell_A(\delta_{x^*})(\delta_z) \tag{ℓ.c.1}$$

$$\ell_A(\delta_0)(\delta_z) = 1 \tag{ℓ.w.1}$$

$$\ell_A(\mathsf{D}_0(\_)(x^*))(\delta_z) = x^*(z) \tag{ℓ.d.1}$$

Note the similarities with some of the basic identities the exponential function e^x satisfies. We will make this connection precise in Section 4 when we show how Laplace distributors correspond to a generalized version of the exponential function in a differential linear category. The last axiom $(\ell.\mu)$ essentially tells us that every Laplace distributor can indeed be described as one of these generalized exponential functions. Somewhat surprisingly, we also get that the Laplace distributor also "co-transforms" the structural rules into their mirror co-structural rules, as we will see in Lemma 7 below.

Now the keen eyed-reader will note that in an arbitrary differential linear closed category, there are always two possible natural transformations of type $!A^* \to (!A)^*$ given by the following composites:

$$!A^* \xrightarrow{\mathsf{w}_{A^*}} I \xrightarrow{\upsilon_I} I^* \xrightarrow{\mathsf{w}^*_A} (!A)^* \qquad\qquad !A^* \xrightarrow{\mathsf{d}_{A^*}} A^* \xrightarrow{\mathsf{d}^*_A} (!A)^*$$

However, since by [2, Lemma 2] we have that $\overline{\mathsf{w}};\mathsf{d} = 0$, the first map won't satisfy $(\ell.\mathsf{d}.1)$, while by [2, Def 9] we have that $\overline{\mathsf{d}};\mathsf{w} = 0$, so the second map won't satisfy $(\ell.\mathsf{w}.1)$. So a Laplace evaluator does not always necessarily exists, and is indeed extra structure. To further justify this fact, in Ex 13 we give an example of a differential linear closed category which does *not* have a Laplace distributor.

Moreover, note that given the type of a Laplace distributor $\ell_A : !A^* \to (!A)^*$, we can uncurry it to get a map of type $!A^* \otimes !A \to I$, which we call the *Laplace evaluator.*

▶ **Definition 2.** *In a differential linear closed category with a Laplace distributor ℓ, the* ***Laplace evaluator*** *is the extranatural transformation $\eth_A : !A^* \otimes !A \to I$ defined as the composite:*

$$\eth_A :\ = \ !A^* \otimes !A \xrightarrow{\ell_A \otimes 1_{!A}} (!A)^* \otimes !A \xrightarrow{\epsilon_{!A,I}} I \tag{11}$$

By extranaturality, we mean that for all maps $f : A \to B$, the following equality holds:

$$(!f^* \otimes 1_A);\ell_A = (1_{!B^*} \otimes !f);\eth_B \tag{12}$$

Of course, since the currying operation is an isomorphism, we could have alternatively and equivalently written this story in terms of the Laplace evaluator and defined the Laplace distributor as its curry, $\ell_A = \lambda(\eth_A)$.

4 Exponential Map

In this section, we show that Laplace distributors correspond precisely to a generalized version of the exponential function e^x on the monoidal unit. The generalization of e^x in a differential category was introduced by the second named author in [20] and was called an !-*differential exponential map.* An !-differential exponential map can be defined for any commutative monoid in a differential category and is axiomatized by analogues of the fact that e^x is its own derivative and is a monoid morphism from addition to multiplication. Since the monoidal unit I in a symmetric monoidal category is canonically a monoid, we can consider an !-differential exponential map on I, which we call an *I-exponential map* for short.

▶ **Definition 3.** *In a differential linear category, an* ***I-exponential map*** *[20, Def 14] is a map $\mathsf{e} : !I \to I$ such that the following diagrams commute:*

$$\begin{tikzcd} !I \otimes !I \arrow[r, "\overline{\mathsf{c}}_I"] \arrow[d, "\mathsf{e}\otimes\mathsf{e}"'] \arrow[dr, phantom, "(\mathsf{e}.\overline{\mathsf{c}})"] & !I \arrow[d, "\mathsf{e}"] \\ I \otimes I \arrow[r, "\rho_I"'] & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "\overline{\mathsf{w}}_I"] \arrow[dr, equal, bend right] \arrow[dr, phantom, "(\mathsf{e}.\overline{\mathsf{w}})"] & !I \arrow[d, "\mathsf{e}"] \\ & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "\overline{\mathsf{d}}_I"] \arrow[dr, equal, bend right] \arrow[dr, phantom, "(\mathsf{e}.\overline{\mathsf{d}})"] & !I \arrow[d, "\mathsf{e}"] \\ & I \end{tikzcd}$$

Using our distribution analogy, let us explain why an I-exponential map is indeed the correct generalization of the exponential function $e^x : \mathbb{R} \to \mathbb{R}$. Note that by its type, an I-exponential map e is a smooth map from I to I, just like how e^x is a smooth function from $\mathbb{R}$ to $\mathbb{R}$. Then the axioms of an I-exponential map are:

$$\mathsf{e}(\delta_{x+y}) = \mathsf{e}(\delta_x)\mathsf{e}(\delta_y) \tag{e.$\overline{\mathsf{c}}$}$$

$$\mathsf{e}(\delta_0) = 1 \tag{e.$\overline{\mathsf{w}}$}$$

$$\mathsf{e}(\mathsf{D}_0(\_)(x)) = x \tag{e.$\overline{\mathsf{d}}$}$$

Now suggestively writing $\mathsf{e}(\delta_x) = e^x$, the three axioms of an I-exponential map give us precisely the well-known identities of the exponential function which are:

$$e^{x+y} = e^x e^y \qquad\qquad e^0 = 1 \qquad\qquad \mathsf{D}_0(e^x)(x) = x$$

Recall that in the previous section, we suggested that the axioms of a Laplace distributor also corresponded to these three e^x identities. Here, we make this precise by showing that there is a bijective correspondence between Laplace distributors and I-exponential maps.

Starting from a Laplace distributor, we get an I-exponential map by considering the Laplace distributor at I and the fact that $I \cong I^*$:

▶ **Proposition 4.** *In a differential linear closed category with a Laplace distributor ℓ, define the map $\mathsf{e}^\ell : !I \to I$ as the following composite:*

$$\mathsf{e}^\ell := \; !I \xrightarrow{\;!\upsilon_I\;} !I^* \xrightarrow{\;\ell_I\;} (!I)^* \xrightarrow{\;\mu_I^*\;} I^* \xrightarrow{\;\upsilon_I^{-1}\;} I \tag{13}$$

Proof. The key to this proof is using the part from the definition of a monoidal coalgebra modality [2, Def 1] which says that d, c, and w are compatible with μ_I since they are monoidal transformations. So, by definition, we have:

$$\mu_I; \mathsf{d}_I = 1_I \qquad\qquad \mu_I; \mathsf{w}_I = 1_I \qquad\qquad \mu_I; \mathsf{c}_I = \rho_I^{-1}; (\mu_I \otimes \mu_I) \tag{14}$$

More precisely, for the calculations in this proof, we will need the dualized versions of the above identities:

$$\mathsf{d}_I^*; \mu_I^* = 1_{I^*} \qquad\qquad \mathsf{w}_I^*; \mu_I^* = 1_{I^*} \qquad\qquad \mathsf{c}_I^*; \mu_I^* = (\mu_I \otimes \mu_I)^*; (\rho_I^{-1})^* \tag{15}$$

So we first compute (e.$\overline{\mathsf{w}}$):

$$\overline{\mathsf{w}}_I; \mathsf{e}^\ell \overset{(13)}{=} \overline{\mathsf{w}}_I; !\upsilon_I; \ell_I; \mu_I^*; \upsilon_I^{-1} \overset{\text{nat.}}{=} \overline{\mathsf{w}}_{I^*}; \ell_I; \mu_I^*; \upsilon_I^{-1} \overset{(\ell.\mathsf{w}.1)}{=} \upsilon_I; \mathsf{w}_I^*; \mu_I^*; \upsilon_I^{-1} \overset{(15)}{=} \upsilon_I; \upsilon_I^{-1} = 1_I$$

So $\overline{\mathsf{w}}_I; \mathsf{e}^\ell = 1_I$. Next we compute (e.$\overline{\mathsf{d}}$):

$$\overline{\mathsf{d}}_I; \mathsf{e}^\ell \overset{(13)}{=} \overline{\mathsf{d}}_I; !\upsilon_I; \ell_I; \mu_I^*; \upsilon_I^{-1} \overset{\text{nat.}}{=} \upsilon_I; \overline{\mathsf{d}}_{I^*}; \ell_I; \mu_I^*; \upsilon_I^{-1} \overset{(\ell.\mathsf{d}.1)}{=} \upsilon_I; \mathsf{d}_I^*; \mu_I^*; \upsilon_I^{-1} \overset{(15)}{=} \upsilon_I; \upsilon_I^{-1} = 1_I$$

