The Recurrence/Transience of Random Walks on a Bounded Grid in an Increasing Dimension

Shuma Kumamoto
Graduate School of Mathematical Science, Kyushu University, Fukuoka, Japan

Shuji Kijima
Faculty of Data Science, Shiga University, Hikone, Japan

Tomoyuki Shirai
Institute of Mathematics for Industry, Kyushu University, Fukuoka, Japan

Abstract

It is celebrated that a simple random walk on $\mathbb{Z}$ and $\mathbb{Z}^2$ returns to the initial vertex v infinitely many times during infinitely many transitions, which is said *recurrent*, while it returns to v only finite times on $\mathbb{Z}^d$ for $d \geq 3$, which is said *transient*. It is also known that a simple random walk on a growing region on $\mathbb{Z}^d$ can be recurrent depending on growing speed for any *fixed* d. This paper shows that a simple random walk on $\{0, 1, \ldots, N\}^n$ with an *increasing* n and a fixed N can be recurrent depending on the increasing speed of n. Precisely, we are concerned with a specific model of a *random walk on a growing graph* (*RWoGG*) and show a phase transition between the recurrence and transience of the random walk regarding the growth speed of the graph. For the proof, we develop a *pausing coupling* argument introducing the notion of *weakly less homesick as graph growing* (*weakly LHaGG*).

2012 ACM Subject Classification Theory of computation → Random walks and Markov chains

Keywords and phrases Random walk, dynamic graph, recurrence, transience, coupling

Digital Object Identifier 10.4230/LIPIcs.AofA.2024.22

Funding *Shuma Kumamoto*: This work is supported by JST SPRING, Grant Number JPMJSP2136.
Shuji Kijima: This work is partly supported by JSPS KAKENHI Grant Number JP21H03396.
Tomoyuki Shirai: This work is partly supported by JSPS KAKENHI Grant Number JP20K20884, JP22H05105, JP23H01077, and JP23K25774.

1 Introduction

The recurrence or transience is a classical and fundamental topic of random walks on *infinite* graphs, see e.g., [16]: let $X_0, X_1, X_2, \ldots$ be a random walk (or a Markov chain) on an infinite state space V, e.g., $V = \mathbb{Z}$, with $X_0 = v$ for $v \in V$. The random walk is said to be *recurrent* at the initial state v if

$$\sum_{t=1}^{\infty} \Pr[X_t = v] = \infty \tag{1}$$

holds, otherwise it is said to be *transient*. Intuitively, (1) means that the random walk is "expected" to return to the initial state infinitely many times. An interesting fact is that a simple random walk on $\mathbb{Z}$ or $\mathbb{Z}^2$ is recurrent, while a simple random walk on $\mathbb{Z}^d$ is transient for $d \geq 3$, cf. [16].

Analysis of random walks on *dynamic graphs* has been developed in several contexts. Random walks in random environment is a popular subject in probability theory, where self-interacting random walks including reinforced random walks and excited random walks have been intensively investigated as a relatively tractable non-Markovian process, see e.g., [9, 5, 15, 29, 30, 21]. The recurrence or transience of a random walk in a random environment

35th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2024).
Editors: Cécile Mailler and Sebastian Wild; Article No. 22; pp. 22:1–22:15
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

is a major topic there, particularly random walks on growing subgraphs of $\mathbb{Z}^d$ or infinitely growing trees are the major targets [11, 12, 18, 1]. In distributed computing, analysis of algorithms including random walks on dynamic graphs attracts increasing attention because networks are often dynamic [7, 22, 2, 28]. Searching or covering networks, related to hitting or cover times of random walks, are major topics there [8, 3, 14, 4, 24, 6, 20].

Existing works. As we stated above, a simple random walk on the infinite integer grid $\mathbb{Z}^d$ is recurrent for $d = 1$ and 2, while it is transient for $d \geq 3$. Dembo et al. [12] investigated a random walk on an infinitely *growing* region of $\mathbb{Z}^d$ and showed a phase transition, that is roughly speaking a random walk is recurrent if and only if $\sum_{t=1}^{\infty} \pi_t(\mathbf{0}) = \infty$ holds under certain conditions, where π_t denotes the stationary distribution of the transition matrix at time t. Huang [18] extended the argument of [12] and gave a similar or essentially the same phase transition for more general graphs. The proofs are based on the edge conductance and the central limit theorem on the assumption that every vertex of the dynamic graph has a degree at most *constant* to time (or the size of the graph). Those arguments are sophisticated and enhanced using the argument of evolving set and the heat kernel by recent works [10, 13].

Kumamoto et al. [23] were concerned with a specific model called *random walk on growing graph* (*RWoGG*), which is parametrized by $\mathfrak{d} \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ representing the growing (inverse) speed of the graph. Then, they investigated a simple random walk on $\{0,1\}^n$ *with an increasing* n, and showed that the random walk is recurrent if $\sum_{n=1}^{\infty} \mathfrak{d}(n)/2^n = \infty$, otherwise transient. Notice that the degree of every vertex of the $\{0,1\}^n$ skeleton graph infinitely grows as $n \to \infty$. They introduced the notion of *less-homesickness as graph growing* (*LHaGG*) and gave a proof by a coupling argument, which is easier than the arguments based on the conductance or heat kernel, for this specific object. However, the proof technique is not simply applicable to a simple random walk on $\{0,\ldots,N\}^n$ with an increasing n (and a fixed N), and it remained as future work.

Result. This paper is concerned with the RWoGG model (see Sec. 2.1), and shows a phase transition by the growing speed regarding a random walk being recurrent/transient for a lazy simple random walk on $\{0,\ldots,N\}^n$ with an increasing n and a fixed N. For this purpose, we introduce the notion of *weakly* less-homesick as graph growing (*weakly* LHaGG; see Sec. 3) and show sufficient conditions for a weakly LHaGG RWoGG to be recurrent (Thm. 2) or transient (Thm. 4). The notion of weakly LHaGG is quite intuitive and natural, but we have to develop a new technique of *pausing coupling* to prove that a lazy simple RWoGG is weakly LHaGG. Then, we give the threshold $\sum_{k=1}^{\infty} \mathfrak{d}(k)/(2N)^k = \infty$ of the phase transition (Thm. 6).

