On Key Parameters Affecting the Realizability of Degree Sequences

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Abstract

Call a sequence $d = (d_1, d_2, \ldots, d_n)$ of positive integers *graphic*, *planaric*, *outer-planaric*, or *forestic* if it is the degree sequence of some arbitrary, planar, outer-planar, or cycle-free graph G, respectively. The two extreme classes of graphic and forestic sequences were given full characterizations. (The latter has a particularly simple criterion: d is forestic if and only if its *volume*, $\sum d \equiv \sum_i d_i$, satisfies $\sum d \leq 2n - 2$.) In contrast, the problems of fully characterizing planaric and outer-planaric degree sequences are still open.

In this paper, we discuss the parameters affecting the realizability of degree sequences by restricted classes of sparse graph, including planar graphs, outerplanar graphs, and some of their subclasses (e.g., 2-trees and cactus graphs). A key parameter is the *volume* of the sequence d, namely, $\sum d$ which is twice the number of edges in the realizing graph. For planar graphs, for example, an obvious consequence of Euler's theorem is that an n-element sequence d satisfying $\sum d > 4n - 6$ cannot be planaric. Hence, $\sum d \leq 4n - 6$ is a necessary condition for d to be planaric. What about the opposite direction? Is there an *upper* bound on $\sum d$ that guarantees that if d is graphic then it is also planaric. Does the answer depend on additional parameters? The same questions apply also to sub-classes of the planar graphs.

A concrete example that is illustrated in the technical part of the paper is the class of outer-planaric degree sequences. Denoting the number of 1's in d by ω_1, we show that for a graphic sequence d, if $\omega_1 = 0$ then d is outer-planaric when $\sum d \leq 3n - 3$, and if $\omega_1 > 0$ then d is outer-planaric when $\sum d \leq 3n - \omega_1 - 2$. Conversely, we show that there are graphic sequences that are not outer-planaric with $\omega_1 = 0$ and $\sum d = 3n - 2$, as well as ones with $\omega_1 > 0$ and $\sum d = 3n - \omega_1 - 1$.

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1 Introduction

Background. Throughout, we consider a a nonincreasing sequence $d = (d_1, \ldots, d_n)$ of n nonnegative integers. The sequence d is *graphic* if it is the degree sequence of some graph G. The graph realization problem concerns deciding, for a given d, if it is graphic, and if so -

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constructing a realizing graph for it. The problem has been studied extensively over the past 60 years, and was given both combinatorial characterizations and construction algorithms, cf. [9, 13, 14]. In particular, by [9], a nonincreasing sequence d is graphic if and only if

$$\sum_{i=1}^{\ell} d_i \leq \ell(\ell-1) + \sum_{i=\ell+1}^{n} \min\{\ell, d_i\}, \qquad \text{for } \ell = 1, \dots, n. \tag{1}$$

Beyond its theoretical interest, realization questions are also relevant in the context of specification-based network design, where the users specify the desired network properties, as well as in some scientific contexts where it is required to discover the unknown structure of a network based on measurements on its various parameters.

The question can be asked also in the context of special graph families. In particular, the realizability of a given sequence d by a *planar* graph was studied in [23] and some of the sequences whose status was left undetermined in [23] were later resolved in [10, 11]. In addition, regular planar graphic sequences were classified in [15], and planar bipartite biregular degree sequences were studied in [1]. Still, the realizability of a given sequence d by a planar graph was not given a complete solution so far.

The same question can be asked with respect to other restricted classes of graphs. One extreme example is that of *trees* and more generally *forests*. Whereas the characterization (1) for graphic degree sequences (realizable by general graphs) is somewhat involved, composed of n different conditions and affected individually by each of the degrees in the sequence, the characterization for *forestic* sequences (namely, ones realizable by a forest) is very simple: an n-element sequence d is forestic if and only if its *volume*, defined as $\sum d \equiv \sum_{i=1}^{n} d_i$, satisfies $\sum d \leq 2n-2$ (with equality if and only if the sequence is realizable by a tree). One might conjecture that restricted graph classes should tend to have simpler characterizations, or ones that depend on fewer parameters.

The current paper is motivated by the (intuitively clear) fact that the volume parameter of the sequence d plays a significant role in its realizability by classes of *sparse* graphs. One direction is easy: if the class $\mathcal{G}$ contains only graphs of M or fewer edges, then clearly, sequences of volume larger than $2M$ cannot have a realization from $\mathcal{G}$. Hence $\sum d \leq 2M$ is a *necessary* condition for realizability by a graph from $\mathcal{G}$. Here, we discuss the converse question: Can one derive *sufficient* conditions for realizability by $\mathcal{G}$ based on the volume parameter?

Indeed, for various low-volume graph classes $\mathcal{G}$, there are known results, guaranteeing that if d is graphic and $\sum d$ is sufficiently small, then d has a realizing graph in $\mathcal{G}$. In particular, consider the following hierarchy of graph classes and corresponding hierarchy of volume bounds.

- Forests [12]: A sequence d is forestic if and only if $\sum d \leq 2n-2$ and $\sum d$ is even.
- Uni-cyclic graphs [6]: These are connected graphs with precisely one cycle. A sequence d is uni-cyclic if and only if $\sum d = 2n$ and $d_1 \leq n-1$.
- Bi-cactus graphs [2]: Here, there is a full characterization that involves two additional parameters, ω_1 and ω_{odd}, where ω_i is the number of i's in d and ω_{odd} is the number of odd degrees in d. In particular, a necessary condition for a sequence d to be realizable by a bi-cactus is that $\sum d < 8n/3$. The sufficient condition is more complex, utilizing also ω_1 and ω_{odd}.
- Cactus graphs [20]: Here, again, there is a full characterization that involves ω_1 and ω_{odd}. In particular, a necessary condition for a sequence d to be realizable by a cactus graph is that $\sum d \leq 3(n-1)$, and the sufficient condition utilizes also ω_1 and ω_{odd}.
- Outer-planar graphs: This is the main technical contribution of the current paper, to be described shortly.

- 2-trees [7]: This class was given a full characterization, where the volume requirement is $\sum d = 4n-6$ but the precise conditions depend also on ω_1 and ω_2, and involve also some exceptions.
- Planar graphs [4]: Full characterization is still out of reach. The natural necessary condition based on volume is $\sum d \le 6n-12$. The sufficient condition given in [4] depends also on ω_1.

Our main technical contribution concerns sequences that can be realized by outer-planar graphs. A graph is *outer-planar* if it can be embedded in the plane such that edges do not intersect each other and additionally, each vertex lies on the outer face of the embedding, i.e., no vertex is fully surrounded inside an internal face (cf. [24]). Call a sequence d *planaric* (respectively, *outer-planaric*) if it is the degree sequence of some planar (resp., outer-planar) graph G.

