Simple Qudit ZX and ZH Calculi, via Integrals

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Abstract

The ZX calculus and ZH calculus use diagrams to denote and compute properties of quantum operations, using “rewrite rules” to transform between diagrams which denote the same operator through a functorial *semantic map*. Different semantic maps give rise to different rewrite systems, which may prove more convenient for different purposes. Using discrete measures, we describe semantic maps for ZX and ZH diagrams, well-suited to analyse unitary circuits and measurements on qudits of any fixed dimension $D > 1$ as a single “ZXH-calculus”. We demonstrate rewrite rules for the “stabiliser fragment” of the ZX calculus and a “multicharacter fragment” of the ZH calculus.

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1 Introduction

The ZX calculus [10, 2, 28, 42] and ZH calculus [3, 40] are systems using annotated graphs (“ZX diagrams” and “ZH diagrams”), to denote tensor networks for quantum computation, and other problems involving tensors over $\mathbb{C}^2$ [16, 19, 39, 18, 17, 26]. They include rewrite rules, to perform computations on diagrams without recourse to exponentially large matrices. Complicated procedures may involve diagrams of mounting complexity to analyse, but the ZX- and ZH-calculi often simplify the analysis of many-qubit procedures. It is also increasingly common to consider versions of the ZX- and ZH-calculi for qudits [20, 44, 48, 6, 41, 27, 33, 32], which promise similar benefits for the analysis of procedures on qudits.

Most treatments of these calculi [2, 3, 4, 5, 6, 11, 14, 21, 22, 23, 24, 25, 28, 29, 30, 33, 37, 38, 40, 42, 43, 45, 46, 47, 48, 49] are “scalar exact”: equational theories, that do not introduce changes by scalar factors. Changes by scalar factors do not matter for some applications (e.g., testing equivalence of unitary transformations), but *are* important for probabilistic processes (e.g., postselection) or to compute specific numerical values [26]. But “scalar exact” treatments may involve frequent accumulation or deletion of *scalar gadgets*: disconnected sub-diagrams which obliquely denote normalisation factors. Presentations of these calculi which avoid such book-keeping, are simpler for instruction and practical use, and may also admit a unified rewrite system (a “ZXH calculus”) incorporating the rules of each [14, 19, 18].

In previous work [14], one of us addressed this issue of bookkeeping of scalars for ZX- and ZH-diagrams on qubits through a carefully constructed semantic map. The result, described as “well-tempered” versions of these calculi, are scalar exact while avoiding the modifications of scalar gadgets for the most often-used rewrites. However, while the rewrite rules of this “well-tempered” notation are simple, the notational convention itself (i.e., the semantics of the generators of the calculi) is slightly unwieldy. Furthermore, it left open how to address similar issues with scalars for versions of these calculi on qudits of dimension $D > 2$.

In this work, we consider how different normalisations of the ZX- and ZH-calculus may be expressed in a more uniform way, by representing operators on qudits (of any fixed dimension $D > 1$) through the use of integrals with respect to a discrete measure.

For a finite set S, let $\#S$ denote its cardinality. Consider a measure $\mu(S) = \#S \cdot \nu^2$ on subsets $S \subseteq \mathbb{Z}$, for $\nu > 0$ to be fixed later. This is a "measure" on sets S (see Section 2.1) which allows us to define a formal notion of integration of functions $f : \mathbb{Z} \to \mathbb{C}$,

$$\int_{x\in S} f(x) \;:=\; \int_{x\in S} f(x)\,\mathrm{d}\mu(x) \;:=\; \sum_{x\in S} f(x)\,\nu^2 \,. \tag{1}$$

(For the sake of brevity, we often use the standard convention of suppressing the differential $\mathrm{d}\mu$, as on left-hand expression below, when the measure of integration is understood.) Such integrals allow us to express sums with certain normalising factors more uniformly, by absorbing the factors into the measure μ by an appropriate choice of $\nu > 0$. For a finite-dimensional Hilbert space $\mathcal{H}$ with standard basis $|x\rangle := \mathbf{e}_x$ for some index set $x \in \mathbf{D}$, we may define the (not-necessarily normalised) *point-mass distributions* $|x\rangle\!\rangle = \nu^{-1}\,|x\rangle \in \mathcal{H}$, and their adjoints $\langle\!\langle x| = |x\rangle\!\rangle^\dagger$. Then, if we have some "state-function" $|f\rangle\!\rangle := \int_{x\in\mathbf{D}} f(x)\;|x\rangle\!\rangle$ for an arbitrary function $f : \mathbb{Z} \to \mathbb{C}$, it is easy to show that

