Limit Laws for Critical Dispersion on Complete Graphs

Umberto De Ambroggio
Department of Mathematics, LMU Munich, Germany

Tamás Makai
Department of Mathematics, LMU Munich, Germany

Konstantinos Panagiotou
Department of Mathematics, LMU Munich, Germany

Annika Steibel
Department of Mathematics, LMU Munich, Germany

Abstract

We consider a synchronous process of particles moving on the vertices of a graph G, introduced by Cooper, McDowell, Radzik, Rivera and Shiraga (2018). Initially, M particles are placed on a vertex of G. In subsequent time steps, all particles that are located on a vertex inhabited by at least two particles jump independently to a neighbour chosen uniformly at random. The process ends at the first step when no vertex is inhabited by more than one particle; we call this (random) time step the *dispersion time*.

In this work we study the case where G is the complete graph on n vertices and the number of particles is $M = n/2 + \alpha n^{1/2} + o(n^{1/2})$, $\alpha \in \mathbb{R}$. This choice of M corresponds to the critical window of the process, with respect to the dispersion time. We show that the dispersion time, if rescaled by $n^{-1/2}$, converges in p-th mean, as $n \to \infty$ and for any $p \in \mathbb{R}$, to a continuous and almost surely positive random variable T_α. We find that T_α is the absorption time of a standard logistic branching process, thoroughly investigated by Lambert (2005), and we determine its expectation. In particular, in the middle of the critical window we show that $\mathbb{E}[T_0] = \pi^{3/2}/\sqrt{7}$, and furthermore we formulate explicit asymptotics when $|\alpha|$ gets large that quantify the transition into and out of the critical window. We also study the random variable counting the *total number of jumps* that are performed by the particles until the dispersion time is reached and prove that, if rescaled by $n \ln n$, it converges to $2/7$ in probability.

2012 ACM Subject Classification Theory of computation → Random walks and Markov chains

Keywords and phrases Random processes on graphs, diffusion processes, stochastic differential equations, martingale inequalities

Digital Object Identifier 10.4230/LIPIcs.AofA.2024.26

Related Version *Full Version*: https://doi.org/10.48550/arXiv.2403.05372

Funding *Konstantinos Panagiotou*: All authors were supported by ERC Grant Agreement 772606-PTRCSP.

1 Introduction

The *dispersion process* introduced by Cooper, McDowell, Radzik, Rivera and Shiraga [2] consists of particles moving on the vertices of a given graph G. A particle is said to be *happy* if there are no other particles occupying the same vertex and *unhappy* otherwise. Initially, $M \geq 2$ (unhappy) particles are placed on some vertex of G. Subsequently, at discrete time steps, all unhappy particles move *simultaneously* and *independently* to a neighbouring

35th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2024).
Editors: Cécile Mailler and Sebastian Wild; Article No. 26; pp. 26:1–26:12
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

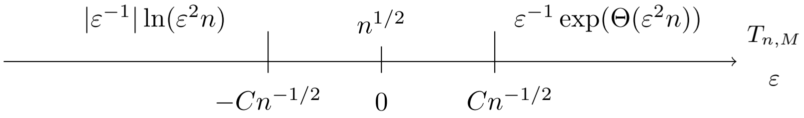

Figure 1 The typical order of $T_{n,M}$ when $M = (1+\varepsilon)n/2$ and $|\varepsilon| = o(1)$. Note that $|\varepsilon^{-1}|\ln(\varepsilon^2 n)$ and $\varepsilon^{-1}\exp(\varepsilon^2 n)$ are in $\Theta(\sqrt{n})$ when $|\varepsilon| = \Theta(n^{-1/2})$, and so the transition into and out of the critical window is smooth.

vertex selected uniformly at random, while the happy particles remain in place. The process terminates at the first time step at which all particles are happy; we call this (random) time step the *dispersion time*.

It is clear that if the number of particles is small – compared to the number of vertices in the graph – then the dispersion time should be small as well. Intuitively, increasing the number of particles makes it more and more difficult for the particles to disperse quickly. This transition from 'fast' to 'slow' dispersion is quite well-understood and sharp when the underlying graph is the complete graph on n vertices with loops, in which case we write $T_{n,M}$ for the dispersion time started with M particles at an arbitrary vertex. The typical order of $T_{n,M}$ changes rather abruptly around $M = n/2$. Indeed, if we write $M = M(n) = (1+\varepsilon)n/2 \in \mathbb{N}$ for some sequence $\varepsilon = \varepsilon(n) \in [-1,1]$, then in [2] it was established that $T_{n,M}$ is typically

- at most logarithmic in n when $\limsup_{n\to\infty} \varepsilon < 0$ and
- at least exponential in n when $\liminf_{n\to\infty} \varepsilon > 0$.

The details of this apparent and abrupt transition from logarithmic to exponential time are obviously of great interest and were investigated further in [3], where the authors studied the typical order and the tails of $T_{n,M}$ when $\varepsilon = o(1)$, that is, when $M = n/2 + o(n)$. In this setting they showed that for any constant $C > 0$, if $\varepsilon \le -Cn^{-1/2}$, then the process typically finishes in $\Theta(|\varepsilon|^{-1}\ln(\varepsilon^2 n))$ steps, while if $\varepsilon \ge Cn^{-1/2}$, then a much larger number $\varepsilon^{-1}\exp(\Theta(\varepsilon^2 n))$ of steps is required. Moreover, within the *critical window* corresponding to the range $|\varepsilon| = O(n^{-1/2})$, they showed that the process typically runs for $\Theta(n^{1/2})$ steps, making the transition into and out of the critical window smooth, see also Figure 1.

In this paper we will perform a fine analysis of the dispersion process within the critical window, that is, when $M = n/2 + O(\sqrt{n})$. Our first main result establishes that the dispersion time, scaled by $n^{-1/2}$, converges in distribution to some continuous and almost surely positive random variable. For a sequence of real-valued random variables $(Z_n)_{n\in\mathbb{N}}$ and a random variable Z we write $Z_n \xrightarrow{d} Z$ to denote that the sequence $(Z_n)_{n\in\mathbb{N}}$ converges to Z in distribution.