Other related works. It is another celebrated fact that a simple random walk on an infinite k-ary tree is transient for $k \geq 2$ [26, 27]. Amir et al. [1] introduced a random walk in a changing environment model and investigated the recurrence and transience of random walks in the model. They gave a conjecture about the conditions of the recurrence and transience regarding the limit of a graph sequence and proved it for trees. Huang's work [18] implicitly implies a phase transition between the recurrence and transience of the random walk on a growing k-ary tree regarding the growing speed of the graph, based on the conductance arguments. Kumamoto et al. [23] explicitly showed the phase transition for a growing k-ary tree under the RWoGG model, where they employ a coupling argument.

There is a lot of work on the recurrence or transience of random walks on growing trees in the context of self-interacting random walks including reinforced random walks and excited random walks, e.g., [19, 17]. They are non-Markovian processes, and in a bit different line from [12, 1, 18, 23] and this paper.

Related to the cover time, which is another major topic on random walks, Cooper and Frieze [8] investigated the covering rate of a random walk on the "web-graph" model, where the graph grows at a constant speed. Kijima et al. [20] introduced the RWoGG model, where the growing (inverse) speed of a graph is parameterized by $\mathfrak{d}\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, and they investigated its covering rate.

Organization. As a preliminary, we describe the model of random walk on growing graph (RWoGG) in Section 2. Section 3 introduces the notion of weakly LHaGG, and presents some general theorems for sufficient conditions that a weakly LHaGG RWoGG is recurrent/transient. Section 4 shows a phase transition between the recurrence and transience of a lazy simple random walk on $\{0, \ldots, N\}^n$ with an increasing n.

2 Preliminaries

2.1 Model

A growing graph is a sequence of (static) graphs $\boldsymbol{\mathcal{G}} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots$ where $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$ for $t = 0, 1, 2, \ldots$ denotes a graph[1] with a finite vertex set $\mathcal{V}_t$ and an edge set $\mathcal{E}_t \subseteq \binom{\mathcal{V}_t}{2}$. For simplicity, this paper assumes[2] $\mathcal{V}_t \subseteq \mathcal{V}_{t+1}$ and $\mathcal{E}_t \subseteq \mathcal{E}_{t+1}$. In this paper, we assume $|\mathcal{V}_\infty| = \infty$, otherwise the subject is trivial; that is always recurrent. A random walk on a growing graph is a Markovian series $X_t \in \mathcal{V}_t$ $(t = 0, 1, 2, \ldots)$.

In particular, this paper is concerned with a specific model, described as follows, cf. [20]. A *random walk on a growing graph* (*RWoGG*), in this paper, is formally characterized by a 3-tuple of functions $\mathcal{D} = (\mathfrak{d}, G, P)$. The function $\mathfrak{d}\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ denotes the duration. For convenience, let $T_n^{\mathfrak{d}} := \sum_{i=1}^{n} \mathfrak{d}(i)$ for $n = 1, 2, \ldots$ and $T_0^{\mathfrak{d}} = 0$. We call the time interval[3] $[T_{n-1}^{\mathfrak{d}}, T_n^{\mathfrak{d}})$ *phase* n for $n = 1, 2, \ldots$; thus $T_{n-1}^{\mathfrak{d}} = \sum_{i=1}^{n-1} \mathfrak{d}(i)$ is the beginning of the n-th phase, but we also say that $T_{n-1}^{\mathfrak{d}}$ is the end of the $(n-1)$-st phase, for convenience. The function $G\colon \mathbb{Z}_{>0} \to \mathfrak{G}$ represents the graph $G(n) = (V(n), E(n))$ for the phase n, where $\mathfrak{G}$ denotes the set of all (static) graphs, i.e., our growing graph $\boldsymbol{\mathcal{G}}$ satisfies $\mathcal{G}_t = G(n)$ for $t \in [T_{n-1}^{\mathfrak{d}}, T_n^{\mathfrak{d}})$. Similarly, the function $P\colon \mathbb{Z}_{>0} \to \mathfrak{M}$ is a function representing the "transition probability" of a random walk on graph $G(n)$ where $\mathfrak{M}$ denotes the set of all transition matrices.

In summary, a RWoGG X_t $(t = 0, 1, 2, \ldots)$ characterized by $\mathcal{D} = (\mathfrak{d}, G, P)$ is temporally a time-homogeneous finite Markov chain according to $P(n)$ with the state space $V(n)$ during the time interval $[T_{n-1}^{\mathfrak{d}}, T_n^{\mathfrak{d}})$. Suppose $X_0 = o$ for $o \in V(1)$. We define the return probability at o by

$$R_{\mathfrak{d}}(t) = \Pr[X_t = o] \ \ (= \Pr[X_t = o \mid X_0 = o]) \tag{2}$$

at each time $t = 0, 1, 2, \ldots$. We say o is recurrent by RWoGG $\mathcal{D} = (\mathfrak{d}, G, P)$ if

$$\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t) = \infty \tag{3}$$

holds, otherwise, i.e., $\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t)$ is upper bounded, o is *transient* by $\mathcal{D}$.

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[1] Every static graph is simple and undirected in this paper, for simplicity of the arguments.
[2] Thus, the current position does not disappear in the next step.
[3] Let $[T_{n-1}^{\mathfrak{d}}, T_n^{\mathfrak{d}}) = \{T_{n-1}^{\mathfrak{d}}, T_{n-1}^{\mathfrak{d}} + 1, \ldots, T_n^{\mathfrak{d}} - 1\}$, for convenience. Notice that $|[T_{n-1}^{\mathfrak{d}}, T_n^{\mathfrak{d}})| = \mathfrak{d}(n)$.