Since it is known that every outer-planar graph has at most $2n-3$ edges, it follows that every sequence d with $\sum d > 4n-6$ cannot be outer-planaric. In fact, as claimed in [3], this bound can be improved if ω_1, the multiplicity of degree 1, is taken into consideration. Specifically, if the sequence d satisfies $\sum d > 4n-6-2\omega_1$ and $d \ne (n-1,1,\ldots,1)$, then it is not outer-planaric (the exceptional sequence has $n-1$ leaves and volume $2n-2$, and is realizable by a star graph). It is also easy to show that this bound is tight, in the sense that there are outerplanar sequences d with $\sum d = 4n-6-2\omega_1$, for ω_1 values in the range $[1, n-2]$.

Focusing on the converse question, we look for a function $f(n,\omega_1)$ guaranteeing that if d is graphic and $\sum d \le f(n,\omega_1)$, then d is *always* outer-planaric. A straightforward such bound is $f(n,\omega_1) = 2n-2$, since as mentioned above, if $\sum d \le 2n-2$ then d has a realization by a forest, hence it is trivially outer-planaric. Here, we present a tight answer to the question, separated into two cases, depending on whether or not the sequence d contains 1's. Specifically, we show that if $\omega_1 = 0$ then the desired property holds when $\sum d \le 3n-3$, and if $\omega_1 > 0$ then the desired property holds when $\sum d \le 3n-\omega_1-2$. Conversely, we show that the set of graphic sequences that are not outer-planaric includes:

(1) sequences with $\omega_1 = 0$ and $\sum d = 3n-2$;

(2) sequences with $\omega_1 > 0$ and $\sum d = 3n-\omega_1-1$.

Related Work. A number of papers studied the outerplanar degree realization problem in the past. *Forcibly* outerplanar graphic sequences, i.e., sequences *all* of whose realizations are outerplanar, were given a full characterization in [8]. The degree sequences of maximal outerplanar graphs with exactly two 2-degree nodes were characterized in [19]. This was also mentioned in [7] independently. The degree sequences of maximal outerplanar graphs with at most four vertices of degree 2 were characterized in [17].

The special class *2-trees* has also received some attention. A graph G is a 2-tree if $G = K_3$ or G has a vertex v with degree 2, whose neighbors are adjacent and $G[V \setminus \{v\}]$ is a 2-tree. Sufficient conditions for a sequence d to have a realization by a 2-tree were given in [18]. The degree sequences of 2-trees were fully characterized in [7]. (The conditions are surprisingly rather involved, and include a number of specific exceptions.) Note that 2-trees have $\sum d = 4n-6$. This implies, using Theorem 1 of [7], that if a sequence d satisfies some specific conditions, then d has a realizing 2-tree. Rengarajan and Veni Madhavan [22] have shown that every 2-tree has a 2-page book embedding. Unfortunately, the class of 2-trees is not hereditary (meaning that a subgraph of a 2-tree is not necessarily a 2-tree), so the result of [7] does not extend to non-maximal degree sequences.

The planar degree realization problem also has a long history. Regular planar graphic sequences and planar bipartite biregular degree sequences were given a classification in [15] and [1] respectively. Schmeichel and Hakimi [23] identified the graphic sequences with $d_1 - d_n = 1$ that are planaric, and did the same for $d_1 - d_n = 2$ with a small number of unresolved exceptions, some of which were later resolved in [10, 11]. Additional studies on special cases of the planaric degree realization problem are discussed in [21].

In [4] it is shown that for every sequence d with $\sum d \leq 4n - 4 - 2\omega_1$, if d is graphic then it is also planaric. Conversely, there are graphic sequences with $\sum d = 4n - 2\omega_1$ that are non-planaric. For the case $\omega_1 = 0$, it is shown therein that d is planaric when $\sum d \leq 4n - 4$, and conversely, there is a graphic sequence with $\sum d = 4n - 2$ that is non-planaric.

A cactus graph is a connected graph in which any edge may be a member of at most one cycle, which means that different cycles do not share edges, but may share one vertex. Rao [20] provided a full characterization for degree sequences realizable by cactus graphs, and also gave a characterization for degree sequences realizable by cactus graphs whose cycles are triangles and for degree sequences realizable by connected graphs whose blocks are cycles of k vertices. Beineke and Schmeichel [5] characterized cacti degree sequences with up to four cycles and also provided a sufficient condition for cactus realization. A characterization for degree sequences realizable by bipartite cactus graphs was shown in [2].

2 Preliminaries and Known Results

Some Terminology. Two necessary (but not sufficient) conditions for a non-increasing sequence $d = (d_1, \ldots, d_n)$ to be graphic are that $\sum d$ is even and $d_1 \leq n - 1$. We refer to sequences that satisfy these two conditions as *standard* sequences. Hereafter, we consider only standard sequences.

For a nonincreasing sequence d of n nonnegative integers, let $\mathsf{pos}(d)$ denote the prefix consisting of the positive integers of d. We use the shorthand a^k to denote a subsequence of k consecutive a's. Denote the *volume* of sequence $d = (d_1, \ldots, d_n)$ by $\sum d \equiv \sum_{i=1}^{n} d_i$.

Trees and Forests. Consider a sequence $d = (d_1, \ldots, d_n)$ of positive integers such that $\sum d$ is an even number. It is known that if $\sum d \leq 2n - 2$, then d is graphic and moreover, $\mathcal{G}(d)$ contains an acyclic graph (forest). In this case, we say that d is *forestic*. If $\sum d = 2n - 2$, then d can be realized by a tree and we say that d is *treeic*. We refer to a vertex of degree one as a *leaf*.

We make use of a special type of realizations for treeic and forestic sequences, known as *caterpillar graphs*. In a caterpillar graph $G = (V, E)$, all non-leaves are arranged on a path which we call the *spine* of G, i.e., the spine $S = (x_1, \ldots, x_s) \subset V$ is an ordered sequence where $(x_i, x_{i+1}) \in E$, for $i = 1, \ldots, s - 1$ (see Figure 1 for an example).

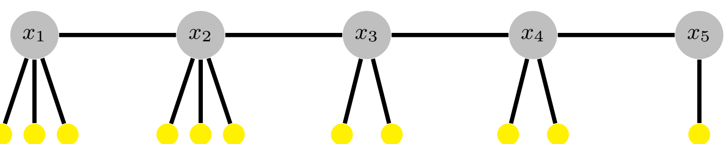

Figure 1 Caterpillar graph with degree sequence $(5, 4^3, 2, 1^{11})$. Leaves are depicted in yellow.

The following (possibly folklore) observation appears in [3]. We provide a proof, since its method will be instrumental in what follows.