▶ **Theorem 1.** *Let $\alpha \in \mathbb{R}$ and $M = M(n) = n/2 + \alpha\sqrt{n} + o(\sqrt{n}) \in \mathbb{N}$. Then there is a continuous and almost surely positive random variable T_α such that, as $n \to \infty$,*

$$n^{-1/2}T_{n,M} \xrightarrow{d} T_\alpha \ .$$

Within the proof of Theorem 1 we derive an explicit description of the distribution of T_α. In order to specify it at this point we need to step back a bit and introduce some notation and present some facts about the process. Let us write U_t for the (random) number of unhappy particles at the end of step t, so that $U_0 = M$, and let us fix some $\delta > 0$. As we will argue in Section 3, U_t drops rather quickly to $\Theta(n^{1/2})$ particles. In particular, with probability at least $1-\delta$, after $t^* \sim \frac{4}{7}\delta n^{1/2}$ steps we have that $U_{t^*} \sim n^{1/2}/\delta$; here and everywhere else "$\sim$" will stand for "$= (1+o(1))$" and asymptotic statements are, unless stated explicitly otherwise, with respect to $n \to \infty$ and uniform in all other parameters. After t^* the process

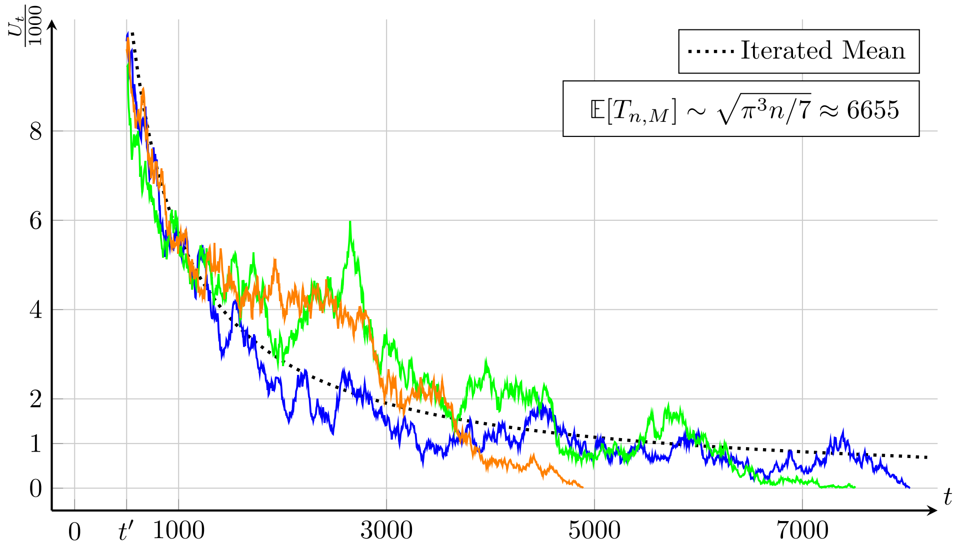

Figure 2 Three sample runs of the dispersion process with $n = 10^7$ and $M = n/2$, where we depict the number of unhappy particles U_t, divided by 1000, at each step t. The trajectory is revealed only after $t' = 500$, where $U_{t'} \approx 10^4 \approx 3\sqrt{n}$ in all cases. The dotted line represents the iterated mean of U_t. For the asymptotics of $\mathbb{E}[T_{n,M}]$ see (4).

$(U_t)_{t \geq t^*}$ of unhappy particles starts fluctuating significantly, see Figure 2 for outcomes of a simulation study when $M = n/2$. In order to get a grip on it, we scale time and space by a factor of $n^{1/2}$ and establish that $(n^{-1/2} U_{t^*+\lfloor s\sqrt{n} \rfloor})_{s \geq 0}$ converges weakly to a diffusion process. Here weak convergence denotes, as usual, convergence in $D([0,T], \mathbb{R})$ for all $T < \infty$, where $D([0,T], \mathbb{R})$ represents the space of all right-continuous functions from $[0,T]$ to $\mathbb{R}$ with left-limits.

▶ **Lemma 2.** *Let $\alpha \in \mathbb{R}$ and $M = M(n) = n/2 + \alpha\sqrt{n} + o(\sqrt{n}) \in \mathbb{N}$. Let $\delta > 0$ and let*

$$T_{n,M,\delta} := \inf\{t > 0 : U_t \leq n^{1/2}/\delta\}$$

be the first step at which there are at most $n^{1/2}/\delta$ unhappy particles. Then, as $n \to \infty$, weakly

$$\left(n^{-1/2}\, U_{T_{n,M,\delta}+\lfloor sn^{1/2} \rfloor}\right)_{s \geq 0} \to X,$$

where X is a logistic branching process starting from $X_0 = \delta^{-1}$. In particular, if we denote by B a standard Brownian motion, then X uniquely satisfies the SDE

$$dX_s = \left(2\alpha X_s - \frac{7}{4} X_s^2\right) ds + \sqrt{X_s} dB_s, \quad s > 0, \qquad \textit{and} \qquad X_0 = \delta^{-1}. \tag{1}$$

For more background on SDEs in general and the specific equation encountered here we refer to Section 2. Let us mention only that stochastic processes satisfying (1) are well-studied and are also called in the literature *logistic Feller diffusions* or *Feller diffusions with logistic growth*. Generally, such processes satisfy an SDE of the form

$$dX_s = (aX_s - cX_s^2)ds + \sqrt{\gamma X_s} dB_s, \; s > 0, \text{ with initial condition } X_0 = x \geq 0, \tag{2}$$

where $a \in \mathbb{R}$ and $c, \gamma > 0$. They appear in the context of population dynamics and stochastically extend the deterministic logistic growth model that describes the evolution of a population under the influences of natural birth, mortality and inter-individual competition. A prime source on the topic is Lambert [7], who provides a thorough and detailed discussion of the properties of solutions to (2).