2.2 Mixing time

Next, we briefly introduce some terminology for random walks on static graphs, or time-homogeneous Markov chains, according to [25]. Suppose that $X_0, X_1, X_2, \dots$ is a random walk on a static graph $G = (V, E)$ characterized by a time-homogeneous transition matrix $P = (P(u, v)) \in \mathbb{R}_{\geq 0}^{V \times V}$ where $P(u, v) = \Pr[X_{t+1} = v \mid X_t = u]$. A transition matrix P is *irreducible* if $\forall u, v \in V$, $\exists t > 0$, $P^t(u, v) > 0$. A transition matrix P is *aperiodic* if $\gcd\{t > 0\,; P^t(v, v) > 0\} = 1$ for any $v \in V$. A Markov chain is *ergodic* if it is irreducible and aperiodic. A probability distribution π over V is a *stationary distribution* if it satisfies $\pi P = \pi$. It is well known that an ergodic P has a unique stationary distribution [25].

Let

$$d(t) := \max_{x \in V} \|P^t(x, \cdot) - \pi\|_{TV}. \tag{4}$$

Then, the *mixing time* of P is given by

$$t_{mix}(\epsilon) := \min\{t\,; d(t) \leq \epsilon\} \tag{5}$$

for $\epsilon \in (0, 1)$. We will use the following fact in the proof of Lemma 3 appearing later.

▶ **Lemma 1.** *Suppose P is ergodic. Let π_v denote the probability of $v \in V$ in the stationary distribution of P. If $t \geq t_{mix}(\frac{\pi_v}{4})$ then $d(t) \leq \frac{\pi_v}{2}$ holds.*

Proof. Let

$$\overline{d}(t') := \max_{x, y \in V} \left\| P^{t'}(x, \cdot) - P^{t'}(y, \cdot) \right\|_{TV}.$$

It is known that $d(t) \leq \overline{d}(t) \leq 2d(t)$ holds (cf. Lemma 4.10 in [25]). For convenience, let $t = t_{mix}(\frac{\pi_v}{4}) + s$ for $s \geq 0$. Then,

$$\begin{aligned}
d(t) &\leq \overline{d}(t) && \text{(by Lem. 4.10 in [25])}\\
&\leq \overline{d}\left(t_{mix}(\tfrac{\pi_v}{4})\right) \overline{d}(s) && \text{(by the submultiplicativity of } \overline{d} \text{ (cf. Lem. 4.11 in [25]))}\\
&\leq \overline{d}\left(t_{mix}(\tfrac{\pi_v}{4})\right) && \text{(since } \overline{d}(s) \leq 1)\\
&\leq 2d\left(t_{mix}(\tfrac{\pi_v}{4})\right) && \text{(since } \overline{d}(t) \leq 2d(t) \text{ (cf. Lem. 4.10 in [25]))}\\
&\leq 2\frac{\pi_v}{4} && \text{(by (5))}
\end{aligned}$$

and we obtain the claim. ◀

3 Recurrence and Transience

This section presents sufficient conditions that RWoGG $\mathcal{D} = (\mathfrak{o}, G, P)$ is recurrent/transient. Let $t_{mix}^{G(k)}(\epsilon)$ denote the mixing time of $P(k)$ and let $\pi_o^{G(k)}$ denote the probability of a vertex $o \in V$ in the stationary distribution of $P(k)$ in the following. In this paper, we are mainly concerned with RWoGG $\mathcal{D} = (\mathfrak{o}, G, P)$ satisfying

$$\sum_{k=1}^{\infty} \tau^*(k) p(k) < \infty \tag{6}$$

where $p(k) := \pi_o^{G(k)}$ and $\tau^*(k) := t_{mix}^{G(k)}\left(\frac{p(k)}{4}\right)$. Roughly speaking, the condition (6) means that the mixing times of $\mathcal{D}$ are not very large.

3.1 Recurrence

This section gives a sufficient condition that o is recurrent by $\mathcal{D}$.

▶ **Theorem 2.** *Suppose $\mathcal{D} = (\mathfrak{d}, G, P)$ satisfies (6). If $\mathfrak{d}$ satisfies*

$$\sum_{k=1}^{\infty} \mathfrak{d}(k)p(k) = \infty \tag{7}$$

then the initial vertex o is recurrent by $\mathcal{D}$ where $p(k) = \pi_o^{G(k)}$.

To prove Theorem 2, we prove the following lemma.

▶ **Lemma 3.** *Any RWoGG $\mathcal{D} = (\mathfrak{d}, G, P)$ satisfies*

$$\sum_{t=1}^{T_n^{\mathfrak{d}}} R_{\mathfrak{d}}(t) \geq \frac{1}{2}\sum_{k=1}^{n} \mathfrak{d}(k)p(k) - \frac{1}{2}\sum_{k=1}^{n} \tau^*(k)p(k) \tag{8}$$

for any $n \geq 1$, where recall $\tau^*(k) := t_{mix}^{G(k)}\left(\frac{p(k)}{4}\right)$ and $p(k) := \pi_o^{G(k)}$.

Proof. We prove the claim by an induction with respect to n. For $n = 1$, we prove

$$\sum_{t=1}^{\mathfrak{d}(1)} R_{\mathfrak{d}}(t) \geq \frac{1}{2}\left(\mathfrak{d}(1) - \tau^*(1)\right)p(1) \tag{9}$$

holds, where recall $T_1^{\mathfrak{d}} = \mathfrak{d}(1)$ by definition. We consider two cases whether $\mathfrak{d}(1) \leq \tau^*(1)$ or not. If $\mathfrak{d}(1) \leq \tau^*(1)$ then the right hand side of (8) ≤ 0. Clearly the left hand side of (8) ≥ 0, and we obtain (8). Suppose $\mathfrak{d}(1) > \tau^*(1)$. Notice that

$$\left|R_{\mathfrak{d}}(t) - \pi_o^{G(1)}\right| \leq 2\frac{p(1)}{4} = \frac{1}{2}p(1)$$

holds for $t \geq \tau^*(1)$ by Lemma 1. It implies

$$R_{\mathfrak{d}}(t) \geq \pi_o^{G(1)} - \frac{1}{2}p(1) = \frac{1}{2}p(1)$$

for $t \geq \tau^*(1)$, where recall $p(1) = \pi_o^{G(1)}$ by definition. Then,

$$\sum_{t=1}^{\mathfrak{d}(1)} R_{\mathfrak{d}}(t) \geq \sum_{t=\tau^*(1)}^{\mathfrak{d}(1)} R_{\mathfrak{d}}(t) \geq \sum_{t=\tau^*(1)}^{\mathfrak{d}(1)} \frac{1}{2}p(1) = \frac{1}{2}(\mathfrak{d}(1) - \tau^*(1))p(1)$$

holds. We obtain (9).