So $\overline{\mathsf{d}}_I; \mathsf{e}^\ell = 1_I$. Lastly, using that Θ is natural and:

$$\Theta_{I,I}; (\rho_I^{-1})^*; \upsilon_I^{-1} = (\upsilon_I^{-1} \otimes \upsilon_I^{-1}); \rho_I \tag{16}$$

which we leave to the reader to check for themselves, we compute (e.$\overline{\mathsf{c}}$):

$$\overline{\mathsf{c}}_I; \mathsf{e}^\ell \overset{(13)}{=} \overline{\mathsf{c}}_I; !\upsilon_I; \ell_I; \mu_I^*; \upsilon_I^{-1} \overset{\text{nat.}}{=} (!\upsilon_I \otimes !\upsilon_I); \overline{\mathsf{c}}_{I^*}; \ell_I; \mu_I^*; \upsilon_I^{-1}$$

$$\overset{(\ell.\mathsf{c}.1)}{=} (!\upsilon_I \otimes !\upsilon_I); (\ell_I \otimes \ell_I); \Theta_{!I,!I}; \mathsf{c}_I^*; \mu_I^*; \upsilon_I^{-1}$$
$$\overset{(15)}{=} (!\upsilon_I \otimes !\upsilon_I); (\ell_I \otimes \ell_I); \Theta_{!I,!I}; (\mu_I \otimes \mu_I)^*; (\rho_I^{-1})^*; \upsilon_I^{-1}$$
$$\overset{\text{nat.}}{=} (!\upsilon_I \otimes !\upsilon_I); (\ell_I \otimes \ell_I); (\mu_I^* \otimes \mu_I^*); \Theta_{I,I}; (\rho_I^{-1})^*; \upsilon_I^{-1}$$
$$\overset{(16)}{=} (!\upsilon_I \otimes !\upsilon_I); (\ell_I \otimes \ell_I); (\mu_I^* \otimes \mu_I^*); (\upsilon_I^{-1} \otimes \upsilon_I^{-1}); \rho_I \overset{(13)}{=} (\mathsf{e}^\ell \otimes \mathsf{e}^\ell); \rho_I$$

So $\overline{\mathsf{c}}_I; \mathsf{e}^\ell = (\mathsf{e}^\ell \otimes \mathsf{e}^\ell); \rho_I$. Therefore, we conclude that e^ℓ is an I-exponential map. ◀

It is worthwhile to remark that we did not need $(\ell.\mu)$ to show that e^ℓ is an I-exponential map. This axiom will be used in Thm 6 when we prove the bijective correspondence. Before that, we must explain how from an I-exponential map we obtain a Laplace distributor, which is constructed using the evaluation map as well.

▶ **Proposition 5.** *In a differential linear closed category, if $\mathsf{e} : !I \to I$ is an I-exponential map then define the map $\ell^{\mathsf{e}}_A : !A^* \to (!A)^*$ as the curry of the following composite:*

$$!A^* \otimes !A \xrightarrow{\mu_{A^*,A}} !(A^* \otimes A) \xrightarrow{!\epsilon_{A,I}} !I \xrightarrow{\mathsf{e}} I \tag{17}$$

In other words, ℓ^{e}_A is the unique map such that the following equality holds:

$$(\ell^{\mathsf{e}}_A \otimes 1_{!A}); \epsilon_{!A,I} = \mu_{A^*,A}; !\epsilon_{A,I}; \mathsf{e} \tag{18}$$

Then ℓ^{e} is a Laplace distributor. Moreover, its induced Laplace evaluator $\partial^{\mathsf{e}} : !A^* \otimes !A \to I$ is precisely the composite (17).

Proof. In this proof, for readability, we omit the subscripts. The key to this proof is that in a symmetric monoidal closed category, the evaluation map is *monic* in its first argument, that is, if $(f \otimes 1); \epsilon = (g \otimes 1); \epsilon$ then $f = g$.

We must first show naturality. So we compute that:

$$(!f^* \otimes 1); (\ell^{\mathsf{e}} \otimes 1); \epsilon \overset{(18)}{=} (!f^* \otimes 1); \mu_{A^*,A}; \epsilon; \mathsf{e} \overset{\text{nat.}}{=} \mu; !(f^* \otimes 1); !\epsilon_{A,I}; \mathsf{e}$$
$$\overset{(10)}{=} \mu; !(1 \otimes f); !\epsilon; \mathsf{e} \overset{\text{nat.}}{=} (1 \otimes !f); \mu; !\epsilon; \mathsf{e} \overset{(18)}{=} (1 \otimes !f); (\ell^{\mathsf{e}} \otimes 1); \epsilon$$
$$= (\ell^{\mathsf{e}} \otimes 1); (1 \otimes !f); \epsilon \overset{(10)}{=} (\ell^{\mathsf{e}} \otimes 1); ((!f)^* \otimes 1); \epsilon$$

Thus we get that $!f^*; \ell^{\mathsf{e}} = \ell^{\mathsf{e}}; (!f)^*$, and therefore ℓ^{e} is indeed a natural transformation.

To prove the first three axioms of a Laplace distributor, we will need the following compatibility relation between μ and the co-structural maps from [2, Prop 2 & Prop 5]:

$$(\overline{\mathsf{w}} \otimes 1); \mu = \rho; \mathsf{w}; \overline{\mathsf{w}} \quad \left(\overline{\mathsf{d}} \otimes 1\right); \mu = (1 \otimes \mathsf{d}); \overline{\mathsf{d}} \quad (\overline{\mathsf{c}} \otimes 1); \mu = (1 \otimes 1 \otimes \mathsf{c}); \tau; (\mu \otimes \mu); \overline{\mathsf{c}} \tag{19}$$

So for $(\ell.\mathsf{w}.1)$, we compute that:

$$(\overline{\mathsf{w}} \otimes 1); (\ell^{\mathsf{e}} \otimes 1); \epsilon \overset{(18)}{=} (\overline{\mathsf{w}} \otimes 1); \mu; !\epsilon; \mathsf{e} \overset{(19)}{=} \rho; \mathsf{w}; \overline{\mathsf{w}}; !\epsilon; \mathsf{e} \overset{\text{nat.}}{=} \rho; \mathsf{w}; \overline{\mathsf{w}}; \mathsf{e} \overset{(\mathsf{e}.\overline{\mathsf{w}})}{=} \rho; \mathsf{w}_A$$
$$\overset{\text{nat.}}{=} (1 \otimes \mathsf{w}); \rho \overset{(7)}{=} (1 \otimes \mathsf{w}); (\upsilon \otimes 1); \epsilon = (\upsilon \otimes 1); (1 \otimes \mathsf{w}); \epsilon \overset{(10)}{=} (\upsilon \otimes 1); (\mathsf{w}^* \otimes 1); \epsilon$$

Thus we get that $\overline{\mathsf{w}}; \ell^{\mathsf{e}} = \upsilon; \mathsf{w}^*$. Then for $(\ell.\mathsf{d}.1)$, we compute that:

$$(\overline{\mathsf{d}} \otimes 1); (\ell^{\mathsf{e}} \otimes 1); \epsilon \overset{(18)}{=} \left(\overline{\mathsf{d}} \otimes 1\right); \mu; !\epsilon; \mathsf{e} \overset{(19)}{=} (1 \otimes \mathsf{d}); \overline{\mathsf{d}}; !\epsilon; \mathsf{e}$$
$$\overset{\text{nat.}}{=} (1 \otimes \mathsf{d}); \epsilon; \overline{\mathsf{d}}; \mathsf{e} \overset{(\mathsf{e}.\overline{\mathsf{d}})}{=} (1 \otimes \mathsf{d}); \epsilon \overset{(10)}{=} (\mathsf{d}^* \otimes 1); \epsilon$$