With Lemma 2 at hand we show in Section 3, see Lemma 13 there, that the first step at which the unhappy particles vanish, divided by $n^{1/2}$, converges in distribution to the *absorption time* of X, that is, the first time when X hits zero. Letting $\delta \to 0$ then yields the claimed statement. In particular, T_α in Theorem 1 is the absorption time of the limiting solution of (1) when the initial condition $X_0 \to \infty$; this limiting process, called *standard logistic branching process*, is well-defined and well-studied, see for example [7] and Section 2.2.

The explicit descriptions of X and T_α pave the way to obtain further bits of information. To achieve this we will exploit the following bounds, stating that $n^{-1/2}T_{n,M}$ has exponential tails, that are an immediate consequence of the main theorems in [3].

▶ **Theorem 3.** *Let $\alpha \in \mathbb{R}$ and $M = n/2 + \alpha\sqrt{n} + o(\sqrt{n}) \in \mathbb{N}$. Then there is a constant $c_\alpha > 0$ such that for all sufficiently large n*

$$\mathbb{P}\big(T_{n,M} \le n^{1/2}/Ac_\alpha\big) \le e^{-A} \quad \textit{and} \quad \mathbb{P}\big(T_{n,M} > Ac_\alpha n^{1/2}\big) \le e^{-A}, \qquad A \ge 1.$$

Together with our Theorem 1 this implies that for any $p \in \mathbb{R}$ we even obtain convergence in $\mathcal{L}^p$, in particular

$$n^{-p/2}\mathbb{E}\big[T_{n,M}^p\big] \sim \mathbb{E}\big[T_\alpha^p\big], \quad p \in \mathbb{R}. \tag{3}$$

We also obtain, without including the proof here, for $M = n/2 + \alpha\sqrt{n} + o(\sqrt{n})$ the series representation

$$\mathbb{E}[T_\alpha] = \lim_{n\to\infty} \mathbb{E}\big[n^{-1/2}T_{n,M}\big] = \frac{\pi^{3/2}}{\sqrt{7}} + \frac{1}{\sqrt{7}}\sum_{m\ge 1}\frac{\Gamma(\frac{m+1}{2})}{m!}\left(\frac{8\alpha}{\sqrt{7}}\right)^m t_m, \quad \alpha \in \mathbb{R},$$

where $\Gamma(\cdot)$ is the Gamma function and

$$t_m := \sum_{k\ge 0}\frac{2}{(\frac{m+1}{2}+2k)(\frac{m+3}{2}+2k)} = H_{(m-1)/4} - H_{(m-3)/4}, \quad m \in \mathbb{N}_0,$$

and $H_x = \sum_{k\ge 1}\big(\frac{1}{k} - \frac{1}{k+x}\big)$ denotes the "x-th harmonic number". Let us highlight the specific case $\alpha = 0$: when we are essentially *at* the critical point, then we obtain the beautiful formula

$$\mathbb{E}[T_0] = \lim_{n\to\infty} \mathbb{E}\big[n^{-1/2}T_{n,n/2+o(\sqrt{n})}\big] = \frac{\pi^{3/2}}{\sqrt{7}}, \tag{4}$$

which is in the interval 2.104 ± 0.001, see also Figure 2. Our methods also allow us to study the behavior of the transition in and out of the critical window, that is, $\mathbb{E}[T_\alpha]$ when $\alpha \to -\infty$ or $\alpha \to \infty$. Indeed we are able to show the following asymptotics

$$\mathbb{E}[T_\alpha] \overset{\alpha\to-\infty}{\sim} \frac{\ln|\alpha|}{|\alpha|} \qquad \text{and} \qquad \mathbb{E}[T_\alpha] \overset{\alpha\to\infty}{\sim} \frac{\sqrt{7\pi}}{8}\frac{e^{16\alpha^2/7}}{\alpha^2}.$$

So, when α gets big, then $\mathbb{E}[T_\alpha]$ behaves (up to polynomial corrections) quadratic exponential in α; already for $\alpha = 3$ we obtain the enormous value $\mathbb{E}[T_3] \approx 5.894 \cdot 10^7$. On the other hand, for negative α we get a moderate polynomial behavior with logarithmic corrections. Note that the large $|\alpha|$ asymptotics presented here are in perfect accordance with the transition in and out of the critical window, see also Figure 1 and the discussion at the beginning of the introduction.

Our second main result addresses the total number of jumps $\sum_{t\geq 0} U_t$ performed by the particles. In contrast to the dispersion time, the total number of jumps, scaled by $n \ln n$, converges to a fixed quantity.

▶ **Theorem 4.** *Let $\alpha \in \mathbb{R}$ and $M = M(n) = n/2 + \alpha\sqrt{n} + o(\sqrt{n}) \in \mathbb{N}$. Then*

$$\frac{1}{n \ln n} \sum_{t \geq 0} U_t \xrightarrow{d} \frac{2}{7} \,.$$

In particular, each of the $M \sim n/2$ particles performs on average typically $\sim \frac{4}{7} \ln n$ jumps before everybody settles, and this is independent of α. Indeed, our aforementioned analysis of the early steps in Section 3, that is, the first $o(n^{1/2})$ steps, shows that there are already $\sim \frac{2}{7} n \ln n$ jumps in those steps of the process. With Lemma 2 and Theorem 1 in mind, it is not surprising that the remaining $\Theta(n^{1/2})$ steps only contribute an additional of $O(n)$ number jumps, as $n^{-1/2}U_t$ is typically bounded for $t = \Theta(n^{1/2})$.

Theorem 1 and Lemma 2 actually suggest that a much stronger statement should be true. We know that $(n^{-1/2}U_{T_{n,M,\delta}+\lfloor sn^{1/2} \rfloor})_{s\geq 0}$ converges weakly to a logistic branching process X, and so the total number of jumps should be close to $n^{1/2}A$, where $A := \int_0^\infty X_s ds$, plus the additional $\frac{2}{7} n \ln n$ jumps from the first $T_{n,M,\delta}$ steps. Thus the variations in the total number of jumps should be linear in n; that is, there should be a (non-trivial) random variable S such that

$$n^{-1}\left(\sum_{t\geq 0} U_t - \frac{2}{7} n \ln n\right) \xrightarrow{d} S.$$

We leave it as an open problem to prove this conjecture.