▶ **Observation 1** ([3]). *Any forestic sequence d can be realized by a forest composed of the union of a caterpillar graph and a matching.*

Proof. First consider a treeic sequence d (such that $\sum d = 2n-2$). Assume there are $n-s$ vertices of degree 1 in d. Denote by d^* the prefix of d that contains all the degrees $d_i > 1$. It follows that $\sum d^* + (n-s) = \sum d = 2n-2$, or equivalently $\sum d^* - 2s + 2 = n - s$. To get the caterpillar realization of d, first arrange s vertices corresponding to the degrees of d^* in a path. The path edges contribute $2s-2$ to the volume $\sum d^*$. The missing $\sum d^* - 2s + 2 = n - s$ degrees are satisfied by attaching the $n-s$ leaves to the path, which now forms the spine of the caterpillar realization. Note that the order of the vertices on the spine can be arbitrary.

Now assume that d is forestic but not treeic, i.e., $\sum d < 2n-2$. In this case, remove pairs of 1 degrees from d until the volume of the reaming sequence d' with n' vertices is $2(n'-2)$. This must happen because each pair removal reduces the volume by 2 while $2n-2$ decreases by 4. Let G' be the caterpilar realization of d' as implied by the first part of the proof. To get the realization of d, add $(n-n')/2$ edges to G' to satisfy the $n-n'$ removed 1 degrees by a matching. ◀

Necessary Conditions for Outer-Planaric Sequences.

▶ **Lemma 2** ([25]). *If $d = (d_1, \ldots, d_n)$ is an outer-planaric degree sequence where $n \geq 2$, then $\sum d \leq 4n - 6$.*

It is known that every outer-planar graph has at least two vertices of degree two or less, and at least three vertices of degree three or less, see for example Sysło [24]. This implies the following necessary condition for outer-planaric sequences.

▶ **Lemma 3** ([24]). *If $d = (d_1, \ldots, d_n)$ is an outer-planaric degree sequence, then*
(i) $d_{n-1} \leq 2$, *and*
(ii) $d_{n-2} \leq 3$.

The Havel-Hakimi Algorithm and Outer-planaric Sequences. The lay-off techniques developed by Havel and Hakimi, and subsequently extended by Kleitmann and Wang, are among the fundamental tools for constructing realizations for degree sequences. One might hope to be able to use such techniques for finding (outer-)planar realizations as well. Unfortunately, it turns out that the lay-off operation does not, in general, preserve planarity or outer-planarity (only graphicity), hence it cannot be applied directly to generate a realizing outer-planar graph for a given outer-planaric sequence.

Nevertheless, we do make use of the lay-off technique, and specifically the *minimum pivot* version of the Havel-Hakimi algorithm [13, 14]. It is used for realizing a nonincreasing degree sequence $d = (d_1, d_2, \cdots, d_n)$ associated with the vertices $v_1, v_2, \ldots, v_n$, and is based on repeatedly performing the following operation, hereafter referred to as the *MP-step*, until all the vertices reach their required degree. Suppose the current sequence of residual nonzero degrees is $d' = (d'_1 \geq d'_2 \geq \cdots \geq d'_h)$ and the corresponding vertices are $v_{i_1}, v_{i_2}, \ldots, v_{i_h}$.

- Pick the vertex $v = v_{i_h}$ with degree d'_h as pivot.
- Set v's neighbors to be the d'_h vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_{d'_h}}$.
- Delete the pivot from d', reduce by 1 the residual degrees of its selected neighbors, and delete from d every vertex whose residual degree became zero.

The key observation is that, in case the MP-step transforms the residual degree sequence d into d', the following holds: d is graphic if and only if d' is graphic.

3 Outer-Planaric Degree Sequences

3.1 Necessary conditions

We first recall that Lemma 2 can be improved if ω_1, the multiplicity of degree 1, is taken into consideration. We add the proof for completeness.

▶ **Lemma 4** ([3]). *If d is an outer-planaric sequence such that $\sum d > 2n-2$, then $\sum d \leq 4n-6-2\omega_1$.*

Proof. Let d be as in the lemma and let $G=(V,E)$ be an outerplanar graph realizing d. Construct a graph G' by deleting from G all the ω_1 vertices of degree 1 together with their incident edges, and subsequently deleting the vertices whose degree was reduced to zero. Notice that the graph G' may contain vertices whose degree became 1. Observe that G' is outerplanar, and let $d' = \deg(G')$. Note that $n' \leq n-\omega_1$, where n' the number of vertices in G'. We have that $\sum d' \leq 4n'-6$ due to Lemma 2. Hence,

$$\sum d = \sum d' + 2\omega_1 \leq 4(n-\omega_1) - 6 + 2\omega_1 = 4n - 6 - 2\omega_1 \ ,$$

where the first equality comes from the fact that ω_1 vertices of degree 1 together with their incident edges are deleted. ◀

We remark that Lemma 4 is false if we drop the assumption $2n-2 < \sum d$. To see this, consider the sequence $d=(a,1^a)$ which is (uniquely) realized by a star graph and therefore outer-planaric. The construction of G' in the proof above, applied to d, yields an isolated vertex. Note that Lemma 2 cannot be applied in this case. Indeed, we have that

$$\sum d = 2a > 2a-2 = 4(a+1) - 6 - 2a = 4n - 6 - 2\omega_1.$$

Observe that sequences of the form $d=(a,1^{a+t})$ are the only sequences where G' has order one or less. If $t \geq 2$ (note that t must be even), the bound can be shown directly with the above calculation. We state the following corollary.

▶ **Corollary 5.** *If $d=(d_1,\ldots,d_n)$ is an outer-planaric sequence such that $d \neq (a,1^a)$, then $\sum d \leq 4n-6-2\omega_1$.*

The next corollary gives us a useful necessary condition for outer-planaric sequences.

▶ **Corollary 6** ([16]). *If $d=(d_1,\ldots,d_n)$ is an outer-planaric sequences, then $d_1+d_2 \leq n+2$.*

3.2 Sufficient Conditions

In this section, we show sufficient condition for outer-planaric sequences.

The collection of *low-volume* (standard) sequences is defined as

$$LV \ = \ LV_1 \cup LV_2 \ ,$$

where

$$LV_1 = \left\{ d \mid \sum d \leq 3n - \omega_1 - 2, \quad d_n = 1, \quad d \text{ is graphic} \right\}$$
$$LV_2 = \left\{ d \mid \sum d \leq 3n - 3, \quad d_n = 2 \right\}$$

Notice that by assuming that the sequences are standard, we implicitly require that $\sum d$ is even and that $d_1 \leq n-1$.

