Early Typical Vertices in Subcritical Random Graphs of Preferential Attachment Type

Peter Mörters
University of Cologne, Germany

Nick Schleicher[1]
University of Cologne, Germany

Abstract

We study the size of the connected component of early typical vertices in a subcritical inhomogeneous random graph with a kernel of preferential attachment type. The principal tools in our analysis are, first, a coupling of the neighbourhood of a typical vertex in the graph to a killed branching random walk and, second, an asymptotic result for the number of particles absorbed at the killing barrier in this branching random walk.

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1 Introduction and statement of results

There is currently a huge demand for models of scale-free networks coming from a variety of application areas, ranging from social sciences, telecommunications to power grids. These applications lead to competing demands on the network models: On the one hand they should be amenable to mathematical and statistical analysis; models like stochastic block models or, more generally, inhomogeneous random graphs have shown to be useful here. On the other hand models should also incorporate network features beyond the scale-free distribution of the degrees. A sensible approach here is to go beyond phenomenological modelling of a scale-free network and observe which network features emerge from basic building principles. Preferential attachment, popularized by Barabási and Albert [2], has shown to be a particularly natural and interesting principle. In this paper we study, from a mathematical point of view, an inhomogeneous random graph model with a kernel that mimics the connection probabilities of preferential attachment models. We show that this model, while having many features of more complicated preferential attachment networks, allows a very fine analysis even in the difficult subcritical case, when despite the scale-free degree distribution the network exhibits only weak connectivity.

[1] Corresponding author

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For the general *inhomogeneous graph model* [4] we take a symmetric kernel

$$\kappa : (0,1]^2 \to (0,\infty)$$

and for each $n \in \mathbb{N}$ we build the graph G_n with vertex set $V_n = \{1, \dots, n\}$ by connecting two distinct vertices $i, j \in V_n$ independently with probability

$$p_{ij} := \frac{1}{n}\left(\kappa\left(\tfrac{i}{n}, \tfrac{j}{n}\right) \wedge n\right).$$

For example, in the *stochastic block model* the interval $(0,1]$ is partitioned into finitely many blocks and κ chosen to be constant on the cartesian product of any pair of blocks. In order to get scale-free networks, however, one uses a kernel κ with a singularity at the origin.

In preferential attachment models vertices arrive one-by-one and attach themselves to existing vertices with a preference for powerful vertices, specifically those which already have a large degree. There are various ways to turn this idea into a proper definition, but they all have in common that the expected degree of a fixed vertex i in a network of n vertices grows, as $n \to \infty$, like $\approx c(n/i)^\gamma$, for some constant c and exponent $\gamma \in (0,1)$. Choosing a connection probability of the nth vertex to each earlier vertex $i < n$ which is proportional to its expected degree at time n and picking the proportionality factor such that the expected number of connections remains bounded from zero and infinity leads to a connection probability $p_{i,n} = \beta n^{\gamma-1} i^{-\gamma}$, for some constant $\beta > 0$, which makes the model an inhomogeneous random graph with kernel

$$\kappa(x,y) = \beta(x \vee y)^{\gamma-1}(x \wedge y)^{-\gamma},$$

where $0 < \gamma < 1$ parametrizes the strength of the preferences of early vertices and $\beta > 0$ is an edge density parameter. We call this model the *inhomogeneous random graph of preferential attachment type* and explore some of its properties here.

The graph has a phase transition in the sense that if and only if the parameters γ and β are big enough, there exists a component of the graph of macroscopic size. More precisely (G_n) has a giant component if the size S_n of the largest connected component in G_n satisfies

$$\frac{S_n}{n} \to \theta > 0 \text{ in probability.}$$

For the inhomogeneous random graph of preferential attachment type we have:

▶ **Theorem 1.** *A giant component exists if and only if*

$$\gamma \geq \frac{1}{2} \quad \textit{or} \quad \beta > \beta_c := \frac{1}{4} - \frac{\gamma}{2}.$$

This is a simplification of the main result in [5]. The proof can be based on taking a weak local limit in the graph, a sketch of the argument can be found in [11] .