Variations on the Theme

Our work opens up opportunities for studying a variety of models that are related to the dispersion process or extensions of it. In a general setting, *happiness* can be defined as a property of individual vertices and particles. More specifically, each vertex may have a *capacity*, which, if exceeded, deems all particles on that vertex as unhappy. On the other side, each particle p may have a *stress level*, which dictates an upper bound on the particles that share a vertex with p so that p is still happy. We leave it as an open problem to study the precise behavior in a general setting, where for example the empirical distributions of the capacities and the stress levels fulfill appropriate convergence properties.

In a different line of research it would be challenging to provide detailed studies of dispersion processes on graphs different than the complete graph. We believe, for example, that our results also hold if the underlying graph is a sufficiently dense Erdős-Rényi random graph $G_{n,p}$, which is obtained by retaining independently each edge of the complete graph on n vertices with probability p. In particular, if, say, $p = \omega(n^{-1/2})$, guaranteeing that the minimum degree is much larger than $\sqrt{n}$, then similar results as in Theorem 1 should hold, as the process finishes after $O(\sqrt{n})$ rounds if the graph is complete. However, it might be the case that even on much sparser graphs the behavior does not change (since, for example, in most steps just an $O(\sqrt{n})$ number of particles move). We consider it as an important and eminent challenge to study the effect of the edge probability p on the distribution of the dispersion time.

Related work

The dispersion process was also studied by Frieze and Pegden [6], who, apart from the dispersion time, also considered the *dispersion distance* on the infinite line. They showed that the dispersion distance is $\Theta(n)$ when there are n particles in the system, improving upon previous results in [2]. A similar setup was considered by Shang [10], who studied the dispersion distance on the infinite line in a non-uniform dispersion process.

Processes where particles move on the vertices of a graph have been widely studied over the past decades; we refer the reader to [2] for references. Concerning processes whose scope is to *disperse* particles on a discrete structure, arguably the best known such model is Internal Diffusion Limited Aggregation (IDLA), see [1, 4, 8]. In this model, particles sequentially start (one at a time) from a specific vertex designated as the origin. Each particle moves randomly until it finds an unoccupied vertex; then it occupies it forever, meaning that it does not move at subsequent process steps.

Another related and well-studied class of models are Activated Random Walks (ARWs) that evolve on the d-dimensional lattice, see [9] for an extensive review. Roughly speaking, we place particles on $\mathbb{Z}^d$, and some of them are initially active while others are asleep. The rules of the process are then as follows. Whenever a particle is alone on a vertex, it falls asleep with a certain rate. On the other hand, active particles jump according to independent random choices, and whenever they encounter a particle that is asleep, they wake it up.

Outline

In Section 2 we present the main tool used during our proof, namely diffusion approximation, and then in Section 3 we include a brief derivation of some of the aforementioned results, primarily Lemma 2 and Theorem 1. The full paper containing all proofs is available at `arXiv:2403.05372`.

2 Probabilistic Preliminaries

2.1 Diffusion Approximation

A main tool that we will use in the proof of Theorem 1 is the concept of *diffusion approximation*, which allows us to approximate a Markov chain $(Y^{(n)})_{n\in\mathbb{N}}$ with values in $\mathbb{R}$, by a continuous-time stochastic process. More specifically, we examine convergence properties of $(Y^{(n)})_{n\in\mathbb{N}}$ to a process satisfying a stochastic differential equation (SDE)

$$dX_s = b(X_s)ds + \sigma(X_s)dB_s, \quad s > 0, \tag{5}$$

where $b, \sigma : \mathbb{R} \to \mathbb{R}$ are suitable functions and B is a 1-dimensional standard Brownian motion. In this section we provide an overview of the necessary results from stochastic calculus. Additionally, we collect some properties of the limit process that will emerge within the proof of Theorem 1. In what follows we denote discrete time by $t \in \mathbb{N}_0$ (so, for example, $Y^{(n)} = (Y_t^{(n)})_{t\in\mathbb{N}_0}$), whereas $s \geq 0$ represents continuous time.

Let us consider (5). A *(weak) solution to* (5) *with initial value* $X_0 = x \in \mathbb{R}$ is a triple $(X, B, \mathcal{P})$, where $\mathcal{P} = (\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P})$ is a filtered probability space with the filtration satisfying the usual conditions, i.e. $(\mathcal{F}_s)_{s\geq 0}$ is right-continuous and complete. Further, $X = (X_s)_{s\geq 0}$ and $B = (B_s)_{s\geq 0}$ are continuous stochastic processes that are adapted to $(\mathcal{F}_s)_{s\geq 0}$ such that

- B is a standard 1-dimensional Brownian motion with respect to $(\mathcal{F}_s)_{s\geq 0}$, i.e. B is a standard Brownian motion and $B_s - B_r$ is independent of $\mathcal{F}_r$ for any $0 \leq r < s$;

- X_s satisfies (5) and the initial condition, i.e.

$$X_s = x + \int_0^s b(X_r)dr + \int_0^s \sigma(X_r)dB_r, \quad s \geq 0.$$

Moreover, we say that there is *(weak) uniqueness* if whenever $(X, B, \mathcal{P})$ and $(\tilde{X}, \tilde{B}, \tilde{\mathcal{P}})$ solve (5) weakly and satisfy $X_0 = \tilde{X}_0$, then X and $\tilde{X}$ have the same law.

In order to get the diffusion approximation to work, we construct a sequence of right-continuous and continuous-time stochastic processes from the given sequence $(Y^{(n)})_{n\in\mathbb{N}}$ of discrete time Markov chains by using constant interpolation between the time points. Then, under appropriate conditions specified in the subsequent theorem, $(Y^{(n)})_{n\in\mathbb{N}}$ converges weakly to the solution of an SDE. With the necessary concepts at hand we are now ready to present our main tool, and we refer for example to [5, Ch. 8] for an extensive treatment.