$$\langle\!\langle z|f\rangle\!\rangle \;:=\; \int_{x\in\mathbf{D}} \langle\!\langle z|x\rangle\!\rangle\; f(x) \;=\; f(z), \tag{2}$$

similar to how Dirac measures are used with integration over $\mathbb{R}$. While a similar result $\langle z|f\rangle = f(z)$ holds if we simply define $|f\rangle = \sum_{x\in D} f(x)\;|x\rangle$, couching this sort of analysis in terms of discrete integrals and point-mass functions $|x\rangle\!\rangle$ allows us to accommodate scalar factors which may arise when manipulating expressions involving operators such as $\sum_{x\in\mathbf{D}} |x\rangle^{\otimes n}\,\langle x|^{\otimes m}$ for $m, n > 1$. This is an example of the sort of operator, for which book-keeping of scalar factors frequently arises in most versions of the ZX- or ZH-calculi.

By introducing the additional layer of abstraction, provided by discrete integrals and their accompanying point-mass functions $|x\rangle\!\rangle$, we describe semantics for ZX- and ZH-diagrams which are simple, and which admits a system of rewrites which largely dispenses with the need for modifications to scalar gadgets in the diagrams. This approach to notation, and the rewrites which we demonstrate, are applicable for generators representing operators on qudits of *any* finite dimension, and enables the two calculi to be used interoperably as a single "ZXH-calculus". We present this approach in the hopes that it facilitates the development of practically useful extensions of these calculi beyond qubits.

Structure of this article. Section 2 sets out number-theoretic preliminaries, some background in string diagrams, and common approaches to defining ZX and ZH calculi. Section 3 introduces discrete measures and integrals on $\mathbf{D}$, including what little measure theory we require, and considers the constraints that follow from a particular treatment of discrete Fourier transforms. Section 4 demonstrates how using such discrete integrals as the basis for a semantic map for ZX and ZH diagrams, leads to convenient representations of particular unitary operators and convenient rewrites for both the ZX and ZH calculi. We frequently refer to the Appendices of the full version [15] of this work, where we provide complete proofs, more details about our constructions, and connections to related subjects.

Related work. As we note above, there is recent and ongoing work [20, 44, 48, 6, 41, 27, 33, 32, 34] on ZX, ZH, and related calculi on qudits of dimension $D > 2$ (though often restricted to the case of D an odd prime). Our work is influenced in particular by Booth and Carette [6], and Roy [33], and we are aware of parallel work by collaborations involving these authors [34, 31]. Our work is distinguished in presenting convenient semantics for both ZX and ZH diagrams for arbitrary $D>1$, notably including the case where D is composite or even.

2 Preliminaries

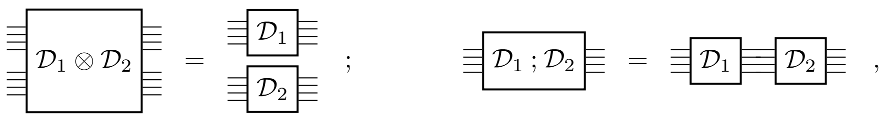
2.1 Number-theoretic preliminaries

Let $D > 1$ be a fixed integer, and $\omega = \mathrm{e}^{2\pi i/D}$. We assume basic familiarity with number theory, in particular with $\mathbb{Z}_D$, the integers modulo D. While it is common to associate $\mathbb{Z}_D$ with the set $\{0, 1, \ldots, D-1\}$ of non-negative "residues" of integers modulo D, one may associate them $\mathbb{Z}_D$ with any contiguous set of residues $\mathbf{D} = \{L, L+1, \ldots, U-1, U\}$ where $U - L + 1 = D$.[1] We may then occasionally substitute $\mathbb{Z}_D$ for $\mathbf{D}$ when this is unlikely to cause confusion: this will most often occur in the context of expressions such as ω^{xy}, which is well-defined modulo D in each of the variables x and y (i.e., adding any multiple of D to either x or y does not change the value of the expression). In such an expression, while we may intend for one of x or y or both may be an element of $\mathbb{Z}_D$ in principle, they would in practise be interpreted as a representative integer $x, y \in \mathbf{D} \subseteq \mathbb{Z}$.

2.2 String diagrams

ZX- and ZH-diagrams are examples of *string diagrams*, which can be described as diagrams composed of dots (or boxes) and wires, where the wires denote objects and the dots/boxes denote maps on those objects.