We now show that all sequences in LV are outer-planaric. For the proof we analyze a slightly large collection than LV_2, namely,

$$LV_2' = \left\{ d \mid \sum d \le 3n-2, \quad d_n = 2 \right\},$$

Notice that $LV_2 \subset LV_2'$. We prove that all the sequences in LV_2' except for a small set EX are outer-planaric. As $LV_2 \subset LV_2' \setminus EX$, we get that all the sequences in LV_2 are outer-planaric. Moreover, we also use LV_2' to prove that all the sequences in LV_1 are outer-planaric.

In order to analyze the collection LV_2', we break it into four sub-collections A, B, C, D, and handle each of them separately. Specifically, we define the following sets.

$$\begin{aligned}
A &= \left\{ (3^s, 2^{n-s}) \mid n \ge 3, \quad 0 \le s \le n-2, \quad \text{and } s \text{ is even} \right\}, \\
B &= \left\{ (d_1, 2^{n-1}) \mid d_1 \ge 4 \quad \text{and} \quad d_1 \text{ is even} \right\}, \\
C^M_{even} &= \left\{ (d_1, d_2, 2^{n-2}) \mid d_1, d_2 \ge 4, \quad d_1 + d_2 = M \quad \text{and} \quad d_1, d_2 \text{ are even} \right\}, \\
C^M_{odd} &= \left\{ (d_1, d_2, 2^{n-2}) \mid d_1 \ge 5, \quad d_2 \ge 3, \quad d_1 + d_2 = M \quad \text{and} \quad d_1, d_2 \text{ are odd} \right\}, \\
C_{even} &= \bigcup_{M \le n+2} C^M_{even}, \\
C_{odd} &= \bigcup_{M \le n+2} C^M_{odd}, \\
C &= C_{even} \cup C_{odd}, \\
D &= \left\{ d \mid d_1 \ge 4, \quad d_3 \ge 3, \quad d_n = 2, \quad \sum d \le 3n-2 \right\}.
\end{aligned}$$

Note that all sequences in A, B, C, D are standard.

▶ **Observation 7.** $LV_2' = A \cup B \cup C \cup D.$

Proof. One can check directly that $A, B, C, D \subseteq LV_2'$, and hence $A \cup B \cup C \cup D \subseteq LV_2'$. For the converse, consider some $d \in LV_2'$. If $d \in LV_2' \setminus D$, then $d_3 = 2$ or $d_1 \le 3$, so d has the form $(d_1, d_2, 2^{n-2})$ or $(3^s, 2^{n-s})$. First assume that $d_3 = 2$ and consider d of the form $(d_1, d_2, 2^{n-2})$. If $d_2 = 2$, then $d \in B$. If $d_1 = d_2 = 3$, then $d \in A$. Otherwise $d_1 \ge 4$ and $d_2 \ge 3$, we have $d_1 + d_2 \le n+2$ since $\sum d \le 3n-2$. In this case, $d \in C$. Next consider d of the form $(3^s, 2^{n-s})$. Since $\sum d \le 3n-2$, $s \le n-2$. In this case, $d \in A$. Combining these cases, we have $LV_2' \subseteq A \cup B \cup C \cup D$. The observation follows. ◀

▶ **Lemma 8.** *Every sequence $d \in A$ is outer-planaric.*

Proof. Let d be as in the lemma. By the definition of A, $d = (3^s, 2^{n-s})$ where s is even and $0 \le s \le n-2$. We construct an outer-planar realization G of d as follows. First, arrange n vertices in a cycle, i.e., let $G = C_n$. If $s = 0$, then G is a valid realization of d (noting that $n \ge 3$ so G is a simple graph). Now suppose $s > 0$. Select one vertex y on the cycle and denote its clockwise (respectively, counter-clockwise) neighbor of distance x by u_x (resp., v_x), for $x = 1, \ldots, s/2$. Observe that the vertices $u_{s/2}$ and $v_{s/2}$ cannot be connected by an edge in C_n, since $s \le n-2$. To complete our construction, we add to $E(G)$ a matching consisting of the edges (v_x, u_x), for $x = 1, \ldots, s/2$ Since these new edges can be placed *inside* the cycle, G has an outer-planar embedding as shown in Figure 2. Verifying that $\deg(G) = d$, the claim follows. ◀

▶ **Lemma 9.** *Every sequence $d \in B$ is outer-planaric.*

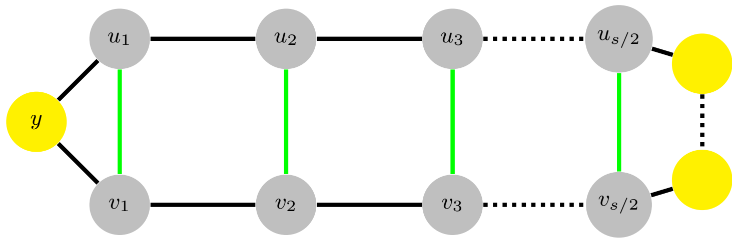

Figure 2 Schematic outer-planar realization of a sequence $d = (3^s, 2^{n-s})$ where s is even and $0 \le s \le n-2$. The yellow vertices have degree two and the gray vertices have degree three. Cycle edges are drawn in black and matching edges are drawn in green.

Proof. Since d_1 is even, d can be realized by the graph $G = (V, E)$ where $V = \{v_1, \dots, v_n\}$ and the edge set E is constructed by the following steps:

(1) $E_1 \leftarrow \{(v_1, v_i) \mid i = 2, \dots, d_1 + 1\}$ connects v_1 to d_1 other vertices.

(2) $E_2 \leftarrow \{(v_{2j}, v_{2j+1}) \mid j = 2, \dots, d_1/2\}$ is a perfect matching among the neighbors of v_1.

(3) Let $W \leftarrow V \setminus \{v_1, \dots, v_{d_1+1}\}$ be the set of remaining vertices. If $W = \emptyset$, then set $E_3 \leftarrow \{(v_2, v_3)\}$. Otherwise (i.e., if $W = \{v_j \mid j = d_1 + 2, \dots, n\}$ where $d_1 + 2 \le n$), $E_3 \leftarrow \{(v_2, v_{d_1+2}), (v_{d_1+2}, v_{d_1+3}), \dots, (v_n, v_3)\}$.

(4) Set $E \leftarrow E_1 \cup E_2 \cup E_3$.