▶ **Theorem 5** (Diffusion Approximation). *Let $b, \sigma : \mathbb{R} \to \mathbb{R}$ be continuous functions and assume that for any $x \in \mathbb{R}$ the SDE (5) possesses a unique solution such that $X_0 = x$. Furthermore, let $h : \mathbb{N} \to \mathbb{R}_+$ be a sequence with $\lim_{n\to\infty} h(n) = 0$ and for all $n \in \mathbb{N}$ let $Y^{(n)} = (Y_t^{(n)})_{t\in\mathbb{N}_0}$ be a discrete-time Markov chain with values in $S^{(n)} \subseteq \mathbb{R}$. Define, for all $t \in \mathbb{N}_0$, $x \in S^{(n)}$*

$$b^{(n)}(x) := \frac{\mathbb{E}\big[Y_{t+1}^{(n)} - x \mid Y_t^{(n)} = x\big]}{h(n)}, \quad a^{(n)}(x) := \frac{\mathbb{E}\big[(Y_{t+1}^{(n)} - x)^2 \mid Y_t^{(n)} = x\big]}{h(n)},$$

and $\gamma_p^{(n)}(x) := \mathbb{E}\big[|Y_{t+1}^{(n)} - x|^p \mid Y_t^{(n)} = x\big]/h(n)$ for $p \geq 2$. Let $a := \sigma^2$ and assume that for all $R < \infty$

$$\lim_{n\to\infty} \sup_{x\in S^{(n)}, |x|\leq R} |b^{(n)}(x) - b(x)| = 0, \quad \lim_{n\to\infty} \sup_{x\in S^{(n)}, |x|\leq R} |a^{(n)}(x) - a(x)| = 0,$$

and

$$\lim_{n\to\infty} \sup_{x\in S^{(n)}, |x|\leq R} \gamma_p^{(n)}(x) = 0 \ \text{ for some } \ p \geq 2.$$

Finally, assume that $Y_0^{(n)} \to x$ as $n \to \infty$. Then $(Y_{\lfloor s/h(n)\rfloor}^{(n)})_{s\geq 0}$ converges weakly to a strong Markov process X that satisfies the SDE (5) with $X_0 = x$.

2.2 The (Standard) Logistic Branching Process

We already discussed in the introduction that the processes that will be relevant here are the so-called logistic branching processes, given by the solution of

$$dX_s = (aX_s - cX_s^2)ds + \sqrt{\gamma X_s}dB_s, \; s > 0,$$

with $X_0 = x \geq 0$, $a \in \mathbb{R}$ and $c, \gamma > 0$, see also (2) (and (1) for the particular case that will appear here). In the remainder of this section we collect some key properties that will be handy. The first one is about the existence and uniqueness of solutions, see [7].

▶ **Lemma 6.** *For all initial states $x \geq 0$ and for all $a \in \mathbb{R}$ and $c, \gamma > 0$, there exists a unique solution $(X_{a,c,\gamma,x}, B_{a,c,\gamma,x}, \mathcal{P}_{a,c,\gamma,x})$ to (2). Moreover, $X_{a,c,\gamma,x}$ is non-negative.*

In what follows it will be convenient to consider a specific choice of the filtered probability space $\mathcal{P}_{a,c,\gamma,x} = (\Omega', \mathcal{F}', (\mathcal{F}'_s)_{s\geq 0}, \mathbb{P}')$ (where all components depend on the parameters a, c, γ, x) from the previous lemma that we construct as follows. Let Ω be the space of all continuous maps $[0, \infty) \to \mathbb{R}$ and let X be the coordinate process given by $X_s(\xi) = \xi(s)$ for all $s \geq 0$ and $\xi \in \Omega$. Additionally, consider the σ-algebra $\mathcal{F} = \sigma\{X_s \mid s \geq 0\}$ and equip the measurable space $(\Omega, \mathcal{F})$ with the filtration $(\mathcal{F}_s)_{s\geq 0}$ given by $\mathcal{F}_s = \sigma\{X_r \mid 0 \leq r \leq s\}$ for all $s \geq 0$, which we may complete and right-continuously extend in order to fulfil the usual conditions. Via the map $\Omega' \ni \xi' \mapsto X_{a,c,\gamma,x}(\xi') \in \Omega$ it is possible to switch from $\mathcal{P}_{a,c,\gamma,x}$ to the canonical probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P}_{a,c,\gamma,x})$, where $\mathbb{P}_{a,c,\gamma,x}$ is the probability measure given by $\mathbb{P}_{a,c,\gamma,x}(A) = \mathbb{P}'((X_{a,c,\gamma,x})^{-1}(A))$ for all $A \in \mathcal{F}$. By this particular choice we obtain that the coordinate process X on $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P}_{a,c,\gamma,x})$ has the same law as $X_{a,c,\gamma,x}$ under $\mathbb{P}'$, i.e. under $\mathbb{P}_{a,c,\gamma,x}$ the process X satisfies (2). The following corollary is now an immediate consequence of Lemma 6, and a similar construction was also performed in [7].

▶ **Corollary 7.** *For all initial states $x \geq 0$ and for all $a \in \mathbb{R}$ and $c, \gamma > 0$, there is a unique solution $(X, B_{a,c,\gamma,x}, \mathcal{P}_{a,c,\gamma,x})$ to (2), where X is the coordinate process and thus independent of a, c, γ, x. Moreover, X is non-negative $\mathbb{P}_{a,c,\gamma,x}$-almost surely, where $\mathbb{P}_{a,c,\gamma,x}$ denotes the probability measure of $\mathcal{P}_{a,c,\gamma,x}$.*