In this paper we are primarily interested in the *subcritical regime*, i.e. when $\gamma < \frac{1}{2}$ and $0 < \beta < \beta_c$. In this case all component sizes are of smaller order than n. Our main result, Theorem 2 below, identifies the component sizes of vertices in a moving observation window, which we call early typical vertices. More precisely, a sequence of vertices $o_n \in V_n$ is called *typical* if $o_n/n \to u$ for some $u > 0$ and our observation window comprises typical vertices with small u, which are the *early typical vertices*. We show that these vertices have a connected component of asymptotic size $Yu^{-\rho_-}$ independent of n, where ρ_- is an explicitly given exponent and Y a positive random variable, whose tail behaviour we also identify.

▶ **Theorem 2.** *Let $S_n(i)$ be the size of the connected component of vertex $i \in V_n$ in the inhomogeneous random graph of preferential attachment type in the subcritical regime. If $o_n \in V_n$ is such that $\frac{o_n}{n} \to u \in (0,1]$, then*

$$\lim_{u \downarrow 0} \lim_{n \to \infty} \mathbb{P}\left(S_n(o_n) \geq u^{-\rho_-} x\right) = \mathbb{P}\left(Y \geq x\right),$$

for all $x > 0$, where

$$\rho_{\pm} = \tfrac{1}{2} \pm \sqrt{(\gamma - \tfrac{1}{2})^2 + \beta(2\gamma - 1)}.$$

and Y is a positive random variable satisfying

$$\mathbb{P}\left(Y \geq x\right) = x^{-(\rho_+/\rho_-)+o(1)} \text{ as } x \to \infty.$$

The remainder of the paper explains the ideas behind the proof of Theorem 2. We first look at the inner limit, when $n \to \infty$, which we investigate using a coupling of the neighbourhood of vertex o_n to a killed branching random walk. This is done in Section 2. In Section 3 we study the number of particles absorbed at the killing boundary of the branching random walk, from which our result follows.

2 Local coupling

The main object in this section is a branching random walk on the real line with a killing barrier at the origin. The branching random walk is started with a particle located in $\log u < 0$ and the displacements of the children of a vertex are given by an independent Poisson point process with intensity

$$\pi(\mathrm{d}y) = \beta(\mathrm{e}^{\gamma y} \mathbb{1}_{y>0} + \mathrm{e}^{(1-\gamma)y} \mathbb{1}_{y<0})\, \mathrm{d}y\,.$$

As π is an infinite measure initially every particle has infinitely many children, but we kill all particles located to the right of the killing barrier together with their offspring. As a result the killed branching process lives entirely on the negative half axis and it turns out that, for parameters $\gamma < \frac{1}{2}$ and $\beta < \beta_c$, the killed branching process becomes extinct after a finite number of generations and its genealogical tree is therefore finite. We denote this marked tree (with the vertex locations as marks) by $\mathcal{T}(u)$ and by $T(u)$ the number of vertices in this tree. The main result of this section is the following proposition.

▶ **Proposition 3.** *If $o_n \in V_n$ is such that $\frac{o_n}{n} \to u \in (0,1]$ and $x > 0$, then*

$$\lim_{n \to \infty} \mathbb{P}\left(S_n(o_n) \geq u^{-\rho_-} x\right) = \mathbb{P}\left(T(u) \geq u^{-\rho_-} x\right).$$

The proof is an adaptation of arguments in [5] to our model. It is based on a coupling of the neighbourhood of vertex o_n in the graph G_n to the killed branching random walk starting with a particle in location $\log u_n$, for a suitable sequence (u_n) with $u_n \to u$, which we will carry out in two steps in the following sections.

2.1 First step: Coupling to a random labelled tree

We first couple our graph to a tree, which we call the *random labelled tree*. Each vertex of this tree carries a label from the set $\{1, \cdots, n\}$, we denote by $\mathbb{T}_n(o)$ the tree with the root labelled by $o \in \{1, \cdots, n\}$. Every vertex with label $i \in \{1, \ldots, n\}$ produces independently for every $j \in \{1, \ldots, i-1, i+1, \ldots, n\}$ exactly one offspring of label j with probability $p_{ij} := \beta(i \vee j)^{\gamma-1}(i \wedge j)^{-\gamma} \wedge 1$. Note that different vertices in $\mathbb{T}_n(o)$ may carry the same label.