Thus we get that $\overline{\mathsf{d}}_{A^*}; \ell^{\mathsf{e}}_A = \mathsf{d}^*_A$. Now for ($\ell$.c.1), we compute that:

$$(\overline{\mathsf{c}} \otimes 1); (\ell^{\mathsf{e}} \otimes 1); \epsilon_{!A,I} \overset{(18)}{=} (\overline{\mathsf{c}} \otimes 1); \mu; !\epsilon; \mathsf{e} \overset{(19)}{=} (1 \otimes \mathsf{c}); \tau; (\mu \otimes \mu); \overline{\mathsf{c}}; !\epsilon; \mathsf{e}$$
$$\overset{\text{nat.}}{=} (1 \otimes \mathsf{c}); \tau; (\mu \otimes \mu); (!\epsilon \otimes !\epsilon); \overline{\mathsf{c}}; \mathsf{e} \overset{(\mathsf{e}.\overline{\mathsf{c}})}{=} (1 \otimes \mathsf{c}); \tau; (\mu \otimes \mu); (!\epsilon \otimes !\epsilon); (\mathsf{e} \otimes \mathsf{e})$$
$$\overset{(18)}{=} (1 \otimes \mathsf{c}); \tau; ((\ell^{\mathsf{e}} \otimes 1) \otimes (\ell^{\mathsf{e}} \otimes 1)); (\epsilon \otimes \epsilon) \overset{\text{nat.}}{=} (1 \otimes \mathsf{c}); ((\ell^{\mathsf{e}} \otimes \ell^{\mathsf{e}}) \otimes 1); \tau; (\epsilon \otimes \epsilon)$$
$$\overset{(9)}{=} ((\ell^{\mathsf{e}} \otimes \ell^{\mathsf{e}}) \otimes 1); (\Theta \otimes 1); (1 \otimes \mathsf{c}); \epsilon \overset{(10)}{=} ((\ell^{\mathsf{e}} \otimes \ell^{\mathsf{e}}) \otimes 1); (\Theta \otimes 1); (\mathsf{c}^* \otimes 1); \epsilon$$

Thus we get that $\overline{\mathsf{c}}_{A^*}; \ell^{\mathsf{e}}_A = (\ell^{\mathsf{e}}_A \otimes \ell^{\mathsf{e}}_A); \Theta_{!A,!A}; \mathsf{c}^*_A$. Lastly, for ($\ell.\mu$), we will need the monoidal associativity axiom from the definition of a monoidal coalgebra modality [2, Def 2]:

$$\alpha; (\mu \otimes 1); \mu = (1 \otimes \mu); \mu; !\alpha \tag{20}$$

Then we compute that:

$$(!\nu \otimes 1); (\xi \otimes 1); ((1 \multimap \ell^{\mathsf{e}}) \otimes 1); \left(\nu^{-1} \otimes 1\right); \epsilon$$
$$\overset{(8)}{=} (!\nu \otimes 1); (\xi \otimes 1); ((1 \multimap \ell^{\mathsf{e}}) \otimes 1); \alpha; (\epsilon \otimes 1); \epsilon$$
$$\overset{\text{nat.}}{=} \alpha; ((!\nu \otimes 1) \otimes 1); ((\xi \otimes 1) \otimes 1); (((1 \multimap \ell^{\mathsf{e}}) \otimes 1) \otimes 1); (\epsilon \otimes 1); \epsilon$$
$$\overset{(5)}{=} \alpha; ((!\nu \otimes 1) \otimes 1); ((\xi \otimes 1) \otimes 1); (\epsilon \otimes 1); \ell^{\mathsf{e}}; \epsilon$$
$$\overset{(8)}{=} \alpha; ((!\nu \otimes 1) \otimes 1); (\mu \otimes 1); (!\epsilon \otimes 1); \ell^{\mathsf{e}}; \epsilon$$
$$\overset{\text{nat.}}{=} \alpha; (\mu \otimes 1); (!(\nu \otimes 1) \otimes 1); (!\epsilon \otimes 1); \ell^{\mathsf{e}}; \epsilon$$
$$\overset{(18)}{=} \alpha; (\mu \otimes 1); (!(\nu \otimes 1) \otimes 1); (!\epsilon \otimes 1); \mu; !\epsilon; \mathsf{e}$$
$$\overset{\text{nat.}}{=} \alpha; (\mu \otimes 1); \mu; !((\nu \otimes 1) \otimes 1)); !(\epsilon \otimes 1); !\epsilon; \mathsf{e}$$
$$\overset{(20)}{=} (1 \otimes \mu); \mu; !\alpha; !((\nu \otimes 1) \otimes 1)); !(\epsilon \otimes 1); !\epsilon; \mathsf{e} \overset{(8)}{=} (1 \otimes \mu); \mu; !\epsilon; \mathsf{e}$$
$$\overset{(18)}{=} (1 \otimes \mu); (\ell^{\mathsf{e}} \otimes 1); \epsilon = (\ell^{\mathsf{e}} \otimes 1); (1 \otimes \mu); \epsilon \overset{(10)}{=} (\ell^{\mathsf{e}} \otimes 1); (\mu^* \otimes 1); \epsilon$$

Thus we get that $!\nu_{A,B}; \xi_{A,B^*}; 1_{!A} \multimap \ell^{\mathsf{e}}_B; \nu^{-1}_{!A,!B} = \ell^{\mathsf{e}}_{A\otimes B}; \mu^*_{A,B}$. Therefore, we conclude that ℓ^{e} is a Laplace distributor. By definition, we then get that (17) is indeed the induced Laplace evaluator. ◀

The constructions from the above two propositions are inverses of each other, thus giving us our desired bijective correspondence.

▶ **Theorem 6.** *For a differential linear closed category, there is a bijective correspondence between Laplace distributors and I-exponential maps.*

Proof. To show that the constructions from Prop 4 and Prop 5 are invereses of each other, we must show that $\mathsf{e}^{\ell^{\mathsf{e}}} = \mathsf{e}$ and $\ell^{\mathsf{e}^{\ell}} = \ell$. For the former, we will need the monoidal unital axiom from the definition of a monoidal coalgebra modality [2, Def 2]:

$$(1_{!A} \otimes \mu_I); \mu_{A,I} = \rho_{!A}; !(\rho^{-1}_A) \tag{21}$$

So we first compute that:

$$(\mathsf{e}^{\ell^{\mathsf{e}}} \otimes 1_I); (\upsilon_I \otimes 1); \epsilon_{I,I} \overset{(13)}{=} (!\upsilon_I \otimes 1_I); (\ell^{\mathsf{e}}_I \otimes 1_I); (\mu^*_I \otimes 1_I); (\upsilon^{-1}_I \otimes 1_I); (\upsilon_I \otimes 1); \epsilon_{I,I}$$
$$(!\upsilon_I \otimes 1_I); (\ell^{\mathsf{e}}_I \otimes 1_I); (\mu^*_I \otimes 1_I); \epsilon_{I,I} \overset{(10)}{=} (!\upsilon_I \otimes 1_I); (\ell^{\mathsf{e}}_I \otimes 1_I); (1_{(!I)^*} \otimes \mu_I); \epsilon_{!I,I}$$

$$
\begin{aligned}
&= (!\upsilon_I \otimes 1_I); (1_{!I^*} \otimes \mu_I); (\ell_I^{\mathsf{e}} \otimes 1_{!I}); \epsilon_{!I,I} \overset{(18)}{=} (!\upsilon_I \otimes 1_I); (1_{!I^*} \otimes \mu_I); \mu_{I^*,I}; !\epsilon_{I^*,I}; \mathsf{e} \\
&\overset{(21)}{=} (!\upsilon_I \otimes 1_I); \rho_{!I^*}; !(\rho_{I^*}^{-1}); !\epsilon_{I^*,I}; \mathsf{e} \overset{\text{nat.}}{=} \rho_{!I}; !\upsilon_I; !(\rho_{I^*}^{-1}); !\epsilon_{I^*,I}; \mathsf{e} \overset{(7)}{=} \rho_{!I}; !\upsilon_I; !\upsilon_I^{-1}; \mathsf{e} = \rho_{!I}; \mathsf{e} \\
&\overset{\text{nat.}}{=} (\mathsf{e} \otimes 1_I); \rho_I \overset{(7)}{=} (\mathsf{e} \otimes 1_I); (\upsilon_I \otimes 1); \epsilon_{I,I}
\end{aligned}
$$

So we get that $\mathsf{e}^{\ell^{\mathsf{e}}}; \upsilon_I = \mathsf{e}; \upsilon_I$, and since υ_I is an isomorphism, we get that $\mathsf{e}^{\ell^{\mathsf{e}}} = \mathsf{e}$. On the other hand, we leave it as an exercise for the reader to check that when taking $B = I$ in $(\ell.\mu)$ it follows that the following diagram commutes:

$$
\begin{tikzcd}
!A^* \arrow[rrr, "\ell_A"] \arrow[d, "\xi_{A,I}"'] & & & (!A)^* \\
!A \multimap !I \arrow[d, "1_{!A} \multimap !\upsilon_I"'] & (\ell.\mu_I) & & \\
!A \multimap !I^* \arrow[r, "1_{!A} \multimap \ell_I"'] & !A \multimap (!I)^* \arrow[rr, "1_{!A} \multimap \mu_I^*"'] & & !A \multimap I^* \arrow[uu, "1_{!A} \multimap \upsilon_I^{-1}"']
\end{tikzcd}
$$