For the rest of this paper we will adopt the above procedure and consider solutions to (2) only with respect to the canonical probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P}_{a,c,\gamma,x})$. Our main object of interest will be the time at which the logistic Feller diffusion X hits zero, which under $\mathbb{P}_{a,c,\gamma,x}$ is given by the stopping time

$$T(\xi) = \inf\{s \geq 0 : \xi(s) = 0\}, \quad \xi \in \Omega.$$

The author of [7] establishes that T is finite $\mathbb{P}_{a,c,\gamma,x}$-almost surely. Moreover, $X_s = 0$ for all $s \geq T$ under $\mathbb{P}_{a,c,\gamma,x}$, i.e. upon hitting zero the process becomes constant, which is why we also refer to T as *absorption time.*

Within our context, it will be necessary to consider solutions to (2) with initial value $x \to \infty$. The required results are covered by the following statement, whose proof can be found in [7] and for which we define the function $\theta : [0, \infty) \to \mathbb{R}$ by

$$\theta(\lambda) := \int_0^\lambda \exp\left(\frac{\gamma}{4c}v^2 - \frac{a}{c}v\right) dv, \quad \lambda \geq 0. \tag{6}$$

▶ **Lemma 8.** *For all $x \geq 0, a \in \mathbb{R}$ and $c, \gamma > 0$, the expectation of T under $\mathbb{P}_{a,c,\gamma,x}$ is finite and*

$$\mathbb{E}_{a,c,\gamma,x}[T] = \frac{1}{c}\int_0^\infty \frac{\theta(\lambda)}{\lambda\theta'(\lambda)}(1 - \exp(-x\lambda))\, d\lambda.$$

In addition, the measures $(\mathbb{P}_{a,c,\gamma,x})_{x\geq 0}$ converge weakly, as $x \to \infty$, to the law $\mathbb{P}_{a,c,\gamma,\infty}$ of the so-called standard logistic branching process*. Under $\mathbb{P}_{a,c,\gamma,\infty}$, the hitting time T is a continuous random variable which is finite almost surely and has finite expectation given by*

$$\mathbb{E}_{a,c,\gamma,\infty}[T] = \sup_{x\geq 0} \mathbb{E}_{a,c,\gamma,x}[T] = \frac{1}{c}\int_0^\infty \frac{\theta(\lambda)}{\lambda\theta'(\lambda)}\, d\lambda.$$

3 Proof Strategy & Some Details

In the following lemma we investigate the early phase of the process. In particular we are interested in the number of steps and the number of jumps until the number of unhappy particles drops to $\Theta(n^{1/2})$.

▶ **Lemma 9.** *Let $\epsilon, \delta > 0, \alpha \in \mathbb{R}$ and $M = n/2 + \alpha\sqrt{n} + o(\sqrt{n}) \in \mathbb{N}$. Let*

$$T_{n,M,\delta} := \inf\left\{t > 0 : U_t \le n^{1/2}/\delta\right\}.$$

Then, for all sufficiently small $\delta > 0$ and all sufficiently large n, with probability at least $1-\delta$,

$$\left|T_{n,M,\delta} - \frac{4}{7}\delta n^{1/2}\right| \le \epsilon\delta n^{1/2} \quad \textit{and} \quad \left|\sum_{0\le t\le T_{n,M,\delta}} U_t - \frac{2}{7}n\ln n\right| \le \epsilon n \ln n.$$

In particular, (roughly) $\frac{4}{7}\delta n^{1/2}$ steps are required to drop below $n^{1/2}/\delta$ unhappy particles, and at this step the accumulated number of unhappy particles, which corresponds to the total number of jumps, is (roughly) $\frac{2}{7}n\ln n$. The lemma is established by considering the number of unhappy particles for a relatively short number of steps, where the change of the process can be precisely controlled by means of martingale concentration, exploiting the subgaussian nature of the increments. We omit the details due to space limitations.

We focus on the late phase, which uses the diffusion approximation toolbox. We write

$$U_{t+1} - U_t = X_{t+1} - Y_{t+1},$$

where X_{t+1} stands for the number of particles that were happy at step t but become unhappy in step $t+1$ (because some particle which was unhappy at time t moved onto their vertex) and Y_{t+1} is the number of unhappy particles at time t that become happy at step $t+1$ (because at time $t+1$ they are alone on the vertex that they occupy). Moreover, define

$$X_{t+1,h} := \mathbb{1}[h \in \mathcal{U}_{t+1}] \quad \text{and} \quad Y_{t+1,u} := \mathbb{1}[u \in \mathcal{H}_{t+1}]$$

where $\mathcal{H}_{t+1}/\mathcal{U}_{t+1}$ is the set of happy/unhappy particles at time $t+1$ and, so that we can write

$$X_{t+1} = \sum_{h\in\mathcal{H}_t} X_{t+1,h} \quad \text{and} \quad Y_{t+1} = \sum_{u\in\mathcal{U}_t} Y_{t+1,u}.$$

It is clear that, given U_t, we can compute $\mathbb{E}[X_{t+1,h}], \mathbb{E}[Y_{t+1,u}]$ and $\mathbb{E}[X_{t+1,h}Y_{t+1,u}]$ for any $h \in \mathcal{H}_t$ and $u \in \mathcal{U}_t$; the details are omitted. With this at hand, we then establish asymptotics of the drift and variation for the number of unhappy particles, which we describe in the following two lemmas.

▶ **Lemma 10.** *Let $\varepsilon = \varepsilon(n) = o(1)$, $u : \mathbb{N} \to \mathbb{N}$ and $M = M(n) := (1+\varepsilon)n/2 \in \mathbb{N}$. Then, uniformly,*

$$\mathbb{E}\big[U_{t+1} - U_t \mid U_t = u\big] = \varepsilon u - \frac{u^2}{n}\left(\frac{7}{4} + \frac{3\varepsilon}{4}\right) + O\left(\frac{u}{n} + \frac{u^3}{n^2}\right).$$

▶ **Lemma 11.** *Let $\varepsilon = \varepsilon(n) = o(1)$ and $u : \mathbb{N} \to \mathbb{N}$ be such that $u = o(n^{2/3})$ and $M = M(n) := (1+\varepsilon)n/2 \in \mathbb{N}$. Then, uniformly,*

$$\mathbb{E}\big[(U_{t+1} - U_t)^2 \mid U_t = u\big] = u + o(\varepsilon u^2 + u).$$

To continue we introduce the (continuous) time-shifted process

$$U'_s := U_{\lfloor s\rfloor + T_{n,M,\delta}}, \quad s \ge 0.$$

By applying Theorem 5 we will show that $(n^{-1/2}U'_{s\sqrt{n}})_{s\ge 0}$ converges weakly to a diffusion. Note that the following lemma is just a reformulation of Lemma 2 in the Introduction, as (7) corresponds to the SDE (1).