Inductively assuming (8) holds for n, we prove it for $n+1$. Noting that $T_{n+1}^{\mathfrak{d}} = T_n^{\mathfrak{d}} + \mathfrak{d}(n+1)$,

$$\begin{aligned}\sum_{t=1}^{T_{n+1}^{\mathfrak{d}}} R_{\mathfrak{d}}(t) &= \sum_{t=1}^{T_n^{\mathfrak{d}}} R_{\mathfrak{d}}(t) + \sum_{t=1}^{\mathfrak{d}(n+1)} R_{\mathfrak{d}}(T_n^{\mathfrak{d}} + t) \\ &\geq \frac{1}{2}\sum_{k=1}^{n} \mathfrak{d}(k)p(k) - \frac{1}{2}\sum_{k=1}^{n} \tau^*(k)p(k) + \sum_{t=1}^{\mathfrak{d}(n+1)} R_{\mathfrak{d}}(T_n^{\mathfrak{d}} + t) \qquad (10)\end{aligned}$$

holds since $\sum_{t=1}^{T_{n+1}^{\mathfrak{d}}} R_{\mathfrak{d}}(t) \geq \frac{1}{2}\sum_{k=1}^{n} \mathfrak{d}(k)p(k) - \frac{1}{2}\sum_{k=1}^{n} \tau^*(k)p(k)$ holds by the inductive assumption. Concerning the third term of (10), we can prove

$$\sum_{t=1}^{\mathfrak{d}(n+1)} R_{\mathfrak{d}}(T_n^{\mathfrak{d}} + t) \geq \frac{1}{2}(\mathfrak{d}(n+1) - \tau^*(n+1))p(n+1)$$

in a similar way as (9). Therefore,

$$\begin{aligned}(10) &\geq \frac{1}{2}\sum_{k=1}^{n} \mathfrak{d}(k)p(k) - \frac{1}{2}\sum_{k=1}^{n} \tau^*(k)p(k) + \frac{1}{2}\left(\mathfrak{d}(n+1) - \tau^*(n+1)\right)p(n+1) \\ &= \frac{1}{2}\sum_{k=1}^{n+1} \mathfrak{d}(k)p(k) - \frac{1}{2}\sum_{k=1}^{n+1} \tau^*(k)p(k)\end{aligned}$$

holds. We obtain the claim. ◀

Now, we prove Theorem 2.

Proof of Theorem 2. Recall the assumption (6),

$$\begin{aligned}\sum_{t=1}^{T_n^{\mathfrak{d}}} R_{\mathfrak{d}}(t) &\geq \frac{1}{2}\sum_{k=1}^{n} \mathfrak{d}(k)p(k) - \frac{1}{2}\sum_{k=1}^{n} \tau^*(k)p(k) && \text{(by Lemma 3)} \\ &\geq \frac{1}{2}\sum_{k=1}^{n} \mathfrak{d}(k)p(k) - \frac{1}{2}\sum_{k=1}^{\infty} \tau^*(k)p(k) && (\tau^*(k)p(k) \geq 0 \text{ for any } k) \\ &= \frac{1}{2}\sum_{k=1}^{n} \mathfrak{d}(k)p(k) - C_1 && \text{(by (6))} \qquad (11)\end{aligned}$$

hold, where C_1 is a positive constant. Thus, the hypothesis $\sum_{k=1}^{\infty} \mathfrak{d}(k)p(k) = \infty$ implies $\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t) = \infty$, which is what we want. ◀

Even if (6) does not hold, Lemma 3 implies that the weaker condition

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\mathfrak{d}(k) - \tau^*(k))p(k) = \infty$$

guarantees the recurrence.

3.2 Weakly less homesick as graph growing

Before giving a sufficient condition for transience, we introduce the notion of weakly less-homesickness as graph growing, which is a relationship between RWoGGs and plays an important role in our analysis. Let $\mathcal{D}_f = (f, G, P)$ and $\mathcal{D}_{f'} = (f', G', P')$ be RWoGGs, and let $R_f(t)$ and $R_{f'}(t)$ respectively denote their return probabilities at time $t = 1, 2, \ldots$. We say $\mathcal{D}_{f'} = (f', G', P')$ is *weakly less-homesick* than $\mathcal{D}_f = (f, G, P)$ at time t if

$$\sum_{k=1}^{t} R_f(k) \geq \sum_{k=1}^{t} R_{f'}(k) \qquad (12)$$

holds.

In particular, this paper is mainly concerned with the weakly less-homesick relation between $\mathcal{D}_f = (f, G, P)$ and $\mathcal{D}_g = (g, G, P)$ with the same P (and G). We say $P\colon \mathbb{Z}_{>0} \to \mathcal{M}$ is *weakly less-homesick as graph growing (weakly LHaGG)*[4] if $\mathcal{D}_g = (g, G, P)$ is weakly less-homesick than $\mathcal{D}_f = (f, G, P)$ whenever

$$\sum_{k=1}^{n} f(k) \geq \sum_{k=1}^{n} g(k) \tag{13}$$

holds for any $n \in \mathbb{Z}_{>0}$, where we remark that G and P are common in $\mathcal{D}_f$ and $\mathcal{D}_g$. The condition (13) implies the graph in $\mathcal{D}_g$ grows faster than $\mathcal{D}_f$, intuitively.

3.3 Transience

Next, we give a sufficient condition that o is transient by $\mathcal{D}$.