From this, we compute that:

$$
\begin{aligned}
&(\ell_A^{\mathsf{e}^{\ell}} \otimes 1_{!A}); \epsilon_{!A,I} \overset{(18)}{=} \mu_{A^*,A}; !\epsilon_{A,I}; \mathsf{e}^{\ell} \overset{(13)}{=} \mu_{A^*,A}; !\epsilon_{A,I}; !\upsilon_I; \ell_I; \mu_I^*; \upsilon_I^{-1} \\
&\overset{(6)}{=} (\xi_{A,I} \otimes 1_{!A}); \epsilon_{!A,!I}; !\upsilon_I; \ell_I; \mu_I^*; \upsilon_I^{-1} \\
&\overset{(5)}{=} (\xi_{A,I} \otimes 1_{!A}); ((1_{!A} \multimap !\upsilon_I) \otimes 1_{!A}); ((1_{!A} \multimap \ell_I) \otimes 1_{!A}); \\
&\qquad ((1_{!A} \multimap !\upsilon_I) \otimes \mu_I^*); \left((1_{!A} \multimap \upsilon_I^{-1}) \otimes 1_{!A}\right); \epsilon_{!A,I} \\
&\overset{(\ell.\mu_I)}{=} (\ell_A \otimes 1_{!A}); \epsilon_{!A,I}
\end{aligned}
$$

So we get that $\ell_A^{\mathsf{e}^{\ell}} = \ell_A$. ◀

So now that we have proven that Laplace distributors do indeed correspond precisely to generalized versions of the exponential functions, let us revisit our distribution intuition for the axioms of a Laplace distributor. So suppose we have an I-exponential map, which recall we wrote as $\mathsf{e}(\delta_x) = e^x$. Then for every linear functional $x^* : A \multimap I$ and $z \in A$, the induced Laplace distributor is given as follows, which also corresponds to the axiom $(\ell.\mu)$:

$$\ell_A(\delta_{x^*})(\delta_z) = e^{x^*(z)} \tag{$\ell.\mu$}$$

Then the three other axioms of a Laplace distributor do indeed correspond to the three main identities of the exponential function:

$$e^{x^*(z)+y^*(z)} = e^{x^*(z)} e^{y^*(z)} \tag{ℓ.c.1}$$

$$e^{0(z)} = 1 \tag{ℓ.w.1}$$

$$\mathsf{D}_0(e^x)(x^*(z)) = x^*(z) \tag{ℓ.d.1}$$

We can also describe the Laplace evaluator on Dirac distributions as:

$$\reflectbox{∂}_A(\delta_{x^*} \otimes \delta_z) = e^{x^*(z)} \tag{22}$$

Moreover, we are also in a position to show that the Laplace distributor also "co-transforms" the structural rules into their mirror co-structural rules in the following sense:

▶ **Lemma 7.** *In a differential linear closed category with a Laplace distributor, the following diagrams commute:*

$$\begin{tikzcd}
\oc A^* \arrow[rr, "\ell_A"] \arrow[d, "\mathsf{c}_{A^*}"'] & & (\oc A)^* \arrow[dd, "\overline{\mathsf{c}}^*_A"] \\
\oc A^* \otimes \oc A^* \arrow[d, "\ell_A \otimes \ell_A"'] & (\ell.\mathrm{c}.2) & \\
(\oc A)^* \otimes (\oc A)^* \arrow[rr, "\Theta_{\oc A, \oc A}"'] & & (\oc A \otimes \oc A)^*
\end{tikzcd}
\qquad
\begin{tikzcd}
\oc A^* \arrow[r, "\ell_A"] \arrow[d, "\mathsf{w}_{A^*}"'] \arrow[dr, phantom, "(\ell.\mathrm{w}.2)"] & (\oc A)^* \arrow[d, "\overline{\mathsf{w}}^*_A"] \\
I \arrow[r, "\upsilon_I"'] & I^*
\end{tikzcd}
\qquad
\begin{tikzcd}
\oc A^* \arrow[r, "\ell_A"] \arrow[dr, "\mathsf{d}_{A^*}"'] & (\oc A)^* \arrow[d, "\overline{\mathsf{d}}^*_A"] \\
(\ell.\mathrm{d}.2) & A^*
\end{tikzcd}$$

Proof. By Thm 6, we now know that $\ell_A : !A^* \to (!A)^*$ is the unique map such that:

$$(\ell_A \otimes 1_{!A}); \epsilon_{!A,I} = \mu_{A^*,A}; !\epsilon_{A,I}; \mathsf{e}^\ell \tag{23}$$

So by using this to our advantage, we can use the same techniques as in the proof of Prop 5 as well as the right side versions of (19), which recall were the compatibility relations between μ and the co-structural maps [2, Prop 2 & Prop 5], to then show that the desired identities hold. Indeed, for (ℓ.d.2), we compute that (omitting subscripts for readability again):

$$(\ell \otimes 1); (\overline{\mathsf{d}}^* \otimes 1); \epsilon \overset{(10)}{=} (\ell \otimes 1); (1 \otimes \overline{\mathsf{d}}); \epsilon = (1 \otimes \overline{\mathsf{d}}); (\ell \otimes 1); \epsilon \overset{(18)}{=} (1 \otimes \overline{\mathsf{d}}); \mu; !\epsilon; \mathsf{e}^\ell$$
$$\overset{(19)}{=} (\mathsf{d} \otimes 1); \overline{\mathsf{d}}; !\epsilon; \mathsf{e}^\ell \overset{\text{nat.}}{=} (\mathsf{d} \otimes 1); \epsilon; \overline{\mathsf{d}}; \mathsf{e}^\ell \overset{(\mathsf{e}.\overline{\mathsf{d}})}{=} (\mathsf{d} \otimes 1); \epsilon$$

So we get that $\ell; \overline{\mathsf{d}}^* = \mathsf{d}$. We can also compute ($\ell$.c.2) and ($\ell$.w.2) via similar computations. ◀

Using our distribution intuition, we see that these identities for the Laplace distributor correspond again to the three main identities of the exponential function, but expressed in a slightly different manner. So for $y, z \in A$ and $x^* \in A$, we get that:

$$e^{x^*(z+y)} = e^{x^*(z)} e^{x^*(y)} \tag{ℓ.c.2}$$
$$e^{x^*(0)} = 1 \tag{ℓ.w.2}$$
$$\mathsf{D}_0(e^{x^*(\_)})(x^*(z)) = x^*(z) \tag{ℓ.d.2}$$

5 Laplace Transformation

Laplace distributors are also particularly interesting when considered in the *isomix star-autonomous* setting. Recall that a **star-autonomous category** [22, Sec 4.8] is a symmetric monoidal category with a chosen object $\bot$, called the *dualizing object*, such that for every object A, writing $A^\bot := A \multimap \bot$, the canonical map $\varrho_A : A \to A^{\bot\bot}$ is an isomorphism. A star-autonomous category whose dualizing object is the monoidal unit $\bot = I$ is called **isomix** [6, Def 6.5]. So in an isomix star-autonomous category, $A^\bot = A^*$ and therefore we have the isomorphism $A \cong A^{**}$, and thus every object is *reflexive*. Then by a **differential linear isomix category** we mean a differential linear closed category whose underlying symmetric monoidal closed category is an isomix star-autonomous category. This is a natural setting to consider since many important categorical models of Classical DiLL are isomix.

Now in an isomix star-autonomous category, we can define a new monoidal product $⅋$ defined as $A ⅋ B :=(A^* \otimes B^*)^*$ [22, Sec 4.8], where I still acts as a unit for $⅋$. Moreover there is also a canonical natural transformation $\mathsf{m}_{A,B} : A \otimes B \to A ⅋ B$ called the **mixor** [6, Def 6.2] and can be defined as the following composite:

$$A \otimes B \xrightarrow{\rho_A \otimes \rho_B} A^{**} \otimes B^{**} \xrightarrow{\Theta_{A^*,B^*}} (A^* \otimes B^*)^* = A ⅋ B \tag{24}$$

The mixor is the categorical interpretation of the mix rule [13]. Moreover, in a differential linear isomix category $\mathcal{C}$, we can also define the functor $? : \mathcal{C} \to \mathcal{C}$ as $?(\_) = (!\_^*)^*$, and it comes equipped with dual versions of the structural maps of !. So we have natural transformations $\mathsf{p}^?_A : ??A \to ?A$, $\mathsf{d}^?_A : A \to ?A$, $\mathsf{c}^?_A : ?A \parr ?A \to ?A$, $\mathsf{w}^?_A : I \to ?A$, $\mu^?_{A,B} : ?(A \parr B) \to ?A \parr ?B$, $\mu^?_I : ?I \to I$, $\overline{\mathsf{c}}^?_A : ?A \to ?A \parr ?A$, $\overline{\mathsf{w}}^?_A : ?A \to I$, and $\overline{\mathsf{d}}^?_A : ?A \to A$. For example, $\mathsf{d}^?_A : A \to ?A$ and $\overline{\mathsf{d}}^?_A : ?A \to A$ are defined as the following composites:

$$\mathsf{d}^?_A := A \xrightarrow{\rho_A} A^{**} \xrightarrow{\mathsf{d}^*_{A^*}} (!A^*)^* = ?A \qquad \overline{\mathsf{d}}^?_A := ?A = (!A^*)^* \xrightarrow{\overline{\mathsf{d}}^*_{A^*}} A^{**} \xrightarrow{\rho^{-1}_A} A \tag{25}$$

In particular, this makes $\mathcal{C}^{op}$ a differential linear isomix category as well with monoidal product $\parr$ and with ? its monoidal differential modality.