▶ **Lemma 12.** *Let $\delta > 0$. As $n \to \infty$, the process $(n^{-1/2}U'_{s\sqrt{n}})_{s\geq 0}$ converges weakly to a process X that satisfies*

$$dX_s = \left(2\alpha X_s - \frac{7}{4}X_s^2\right) ds + \sqrt{X_s} dB_s, \quad s > 0, \quad \textit{and} \quad X_0 = \delta^{-1}. \tag{7}$$

Proof. We will apply Theorem 5 with $h = h(n) = n^{-1/2}$ and $Y_t^{(n)} := n^{-1/2}U'_t$ for $t \in \mathbb{N}_0$. First, note that it is necessary to extend the SDE (7) in a way that it has a unique solution not only for all initial values $x \geq 0$, but for all $x \in \mathbb{R}$. To this end, write $a^+ = \max\{a, 0\}$ for $a \in \mathbb{R}$ and consider the SDE

$$dX_s = \left(2\alpha X_s^+ - \frac{7}{4}(X_s^+)^2\right) ds + \sqrt{X_s^+} dB_s, \quad s > 0, \text{ with } X_0 = x \in \mathbb{R}. \tag{8}$$

Note that if the initial value x is negative, then $X = x$ uniquely satisfies this SDE. For $x \geq 0$, recall that Corollary 7 guarantees the existence of a unique solution $(X, B_{2\alpha,7/4,1,x}, \mathcal{P}_{2\alpha,7/4,1,x})$ to (7) with $X_0 = x$ and such that $X \geq 0$ almost surely. Hence, if $x \geq 0$, (8) coincides with (7) with initial value $X_0 = x$ and we conclude that (8) possesses a unique solution for all $x \in \mathbb{R}$.

Next, we employ Lemmas 10 and 11 with $\varepsilon(n) = 2\alpha n^{-1/2} + o(n^{-1/2})$, as $M = n/2 + \alpha n^{1/2} + o(n^{1/2})$. For this purpose, let $R < \infty$ and consider $x \in S^{(n)} \subseteq \{0, n^{-1/2}, 2n^{-1/2}, ..., n^{1/2}\}$ with $|x| \leq R$. Then, Lemma 10 with $u = xn^{1/2}$ implies that

$$b^{(n)}(x) = \frac{\mathbb{E}\left[n^{-1/2}U'_{t+1} - n^{-1/2}U'_t \mid n^{-1/2}U'_t = x\right]}{n^{-1/2}} = 2\alpha x - \frac{7}{4}x^2 + o\left(R + R^3\right).$$

Further, as $xn^{1/2} = o(n^{2/3})$ due to $|x| \leq R$, it follows from Lemma 11 with $u = xn^{1/2}$ that

$$a^{(n)}(x) = \frac{\mathbb{E}\left[(n^{-1/2}U'_{t+1} - n^{-1/2}U'_t)^2 \mid n^{-1/2}U'_t = x\right]}{n^{-1/2}} = x + o\left(R^2\right). \tag{9}$$

We therefore obtain that for any $R < \infty$

$$\lim_{n\to\infty} \sup_{x\in S^{(n)}, |x|\leq R} \left| b^{(n)}(x) - \left(2\alpha x - \frac{7}{4}x^2\right) \right| = 0 \text{ and } \lim_{n\to\infty} \sup_{x\in S^{(n)}, |x|\leq R} |a^{(n)}(x) - x| = 0. \tag{10}$$

Moreover, we show

$$\lim_{n\to\infty} \sup_{x\in S^{(n)}, |x|\leq R} |\gamma_3^{(n)}(x)| = 0, \tag{11}$$

and

$$U'_0 = U_{T_{n,M,\delta}} \sim n^{1/2}/\delta \text{ with probability } 1 - o(1). \tag{12}$$

The last two facts, whose proof is omitted here, together with (10) and the existence of a unique solution to (8) guarantee that we can apply Theorem 5 to conclude that $(n^{-1/2}U'_{s\sqrt{n}})_{s\geq 0}$ converges weakly to a process X that satisfies (7) with $X_0 = 1/\delta$, and the proof is finished. ◀

Recall from Corollary 7 that $(X, B_{2\alpha,7/4,1,x}, \mathcal{P}_{2\alpha,7/4,1,x})$ represents a solution of (7) with initial value $x \geq 0$ and that the corresponding hitting time of zero is given by

$$T = \inf\{s \geq 0 : X_s = 0\} \tag{13}$$

under the probability measure $\mathbb{P}_{2\alpha,7/4,1,x}$. The next statement asserts that $n^{-1/2}T'_{n,M,\delta}$, where $T'_{n,M,\delta} := T_{n,M} - T_{n,M,\delta}$, converges in distribution to T under $\mathbb{P}_{2\alpha,7/4,1,1/\delta}$.