▶ **Theorem 4.** *Suppose RWoGG $\mathcal{D} = (\mathfrak{d}, G, P)$ is weakly LHaGG and satisfies* (6)*. If $\mathfrak{d}$ satisfies*

$$\sum_{k=2}^{\infty} \mathfrak{d}(k) p(k-1) < \infty \tag{14}$$

then the initial vertex o is transient by $\mathcal{D}$ where $p(k) = \pi_o^{G(k)}$.

To prove Theorem 4, we prove the following lemma.

▶ **Lemma 5.** *Suppose RWoGG $\mathcal{D} = (\mathfrak{d}, G, P)$ is weakly LHaGG. Let*

$$g(k) := \max\left(\mathfrak{d}(k), \tau^*(k)\right) \tag{15}$$

for $k \geq 1$. Then, the sum of return probabilities of the RWoGG $\mathcal{D}_g = (g, G, P)$ satisfies

$$\sum_{t=1}^{T^g_{n+1}} R_g(t) \leq g(1) + \frac{3}{2} \sum_{k=2}^{n+1} g(k) p(k-1) \tag{16}$$

for $n \geq 1$.

Proof. We prove the claim for each fixed $n = 1, 2, \ldots$. Let

$$f(k) := \begin{cases} g(k) & (k \leq n-1) \\ \infty & (k = n). \end{cases} \tag{17}$$

Let Z_t $(t = 0, 1, 2, \ldots)$ denote a RWoGG $\mathcal{D}_f = (f, G, P)$ with $Z_0 = o$. Let $R_f(t)$ denote the return probability of Z_t, i.e., $R_f(t) = \Pr[Z_t = o] = \Pr[Z_t = o \mid Z_0 = o]$. Clearly, $T^f_n \geq T^g_n$ holds for any $n \geq 1$, hence the weakly LHaGG assumption implies

$$\sum_{t=1}^{T} R_g(t) \leq \sum_{t=1}^{T} R_f(t) \tag{18}$$

for any T.

[4] Strictly speaking, weakly LHaGG should be a property of the sequence of transition matrices $P(1), P(2), P(3), \ldots$. For convenience of the notation, we say $\mathcal{D} = (f, G, P)$ is weakly LHaGG, in this paper.

Suppose $Z_{T^g_{n-1}} = v$. Then, Z_t for $t \in (T^g_n, T^g_{n+1}]$ is nothing but a time-homogeneous random walk according to $P(n)$ with the "initial state" $Z_{T^g_{n-1}} = v$. For convenience, let $t = T^g_{n-1} + t'$, then

$$t' \geq g(n) \geq \tau^*(n) \tag{19}$$

by (15) and (17). This implies

$$\begin{aligned}
\sum_{t=1}^{T^g_{n+1}} R_g(t) &= \sum_{k=1}^{n+1} \sum_{t=T^g_{k-1}+1}^{T^g_k} R_f(t) \\
&= \sum_{t=1}^{T^g_1} R_f(t) + \sum_{k=2}^{n+1} \sum_{t=T^g_{k-1}+1}^{T^g_k} R_f(t) \\
&\leq g(1) + \sum_{k=1}^{n} \sum_{t=T^g_k+1}^{T^g_{k+1}} R_f(t) \\
&= g(1) + \sum_{k=1}^{n} \sum_{t=T^g_k+1}^{T^g_{k+1}} \sum_{v \in V(k-1)} \Pr\left[Z_t = o \mid Z_{T^g_{k-1}} = v\right] \Pr\left[Z_{T^g_{k-1}} = v \mid Z_0 = o\right] \\
&= g(1) + \sum_{k=1}^{n} \sum_{t'=1}^{g(k+1)} \sum_{v \in V(k-1)} \Pr\left[Z_{T^g_k+t'} = o \mid Z_{T^g_{k-1}} = v\right] \Pr\left[Z_{T^g_{k-1}} = v \mid Z_0 = o\right] \\
&\leq g(1) + \sum_{k=1}^{n} \sum_{t'=1}^{g(k+1)} \max_{v \in V(k-1)} \Pr\left[Z_{T^g_k+t'} = o \mid Z_{T_{k-1}} = v\right] \\
&\leq g(1) + \sum_{k=1}^{n} \sum_{t'=1}^{g(k+1)} \left(p(k) + \frac{p(k)}{2}\right) \qquad \text{(by (19))} \\
&= g(1) + \sum_{k=1}^{n} \sum_{t'=1}^{g(k+1)} \frac{3}{2} p(k) \\
&= g(1) + \frac{3}{2} \sum_{k=1}^{n} g(k+1) p(k) \\
&= g(1) + \frac{3}{2} \sum_{k=2}^{n+1} g(k) p(k-1)
\end{aligned} \tag{20}$$

holds. The claim is clear by (18) and (20). ◀

Now, we are ready to prove Theorem 4.

Proof of Theorem 4. Let

$$g(k) := \max\left(\mathfrak{d}(k), \tau^*(k)\right)$$

for $k \geq 1$. Notice that $g(k) \leq \mathfrak{d}(k) + \tau^*(k)$ holds. We calculate $\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t)$ using Lemma 5:

$$\begin{aligned}
\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t) &= \lim_{n\to\infty} \sum_{t=1}^{T^g_{n+1}} R_{\mathfrak{d}}(t) \le \lim_{n\to\infty} \sum_{t=1}^{T^g_{n+1}} R_g(t) \qquad \text{(since weakly LHaGG)} \\
&\le \lim_{n\to\infty} \left\{ g(1) + \frac{3}{2} \sum_{k=2}^{n+1} g(k)p(k-1) \right\} \qquad \text{(by Lemma 5)} \\
&= g(1) + \frac{3}{2} \sum_{k=2}^{\infty} g(k)p(k-1) \qquad (21) \\
&\le \mathfrak{d}(1) + \tau^*(1) + \frac{3}{2} \sum_{k=2}^{\infty} \mathfrak{d}(k)p(k-1) + \frac{3}{2} \sum_{k=2}^{\infty} \tau^*(k)p(k-1) \\
&\le \frac{3}{2} \sum_{k=2}^{\infty} \mathfrak{d}(k)p(k-1) + C \qquad \text{(by (6))}
\end{aligned}$$

holds with some constant C. Now it is easy to see that (14) implies $\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t) < \infty$, meaning that $\mathcal{D} = (\mathfrak{d}, G, P)$ is transient. ◀