In the string diagrams which we consider in this article, diagrams are composed of dots or boxes, and wires. These diagrams can be described as being a composition of "generators", which typically consist of one (or zero) dots/boxes with some amount of meta-data, and any number (zero or more) directed wires, where the direction is usually represented by an orientation in the diagram. (In this article, wires are oriented left-to-right, though they are also allowed to bend upwards or downwards.) For any two diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$, we may define composite diagrams $\mathcal{D}_1 \otimes \mathcal{D}_2$ and $\mathcal{D}_1 \,;\, \mathcal{D}_2$, represented schematically by



(3)

which we call the "parallel" and "serial" composition of $\mathcal{D}_1$ and $\mathcal{D}_2$. In the latter case we require that the number of output wires of $\mathcal{D}_1$ (on the right of $\mathcal{D}_1$) equal the number of input wires of $\mathcal{D}_2$ (on the left of $\mathcal{D}_2$), for the composition to be well-defined.

String diagrams may be used to denote maps in a monoidal category $\mathbf{C}$ (in which objects can be aggregated to form composite objects through a parallel product, which we denote by "$\otimes$"). This is done through a *semantic map* $[\![\,\cdot\,]\!]$ which maps each generator to a map in $\mathbf{C}$. This semantic map is defined to be consistent with respect to composition, in the sense that

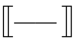
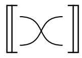
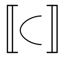
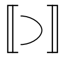
$$\Big[\!\!\Big[\mathcal{D}_1 \otimes \mathcal{D}_2 \Big]\!\!\Big] = [\![\mathcal{D}_1]\!] \otimes [\![\mathcal{D}_2]\!], \qquad \Big[\!\!\Big[\mathcal{D}_1 \,;\, \mathcal{D}_2 \Big]\!\!\Big] = [\![\mathcal{D}_2]\!] \circ [\![\mathcal{D}_1]\!]. \tag{4}$$

[1] The reader may wonder why we do not simply adopt the conventional choice of $\mathbf{D} = \{0, 1, \ldots, D-1\}$. There are multiple reasons, the simplest of which being that allowing for the index to include negative integers may be useful for representing certain "quantum numbers" in application to physics.

Note the reversal of the order for sequential composition, which is just an artefact of the difference in orientation of diagrams (left-to-right), and the conventional right-to-left application order of functions that is common e.g. in quantum information theory.

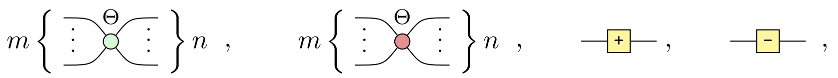
2.3 Preliminary remarks on ZX and ZH diagrams

ZX and ZH diagrams are string diagrams which denote multi-linear operators on some finite-dimensional vector space $\mathcal{H} \cong \mathbb{C}^{\mathbf{D}}$, equipped with a standard basis $|x\rangle$ for $x \in \mathbf{D}$ and functionals $\langle x| = |x\rangle^\dagger$. (For $\mathbf{D}$ a set of D consecutive integers, we may use arithmetic expressions, such as $|x+y\rangle$, to index basis vectors; specifically in the labels of "kets" and "bras", such expressions can be understood to be evaluated mod D.) The parallel product in this case is the usual tensor product $\otimes$, and the sequential product is composition of operators. For a generator $\mathcal{D}$ with m input wires and n output wires, one assigns an operator $[\![\mathcal{D}]\!] : \mathcal{H}^{\otimes m} \to \mathcal{H}^{\otimes n}$. To represent string diagrams to represent maps in which some of the "parallel" operands are being permuted or unaffected, we also consider generators consisting only of wires. We consider four such generators, to which we assign semantics as follows:

 $= \sum_{x \in \mathbf{D}} |x\rangle\langle x|\,,$  $= \sum_{x,y \in \mathbf{D}} |y,x\rangle\langle x,y|\,,$  $= \sum_{x \in \mathbf{D}} |x,x\rangle\,,$  $= \sum_{x \in \mathbf{D}} \langle x,x|\,.$ (5)

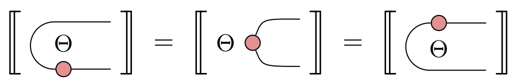
ZX and ZH diagrams are designed with different priorities, but have common features. ZX diagrams are effective for representing operations generated by single-qubit rotations and controlled-NOT gates; in most cases (excepting, e.g., Refs. [46, 6]) it rests on the unitary equivalence of two conjugate bases. ZH diagrams were developed to facilitate reasoning about quantum circuits over the Hadamard-Toffoli gate set [36, 1]. Both were originally defined so that the semantics is preserved by a change in the presentation of the underlying graph, which preserves the connectivity of the diagram [10, 12].