We now use a depth first search on the graph G_n to couple the connected component of o_n to the random labelled tree $\mathbb{T}_n(o_n)$. Sequences (b_n) and (c_n) with $b_n, c_n \in \{1, \dots, n\}$, which we specify later, are used to stop the coupling when certain bad events occur. The coupling of the random labelled tree $\mathbb{T}_n(o_n)$ and the connected component of o_n in the graph G_n is defined on a probability space of Bernoulli variables. For every unordered pair $\{i, j\}$ of distinct labels in $\{1, \dots, n\}$ we generate a sequence $P_{ij}^{(1)}, P_{ij}^{(2)}, \dots,$ of independent Bernoulli variables $(P_{ij}^{(k)})_k$ with parameter p_{ij}. We classify all labels into one of the following categories:

Unseen labels that have not been seen in the construction,

Active labels that have been seen but not yet explored,

Passive labels that have been seen and explored.

Initially, we set $k(\{i, j\}) = 1$ for every unordered pair $\{i, j\}$ of distinct labels. We declare o_n active and all other labels unseen. In every further step, if there are no active labels left we stop and declare the coupling as *successful*. Otherwise we pick the smallest active label, say i, declare it as passive and explore it. This means that, for every $j \in \{1, \dots, i-1, i+1, \dots, n\}$,

- if j is unseen then $k(\{i, j\}) = 1$. We form an edge between i and j in G_n and simultaneously create a child of i with label j in $\mathbb{T}_n(o_n)$ if and only if $P_{ij}^{(1)} = 1$. If we formed an edge in this way we declare the label j as active;
- otherwise, if j is active or passive, and $P_{ij}^{(k(\{i,j\}))} = 1$ we stop and declare the coupling *unsuccessful*, if $P_{ij}^{(k(\{i,j\}))} = 0$ we change neither graph nor tree;
- we increase $k(\{i, j\})$ by one.

If after this step

- one of the active labels has $j \leq b_n$, or
- the total number of active or passive labels exceeds c_n,

then we stop and declare the coupling *unsuccessful*. If we have not stopped we continue the exploration, again with the smallest active label (if there is any).

Observe that this procedure couples the connected component of o_n in the graph G_n based on the variables $(P_{ij}^{(1)} : \{i, j\} \subset \{1, \dots, n\})$ and the random labelled tree $\mathbb{T}_n(o_n)$ based on the variables $(P_{ij}^{(k)} : \{i, j\} \subset \{1, \dots, n\}, k \in \mathbb{N})$ in such a way that for a successful coupling the rooted graph given as the connected component of o_n in G_n coincides with the coupled labelled tree $\mathbb{T}_n(o_n)$. As we are in the subcritical regime we do not expect to see too many labels or very small labels. Hence, for a suitable choice of the sequences (b_n) and (c_n), we expect unsuccessful coupling to be unlikely.

The main technical result of this section confirms this intuition.

▶ **Proposition 4.** *Suppose that $u \in (0, 1]$ and $\frac{o_n}{n} \to u$. If*

$$\lim_{n\to\infty} c_n = \infty \text{ and } \lim_{n\to\infty} \frac{c_n^2}{b_n} = 0 \text{ and } \lim_{n\to\infty} \frac{c_n b_n^\gamma}{n^\gamma} = 0 \,,$$

then with high probability the coupling is successful.

The simple proof is omitted.