4 Random Walk on Growing Dimension Boxes

This section is concerned with a random walk on growing dimension boxes $\mathcal{D} = (\mathfrak{d}, G, P)$. Let $G(n) = (V(n), E(n))$ be a graph given by

$$\begin{aligned}
V(n) &:= \{0, \ldots, N\}^{n_0+n-1} \\
E(n) &:= \{(x, y)\, ;\, x, y \in V(n), \|x - y\|_1 = 1\}
\end{aligned}$$

where n and N are (fixed) positive integers. Let $o \in V(n)$ denote the origin vertex. Let $P^{G(n)}$ for $n \ge 1$ denote the transition probability of a lazy simple random walk on the static graph $G(n)$, which is given by

$$P^{G(n)}_{x,y} = \begin{cases} \frac{1}{2} & (\text{if } x = y) \\ \frac{1}{4(n_0+n-1)} & (\text{if } \|x-y\|_1 = 1,\ \ x_k \neq y_k \text{ and } x_k \notin \{0, N\}) \\ \frac{1}{2(n_0+n-1)} & (\text{if } \|x-y\|_1 = 1,\ \ x_k \neq y_k \text{ and } x_k \in \{0, N\}) \\ 0 & (\text{otherwise}) \end{cases}$$

for $x, y \in V(n)$. Then, we are concerned with a RWoGG X_t ($t = 0, 1, 2, \ldots$) according to $\mathcal{D} = (\mathfrak{d}, G, P)$. If the graph grows at time t, we assume $X_t = (x_1, \ldots, x_{n_0+n-1}) = (x_1, \ldots, x_{n_0+n-1}, 0)$.

▶ **Theorem 6.** *If $\mathcal{D} = (\mathfrak{d}, G, P)$ satisfies*

$$\sum_{k=1}^{\infty} \frac{\mathfrak{d}(k)}{(2N)^k} = \infty$$

then o is recurrent, otherwise o is transient.

We will prove Theorem 6 based on Theorems 2 and 4. As a preliminary step, we remark two facts. One is about the stationary distribution of $P^{G(n)}$, and it is not difficult to observe that

$$p(n) = \frac{1}{(2N)^{n_0+n-1}} \qquad (22)$$

holds. The other is about the mixing time of $P^{G(n)}$, and we can prove

$$\tau^*(n) \leq 8N^2 \log_2(2N)(n_0+n-1)^3 \tag{23}$$

by a standard coupling technique. Therefore, random walk on growing boxes satisfies (6). Then, it is not difficult to see that Theorem 6 follows from the following Lemma 7.

▶ **Lemma 7.** *Random walk on growing dimension boxes is weakly LHaGG.*

Before the proof of Lemma 7, we prove Theorem 6.

Proof of Theorem 6. Notice that $\mathcal{D}$ satisfies (23) and it is weakly LHaGG by Lemma 7, meaning that $\mathcal{D}$ satisfies the hypotheses of Theorems 2 and Theorem 4. By (22),

$$p(n) = \frac{1}{(2N)^{n_0-1}} \frac{1}{(2N)^n}$$

where remark that $\frac{1}{(2N)^{n_0-1}}$ is a constant since N and n_0 are constants. If $\sum_{k=1}^{\infty} \frac{\mathfrak{d}(k)}{(2N)^k} = \infty$ then

$$\sum_{k=1}^{\infty} \mathfrak{d}(k)p(k) = \frac{1}{(2N)^{n_0-1}} \sum_{k=1}^{\infty} \frac{\mathfrak{d}(k)}{(2N)^k} = \infty$$

holds, which implies o is recurrent by Theorem 2. Similarly, if $\sum_{k=1}^{\infty} \frac{\mathfrak{d}(k)}{(2N)^k} \leq C$ holds for some constant then

$$\sum_{k=2}^{\infty} \mathfrak{d}(k)p(k-1) = \frac{1}{(2N)^{n_0-2}} \sum_{k=2}^{\infty} \frac{\mathfrak{d}(k)}{(2N)^k} \leq \frac{1}{(2N)^{n_0-2}} C$$

holds, which implies o is transient by Theorem 4. ◀

4.1 Proof of Lemma 7

We prove Lemma 7 by an artificial coupling. Due to the page limitation, we here explain a proof sketch.

Let $\boldsymbol{X} = X_0, X_1, \ldots$ be a RWoGG according to $\mathcal{D}_f = (f, G, P)$, and let $R_f(t)$ $(t = 0, 1, 2, \ldots)$ denote its return probability. Similarly, let $\boldsymbol{Y} = Y_0, Y_1, \ldots$ be a RWoGG according to $\mathcal{D}_g = (g, G, P)$, and let $R_g(t)$ $(t = 0, 1, 2, \ldots)$ denote its return probability. Note that $X_0 = Y_0 = o$. Suppose that

$$\sum_{k=1}^{n} f(k) \geq \sum_{k=1}^{n} g(k) \tag{24}$$

holds for any $n \geq 1$. Then, we couple $\boldsymbol{X}$ and $\boldsymbol{Y}$ time *asynchronously*, so that $X_t \leq Y_t$ holds for any $t = 0, 1, 2, \ldots$, which is established in three steps by the following Lemmas 8–10.

▶ **Lemma 8.** *Suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ satisfy*

$$X_t = o, \ Y_{t'} = o,$$

for $t \leq t'$. Then, there is a coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that

$$\min\{r \ ; \ r \geq 0, \ X_{t+r} \neq o\} = \min\{r \ ; \ r \geq 0, \ Y_{t'+r} \neq o\}, \tag{25}$$

i.e., X and Y stay at the origin vertex o for exactly the same r steps, where we define $\min \emptyset = \infty$ for convenience.