Now suppose that we have a Laplace distributor $\ell_A : !A^* \to (!A)^*$, and consider its induced Laplace evaluator $\partial_A : !A^* \otimes !A \to I$. Currying the $!A$ gives us back the Laplace distributor. On the other hand, if we curry the $!A^*$ instead, we obtain a map of type $!A \to (!A^*)^* = ?A$, which we call the *Laplace transformation*. Alternatively, the Laplace transformation is the dual of the Laplace distributor, up to the reflexivity isomorphism. Moreover, where the mixor gave a mix rule from the multiplicative conjunction $\otimes$ to the multiplicative disjunction $\parr$, the Laplace transformation provides a mix rule from the exponential conjunction ! to the exponential disjunction ?.

▶ **Definition 8.** *In a differential linear isomix category with a Laplace distributor ℓ, its associated* ***Laplace transformation*** *is the natural transformation $\mathcal{L}_A : !A \to ?A$ defined as:*

$$\mathcal{L}_A : = !A \xrightarrow{\varrho_{!A}} (!A)^{**} \xrightarrow{\ell^*_A} (!A^*)^* = ?A \tag{26}$$

or equivalently, as the unique map such that the following equality holds:

$$(\mathcal{L}_A \otimes 1_{!A^*}); \epsilon_{!A^*,I} = \sigma_{!A,!A^*}; \partial_A \tag{27}$$

Using our distribution intuition, an element of $?A$ is a smooth map $A^* \to I$. Then for every $z \in A$, $\mathcal{L}(\delta_z) : A^* \to I$ is the smooth map defined as:

$$\mathcal{L}(\delta_z)(x^*) = e^{x^*(z)} \tag{28}$$

The induced I-exponential map can also nicely be described in terms of the Laplace transformation:

▶ **Lemma 9.** *For a differential linear isomix category with a Laplace distributor, its induced Laplace transformation is equal to the following composite:*

$$\mathcal{L}_A : = !A \xrightarrow{!\varrho_A} !A^{**} \xrightarrow{\ell_{A^*}} (!A^*)^* = ?A \tag{29}$$

and furthermore, the induced I-exponential map is equal to the following composite:

$$\mathsf{e}^\ell := !I \xrightarrow{\mathcal{L}_I} ?I \xrightarrow{\mu^?_I} I \tag{30}$$

Proof. It is straightforward to check that (26) and (29) both satisfy (27), and therefore must be equal. Then (30) follows from the fact that $\mu^?_I = !(\upsilon^*_I); \mu^*_I; \upsilon^{-1}_I$ and that $!\rho_I; !(\upsilon^*_I) = !\upsilon_I$. ◀

Moreover, the analogues of the diagrams of a Laplace distributor in Def 1 and Lemma 7 have nice representations for the Laplace transformation, which is proven easily through the reflexivity isomorphism.

▶ **Proposition 10.** *In a differential linear isomix category with a Laplace distributor, the following diagrams commute:*

$$\begin{tikzcd}
!A \otimes !A \arrow[rr, "\overline{\mathsf{c}}_A"] \arrow[d, "\mathsf{m}_{!A,!A}"'] & & !A \arrow[dd, "\mathcal{L}_A"] \\
!A \parr !A \arrow[d, "\mathcal{L}_A \parr \mathcal{L}_A"'] \arrow[rr, phantom, "(\mathcal{L}.\mathsf{c}.1)"] & & \\
?A \parr ?A \arrow[rr, "\mathsf{c}^?_A"'] & & ?A
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\overline{\mathsf{w}}_A"] \arrow[dr, bend right, "\mathsf{w}^?_A"'] \arrow[dr, phantom, "(\mathcal{L}.\mathsf{w}.1)"] & !A \arrow[d, "\mathcal{L}_A"] \\
& ?A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\overline{\mathsf{d}}_A"] \arrow[dr, bend right, "\mathsf{d}^?_A"'] \arrow[dr, phantom, "(\mathcal{L}.\mathsf{d}.1)"] & !A \arrow[d, "\mathcal{L}_A"] \\
& ?A
\end{tikzcd}$$

$$\begin{tikzcd}
!A \arrow[rr, "\mathsf{c}_A"] \arrow[dd, "\mathcal{L}_A"'] & & !A \otimes !A \arrow[d, "\mathsf{m}_{!A,!A}"] \\
& (\mathcal{L}.\mathsf{c}.2) & !A \parr !A \arrow[d, "\mathcal{L}_A \parr \mathcal{L}_A"] \\
?A \arrow[rr, "\overline{\mathsf{c}}^?_A"'] & & ?A \parr ?A
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mathcal{L}_A"] \arrow[dr, bend right, "\mathsf{w}_A"'] \arrow[dr, phantom, "(\mathcal{L}.\mathsf{w}.2)"] & ?A \arrow[d, "\overline{\mathsf{w}}^?_A"] \\
& I
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mathcal{L}_A"] \arrow[dr, bend right, "\mathsf{d}_A"'] \arrow[dr, phantom, "(\mathcal{L}.\mathsf{d}.2)"] & ?A \arrow[d, "\overline{\mathsf{d}}^?_A"] \\
& A
\end{tikzcd}$$

Proof. Clearly, ($\mathcal{L}$.c.1), ($\mathcal{L}$.w.1), etc. are precisely the duals (in the sense of applying the contravariant functor $*$) of (ℓ.c.1), (ℓ.w.1), etc. up to the reflexivity isomorphism. ◀

So we clearly see how the Laplace transformation does indeed transform the co-structural rules of ! into the structural rules of ?, where the latter are the dual of the structural rules of !. Moreover, we note that ($\mathcal{L}$.c.1 & 2) and ($\mathcal{L}$.w.1 & 2) say that the Laplace transformation is a morphism from a $\otimes$-(co)monoid to a $\parr$-(co)monoid. This is a key idea for the exponential modality in *dagger* linear logic [8].

Another natural question to ask is if from a Laplace distributor ℓ, we can get isomix star-autonomy. To answer this, note that the functor ? can be defined for any symmetric monoidal closed category. However, ? will not have all the structural maps above since $\parr$ is no longer necessarily a monoidal product. Nevertheless, for any differential linear closed category with a Laplace distributor, we can always get a natural transformation of type $\mathcal{L}_A : !A \to ?A$. If there is also a map of dual type which agrees on the A parts of $!A$ and $?A$ then we do get isomix star-autonomy. This is particularly the case when $\mathcal{L}_A : !A \to ?A$ is an isomorphism. In calculus, that the Laplace transform is reversible is particularly important: this makes it a tool to go from the differential world to the polynomial world and back.

▶ **Proposition 11.** *Let $\mathcal{C}$ be a differential linear closed category with a Laplace distributor ℓ and induced natural transformation $\mathcal{L}_A : !A \to ?A$ as defined in (29). If there is a natural transformation $\mathcal{L}^\bullet : ?A \to !A$ such that the following equalities hold:*

$$\overline{\mathsf{d}}_A; \mathcal{L}_A; \mathcal{L}^\bullet_A; \mathsf{d}_A = 1_A \qquad\qquad \mathsf{d}^*_{A^*}; \mathcal{L}^\bullet_A; \mathcal{L}_A; \overline{\mathsf{d}}^*_{A^*} = 1_{A^{**}} \tag{31}$$

then $\mathcal{C}$ is isomix star-autonomous. In particular, if $\mathcal{L}$ is an isomorphism, then $\mathcal{C}$ is isomix star-autonomous.

Proof. The key to this proof is that by definition of the co-dereliction [2, Def 9], we have that $\overline{\mathsf{d}}; \mathsf{d} = 1$. From this, naturality, ($\mathcal{L}$.d.1), and (ℓ.d.1), we get that $\rho = \overline{\mathsf{d}}; \mathcal{L}; \overline{\mathsf{d}}^*$. Then define $\rho^{-1} = \mathsf{d}^*; \mathcal{L}^\bullet; \mathsf{d}$. Then from (31) and ($\ell$.d.1 & 2), one easily checks that $\rho; \rho^{-1} = 1$ and $\rho^{-1}; \rho = 1$. So we conclude that $\mathcal{C}$ is isomix star-autonomous as desired. Now if $\mathcal{L}$ was an isomorphism to begin with, setting $\mathcal{L}^\bullet = \mathcal{L}^{-1}$, it follows from ($\ell$.d.1 & 2) that (31) holds. ◀

6 Examples

In this section, we give examples of Laplace distributors/evaluators, I-exponential maps, and Laplace transformations in well-known and important examples of differential categories.