2.2 Second step: Coupling to the killed branching random walk

The second step is to couple $\mathbb{T}_n(o_n)$ to the marked tree $\mathcal{T}(u_n)$ of the killed branching random walk started in position $\log u_n$, for a suitable $u_n \to u$. For this purpose, we have to map labels from $\{1, \cdots, n\}$ to positions in $(-\infty, 0]$. We do this using the following map

$$\phi_n \colon \{1, \dots, n\} \to (-\infty, 0] \quad , \quad i \mapsto -\sum_{j=i+1}^{n} \frac{1}{j} \, .$$

Note that the youngest vertex is mapped to the origin, and older vertices are placed to the left with decreasing intensity. Conversely, we define the projection

$$\begin{aligned} \pi_n \colon & (-\infty, 0] \to \{1, \dots, n\} \\ & t \mapsto \min\{m \colon t \le \phi_n(m)\}. \end{aligned}$$

We pick $u_n = \exp(\phi_n(o_n))$ and note that $\frac{o_n}{n} \to u$ implies $u_n \to u$. We now couple $\mathcal{T}(u_n)$ to the random labelled tree $\mathbb{T}_n(o_n)$. We obtain from $\mathcal{T}(u_n)$ the *projection with labels in* $\{1, \dots, n\}$ by taking the particles of $\mathcal{T}(u_n)$ and give each of them the label obtained by applying π_n to its position. However, the process thus obtained is not equal to $\mathbb{T}_n(o_n)$ in law. For example, a particle could have several children with the same label. A more careful coupling is therefore required.

▶ **Proposition 5.** *Suppose that $\frac{o_n}{n} \to u \in (0, 1]$ and $u_n = \exp(\phi_n(o_n))$. If $\mathcal{T}(u_n)$ contains no more than c_n vertices and no vertex in location to the left of $\phi_n(b_n)$ where*

$$\lim_{n\to\infty} \frac{c_n}{b_n^{1-\gamma}} = 0 \text{ and } \lim_{n\to\infty} \frac{c_n n^{\gamma}}{b_n^{\gamma+1}} = 0,$$

then it can be coupled to $\mathbb{T}_n(o_n)$ so that with high probability the projection of $\mathcal{T}(u_n)$ with labels in $\{1, \dots, n\}$ agrees with $\mathbb{T}_n(o_n)$.

We sketch the proof of Proposition 5. We start with the root, which is positioned at $\log(u_n)$. By assumption it will be projected to o_n. We now go through the particles in the lexicographical order of $\mathcal{T}(u_n)$. A vertex at location $t \in (-\infty, 0]$ is projected to $i = \pi_n(t)$ if

$$-\sum_{k=i}^{n} \frac{1}{k} < t \le -\sum_{k=i+1}^{n} \frac{1}{k}.$$

When t branches the number of children with label in $j \in \{1, \dots, n\}$ in the projection is Poisson distributed with parameter

$$\pi\Big(\big(-t - \sum_{k=j}^{n} \frac{1}{k}, -t - \sum_{k=j+1}^{n} \frac{1}{k}\big)\Big).$$

If $i < j$ this is roughly

$$\pi\Big(\big(\sum_{k=i}^{j-1} \frac{1}{k}, \sum_{k=i}^{j} \frac{1}{k}\big)\Big) \approx \beta \int_{\log \frac{j-1}{i}}^{\log \frac{j}{i}} \mathrm{e}^{\gamma y} \, \mathrm{d}y = (\beta/\gamma)\big((\tfrac{j}{i})^{\gamma} - (\tfrac{j-1}{i})^{\gamma}\big) \approx \beta(\tfrac{1}{i})(\tfrac{j}{i})^{\gamma-1} = p_{ij},$$

and if $i > j$ this is roughly

$$\begin{aligned} \pi\Big(\big(-\sum_{k=j}^{i} \frac{1}{k}, -\sum_{k=j+1}^{i} \frac{1}{k}\big)\Big) &\approx \beta \int_{\log \frac{j}{i}}^{\log \frac{j+1}{i}} \mathrm{e}^{(1-\gamma)y} \, \mathrm{d}y = (\beta/1-\gamma)\big((\tfrac{j+1}{i})^{1-\gamma} - (\tfrac{j}{i})^{1-\gamma}\big) \\ &\approx \beta(\tfrac{1}{i})(\tfrac{j}{i})^{-\gamma} = p_{ij}. \end{aligned}$$

Making this more precise, if $i \geq b_n$ one can couple the Poisson random variable to a Bernoulli random variable with parameter p_{ij} with an error probability that can be summed to at most $n^\gamma b_n^{-\gamma-1}$ over all $j \in \{i+1, \ldots, n\}$, resp. $b_n^{\gamma-1}$ over all $j \in \{b_n, \ldots, i-1\}$. Summing these error probabilities over the at most c_n vertices i projected gives the result.