Proof. Notice that each of X and Y remains at the origin vertex o with probability $\frac{1}{2}$, and leaves the origin vertex o with probability $\frac{1}{2}$ independent of dimensions. Then, we can construct a coupling of X and Y. ◀

▶ **Lemma 9.** *Suppose that $|X_t| = |Y_{t'}| = 1$, where $t \leq t'$. Then, there is a coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that*

$$\min\{r \ ; \ r > 0, \ X_{t+r} = o\} \leq \min\{r \ ; \ r > 0, \ Y_{t'+r} = o\}, \tag{26}$$

i.e., X returns to the origin vertex o in a fewer steps than Y.

Sketch of proof. Without loss of generality, we may assume[5] that $X_t = (X_t^1, X_t^2, \ldots, X_t^{n_t}) = (1, 0, \ldots, 0)$ and $Y_{t'} = (Y_{t'}^1, Y_{t'}^2, \ldots, Y_{t'}^{m_{t'}}) = (1, 0, \ldots, 0)$, where we remark n_t and m_t respectively denote the dimensions of X_t and Y_t.

Let $I(t) \in \{1, \ldots, n_t\}$ denote the index selected in the transition from X_{t-1} to X_t, and let $J(t) \in \{1, \ldots, m_t\}$ denote the index selected in the transition from Y_{t-1} to Y_t. For example, when $X_{t-1} = (0, 0, 0)$, $I(t) = 1$ and $X_t^1 = X_{t-1}^1 + 1$ then $X_t = (1, 0, 0)$. Then, we couple $\{I(t + r)\}_{r=1,2,\ldots}$ and $\{J(t + r)\}_{r=1,2,\ldots}$. For $\theta_{t+r} \in \{1, 2, \ldots, n_{t+r}\}$, let

$$\Psi_k(\theta_{t+r}) := \left\{ \{\omega_{t+r}\}_{r\in\mathbb{N}} \,\middle|\, \omega_{k'} > n_{t+r} \text{ for } k' < k \text{ and } \ \omega_k = \theta_{t+r} \right\}$$

and

$$\widetilde{\Psi}(t + r) := \left\{ \{\omega_{t+r}\}_{r\in\mathbb{N}} \,\middle|\, \omega_{k'} > n_{t+r} \text{ for } k' \geq 1 \ \right\}$$

for $r \geq 1$. Let $\boldsymbol{W} = \{W_s\}_{s\in\mathbb{N}}$ satisfy $W_s := J(t' + s)$ for $s \geq 1$. Suppose $I(t + r) = \theta_{t+r}$. Let $s_1 = k$ such that $\boldsymbol{W} \in \Psi_k(\theta_{t+1})$, and let $S(1) := s_1$. Recursively, let $s_r = k'$ such that $\boldsymbol{W} \in \Psi_{k'}(\theta_{t+r})$ for $r \geq 2$, and let $S(r) := S(r - 1) + s_r$. Let $\Psi(\theta_{t+r}) := \bigcup_{k=1}^{\infty} \Psi_k(\theta_{t+r})$.

Firstly, we claim that

$$\Pr[I(t + r) = \theta_{t+r}] = \Pr\left[\boldsymbol{W} \in \Psi(\theta_{t+r})\right] + \frac{1}{n_{t+r}} \Pr\left[\boldsymbol{W} \in \widetilde{\Psi}(t + r)\right]. \tag{27}$$

Clearly,

$$\Pr[I(t + r) = \theta_{t+r}] = \frac{1}{n_{t+r}}$$

holds for the left-hand-side of (27). Notice that

$$t + r \leq t' + S(r) \tag{28}$$

holds. Then, we can prove

$$\Pr\left[\boldsymbol{W} \in \Psi(\theta_{t+r})\right] + \frac{1}{n_{t+r}} \Pr\left[\boldsymbol{W} \in \widetilde{\Psi}(t + r)\right] = \frac{1}{n_{t+r}} \tag{29}$$

holds, which implies (27).

Next, we prove for any r and $i \leq n_{t+r}$ that

$$X_{t+r}^i \leq Y_{t'+s}^i \tag{30}$$

for $S(r) \leq s < S(r + 1)$. We consider two cases whether $i \leq n_{t+1}$ or not.

[5] Suppose that $|Y_t| = |Y_t'| = 1$. Let $Y_t^i = Y_t'^j = 1$. There is the coupling of $\boldsymbol{Y}$ and $\boldsymbol{Y'}$ such that $Y_t = o$ if and only if $Y_t' = o$.

(i) Consider the case $i \leq n_{t+1}$. Recall that $X^i_t = Y^i_{t'}$ for $i \leq n_t$. We inductively prove that

$$X^i_{t+r} = Y^i_{t'+s} \tag{31}$$

for $S(r) \leq s < S(r+1)$ with respect to r. Notice that

$$Y^i_{t'+S(r)} = Y^i_{t'+s} \tag{32}$$

for $S(r) \leq s < S(r+1)$ since $I(t'+s) > n_{t+r+1}$ for $S(r) < s < S(r+1)$. Suppose i is chosen l times for $X_t, \ldots, X_{t+r}$ in the r steps, i.e.,

$$|\{r' \mid I(t+r') = i,\ 1 \leq r' \leq r\}| = l.$$

Notice that i is chosen l times for $Y_{t'} \ldots, Y_{t'+S(r')}$ in the $S(r')$ steps. Then,

$$\begin{aligned}&\Pr\left[X^i_{t+r'} - X^i_{t+r'-1} = z \;\middle|\; I(t+r') = i\right]\\ &= \Pr\left[Y^i_{t'+S(r')} - Y^i_{t'+S(r')-1} = z \;\middle|\; J(t'+S(r')) = i\right]\end{aligned}$$

holds for $z \in \{-1, 0, 1\}$. The inductive assumption $X^i_{t+r'-1} = Y^i_{t'+S(r')-1}$ implies

$$X^i_{t+r'} = Y^i_{t'+S(r')}. \tag{33}$$

We obtain (31).