▶ **Example 12** (Relations). An important model of DiLL is the relational model. So let REL be the category whose objects are sets X and where a map $R : X \to Y$ is a relation, that is, a subset $R \subseteq X \times Y$. REL is a differential linear isomix category; for full details, see [18, Sec IV.A]. In particular, for a set X, $!X = \mathfrak{M}_f(X)$ is the set of finite multisets of X. Moreover, the monoidal unit is a chosen singleton $I = \{*\}$, and (up to isomorphism) we may associate $X^* = X$. From this point of view $?X = !X$, and we trivially see that the identity:

$$1_{!X} = \{(B, B) | \forall B \in !X\} \subseteq !X \times !X$$

is a Laplace distributor and its induced Laplace transformation. Moreover, the induced $\{*\}$-exponential map is the one that relates every bag to the single element, in other words:

$$\mathsf{e} = !\{*\} \times \{*\}$$

REL is also an example of both the more general settings described in Ex 15 and Ex 16 below.

▶ **Example 13** (Weighted Relations). The relation model can be generalized by considering *weighted* relations [23, Sec III] over a *complete commutative semiring*. Recall that a commutative semiring R is complete if sums of elements of R indexed by arbitrary sets are well-defined in R, and these sums satisfy natural distributivity and partition axioms [23, Sec III.C]. For a complete commutative semiring R, define the category R^Π whose objects are sets X, and where a map from X to Y is a function $f : X \times Y \to R$. Composition of $f : X \times Y \to R$ and $g : Y \times Z \to R$ is defined as $(f;g)(x,z) = \sum_{y \in Y} f(x,y)g(y,z)$, which is well-defined since R is complete. The identity is the Kronecker delta function $\delta_X : X \times X \to R$ defined as $\delta_X(x,y) = 0$ if $x \neq y$ and $\delta_X(x,x) = 1$. Then R^Π is a differential linear isomix category; for full details, see [18, Sec IV.B]. In particular, as in Ex 12, the modality is $!X = \mathfrak{M}_f(X)$, the monoidal product is the Cartesian product of sets $\times$ (which is not the categorical product), and the monoidal unit is $I = \{*\}$. However, in general, R^Π will not have an $\{*\}$-exponential map. Suppose that we do have an $\{*\}$-exponential map, so a function $\mathsf{e} : !\{*\} \times \{*\} \to R$. First note that elements of $!\{*\}$ can be associated with the natural numbers: so for every $n \in \mathbb{N}$, let $[n]$ be the finite multiset with n copies of $*$. Now the co-dereliction is:

$$\overline{\mathsf{d}}_{\{*\}}(*, [n]) = \delta_{\mathbb{N}}(1, n)$$

Then (e.$\overline{\mathsf{d}}$) would give us that: $\mathsf{e}([1], *) = 1$. On the other hand, the co-contraction is: $\overline{\mathsf{c}}_{\{*\}}(([n],[m]),[k]) = \binom{n+m}{m}\delta_{\mathbb{N}}(n+m,k)$. So (e.$\overline{\mathsf{c}}$) would give us that:

$$\binom{n+m}{m}\mathsf{e}([n+m], *) = \mathsf{e}([n], *)\mathsf{e}([m], *)$$

Now, taking $n = m = 1$ in this last equality, we would get that $2\mathsf{e}([2], *) = 1$, which says that $\mathsf{e}([2], *)$ is an inverse of 2 in R. However, 2 is not always a unit in an arbitrary semiring. For example, $R = \mathbb{N} \sqcup \{\infty\}$ is a complete commutative semiring for which 2 is not invertible. Therefore, $(\mathbb{N} \sqcup \{\infty\})^\Pi$ is a differential linear closed/isomix category which does not have an $\{*\}$-exponential map. Now if every n is invertible in R, then R^Π does have a $\{*\}$-exponential map given by the function $\mathsf{e} : !\{*\} \times \{*\} \to R$ defined as: $\mathsf{e}([n], *) = \frac{1}{n!}$. That is indeed an

$\{*\}$-exponential map follows from the fact that R^Π is an example of the general settings described in Ex 15 and Ex 16. Moreover, this example also recaptures Ex 12 since by taking the Boolean semiring $\mathbb{B} = \{0, 1\}$, we get back $\mathbb{B}^\Pi \cong \mathsf{REL}$, and since $1 + 1 = 1$ in $\mathbb{B}$, the factor $\frac{1}{n!}$ disappears in the descriptions of the $\{*\}$-exponential map in REL.

▶ **Example 14** (Convenient Vector Spaces)**.** Throughout the paper, we used distributions for intuition. We can make this precise by considering the differential category of *convenient spaces*, introduced by Blute, Ehrhard, and Tasson in [1]. Briefly, a **convenient vector space** [1, Def 2.9] is a special kind of locally convex vector space which in particular has a bornology for which it is Mackey complete. This allows us to define smooth functions (in the usual analysis sense) between convenient vector spaces E and F. Let $\mathcal{C}^\infty(E, F)$ be the set of smooth functions between them, which is itself a convenient vector space. Also, the reals $\mathbb{R}$ is a convenient vector space, and for a convenient vector space E, we let E^* be the vector space of *linear* smooth functions $E \to \mathbb{R}$, which is again a convenient vector space. Now, for every convenient vector space E, we have a smooth function $\delta : E \to \mathcal{C}^\infty(E, \mathbb{R})^*$ which maps $x \in E$ to its associated Dirac distribution $\delta_x \in \mathcal{C}^\infty(E, \mathbb{R})^*$. Then CON, the category of convenient vector spaces and linear smooth functions between them, is a differential linear closed category, where $!E$ is the Mackey completion of $\delta(E) \subseteq \mathcal{C}^\infty(E, \mathbb{R})^*$ [1, Def 5.2]. Moreover, for every smooth function $f : E \to F$ there is a unique linear smooth function $f^\sharp : !E \to F$ such that $f(x) = f^\sharp(\delta_x)$ [1, Thm 5.5]. Now the monoidal unit in CON is $\mathbb{R}$ and since the classical exponential function $e^z : \mathbb{R} \to \mathbb{R}$ is smooth, there exists a unique linear smooth function $\mathsf{e} : !\mathbb{R} \to \mathbb{R}$ such that:

$$e^z = \mathsf{e}(\delta_z)$$

From this, it immediately follows that $\mathsf{e} : !\mathbb{R} \to \mathbb{R}$ is an $\mathbb{R}$-exponential map. Therefore, the Laplace distributor and Laplace evaluator are precisely the unique linear smooth functions such that:

$$\ell_E(\delta_{x^*}) = e^{x^*(-)} \qquad\qquad \partial_A(\delta_{x^*} \otimes \delta_z) = e^{x^*(z)}$$

As such, this is the model which properly interprets the Laplace transform as operating on Dirac distributions as discussed in Section 3.

▶ **Example 15** (Countable Sums)**.** In calculus, the exponential function can be written out as the power series:

$$e^x = \sum_{n \in \mathbb{N}} \frac{x^n}{n!}$$

In a setting where we have countable sums and can scalar multiply by positive rationals $\mathbb{Q}_{\geq 0}$, the same formula holds for constructing an I-exponential map. So suppose that we are in a $\mathbb{Q}^{\boldsymbol{\Sigma}}_{\boldsymbol{\geq 0}}$**-differential linear closed/isomix category** as was considered in [18, Sec III.E], which means that each homset is a countably complete $\mathbb{Q}_{\geq 0}$-module. Now for every $n \in \mathbb{N}$, let $\mathsf{c}^n_A : !A \to !A^{\otimes^n}$ be the map which co-multiplies $!A$ into n-copies of $!A$, and then define $\mathsf{d}^n_A : !A \to A^{\otimes^n}$ as the composite:

$$\mathsf{d}^n_A := \; !A \xrightarrow{\mathsf{c}^n_A} !A \otimes \ldots \otimes !A \xrightarrow{\mathsf{d}_A \otimes \ldots \otimes \mathsf{d}_A} A \otimes \ldots \otimes A$$

With these maps, we can construct an I-exponential map defined as follows:

$$\mathsf{e} := \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(!I \xrightarrow{\mathsf{d}^n_I} I^{\otimes^n} \xrightarrow{\cong} I \right)$$