Proof of Proposition 3. Observe that it is possible to satisfy all the conditions on c_n and b_n imposed in Proposition 4 and 5. If the coupling of the labelled tree $\mathbb{T}_n(o_n)$ and the connected component of o_n in G_n and simultaneously with the branching process $\mathcal{T}(u_n)$ is successful we have $S_n(o_n) = T(u_n)$ and the result follows because

$$\lim_{n\to\infty} \mathbb{P}\left(S_n(o_n) \geq u^{-\rho_-} x\right) = \lim_{n\to\infty} \mathbb{P}\left(T(u_n) \geq u^{-\rho_-} x\right) = \mathbb{P}\left(T(u) \geq u^{-\rho_-} x\right),$$

using stochastic continuity of the family $(T(u) \colon u \in (0,1])$ in the last step. ◀

3 The killed branching random walk

In this section we complete the proof of Theorem 2 by showing the following result about the killed branching random walk.

▶ **Proposition 6.** *Under the conditions of Theorem 2, for every $x > 0$,*

$$\lim_{u\downarrow 0} \mathbb{P}\left(T(u) \geq x u^{-\rho_-}\right) = \mathbb{P}\left(Y \geq x\right),$$

where Y is a positive random variable satisfying

$$\mathbb{P}\left(Y \geq x\right) = x^{-(\rho_+/\rho_-)+o(1)} \text{ as } x \to \infty.$$

The proof uses arguments from Aidekon et al. [1] in our setup. The role of the exponents $\rho_\pm$ will become clear in Section 3.1, while in Section 3.2 we will use a famous law of large number for general branching processes due to Nerman [12] to obtain the desired asymptotic.

3.1 Background on branching random walks without killing

Consider the marked tree we get from the branching random walk (without killing) with displacements given by a Poisson process Π with intensity π, where the mark of a particle x (identified, for example, by its Ulam-Harris label) corresponds to its location τ_x.

We define

$$\psi(\alpha) = \mathbb{E}\Big[\sum_{x\in\Pi} \mathrm{e}^{-\alpha\tau_x}\Big].$$

We can calculate $\psi(\alpha)$ for $\gamma < \alpha < 1-\gamma$ exactly with Campbell's formula [8],

$$\psi(\alpha) = \mathbb{E}\Big[\sum_{x\in\Pi} \mathrm{e}^{-\alpha\tau_x}\Big] = \int \mathrm{e}^{-\alpha t}\pi(\mathrm{d}t) = \beta \int_0^\infty \mathrm{e}^{(\gamma-\alpha)t}\,\mathrm{d}t + \beta \int_{-\infty}^0 \mathrm{e}^{(1-\gamma-\alpha)t}\,\mathrm{d}t$$
$$= \frac{\beta}{\alpha-\gamma} + \frac{\beta}{1-\gamma-\alpha},$$

otherwise, for $\alpha \notin (\gamma, 1-\gamma)$, we have $\psi(\alpha) = \infty$. There exists α with $\psi(\alpha) < 1$ if and only if $\gamma < \frac{1}{2}$ and $\beta < \frac{1}{4} - \frac{\gamma}{2}$, i.e. in the subcritical regime for the inhomogeneous random graph. This is also the exact condition for the branching random walk with killing barrier at the origin to suffer extinction in finite time almost surely.

If there exists α with $\psi(\alpha) < 1$, by continuity, there exist two real numbers $\gamma < \rho_- < \rho_+ < 1 - \gamma$ with $\psi(\rho_-) = \psi(\rho_+) = 1$. We can calculate both values explicitly

$$\rho_\pm = \tfrac{1}{2} \pm \sqrt{(\gamma - \tfrac{1}{2})^2 + \beta(2\gamma - 1)}.$$

Because $\psi(\rho_-) = 1$ we obtain a nonnegative martingale (W_n) by letting W_n be the sum of $\mathrm{e}^{-\rho_- \tau_x}$ over all the particles x (with position denoted τ_x) in the nth generation of the branching random walk. By Biggins' theorem for branching random walks, see e.g. [3, 10], the martingale limit W is strictly positive if and only if the following two conditions hold,

(i) $\log(\psi(\rho_-)) - \frac{\rho_- \psi'(\rho_-)}{\psi(\rho_-)} > 0\,,$

(ii) $\mathbb{E}[W_1 \log W_1] < \infty$.