ZX Diagrams. We define the following ZX generators on qudits with state-space $\mathcal{H}$,



(6)

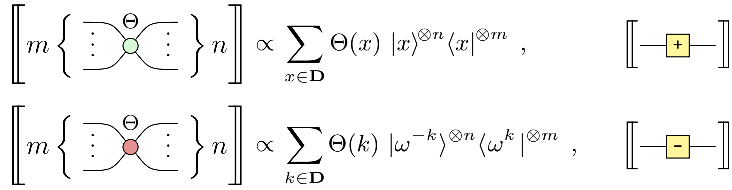
where $m, n \in \mathbb{N}$, and for any function $\Theta : \mathbb{Z} \to \mathbb{C}$. (Our approach of using functions mildly extends the approach of Wang [48], who prefers to parameterise the generators with vectors indexed from 1. For the constant function $\Theta(x) = 1$, we may omit the label Θ entirely.) We call these generators "green dots", "red dots", "Hadamard plus boxes", and "Hadamard minus boxes". The usual approach to assigning semantics to ZX generators is by considering the green and red dots to represent similar operations, subject to different (conjugate) choices of orthonormal basis, and a unitary "Hadamard" gate relating the two bases. One defines a semantic map $[\![\,\cdot\,]\!]$ in which the (lighter-coloured) "green" dots are mapped to an action on the basis $|x\rangle$, and the (darker-coloured) "red" are mapped to an action on the basis $|\omega^x\rangle$, where $|\omega^k\rangle = \frac{1}{\sqrt{D}} \sum_x \omega^{-kx} |x\rangle$ for $k \in \mathbf{D}$ (and where again $\omega = \mathrm{e}^{2\pi i/D}$).[2] The conventional choice would be, for a green dot, to assign an interpretation such as $\sum_{x \in \mathbf{D}} \Theta(x)\, |x\rangle^{\otimes n} \langle x|^{\otimes m}$; and for a red dot, to assign the interpretation $\sum_{x \in \mathbf{D}} \Theta(x)\, |\omega^x\rangle^{\otimes n} \langle \omega^x|^{\otimes m}$. Unfortunately, for $D > 2$, such a conventional interpretation does not yield a "flexsymmetric" [7] calculus, in effect because $\langle \omega^a|^{\mathsf{T}} = |\omega^a\rangle^* = |\omega^{-a}\rangle$. In particular, the conventional approach described just above would mean that the equality



(7)

[2] In the notation of Ref. [6], we have $|\omega^k\rangle = |k{:}X\rangle$, up to a relabeling of the basis elements of $\mathcal{H}$.

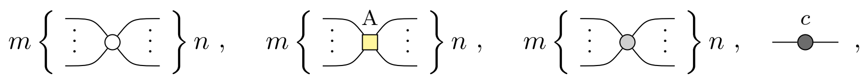
would not hold: the first diagram would denote $\sum_x \Theta(x)\,|\omega^{-x},\omega^{x}\rangle$, the second would denote $\sum_x \Theta(x)\,|\omega^{x},\omega^{x}\rangle$, and the third would denote $\sum_x \Theta(x)\,|\omega^{x},\omega^{-x}\rangle$. This represents a way in which such a calculus would fail to have the useful syntactic property that "only the connectivity matters" [10, 12]; and other inconveniences would also arise, which would make these diagrams more difficult to work with. To avoid this problem, we endorse the convention adopted by Refs. [6, 41] of involving a generator which is related to the green dot by *different* unitary transformations on the inputs and outputs, but which differ only by a permutation. We then interpret the generators of Eqn. (6) as operators using a model $[\![\,\cdot\,]\!]$ which satisfies



$$\propto \sum_{x\in\mathbf{D}} \Theta(x)\;|x\rangle^{\otimes n}\langle x|^{\otimes m}\,, \qquad \propto \sum_{x,k\in\mathbf{D}}\omega^{kx}\,|x\rangle\langle k|$$
$$\propto \sum_{k\in\mathbf{D}} \Theta(k)\;|\omega^{-k}\rangle^{\otimes n}\langle \omega^{k}|^{\otimes m}\,, \qquad \propto \sum_{x,k\in\mathbf{D}}\omega^{-kx}\,|x\rangle\langle k| \tag{8}$$