▶ **Lemma 13.** *Let $\delta > 0$. Then, as $n \to \infty$, $n^{-1/2} T'_{n,M,\delta} \xrightarrow{d} T$, with T given by* (13) *under the probability measure $\mathbb{P}_{2\alpha,7/4,1,1/\delta}$.*

Proof. Let $s \geq 0$ and recall from Section 2.1 that, under $\mathbb{P}_{2\alpha,7/4,1,1/\delta}$, $X_s = 0$ is equivalent to $T \leq s$. Similarly, $U_{\lfloor s \rfloor} = 0$ if and only if $T_{n,M} \leq s$, from which we obtain that $T'_{n,M,\delta} = T_{n,M} - T_{n,M,\delta} \leq s$ if and only if $U'_s = U_{\lfloor s \rfloor + T_{n,M,\delta}} = 0$. Hence,

$$\mathbb{P}_{2\alpha,7/4,1,1/\delta}(X_s = 0) = \mathbb{P}_{2\alpha,7/4,1,1/\delta}(T \leq s) \quad \text{and} \quad \mathbb{P}(U'_s = 0) = \mathbb{P}(T'_{n,M,\delta} \leq s).$$

As Lemma 12 entails $\lim_{n\to\infty} \mathbb{P}\big(n^{-1/2} U'_{s\sqrt{n}} = 0\big) = \mathbb{P}_{2\alpha,7/4,1,1/\delta}(X_s = 0)$, we conclude

$$\lim_{n\to\infty} \mathbb{P}(n^{-1/2} T'_{n,M,\delta} \leq s) = \mathbb{P}_{2\alpha,7/4,1,1/\delta}(T \leq s).$$ ◀

With the convergence in distribution shown in the previous lemma at hand, we are now in the position to prove Theorem 1.

Proof of Theorem 1. Recall from Lemma 9 that $T_{n,M,\delta} \leq \delta n^{1/2}$ with probability at least $1 - \delta$ for δ sufficiently small and n large enough. As $T_{n,M,\delta}$ is non-negative, this implies that

$$n^{-1/2} T'_{n,M,\delta} \leq n^{-1/2} T_{n,M} \leq \delta + n^{-1/2} T'_{n,M,\delta} \text{ with probability at least } 1 - \delta.$$

Applying Lemma 13, we therefore obtain that for all $s \geq 0$

$$\lim_{n\to\infty} \mathbb{P}(n^{-1/2} T_{n,M} \leq s) \leq \lim_{n\to\infty} \mathbb{P}(n^{-1/2} T'_{n,M,\delta} \leq s) = \mathbb{P}_{2\alpha,7/4,1,1/\delta}(T \leq s)$$

and

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\big(n^{-1/2} T_{n,M} \geq s\big) &\leq \lim_{n\to\infty} \mathbb{P}\big(n^{-1/2} T_{n,M} \geq s, T_{n,M,\delta} \leq \delta n^{1/2}\big) + \delta \\ &\leq \mathbb{P}_{2\alpha,7/4,1,1/\delta}\big(T \geq s - \delta\big) + \delta.\end{aligned}$$

Note that $\mathbb{P}_{2\alpha,7/4,1,x_1}(T \geq \tau) \geq \mathbb{P}_{2\alpha,7/4,1,x_2}(T \geq \tau)$ for all $x_1 > x_2 \geq 0$ and $\tau \geq 0$, since X is almost surely continuous and needs a positive and finite amount of time to drop from x_1 to x_2. So, since according to Lemma 8 we have $\lim_{\delta\to 0} \mathbb{P}_{2\alpha,7/4,1,1/\delta} = \mathbb{P}_{2\alpha,7/4,1,\infty}$ and T is continuous, it therefore follows that

$$\lim_{\delta\to 0} \mathbb{P}_{2\alpha,7/4,1,1/\delta}\big(T \geq s - \delta\big) \leq \lim_{\delta\to 0} \mathbb{P}_{2\alpha,7/4,1,\infty}\big(T \geq s - \delta\big) = \mathbb{P}_{2\alpha,7/4,1,\infty}\big(T \geq s\big),$$

which yields

$$\lim_{n\to\infty} \mathbb{P}(n^{-1/2} T_{n,M} \leq s) = \mathbb{P}_{2\alpha,7/4,1,\infty}(T \leq s), \quad s \geq 0.$$

Thus, $n^{-1/2} T_{n,M} \xrightarrow{d} T_\alpha$, where T_α satisfies

$$\mathbb{P}(T_\alpha \leq s) = \mathbb{P}_{2\alpha,7/4,1,\infty}(T \leq s), \quad s \geq 0.$$

Moreover, $\mathbb{P}_{2\alpha,7/4,1,\infty}(T > 0) = 1$ implies that T_α is positive almost surely, and this completes the proof of Theorem 1. ◀

References

1 K. Bringmann, F. Kuhn, K. Panagiotou, U. Peter, and H. Thomas. Internal DLA: Efficient simulation of a physical growth model. In *Automata, Languages, and Programming*, pages 247–258, Berlin, Heidelberg, 2014. Springer Berlin Heidelberg.

2 C. Cooper, A. McDowell, T. Radzik, N. Rivera, and T. Shiraga. Dispersion processes. *Random Structures Algorithms*, 53(4):561–585, 2018. doi:10.1002/rsa.20822.

3 U. De Ambroggio, T. Makai, and K. Panagiotou. Dispersion on the complete graph, 2023. An extended abstract appeared in the *Proceedings of EUROCOMB '23*. arXiv:2306.02474.

4 P. Diaconis and W. Fulton. A growth model, a game, an algebra, Lagrange inversion, and characteristic classes. *Rend. Sem. Mat. Univ. Politec. Torino*, 49(1):95–119, 1991.

5 R. Durrett. *Stochastic calculus – a practical introduction.* Probability and Stochastics Series. CRC Press, Boca Raton, FL, 1996.

6 A. Frieze and W. Pegden. A note on dispersing particles on a line. *Random Structures Algorithms*, 53(4):586–591, 2018. doi:10.1002/rsa.20821.

7 A. Lambert. The branching process with logistic growth. *Ann. Appl. Probab.*, 15(2):1506–1535, 2005. doi:10.1214/105051605000000098.

8 G. F. Lawler, M. Bramson, and D. Griffeath. Internal diffusion limited aggregation. *Ann. Probab.*, 20(4):2117–2140, 1992.

9 L. T. Rolla. Activated Random Walks on $\mathbb{Z}^d$. *Probability Surveys*, 17:478–544, 2020. doi:10.1214/19-PS339.

10 Y. Shang. Longest distance of a non-uniform dispersion process on the infinite line. *Inform. Process. Lett.*, 164:106008, 5, 2020. doi:10.1016/j.ipl.2020.106008.