Note that Steps (1) and (3) can be performed based on d since $d_1 \le n-1$. More specifically, Step (1) yields well-defined edges, i.e., $E_1 \subseteq V \times V$, and in Step (3), W is well-defined. Step (2) can be performed based on d i.e., $E_2 \subseteq V \times V$, since d_1 is even. Figure 3 illustrates these steps. Note that every vertex is located on the outer face, so G is outerplanar. As G realizes the degree sequence d, it follows that d is outer-planaric. ◀

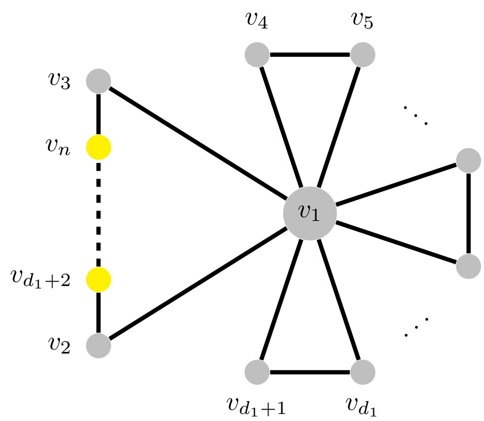

Figure 3 An illustration of the graph G constructed in the proof of Lemma 9 where the set W (vertices in yellow) is not empty.

Define $EX = C^{n+2}_{even} = \{(d_1, d_2, 2^{n-2}) \mid d_1, d_2 \ge 4, \ \ d_1 + d_2 = n+2 \ \text{ and } \ d_1, d_2 \text{ are even}\}$.

▶ **Lemma 10.** *Let $d \in C \setminus EX$. Then d is outer-planaric.*

Proof. Suppose $d \in C \setminus EX$. Since $\sum d \le 3n-2$, necessarily $d_1 + d_2 \le n+2$.

First suppose $d \in C_{odd}$. Construct a realizing graph G on the vertex set $V = \{v_1, \dots, v_n\}$ by taking the following steps. Let v_1 and v_2 be the vertices with degree d_1 and d_2 respectively.

(1) $E_1 = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4)\}$. These edges connect v_1 to v_2 and also connect v_1 and v_2 to two common neighbors, v_3 and v_4.
(2) $E_2 \leftarrow \{(v_1, v_i) \mid i = 5, \ldots, d_1 + 1\}$ connects $d_1 - 3$ extra neighbors from the remaining vertices to v_1, and $E_3 \leftarrow \{(v_{5+2j}, v_{6+2j}) \mid j = 0, \ldots, \frac{d_1-5}{2}\}$ sets a perfect matching among the vertices $\{v_i \mid i = 5, \ldots, d_1 + 1\}$.
(3) If $d_2 = 3$, then $E_4, E_5 \leftarrow \emptyset$.
Otherwise, $E_4 \leftarrow \{(v_2, v_i) \mid i = d_1 + 2, \ldots, d_1 + d_2 - 2\}$ connects $d_2 - 3$ extra neighbors from the remaining vertices to v_2, and $E_5 \leftarrow \{(v_{d_1+2+2j}, v_{d_1+3+2j}) \mid j = 0, \ldots, \frac{d_2-5}{2}\}$ is a perfect matching among the vertices $\{v_i \mid i = d_1 + 2, \ldots, d_1 + d_2 - 2\}$.
(4) Let $W \leftarrow V \setminus \{v_1, \ldots, v_{d_1+d_2-2}\}$ be the set of remaining vertices. If $W = \emptyset$, then set $E_6 \leftarrow \{(v_1, v_3)\}$
Otherwise, plant the vertices of W on the edge (v_1, v_3), or more formally, replace (v_1, v_3) with the edges in the path $(v_1, v_{d_1+d_2-1}, \ldots, v_n, v_3)$, namely with the edge set $E_6 \leftarrow \{(v_1, v_{d_1+d_2-1}), (v_{d_1+d_2-1}, v_{d_1+d_2}), \ldots, (v_n, v_3)\}$
(5) Set $E \leftarrow (E_1 \setminus \{(v_1, v_3)\} \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$.

Note that Step (1) can be performed based on d, in the sense that no vertex is assigned more edges than its degree, since $d_1, d_2 \geq 3$ and $d_3 = d_4 = 2$. Steps (2) and (3) use $d_1 + d_2 - 6$ distinct vertices (from v_5 to $v_{d_1+d_2-2}$) to serve as the remaining neighbors of v_1 and v_2. Since $d_1 + d_2 \leq n + 2$, Steps (2) and (3) can be performed based on d, i.e., $d_1 + d_2 - 2 \leq n$ so $E_2 \cup E_3 \subseteq V \times V$. Besides, the matchings in Steps (2) and (3) can be performed based on d, i.e., $E_4 \subseteq V \times V$, since d_1 and d_2 are both odd. Figure 4 illustrates the steps of the construction. One can check that constructed graph G realizes d and is outer-planaric, since all vertices are located on the outer face.

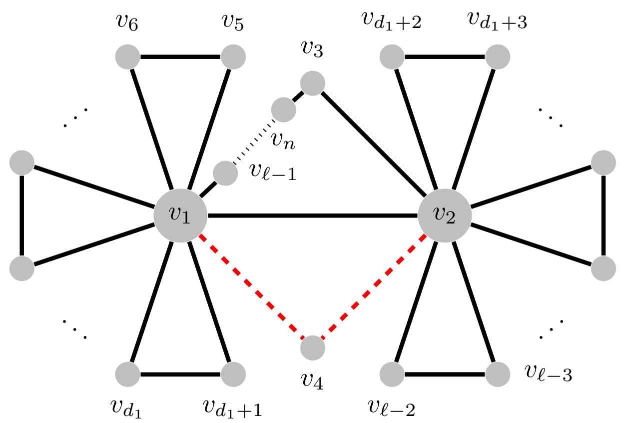

Figure 4 An illustration of the graph G constructed in the proof of Lemma 10 for the case where $d \in C_{odd}$. Let $\ell = d_1 + d_2$. The graph for the case where $d \in C_{even}$ is obtained by removing the red dashed edges.

Now suppose $d \in C_{even} \setminus EX$. Hence, $d_1 + d_2 \leq n + 1$. Let $d' = (d_1 + 1, d_2 + 1, 2^{n-1})$. Notice that $n' = n + 1$ and that $d'_1 + d'_2 = d_1 + d_2 + 2 \leq n + 1 + 2 = n' + 2$. Hence, $d' \in C_{odd}$. Construct a realizing outerplanar graph G' for the sequence d'. Let G be the graph we get by removing v_4 and its two edges the edges (v_1, v_4) and (v_2, v_4) from G'. (The dashed red edges in Figure 4.) Observe that G realizes d and it is outer-planar. ◀

▶ **Lemma 11.** *Let $d \in EX$. Then d is not outer-planaric.*

Proof. Recall that $d = (d_1, d_2, 2^{n-2})$, where $4 \le d_1 \le n-1$, $d_2 \ge 3$, $d_1 + d_2 = n+2$, and both d_1 and d_2 are even. Assume towards contradiction that d is outer-planaric, and let $G = (V, E)$ be a realizing outerplanar graph for it. Let v_1 and v_2 be the vertices of degree d_1 and d_2 in G, respectively.