The first one holds as $\psi(\rho_-) = 1$ and $\psi'(\rho_-) < 0$. For the second condition it suffices to check $\mathbb{E}[W_1^\alpha] < \infty$ for some $\alpha > 1$. For this we define

$$f(x, \Pi) = \mathrm{e}^{-\rho_- \tau_x} \big(\sum_{y \in \Pi} \mathrm{e}^{-\rho_- \tau_y}\big)^{\alpha - 1}.$$

Then $\mathbb{E}[W_1^\alpha] = \mathbb{E}[\int f(x, \Pi)\, \Pi(dx)]$ and by Mecke's equation [8, Theorem 4.1] we get

$$\begin{aligned}\mathbb{E}[W_1^\alpha] &= \int \mathbb{E}[f(x, \Pi + \delta_x)]\, \pi(\mathrm{d}x) = \int \mathrm{e}^{-\rho_- x} \mathbb{E}\big[(\mathrm{e}^{-\rho_- x} + \int \mathrm{e}^{-\rho_- t}\, \Pi(\mathrm{d}t))^{\alpha-1}\big]\, \pi(\mathrm{d}x) \\ &\le 2^{\alpha - 1} \Big(\int \mathrm{e}^{-\alpha \rho_- x} \pi(\mathrm{d}x) + \mathbb{E}\big[(\int \mathrm{e}^{-\rho_- t}\, \Pi(\mathrm{d}t))^{\alpha - 1}\big]\, \psi(\rho_-) \Big).\end{aligned}$$

The right summand is finite if $1 < \alpha \le 2$ because in this case, by Jensen's inequality, the expectation is bounded by one. The left summand is equal to $\psi(\alpha\rho_-)$ which is finite for $1 < \alpha < \frac{1-\gamma}{\rho_-}$. Hence W is strictly positive.

3.2 Convergence of the total number of particles

We now introduce the setting of general branching processes as used in Nerman [12]. Let ξ be a point process on $[0, \infty)$. The points represent the ages at which an individual gives birth to another particle. We denote by $\mu = \mathbb{E}[\xi]$ the intensity measure of the point process. The following conditions have to be met:

(i) μ is not concentrated on any lattice,

(ii) there exists an $\alpha \in (0, \infty)$ such that $\int_0^\infty \mathrm{e}^{-\alpha t} \mu(\mathrm{d}t) = 1$ and

(iii) we have $\int_0^\infty t \mathrm{e}^{-\alpha t} \mu(\mathrm{d}t) < \infty$.

α is called the *Malthusian parameter*. A continuous-time branching process where every individual x (identified, again, by its Ulam-Harris label) gives birth to a single new individual at the times given by adding to its own birth time σ_x the points of an independent copy ξ_x of a point process as above, is called a *Crump-Mode-Jagers* or *general branching process*. We denote by $\mathcal{T}$ the set of all particles that exist in the general branching process.

The set-up of [12] allows to also include a time dependent characteristic for each particle x, but in our case it suffices to consider a random variable X_x, which may depend on ξ_x but is independent for each particle and distributed like some X. We sum X_x over all particles born before time t,

$$Z_t^X := \sum_{x \in \mathcal{T}, \sigma_x < t} X_x.$$

The following result is [12, Theorem 3.1] formulated in our set-up.