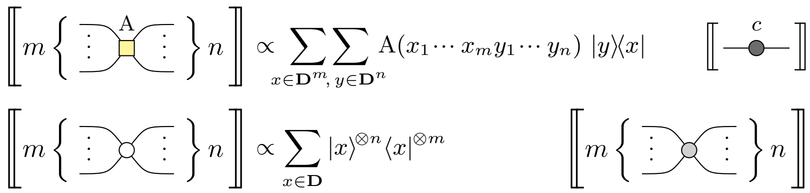
so that the "Hadamard" plus and minus boxes are proportional to the quantum Fourier transform $|\omega^k\rangle \mapsto |k\rangle$ (i.e., the inverse discrete Fourier transform), and its adjoint.

ZH Diagrams. We define the following ZH generators on qudits with Hilbert space $\mathcal{H}$,



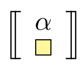
(9)

where $m,n\in\mathbb{N}$, $c\in\mathbb{Z}$, and for any function $\mathrm{A}:\mathbb{Z}\to\mathbb{C}$. (If $\mathrm{A}(t)=\alpha^t$ for some $\alpha\in\mathbb{C}^\times$, we may write the scalar α in place of A, consistent with the notation for ZH generators in Refs. [3, 14]. Following Roy [33], we later define a further short-hand notation for $\mathrm{A}(t)=\chi_c(t)=\exp(2\pi i c t/D)$ with $c\in\mathbb{Z}$.) We call these generators "white dots", "H-boxes", "gray dots", and "generalised-not dots".[3] We interpret the generators of Eqn. (9) as operators using a model $[\![\,\cdot\,]\!]$ which satisfies the following:



$$\propto \sum_{x\in\mathbf{D}^m,\,y\in\mathbf{D}^n}\mathrm{A}(x_1\cdots x_m y_1\cdots y_n)\;|y\rangle\!\langle x| \qquad \propto \sum_{x\in\mathbf{D}}|{-c}{-x}\rangle\langle x|$$
$$\propto \sum_{x\in\mathbf{D}}|x\rangle^{\otimes n}\langle x|^{\otimes m} \qquad \propto \sum_{\substack{x\in\mathbf{D}^m,\,y\in\mathbf{D}^n\\ \sum_h x_h+\sum_k y_k\equiv 0}}|y\rangle\!\langle x|\,, \tag{10}$$

where for the gray dots, we constrain the indices $x\in\mathbf{D}^m$ and $y\in\mathbf{D}^n$, so that the sum of their entries is 0 mod D; and for the not-dots we interpret the index of the vector $|-c-x\rangle$ modulo D. By contrast, note that for the H-boxes, we consider the products of the input and output labels $x_1,\ldots,x_m,y_1,\ldots,y_m$ as *integers*,[4] i.e., elements of $\mathbb{Z}$, whose product is the argument of A in the the expression $\mathrm{A}(x_1\cdots y_m)$. In particular, we fix the semantics so that



$$= \sum_{(\text{singleton})}\alpha^{(\text{empty product})}\cdot 1 \;=\; \alpha^1 \;=\; \alpha, \tag{11}$$

again using the short-hand that $\alpha\in\mathbb{C}^\times$ stands for the function $\mathrm{A}(t)=\alpha^t$.

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[3] We follow Ref. [14] in considering the gray and not dots to be (primitive) generators, rather than gadgets or "derived generators", e.g., as in Refs. [3, 33].

[4] Note that our use of the index-set $x\in\mathbf{D}=\{L,L+1,\ldots,U-1,U\}$ means that the exponential function $t\mapsto\alpha^t$ for $\alpha=0$ is not well-defined if $L<0$. We may instead consider a function $\mathbf{X}_{\{0\}}:\mathbb{Z}\to\mathbb{C}$ given by $\mathbf{X}_{\{0\}}(t)=1$ for $t=0$, with $\mathbf{X}_{\{0\}}(t)=0$ otherwise: this is substitution is adequate, e.g., for applications to counting complexity [16, 26].

Remarks on semantics, and rewriting systems. Eqns. (8) and (10) describe not one semantic map $[\![\,\cdot\,]\!]$ for ZX diagrams or ZH diagrams, but rather the conventional approach (with minor elaborations) to choosing such semantic maps. A specific semantic map determines which pairs of diagrams have the same semantics, and therefore which *diagrammatic rewrites* are *sound* (i.e., which local transformations one may perform to a diagram without changing its semantics). We suggest that rewrite systems, in which the most commonly used diagrammatic rewrites can be expressed simply, are to be preferred over others. However, this depends on obtaining a semantic map $[\![\,\cdot\,]\!]$ for which such a rewrite system is sound.