Checking that this is an I-exponential map is the same proof that checking that e^x satisfies the analogue identities using its power series. Then, the induced Laplace distributor, Laplace evaluator, and Laplace transformation are given as follows:

$$\ell_A := \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(!A^* \xrightarrow{\mathsf{d}^n_{A^*}} A^* \otimes \ldots \otimes A^* \xrightarrow{\Theta_{A,\ldots,A}} (A \otimes \ldots \otimes A)^* \xrightarrow{\mathsf{d}^{n\,*}_A} (!A)^* \right)$$

$$\reflectbox{ℓ}_A := \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(!A^* \otimes !A \xrightarrow{\mathsf{d}^n_{A^*} \otimes \mathsf{d}^n_A} (A^*)^{\otimes^n} \otimes A^{\otimes^n} \xrightarrow{\cong} (A^* \otimes A)^{\otimes^n} \xrightarrow{\epsilon^{\otimes^n}_{A,I}} I^{\otimes^n} = I \right)$$

$$\mathcal{L}_A := \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(!A \xrightarrow{\mathsf{d}^n_A} A \otimes \ldots \otimes A \xrightarrow{\mathsf{m}_{A,\ldots,A}} A \parr \ldots \parr A \xrightarrow{\mathsf{d}^{?\,n}_A} ?A \right)$$

▶ **Example 16** (Co-digging). In [18], the authors introduced the notion of *co-digging* for differential categories, which is the co-structural version of digging. Briefly, **co-digging** [18, Def III.3] for a differential linear category is a natural transformation of dual type of the digging, $\overline{\mathsf{p}}_A : !!A \to !A$, which satisfies the dual axioms of the digging. Using our distribution intuition, the co-digging corresponds to the notion of convolution exponential for distributions:

$$\overline{\mathsf{p}}_A(\delta_{\delta_z}) = \sum_{n \in \mathbb{N}} \frac{\delta_{nx}}{n!} \tag{32}$$

See [18, Sec III.C] for more details. The co-digging always induces an I-exponential map $\overline{\mu}_I : !I \to I$ [18, Lemma III.4] defined as the composite:

$$\overline{\mu}_I := \; !I \xrightarrow{!\overline{\mathsf{w}}_I} !!I \xrightarrow{\overline{\mathsf{p}}_I} !I \xrightarrow{\mathsf{w}_I} I \tag{33}$$

Therefore, every differential linear closed/isomix category with co-digging has a Laplace distributor/transformation. Furthermore, if $\overline{\mathsf{p}}$ and μ are compatible in the sense that the following diagram commutes:

$$\begin{tikzcd}
!!A \otimes !B \arrow[r, "1_{!A} \otimes \mathsf{p}_B"] \arrow[d, "\overline{\mathsf{p}}_A \otimes 1_{!B}"'] & !!A \otimes !!B \arrow[r, "\mu_{!A,!B}"] & !\,(!A \otimes !B) \arrow[r, "!(\mu_{A,B})"] & !!(A \otimes B) \arrow[d, "\overline{\mathsf{p}}_{!A,!B}"] \\
!A \otimes !B \arrow[rrr, "\mu_{A,B}"'] & & & A \otimes B
\end{tikzcd} \tag{34}$$

which is the same kind of compatiblity as those in (19) − then using the same techniques as in the proof of Prop 5, we also get that the Laplace distributor/transformation also (co-)transforms the co-digging into the digging:

$$\begin{tikzcd}
!!A^* \arrow[r, "\overline{\mathsf{p}}_{A^*}"] \arrow[d, "!(\ell_A)"'] & !A^* \arrow[dd, "\ell_A"] \\
!(!A)^* \arrow[d, "\ell_{!A}"'] \arrow[r, phantom, "(\ell.\mathsf{p}.1)"] & \\
(!!A)^* \arrow[r, "\mathsf{p}^*_A"'] & (!A)^*
\end{tikzcd}
\qquad
\begin{tikzcd}
!A^* \arrow[r, "\ell_A"] \arrow[d, "\mathsf{p}_{A^*}"'] & (!A)^* \arrow[dd, "\overline{\mathsf{p}}^*_A"] \\
!!A^* \arrow[d, "!(\ell_A)"'] \arrow[r, phantom, "(\ell.\mathsf{p}.2)"] & \\
!(!A)^* \arrow[r, "\ell_{!A}"'] & (!!A)^*
\end{tikzcd}
\qquad
\begin{tikzcd}
!!A \arrow[r, "\overline{\mathsf{p}}_A"] \arrow[d, "!(\mathcal{L}_A)"'] & !A \arrow[dd, "\mathcal{L}_A"] \\
!?A \arrow[d, "\mathcal{L}_{?A}"'] \arrow[r, phantom, "(\mathcal{L}.\mathsf{p}.1)"] & \\
??A \arrow[r, "\mathsf{p}^?_A"'] & ?A
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mathcal{L}"] \arrow[d, "\mathsf{p}_A"'] & ?A \arrow[dd, "\overline{\mathsf{p}}^?_A"] \\
!!A \arrow[d, "!(\mathcal{L}_A)"'] \arrow[r, phantom, "(\mathcal{L}.\mathsf{p}.2)"] & \\
!?A \arrow[r, "\mathcal{L}_{?A}"'] & ??A
\end{tikzcd}$$

These diagrams amount to interpreting the exponential function of the exponential function, e^{e^z}. So, in particular, in terms of the Laplace distributor:

$$e^{e^{x^*(z)}} = \sum_{n\in\mathbb{N}} \frac{e^{nx^*(z)}}{n!} \tag{ℓ.p.1}$$

$$e^{e^{x^*(z)}} = \sum_{n\in\mathbb{N}} \frac{e^{x^*(nz)}}{n!} \tag{ℓ.p.2}$$

Interesting examples of models with co-digging can be found in [18, Sec IV], which include the (weighted) relational model described above.

▶ **Example 17** (Finiteness Spaces)**.** Finiteness spaces [10] are a well-known refinement of REL, giving a vectorial model of DiLL [11]. For a set X, and a subset of its powerset $F \subseteq \mathcal{P}(X)$, we denote by $F^\perp \subseteq \mathcal{P}(X)$ the subsets of $U \subseteq X$ such that for all $V \in F$, $U \cap V$ is finite. Then a **finiteness space** [10, Sec 1] is a pair $\mathcal{X} = (|\mathcal{X}|, \mathfrak{F}(\mathcal{X}))$ of a set $|\mathcal{X}|$ and a subset $\mathfrak{F}(\mathcal{X}) \subseteq \mathcal{P}(|\mathcal{X}|)$ which verifies the finiteness condition $\mathfrak{F}(\mathcal{X})^{\perp\perp} = \mathfrak{F}(\mathcal{X})$. Given a field k, every finiteness space $\mathcal{X}$ generates a linear topological k-vector space $k\langle\mathcal{X}\rangle$ defined as the set of all families $x \in k^{|\mathcal{X}|}$ such that $\mathsf{supp}(x) = \{a \in |X| | x_a \neq 0\} \in \mathfrak{F}(X)$ [10, Sec 3]. Then let FIN^k be the category whose objects are finiteness spaces and where a map from $\mathcal{X}$ to $\mathcal{Y}$ is a linear continuous function $M : k\langle\mathcal{X}\rangle \to k\langle\mathcal{Y}\rangle$, which can be described as an $|\mathcal{X}| \times |\mathcal{Y}|$ matrix $M_{x,y}$. Then FIN^k is a differential linear isomix category; see [11, Sec 5] for full details. The finiteness space $!\mathcal{X}$ has as carrier $|!\mathcal{X}| := \mathfrak{M}_f(|\mathcal{X}|)$, the finite multisets over $|\mathcal{X}|$, and as finiteness structure $\mathfrak{F}(!\mathcal{X}) \subseteq \mathcal{P}(\mathfrak{M}_f(|\mathcal{X}|))$ the collection of all sets of multisets on $|X|$ whose union is in $\mathfrak{F}(\mathcal{X})$ [10, Sec 1.1]. On the other hand, the monoidal unit is $I = (\{*\}, \mathcal{P}(*))$. Then FIN^k has an I-exponential map given by the exponential function described by Ehrhard in [10, Lemma 19], that is, the linear continuous function $\mathsf{e} : k\langle !I\rangle \to k\langle I\rangle$ whose associated matrix is:

$$e_{n,*} = \frac{1}{n!}$$

Now for a finiteness space $\mathcal{X}$, its dual is the finiteness space $\mathcal{X}^* = (|\mathcal{X}|, \mathfrak{F}(\mathcal{X})^\perp)$. Then both the induced Laplace distributor $\ell_\mathcal{X} : k\langle !\mathcal{X}^*\rangle \to k\langle (!X)^*\rangle$ and Laplace transformation $\mathcal{L}_\mathcal{X} : k\langle !X\rangle \to k\langle ?\mathcal{X}\rangle$ have the same associated matrix with coefficients indexed by multisets $m, m' \in \mathfrak{M}_f(|\mathcal{X}|)$:

$$(\ell_\mathcal{X})_{m,m'} = (\mathcal{L}_\mathcal{X})_{m,m'} = e^{m'(m)} = e^{\sum_{x\in|\mathcal{X}|} m(x)m'(x)}$$

which is well-defined thanks to the orthogonality condition.