Let $k \le d_2$ be the number of common neighbors of v_1 and v_2. If v_1 and v_2 are not adjacent, then

$$n - 2 = (d_1 - k) + (d_2 - k) + k = d_1 + d_2 - k = n + 2 - k \ .$$

Hence, $k = 4$. In this case, G contains a subgraph homeomorphic to $K_{2,3}$, and therefore it is not outer-planar [24], leading to a contradiction.

Otherwise, if v_1 and v_2 are adjacent, then

$$n - 2 = (d_1 - k - 1) + (d_2 - k - 1) + k = d_1 + d_2 - 2 - k = n - k \ .$$

Hence, $k = 2$. Consequently, denoting the two common neighbors of v_1 and v_2 in G by v_3 and v_4, let $N'[v_1]$ (respectively, $N'[v_2]$) be the set of neighbours of v_1 (resp., v_2) except for v_3, v_4 and v_2 (resp., v_1). See Figure 5. $|N'[v_1]|$ and $|N'[v_2]|$ are odd, since d_1, d_2 are even. Since the vertices in $N'[v_1] \cup N'[v_2]$ all have degree 2, there must exist (at least) one vertex in $N'[v_1]$ and one vertex in $N'[v_2]$ that are connected. (In Figure 5, these are marked as v_5 and v_6.) It follows that there are three paths from v_1 to v_2, the first contains v_3, the second contains v_4, and the third contains v_5 and v_6. In this situation, in any planar embedding of G, at least one of the common neighbors v_3 or v_4 or the pair v_5 and v_6 is not in the outer face. (In Figure 5, the vertex v_3 is blocked.) This leads to a contradiction. ◀

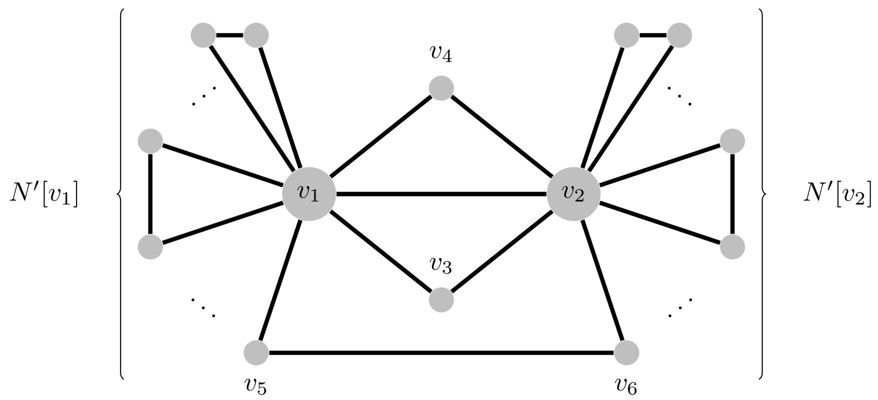

Figure 5 The non outer-planar graph G constructed in the proof of Lemma 11. $N'[v_1]$ and $N'[v_2]$ are, respectively, the vertex sets to the left of v_1 and to the right of v_2.

To prove the next lemma, we construct an outer-planar graph by starting from a caterpillar graph and adding a matching, increasing the degree of each vertex by one. The next observation describes a part of the construction used repetitively.

▶ **Observation 12.** *Let $G = (V, E)$ be a caterpillar graph with an outer-planar embedding as depicted in Figure 1. If $L \subseteq V$ is an even set of leaf vertices that appear consecutively in the embedding, then one can add a matching between the vertices of L such that the resulting graph has an outer-planar embedding.*

Proof. Let $L = \{\ell_1, \dots, \ell_h\}$ be the leaf vertices in consecutive order, for even h. Note that vertices in L are pairwise non-adjacent. We add the matching edges (ℓ_{2i-1}, ℓ_{2i}), for $i = 1, \dots, h/2$, i.e., we add every second edge of the path $(\ell_1, \dots, \ell_h)$. ◀

The next lemma proves the most general case where $d_n \geq 2$.

▶ **Lemma 13.** *Every sequence $d \in D$ is outer-planaric.*

Proof. Consider $d \in D$. Denote $k = \sum d - (2n-2)$. Observe that k is even and that $k \geq 2$, since $\sum d \geq 2n$. Also, $k \leq 3n - 2 - (2n-2) = n$.

Our proof consists of two major steps:

(1) Create a treeic sequence d' from d and find an (outer-planar) caterpillar realization G of d' as described in Observation 1.

(2) Modify G by adding edges such that $\deg(G) = d$ and G remains outer-planar.

For step (1), construct the sequence $d' = (d'_1, d'_2, \ldots, d'_n)$ as follows:

$$d'_i = \begin{cases} d_i, & i \leq n-k, \\ d_i - 1, & i > n-k. \end{cases}$$

d' is well defined since $k \leq n$, and also note that $\sum d' = \sum d - k = 2n - 2$.

We first assume that $\omega'_2 \leq 2$.

Relying on Observation 1, we find a caterpillar realization $G = (V, E)$ of d'. Let $S = \{v_1, \ldots, v_s\} \subseteq V$ be the vertices on the spine of G. Note that $|S| \geq 3$ since $d_3 \geq 3$, implying that (at least) the three highest-degree vertices are non-leaves. Let ℓ denote the number of leaves in the caterpillar. Notice that $\ell + s = n$. Also, observe that $k \geq \ell$. Out goal is to increase the degree of all leaves and of $k - \ell$ vertices in the spine by 1 while maintaining outer-planarity.

As a preliminary step we rearrange the spine such that vertices high degree vertices are closer to the center of the spine. Specifically, if s is even, the order is $v_{s-1}, \ldots, v_3, v_1, v_2, \ldots, v_s$, and if s is odd, the order is $v_{s-1}, \ldots, v_2, v_1, v_3, \ldots, v_s$. Notice that only v_s and v_{s-1} may be of degree 2. We refer to such a construction as an *ordered caterpillar*.

For step (2), we consider two cases:

Case A: $k < n$.

Case A.1: ℓ and $k - \ell$ are even.

We add a perfect matching of consecutive leaves as described in Observation 12. We also add the (matching) edges (v_i, v_{i-1}), for $i = s, s-2, \ldots, s-(k-\ell)+2$ between the vertices on the spine. The latter is feasible since $k < n$ (or $k - \ell < s$).