- Some authors (e.g., [10, 12]) prefer to define semantics only up to proportionality, in which case Eqns. (8) and (10) suffice to determine when two diagrams are equivalent up to an neglected scalar factor. This has the virtue of simplicity, but does not provide the precision needed for all applications one might wish to consider for these calculi.
- Most "scalar exact" treatments of ZX and ZH fix a map $[\![\,\cdot\,]\!]$ by replacing the proportionalities in Eqns. (8) and (10) with equalities – except for the "Hadamard" boxes of Eqn. (8), where a factor of $1/\sqrt{D}$ is used to yield unitary operators. However, the rewrites in those systems often involve book-keeping of auxiliary sub-diagrams ("scalar gadgets").
- In the case $D = 2$, Ref. [14] presents a different, unified semantic map $[\![\,\cdot\,]\!]_\nu$ for both ZX and ZH diagrams, in order to support rewrites involving fewer scalar gadgets. However, the scalar factors involved in those semantics could be considered non-obvious, and does not provide insights into how one would achieve the same goal for arbitrary $D \geqslant 2$.

The aim of this work is to extend the results of Ref. [14], providing a simple approach to fixing a semantic map $[\![\,\cdot\,]\!]$ for both ZX and ZH diagrams for arbitrary $D \geqslant 2$, which supports a set of diagrammatic rewrites without much use of scalar gadgets.

3 Discrete integrals

Our main theoretical contribution is to demonstrate how discrete integrals provide a way to fix a semantic map for ZX and ZH diagrams, with favourable properties. In this section, we introduce discrete measures and discrete integrals independently of string diagrams, and consider the constraints on discrete measures obtained through a particular representation of discrete Fourier transforms.

3.1 Introducing discrete measures and discrete integrals

We begin by introducing more fully the concepts first described on page 2. For a set $\mathbf{X}$, let $\wp(\mathbf{X})$ be the power-set of $\mathbf{X}$. We may define a *σ-algebra on* $\mathbf{X}$ to be a set $\Sigma \subseteq \wp(\mathbf{X})$ which contains $\mathbf{X}$, which is closed under set complements ($S \in \Sigma \iff \mathbf{X} \setminus S \in \Sigma$), and which is closed under countable unions (if $S_1, S_2, \ldots \in \Sigma$, then $S_1 \cup S_2 \cup \cdots \in \Sigma$). – The purpose of defining Σ is to allow the notion of a *measure* $\mu : \Sigma \to \mathbb{R} \cup \{+\infty\}$ to be defined, where the sets $S \in \Sigma$ are the ones which have a well-defined measure. Such a function μ is a measure, if and only if $\mu(\varnothing) = 0$, $\mu(S) \geqslant 0$ for all $S \in \Sigma$, and if

$$\mu\big(S_1 \cup S_2 \cup \cdots\big) = \mu(S_1) + \mu(S_2) + \cdots \tag{12}$$

for any sequence of disjoint sets $S_j \in \Sigma$. An example is the σ-algebra Σ consisting of all countable unions of intervals over $\mathbb{R}$, with μ defined by assigning $\mu(J) = b - a$ to any interval $J = (a, b)$, $J = (a, b]$, $J = [a, b)$, or $J = [a, b]$ for $a \leqslant b$. A somewhat more exotic measure is the Dirac distribution μ_δ on $\mathbb{R}$, for which $\mu_\delta(S) = 1$ if $0 \in S$, and $\mu_\delta(S) = 0$ otherwise. (We remark on the Dirac distribution and related concepts in Appendix [15, Appendix D].) However, we are mainly interested in measures μ defined on subsets $S \subseteq \mathbf{D}$, for which $\mu(S) \propto \#S$.