▶ **Example 18** (Köthe Spaces)**.** Köthe spaces [9] are a model of DiLL based on spaces of sequences. They are studied independently in functional analysis and correspond to a non-discrete version of finiteness spaces. Let k be the field of real or complex numbers. For a denumerable set X, for sequences $a, b \in k^X$ define the orthogonality relation $a \perp b$ if and only if $\sum_{x\in X} |a_x b_x|$ converges. A Köthe space is pair $\mathcal{X} = (|\mathcal{X}|, E_\mathcal{X})$ of a carrier set $|\mathcal{X}|$ and a subspace $E_\mathcal{X} \subseteq k^{|\mathcal{X}|}$, such that $E_\mathcal{X}^{\perp\perp} = E_X$. Then we get a differential linear isomix category of Köthe spaces, which is similar to the finiteness space model described above. In particular, the exponential function corresponds to taking a converging sequence $a \in k^\mathbb{N}$ to $\sum_{n\in\mathbb{N}} \frac{1}{n!} a_n$. The Laplace distributor/transformation is expressed similarly to finiteness spaces.

▶ **Example 19** (Fréchet and DF spaces)**.** Fréchet spaces are metrizable and complete locally convex topological vector spaces. They enjoy a nice duality theory with DF-spaces. When adding the constraint that these spaces must be nuclear [16], one obtains a model of polarized first-order DiLL [17]: the tensor product of two nuclear DF-space is a nuclear DF-space; nuclear DF or Fréchet spaces are isomorphic to their double duals; $?\mathbb{R}^n = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is nuclear DF; and $!\mathbb{R}^n = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$ is nuclear Fréchet. A nuclear Fréchet spaces N is, in fact, a projective limit of Banach Spaces $N = \bigcap_n N_p$. This construction is taken to the higher-order level in [14], using functions whose exponential growth is bounded. For a Young function θ and for a Banach space B, let $Exp(B, \theta, m)$ denote the Banach space of holomorphic functions from B to $\mathbb{C}$ such that $|f(z)| \leq Ke^{\theta(m||z||)}$. Then, one defines the space of functions with exponential growth of minimal order on N as the inductive limit $\mathcal{G}_{\theta^*}(N)$ of spaces $\mathrm{Exp}(\theta, m, p)$, and also the space of functions with exponential growth of arbitrary order on $N' = \bigcup_p (N_p)' = \bigcup_p N'_{-p}$ as the projective limit $\mathcal{F}_\theta(N')$ of spaces $\mathrm{Exp}(\theta, m, -p)$. In this higher-order setting, the Laplace transform has a finer meaning than in the other examples. Indeed, it transforms distributions on one type of function into another type of function and makes the index θ change:

$$\mathcal{L} : \begin{cases} \mathcal{F}'_\theta(N') & \simeq & \mathcal{G}_{\theta^*}(N) \\ \phi & \mapsto & \left(\ell \in N' \mapsto \phi(x \in N' \mapsto e^{\ell(x)} \in \mathbb{C})\right) \end{cases}$$

where $\theta^* := sup_{t \geq 0}(tx - \theta(t))$ is the convex conjugate of θ. Details about this construction can be found in [18, Sec V]. This opens up fascinating questions on Laplace transforms in polarized differential linear categories [4] or even graded differential linear categories [21].

7 Future Work

In this paper, we gave a new point of view on exponential functions in differential categories, and on the exponential connectives ! and ? in DiLL, thanks to the categorification of the Laplace transform. We defined the Laplace distributor as a transformation from the exponential $!A$ to its dual which transforms co-structural rules into structural rules. We related this new distributor to the presence of an exponential scalar function, and to the involutivity of the duality. We presented several examples, as well as one counter-example. We conclude this paper with a brief discussion of interesting potential future work.

A natural path to consider is generalizing this story from isomix star-autonomous categories to *linearly distributive categories* [5]. Indeed, the diagrams in Prop 10 can easily be written down in a linearly distributive category with the proper notion of exponentials [7, 8]. So one could study Laplace transformations in a linearly distributive setting. However, the linearly distributive generalization of differential categories has not yet been properly defined or studied. So hopefully the story of this paper will motivate the development of such a theory.

Work is also needed on concrete models of Laplace transforms. The original intuition for the categorification of the Laplace transform came from higher-order work in functional analysis [14, 18], in which two kinds of functions with different exponential growth model the two types of exponential connectives, applying to formulas with different polarities [19]. The Laplace transformation then changes distributions on one type of function into distributions on the other type of function. Understanding the categorical interplay between the Laplace transformation and polarity might lead to a better axiomatization of differential linear star-autonomous linear categories.

References

1 R. Blute, T. Ehrhard, and C. Tasson. A convenient differential category. *Cahiers de topologie et géométrie différentielle catégoriques*, 2012.

2 R. F. Blute, J. R. B. Cockett, J.-S. P. Lemay, and R. A. G. Seely. Differential categories revisited. *Applied Categorical Structures*, 2020.

3 R. F. Blute, J. R. B. Cockett, and R. A. G. Seely. Differential categories. *Math. Struct. Comput. Sci.*, 2006.

4 F. Breuvart, M. Kerjean, and S. Mirwasser. Unifying graded linear logic and differential operators. In *8th International Conference on Formal Structures for Computation and Deduction, FSCD*, 2023.

5 J. R. B. Cockett and R. A. G. Seely. Weakly distributive categories. *Journal of Pure and Applied Algebra*, 114(2), 1997.

6 J. R. B. Cockett and R.A.G. Seely. Proof theory for full intuitionistic linear logic, bilinear logic, and mix categories. *Theory and Applications of categories*, 3(5), 1997.

7 J. R. B. Cockett and R.A.G. Seely. Linearly distributive functors. *Journal of Pure and Applied Algebra*, 143(1-3), 1999.

8 J.R.B. Cockett and P. V. Srinivasan. Exponential modalities and complementarity (extended abstract). In Kohei Kishida, editor, Proceedings of the Fourth International Conference on *Applied Category Theory,* Cambridge, United Kingdom, 12-16th July 2021, volume 372 of *Electronic Proceedings in Theoretical Computer Science*. Open Publishing Association, 2022.

9 T. Ehrhard. On Köthe Sequence Spaces and Linear Logic. *Mathematical Structures in Computer Science*, 12(5), 2002.

10 T. Ehrhard. Finiteness spaces. *Mathematical Structures in Computer Science*, 15(4), 2005.

11 T. Ehrhard. An introduction to differential linear logic: proof-nets, models and antiderivatives. *Mathematical Structures in Computer Science*, 2017.

12 T. Ehrhard and L. Regnier. Differential interaction nets. *Theoretical Computer Science*, 364(2), 2006.

13 A. Fleury and C. Retoré. The mix rule. *Mathematical Structures in Computer Science*, 4(2), 1994.

14 R. Gannoun, R. Hachaichi, H. Ouerdiane, and A. Rezgui. Un théorème de dualité entre espaces de fonctions holomorphes à croissance exponentielle. *Journal of Functional Analysis*, 171(1), 2000.

15 J.-Y. Girard. Linear Logic. *Theoretical Computer Science*, 50(1), 1987.

16 A. Grothendieck. Produits tensoriels topologiques et espaces nucléaires. *Memoirs of the AMS*, 16, 1966. Publisher: American Mathematical Society.

17 M. Kerjean. A Logical Account for Linear Partial Differential Equations. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018*, 2018.

18 M. Kerjean and J.-S. P. Lemay. Taylor Expansion as a Monad in Models of DiLL. In *2023 38th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, 2023.

19 O. Laurent. *Etude de la polarisation en logique*. Thèse de Doctorat, Université Aix-Marseille II, March 2002.

20 J.-S. P. Lemay. Exponential functions in cartesian differential categories. *Applied Categorical Structures*, 29, 2021.

21 J.-S. P. Lemay and J.-B. Vienney. Graded differential categories and graded differential linear logic. In *Mathematical Foundations of Programming Semantics*, 2023.

22 P.-A. Melliès. Categorical semantics of linear logic. *Société Mathématique de France*, 2008.

23 C.-H. L. Ong. Quantitative semantics of the lambda calculus: Some generalisations of the relational model. In *31st Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, 2017.