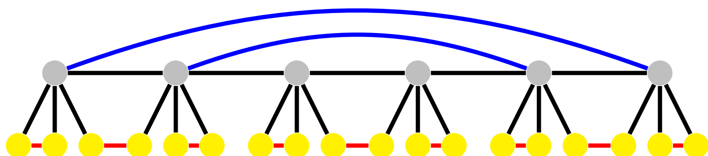

Figure 6 Illustration of the construction in Case A.1.

Case A.2: If ℓ and $k - \ell$ are odd, we add one leaf of v_{s-1} to the spine. Since $\ell - 1$ is even, we can add edges as in case A.1.

Case B: $k = n$.

Observe that n must be even, since k is even. Also, in this case, $s = k - \ell$. Let m be the number of leaves connected to spine vertices to the left of v_1 up to v_1 (including v_1).

Case B.1: s and ℓ are odd.

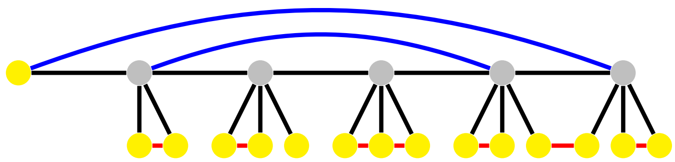

Figure 7 Illustration of the construction in Case A.2.

Case B.1(i): m is even.
Add a perfect matching of the consecutive leaves connected to $v_{s-1}, v_{s-3}, \ldots, v_1$ as described in Observation 12, add an edge connecting v_1 to the leftmost leaf of the vertex to its right (v_2 or v_3), and then add perfect matching of the rest of the leaves. In addition, add the (matching) edges (v_i, v_{i-1}), for $i = s, s-2, \ldots, 3$ between the vertices on the spine.

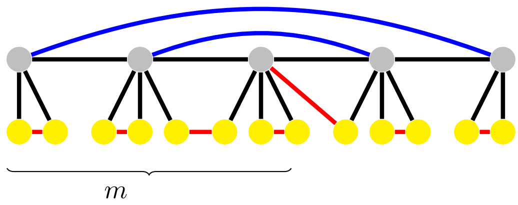

Figure 8 Illustration of the construction in Case B.1(i).

Case B.1(ii): m is odd.
Add the leftmost leaf of v_{s-1} and the rightmost leaf of v_s to the spine. Now use the same construction as in the case B.1(i).

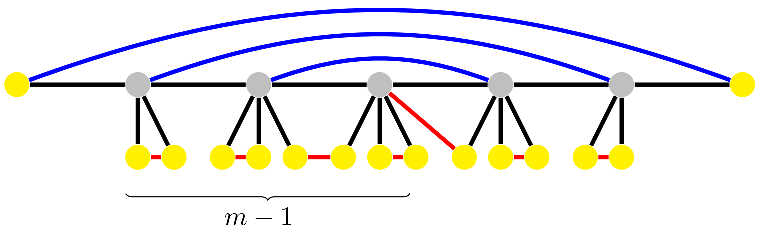

Figure 9 Illustration of the construction in Case B.1(ii).

Case B.2: s and ℓ are even.
Case B.2(i): m is odd.
Add the leftmost leaf of v_{s-1} to the spine, and use the construction of case B.1(i).
Case B.2(ii): m is even.
Add the edges (v_i, v_{i-1}), for $i = s-1, s-3, \ldots, 3$ between the vertices on the spine. Also, add the edge (v_{s-1}, v_s). In addition, replace the edge (v_{s-1}, u) with the edge (v_1, u), where u be the leftmost leaf of v_{s-1}. Add a perfect matching of consecutive leaves as described in Observation 12.

It remains to consider the case where $\omega_2' > 2$. There are two options regarding the 2's in d'. Either all of them appear in d, or all of them originate from 3's in d.

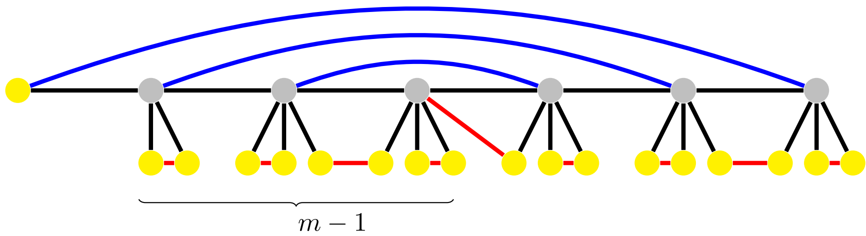

Figure 10 Illustration of the construction in Case B.2(i).

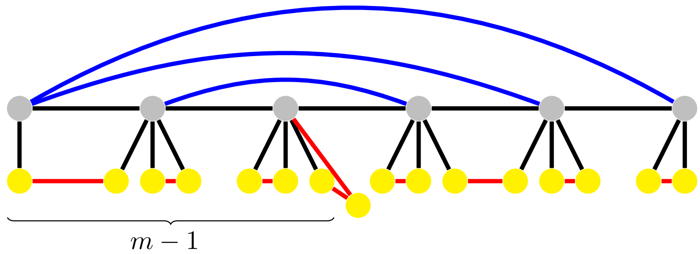

Figure 11 Illustration of the construction in Case B.2(ii).

In the first case, we obtain a sequence d'' by removing all 2's from d', and construct an outer-planar embedding for the first n'' entries by using the construction for case where $\omega_2'' \leq 2$. Consider any edge (x, y) connecting two former leaves in the above construction. We replace the edge with a path of length $\omega_2' + 1$.

In the second case, we remove $t = 2\lfloor\omega_2'/2\rfloor$ 2's from d', and construct an outer-planar embedding for the first n'' entries by using the construction for case where $\omega_2'' \leq 2$. Consider any edge (x, y) connecting two former leaves in the above construction. We replace (x, y) with a path $(x = u_0, u_1, \ldots, u_t, u_{t+1} = y)$. Then, we add the edges $\{(u_i, u_{t+1-i}) : i = 0, 1, \ldots, t/2 - 1\}$, but not in the outer-face. ◀

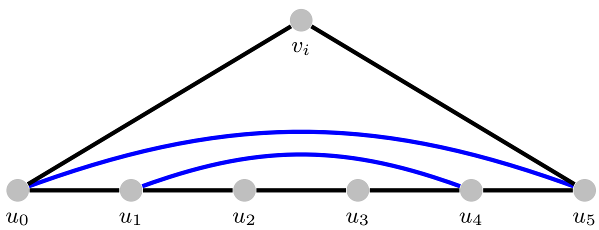

Figure 12 Illustration for the case where $\omega_2' > 2$, where $t = 4$.

Combining Lemmas 8, 9, 10, 11, 13 and Observation 7, we have the following corollary.