For the set $\mathbf{D} = \{L, L+1, \ldots, U-1, U\}$, consider the σ-algebra $\mathcal{B} = \wp(\mathbf{D})$ consisting of all subsets of $\mathbf{D}$. Define the measure $\mu : \mathcal{B} \to \mathbb{R}$ on this σ-algebra given by $\mu(S) \;=\; \#S \cdot \nu^2$, where $\nu > 0$ can in principle be chosen freely. This presents $\mathbf{D}$ as a measure space, the purpose of which is to allow us to define (multi-)linear operators on $\mathcal{H}$ as arising from integrals with respect to that measure. For a function $f : \mathbb{Z} \to \mathbb{C}$, we may define a notion of integration of f over a subset $S \subseteq \mathbf{D}$:

$$\int\limits_{x \in S} f(x)\, \mathrm{d}\mu(x) \;=\; \sum_{x \in S} f(x)\, \mu(\{x\}) \;=\; \sum_{x \in S} f(x)\, \nu^2 \,. \tag{13}$$

We may apply this notion of integration to operator-valued functions, as is typical for wave-functions in quantum mechanics. For instance, one may define

$$\int\limits_{x \in S} f(x)\; |x\rangle\; \mathrm{d}\mu(x) \;=\; \nu^2 \sum_{x \in S} f(x)\; |x\rangle \,. \tag{14}$$

In the usual approach to describing wave-functions over $\mathbb{R}$, one takes $|x\rangle$ to represent a point-mass distribution (i.e., not a vector $\boldsymbol{v} \in \mathbb{C}^{\mathbb{R}}$ for which $v_x = 1$), so that the equality

$$\langle z| \Biggl[\; \int\limits_{x \in \mathbb{R}} f(x)\; |x\rangle\; \mathrm{d}x \Biggr] = \int\limits_{x \in \mathbb{R}} f(x)\, \delta_z(x)\, \mathrm{d}x \;=\; f(z), \tag{15}$$

holds. Here $\delta_z(x)$ is a shifted Dirac distribution (see Ref. [15, Appendix D.1] for more details).[5] To avoid notational confusion, we prefer to reserve the symbol "$|x\rangle$" to represent a unit-norm standard basis vector in $\mathcal{H}$ (i.e., a vector $\boldsymbol{v} \in \mathcal{H}$ such that $v_x = 1$), and introduce a symbol "$|x\rangle\!\rangle$" which denotes the vector $|x\rangle\!\rangle \;=\; \frac{1}{\nu}\,|x\rangle$, specifically so that

$$\langle\!\langle z| \Biggl[\; \int\limits_{x \in \mathbf{D}} f(x)\; |x\rangle\!\rangle\; \mathrm{d}\mu(x) \Biggr] \;=\; \int\limits_{x \in \mathbf{D}} f(x)\; \langle\!\langle z|x\rangle\!\rangle\; \mathrm{d}\mu(x) \;=\; \nu^2 \sum_{x \in \mathbf{D}} f(x) \frac{\langle z|x\rangle}{\nu^2} \;=\; f(z)\,, \tag{16}$$

and also

$$\int\limits_{x \in \mathbf{D}} |x\rangle\!\rangle\langle\!\langle x|\; \mathrm{d}\mu(x) \;=\; \nu^2 \sum_{x \in \mathbf{D}} \frac{|x\rangle\,\langle x|}{\nu^2} \;=\; \sum_{x \in \mathbf{D}} |x\rangle\,\langle x| \;=\; \mathbf{1}\,. \tag{17}$$

The notation "$|x\rangle\!\rangle$" provides us the flexibility to consider which measures $\mu : \mathcal{B} \to \mathbb{R}$ are best suited for defining convenient semantics for ZX and ZH generators, while retaining the features provided by Dirac distributions over $\mathbb{R}$, and without constraining ν.

For maps U and V described in this way, one may analyse compositions UV in the same way that one would do if U and V were given by sums of operators: by using the expressions for U and V in terms of discrete integrals, manipulating expressions within the integrals, and well-judged use of algebraic identities such as Eqns. (16) and (17). For instance, if we have

$$U \;=\; \iint\limits_{x,y \in \mathbf{D}} u_{x,y}\; |x\rangle\!\rangle\langle\!\langle y| \qquad\qquad V \;=\; \iint\limits_{w,z \in \mathbf{D}} v_{w,z}\; |w\rangle\!\rangle\langle\!\langle z| \tag{18}$$

then we may express the composite operation UV by

[5] While it is not necessary to understand our results, readers who are interested in connections between the integrals and measures presented here with integration over compact groups, may be interested in remarks which we make in Ref. [15, Appendix D.3].