(ii) Consider the case $i > n_{t+1}$. Suppose i is chosen l times for $X_t, \ldots, X_{t+r}$ in the r steps, i.e.,

$$|\{r' \mid I(t+r') = i,\ 1 \leq r' \leq r\}| = l,$$

and let r'' denote the minimum satisfying $I(t+r'') = i$ for $1 \leq r'' \leq r$. Clearly, $X^i_{t+r''-1} \leq Y^i_{t'+S(r'')-1}$. If $X^i_{t+r''-1} = Y^i_{t'+S(r'')-1}$, we can prove (30) for any s satisfying $S(r) \leq s < S(r+1)$ in a similar way as the case (i).

Thus, we consider the case $X^i_{t+r''-1} < Y^i_{t+S(r'')-1}$. Then, we can couple the transitions $X^i_{t+r'-1} \mapsto X^i_{t+r'}$ and $Y^i_{t+S(r')-1} \mapsto Y^i_{t+S(r')}$ such that

$$\begin{aligned}&\Pr\left[X^i_{t+r'} - X^i_{t+r'-1} = 0 \;\middle|\; I(t+r') = i\right]\\ &= \Pr\left[\left|Y^i_{t'+S(r')} - Y^i_{t'+S(r')-1}\right| = 1 \;\middle|\; J(t'+S(r')) = i\right] = \frac{1}{2}\\ &\Pr\left[\left|X^i_{t+r'} - X^i_{t+r'-1}\right| = 1 \;\middle|\; I(t+r') = i\right]\\ &= \Pr\left[Y^i_{t'+S(r')} - Y^i_{t'+S(r')-1} = 0 \;\middle|\; J(t+S(r')) = i\right] = \frac{1}{2}\end{aligned}$$

hold. Recall that $X^i_{t+r''-1} < Y^i_{t+S(r'')-1}$ implies $X^i_{t+r''-1} + 1 \leq Y^i_{t+S(r'')-1}$. Thus, the coupling implies $X^i_{t+r''} \leq Y^i_{t'+S(r'')}$. For other $r''' \in \{r' \;;\; I(t+r') = i,\ 1 \leq r' \leq r\}$, we can inductively prove (30) in a similar way.

Therefore, if Y returns to the origin vertex o at time $t' + S(r)$ then X returns to the origin vertex o before the time $t + r$ by (30). Clearly $t + r \leq t' + S(r)$ by (28). We obtain the claim. ◀

▶ **Lemma 10.** *Let*

$$\begin{aligned}\tau_o &:= \min\left\{r \;;\; r > 0,\ X_{t+r} = o\right\},\\ \tau'_o &:= \min\left\{r \;;\; r \geq 0,\ Y_{t'+S(\tau_o)+r} = o\right\}.\end{aligned}$$

If $Y_{t'+S(\tau_o)} \neq o$ then there is a coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $X_{t+\tau_o} = Y_{t'+S(\tau_o)+\tau'_o} = o$, i.e., X stops its time until Y returns to the origin vertex o.

Proof. Since

$$\Pr\left[\tau_o' < \infty \text{ or } \tau_o' = \infty \;\middle|\; Y_{t'+S(\tau_o)} \neq o\right] = 1$$

holds for any $Y_{t'+S(\tau_o)} \in \{0,1,\dots,N\}^{m_{t'+S(\tau_o)}}$. ◀

We prove Lemma 7 using Lemmas 8–10.

Proof of Lemma 7. Let

$$\tau_o^x(n) := \min\{t \;;\; t > \tau_o^x(n-1)\;,\; X_t = o\}\,,$$
$$\tau_o^y(n) := \min\{t \;;\; t > \tau_o^y(n-1)\;,\; Y_t = o\}$$

for $n \geq 1$. For convenience, let $\tau_o^x(0) := 0$ and $\tau_o^y(0) := 0$. To begin with, we prove that there is a coupling of X and Y such that

$$\tau_o^x(n) \leq \tau_o^y(n) \tag{34}$$

for any $n \geq 0$. For $n = 0$, (34) is obvious. Inductively assuming that (34) holds for n, we prove it for $n+1$. If $\tau_o^y(n+1) = \infty$ then we have

$$\tau_o^x(n+1) \leq \tau_o^y(n+1)$$

clearly, and we obtain (34) in this case. Then, we consider the case of $\tau_o^y(n+1) < \infty$. By Lemma 10, X can stop at the vertex o at time $\tau_o^x(n)$ by the time $\tau_o^y(n)$. Therefore, we can consider the coupling of $X_{\tau_o^x(n)}$ and $Y_{\tau_o^y(n)}$. By Lemma 8, there exists $t_n \geq 1$ such that

$$X_{\tau_o^x(n)+s} = o \quad \text{and} \quad Y_{\tau_o^y(n)+s} = o \tag{35}$$

for any s satisfying $0 \leq s < t_n$, and

$$X_{\tau_o^x(k)+t_n} \neq o \quad \text{and} \quad Y_{\tau_o^y(n)+t_n} \neq o \tag{36}$$

hold. If $t_n > 1$ then (35) implies

$$\tau_o^x(n+1) = \tau_o^x(n) + 1 \quad \text{and} \quad \tau_o^y(n+1) = \tau_o^y(n) + 1.$$

This means that we have

$$\tau_o^x(n+1) \leq \tau_o^y(n+1)$$

by the inductive assumption (34), and we obtain the equation (34) in the case $t_n > 1$. Then, we consider the case $t_n = 1$. Notice that (35) and (36) imply

$$|X_{\tau_o^x(k)+t_n}| = 1, \; |Y_{\tau_o^y(k)+t_n}| = 1,$$

and hence Lemma 9 implies that there is a coupling of X and Y such that

$$\tau_o^x(n+1) \leq \tau_o^y(n+1) \tag{37}$$

holds. Therefore, we obtain (34) in the case $t_n = 1$. Thus, we obtain (34) for any $n+1$. It is not difficult to see from (34) that the random walk on growing dimension boxes is weakly LHaGG. ◀

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