▶ **Corollary 14.** *Let $d \in LV_2'$.*

1. *The sequence d is outer-planaric except for $d \in EX$.*
2. *If $d \in LV_2$, then d is outer-planaric.*

Finally, we deal with sequences where $d_n = 1$.

▶ **Lemma 15.** *Every sequence $d \in LV_1$ is outer-planaric.*

Proof. Consider $d \in LV_1$. If $\sum d \le 2n-2$, then the claim follows from Observation 1. So now assume that $\sum d > 2n-2$. Applying the MP-step of the HH method on d (see Section 2) with pivot 1 connecting to the largest degree ω_1 times yields a new sequence d', with $n' := |\mathsf{pos}(d')| = n - \omega_1$ nonzero degrees.

We claim that all the degrees in $\mathsf{pos}(d')$ are at least 2. If $d'_{n'} = 1$, then the other degrees in $\mathsf{pos}(d')$ have value 1 or 2 by the MP-step of the HH method. Therefore, $\sum \mathsf{pos}(d') \le 2(n'-1)+1 = 2n-2\omega_1-1$, and consequently $\sum d = \sum d' + 2\omega_1 \le 2n-1$, contradicting the assumption on $\sum d$. Furthermore, by the MP-step of the HH method, $\mathsf{pos}(d')$ is graphic, so $d'_1 \le n'-1$. Also, $\sum \mathsf{pos}(d') = \sum d - 2\omega_1 \le 3(n-\omega_1) - 2 = 3n'-2$. Hence, $\mathsf{pos}(d') \in LV'_2$. This and Corollary 14 imply that $\mathsf{pos}(d')$ is outer-planaric except for $\mathsf{pos}(d') \in EX$.

Notice that if $\mathsf{pos}(d')$ can be realized by an outer-planar graph G', then d is also outer-planaric, since a realizing graph G can be obtained from G' by inserting $d_i - d'_i$ new leaves to each vertex of degree d'_i. This completes the proof of the lemma for all cases except when $\mathsf{pos}(d') \in EX$, in which case it may be that d is outer-planaric yet $\mathsf{pos}(d')$ is not. For example, the sequence $d = (8, 4, 2^6, 1^2)$ is outer-planaric, as illustrated in Figure 13, but for the sub-sequence d' obtained by applying the MP-step of the HH method on d, the positive prefix $\mathsf{pos}(d') = (6, 4, 2^6)$ is not outer-planaric, by Lemma 11. So it remains to show that d is outer-planaric even if $\mathsf{pos}(d') \in EX$.

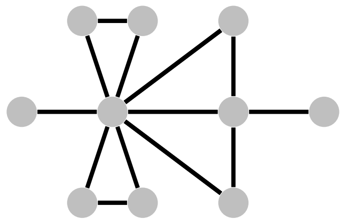

Figure 13 Outer-planar realization of the sequence $d = (8, 4, 2^6, 1^2)$.

A sequence $\mathsf{pos}(d') \in EX$ has the form $\mathsf{pos}(d') = (d'_1, d'_2, 2^{n'-2})$ where $d'_1 + d'_2 = n'+2$ and d'_1 and d'_2 are even. Since $d'_2 \ge 3$, we have $d'_2 \ge 4$.

Convert d' to $d'' = (d'_1+1, d'_2-1, 2^{n'-2})$. Note that d'' is non-increasing since $d'_2 \ge 4$. Then $d''_1 + d''_2 = d'_1 + d'_2 = n'+2$. As $d''_2 \ge 3$, we have $d''_1 \le n'-1 = n''-1$. Also, $\sum d'' = \sum \mathsf{pos}(d') \le 3n'-2 = 3n''-2$, so $d'' \in LV'_2$. One can check that $d'' \notin EX$, since d''_1, d''_2 are odd. By Corollary 14 (1), d'' is outer-planaric. Returning to our example of the outer-planaric $d = (8, 4, 2^6, 1^2)$ where $\mathsf{pos}(d') = (6, 4, 2^6)$ is not outer-planaric, the conversion yields $d'' = (7, 3, 2^6)$, which is outer-planaric by the construction in Lemma 10.

By the construction of d' and d'', $d_i \ge d'_i$ for $1 \le i \le n'$ and $d'_i \ge d''_i$ for $2 \le i \le n'$. As $d_1 > d'_1$, we have $d_i \ge d''_i$ for any $1 \le i \le n'$. Let G'' be an outer-planaric realizing graph for d''. Insert $d_i - d''_i$ leaves to the vertex with degree d''_i in G'' for any $1 \le i \le n'$. This yields an outer-planar graph G with degree sequence d. The lemma follows. ◀

In summary, combining Corollary 14 and Lemma 15, we get the following.

▶ **Theorem 16.** *Consider a nonincreasing n-integer graphic sequence d. If*
(1) $\omega_1 = 0$ *and* $\sum d \le 3n-3$*, or*
(2) $\omega_1 > 0$ *and* $\sum d \le 3n - \omega_1 - 2$*,*
then d is outer-planaric.

Finally, we complement the positive results of Corollary 14 and Lemma 15 by tight negative examples. Let us first consider the case of $\omega_1 = 0$. A tight example is any sequence in EX, which is not outer-planaric by Lemma 11. The volume of any sequence $d \in EX$ is

$\sum d = 3n - 2$. Next consider the case of $\omega_1 > 0$. A tight example is $d' = (3^{n-1}, 1)$ for even $n \geq 6$, which is planaric (See Figure 14) but not outer-planaric by Lemma 3. This sequence satisfies that $\sum d' = 3n - \omega_1 - 1$.

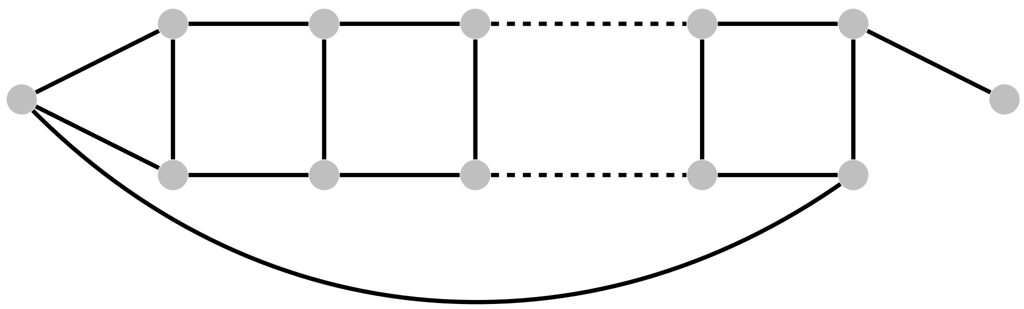

Figure 14 A non-outer-planar realization of the sequence $d = (3^{n-1}, 1)$.

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