▶ **Proposition 7.** *Suppose that* $\mathbb{E}[X] < \infty$*, then*

$$\mathrm{e}^{-\alpha t} Z_t^X \to Y \text{ in probability, as } t \to \infty,$$

where Y *is a finite non-negative random variable.*

We now have all the tools to prove Proposition 6. In order to use Proposition 7 we need to derive a suitable ξ and X from π. For this purpose we take a branching random walk started with a particle at the origin, with offspring displacements given by a Poisson point process with intensity π. We do not kill particles, but we only allow particles at locations in $(-\infty, 0]$ to branch, leaving the particles in $(0, \infty)$ frozen. We let $X \geq 1$ be the total number of branching particles including the particle at the origin, which is finite because the branching random walk with killing barrier at 0 becomes extinct almost surely. We let ξ be the point process of locations of the frozen (non-branching) particles, see Figure 1.

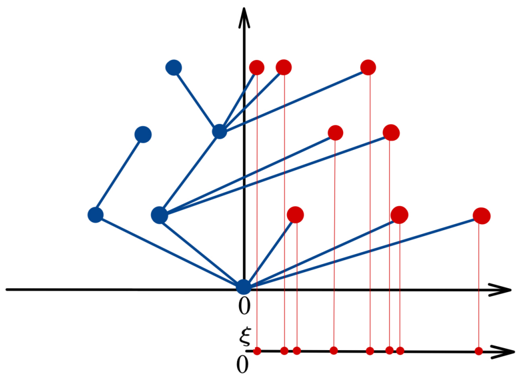

Figure 1 Branching particles are marked in blue, there are $X = 6$ in total. The positions on $[0, \infty)$ of the frozen particles, which are marked in red, yield the point process ξ.

▶ **Proposition 8.** *We have* $\mathbb{E}[X] < \infty$ *and* ξ *satisfies the conditions above for the Malthusian parameter* $\alpha = \rho_-$*. Moreover, for* $t = -\log u$*, we have*

$$Z_t^X \stackrel{d}{=} T(u).$$

Proof. Shifting all particle positions by $t = -\log u$ the killed branching random walk $\mathcal{T}(u)$ becomes a branching random walk $\mathcal{T}'(u)$ started with a particle at the origin, with displacements given by a Poisson point process with intensity π, with a killing barrier at $t = -\log u$. We now construct a coupling of $\mathcal{T}'(u)$ and the general branching process with ξ and X, so that the identity $Z_t^X = T'(u)$ holds, where $T'(u)$ is the number of particles of $\mathcal{T}'(u)$, which has the same law as $T(u)$.

To construct the coupling, we divide the descendants of a particle $x \in \mathcal{T}'(u)$ into branching particles to its left and frozen particles to its right, just as above. The positions of the frozen particles are the birth times of its children in the general branching process, the number of branching particles is the characteristic X_x. In this way the total progeny $T'(u)$ of the killed branching random walk equals Z_t^X, see Figure 2.

We now check that $\mathbb{E}X < \infty$. We pick $\alpha > 0$ with $\psi(\alpha) < 1$ and give a branching particle x in position $\tau_x \leq 0$ the weight $\mathrm{e}^{-\alpha\tau_x} \geq 1$. Then the expected sum over all weights of branching particles in generation n is bounded by $\psi(\alpha)^n$. Hence the total weight summed over all branching particles, and in particular the total number X of such particles, has an expectation which is bounded by $\frac{1}{1-\psi(\alpha)}$.

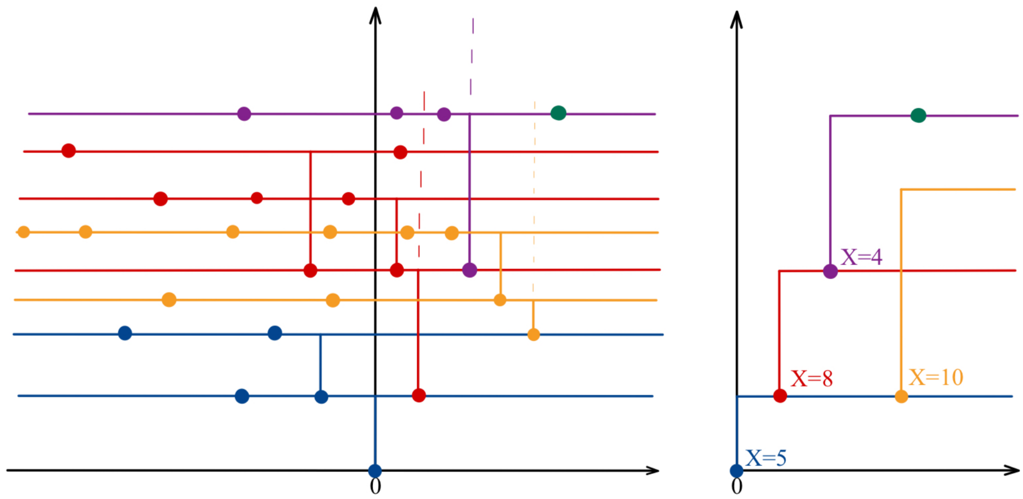

Figure 2 On the left the branching random walk, on the right the associated general branching process and the characteristics of each particle.

It remains to check that ρ_- is the Malthusian parameter associated to ξ. To this end we construct a martingale (M_n) as follows: We start with a particle at the origin and $M_0 = 1$. In every step we replace the leftmost particle in $(-\infty, 0]$ by its offspring chosen with displacements according to a Poisson process of intensity π and leave all other particles alive. Particles in $(0, \infty)$ never branch and remain alive but frozen. If there is no particle in $(-\infty, 0]$ the process stops and the positions of the frozen particles make up ξ. The random variable M_n is obtained as the sum of all particles x alive after the nth step weighted with $\mathrm{e}^{-\rho_- \tau_x}$, where τ_x is the position of particle x. Because $\psi(\rho_-) = 1$ the process (M_n) is indeed a martingale, and it clearly converges almost surely to $\int_0^\infty \mathrm{e}^{-\rho_- t} \xi(\mathrm{d}t)$. Now take $\alpha > \rho_-$ with $\psi(\alpha) < 1$. The martingale (M_n) is dominated by the random variable given as the sum over all branching particles x (with nonpositive position τ_x) weighted with $\mathrm{e}^{-\alpha \tau_x}$ and all frozen particles x (with positive position τ_x) weighted with $\mathrm{e}^{-\rho_- \tau_x}$. This random variable is integrable, as the sum of weights of frozen particles born from a single particle x in position $\tau_x < 0$ is independent with expectation bounded by $\mathrm{e}^{-\alpha \tau_x}$ and the expected sum over these bounds for all branching particles is itself bounded by $\frac{1}{1-\psi(\alpha)}$, as above. We thus get (ii) from dominated convergence and hence ρ_- is the Malthusian parameter. Condition (i) is obvious and (iii) is easy to check. ◀

To complete the proof of Proposition 6 we combine Proposition 7 and 8 to obtain

$$u^{\rho_-} T(u) \to Y \text{ in distribution, as } u \downarrow 0.$$

By [12, Theorem 6.3] the ratios of two cumulative characteristics of the same general branching process converges to a constant. Hence we get, as in [1, Lemma 21], that the limit Y is a constant multiple of the positive martingale limit W. In particular, W and Y share the same tail behaviour at infinity, which by [9, Theorem 2.2] applied to $\chi = \rho_+/\rho_-$ is given by

$$\mathbb{P}(W \geq x) = x^{-(\rho_+/\rho_-)+o(1)}.$$

4 Outlook

One of our principal aims is to find the size of the largest component in the subcritical inhomogeneous random graph of preferential attachment type. In rank one models like the configuration model or the inhomogeneous random graph with product kernel this component is known to have a size of the order of the largest degree in the graph, in our language n^γ, see [7, 6]. However, for the inhomogeneous random graph of preferential attachment type we expect this to be considerably larger because in this model powerful vertices are less well connected so that exploration beyond the first generation is still relevant. We heuristically derive a conjecture from our Theorem 2: Suppose we were allowed to let $n \to \infty$ and $u \to 0$ simultaneously. At best we could be allowed $u \approx \frac{c}{n}$. Then our hypothetic result would give that the most powerful vertices (with index o_n independent of n) would have a connected component of size n^{ρ_-}. Our conjecture is therefore that this is the right order for the size of the largest component. Verifying this conjecture is subject of ongoing work of the authors.

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