$$
\begin{aligned}
UV \;&=\; \Biggl[\;\iint\limits_{x,y\in\mathbf{D}} u_{x,y}\;|x\rangle\!\rangle\langle\!\langle y|\Biggr]\Biggl[\;\iint\limits_{w,z\in\mathbf{D}} v_{w,z}\;|w\rangle\!\rangle\langle\!\langle z|\Biggr] \\
&=\; \iiiint\limits_{w,x,y,z\in\mathbf{D}} u_{x,y}v_{w,z}\;|x\rangle\!\rangle\langle\!\langle y|w\rangle\!\rangle\langle\!\langle z| \;=\; \iint\limits_{x,z\in\mathbf{D}}\Biggl(\;\iint\limits_{w,y\in\mathbf{D}} u_{x,y}v_{w,z}\;\langle\!\langle y|w\rangle\!\rangle\Biggr)|x\rangle\!\rangle\langle\!\langle z| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;=\; \iint\limits_{x,z\in\mathbf{D}}\Biggl(\;\int\limits_{y\in\mathbf{D}} u_{x,y}v_{y,z}\Biggr)|x\rangle\!\rangle\langle\!\langle z|\;.
\end{aligned}
\tag{19}
$$

Composition of such integrals generalises straightforwardly for tensors of any signature: examples can be seen in Appendix A.1 of Ref. [15] (and Appendix A.3 makes heavy use of such analysis of compositions to prove the soundness of various diagrammatic rewrites of ZX and ZH diagrams.) The only distinction between this approach and one expressed directly in terms of summation, are the scalar factors which are subsumed in the integral notation and operators such as $|x\rangle\!\rangle\langle\!\langle z|$, both of which are governed by the choice of measure μ.

3.2 Constraints on normalisation motivated by the Fourier transform

Having defined the discrete measure (and discrete integrals) over $\mathbf{D}$, and the corresponding point-mass distributions $|x\rangle\!\rangle$ to satisfy Eqn. (2), we may consider how this might influence our approach to analysis of complex-valued functions over $\mathbf{D}$ (or $\mathbb{Z}_D$, using a similar measure).

In analogy to a common representation[6] of the Fourier transform of functions on $\mathbb{R}$, we may describe the (discrete) Fourier transform of a function $f : \mathbb{Z}_D \to \mathbb{C}$ by

$$
\hat{f}(k) \;=\; \int\limits_{x\in\mathbb{Z}_D} \mathrm{e}^{-2\pi i k x/D}\, f(x)\;\mathrm{d}\mu(x). \tag{20}
$$

In principle, the domain $\mathbb{Z}_D$ of $\hat{f}$ indexes a character $\chi_k(x) = \mathrm{e}^{-2\pi i k x/D}$ in the dual group $\widehat{\mathbb{Z}_D}$. The dual group $\widehat{\mathbb{Z}_D}$ can itself be assigned a measure $\tilde{\mu}$ which is in principle *independent* of μ. As $\mathbb{Z}_D$ is a finite abelian group, we use the fact that there is an isomorphism $\varepsilon : \widehat{\mathbb{Z}_D} \to \mathbb{Z}_D$ to describe $\hat{f}$ as a function $\mathbb{Z}_D \to \mathbb{C}$. The isomorphism ε induces a measure $\mu' = \tilde{\mu} \circ \varepsilon^{-1}$ on $\mathbb{Z}_D$, which may differ from μ and which would be relevant to any integrals involving the argument of $\hat{f}$.[7] – Note that there are different conventions for normalising the Fourier transform (over $\mathbb{R}$ or $\mathbb{Z}_D$): one might consider modifying Eqn. (20) to include a non-trivial scalar factor on the right-hand side. This is related to the questions of whether we take the Fourier transform $f \mapsto \hat{f}$ to preserve the ℓ_2-norm $\|f\|_2 = \Bigl(\int_x |f(x)|^2\,\mathrm{d}\mu(x)\Bigr)^{1/2}$, and whether we take μ' to differ from μ. We simply adopt the convention of defining the Fourier transform of $f : \mathbb{Z}_D \to \mathbb{C}$ as in Eqn. (20), and consider the constraints that this imposes on these other considerations.

---

[6] We emulate the presentation of the Fourier transform in terms of an oscillation frequency k (including the minus sign in the exponent, which for historical reasons is absent in the definition of the quantum Fourier transform). The main difference between Eqn. (20) and the usual Fourier transform over $\mathbb{R}$ is the factor of $1/D$ in the exponent: this can be shown to arise from a representation of functions $f : \mathbb{Z}_D \to \mathbb{C}$ in terms of discrete distributions on $\mathbb{R}$ (see Ref. [15, Appendix D.3.4]).

[7] The precise relationship between μ and μ', corresponds to the question in physics of the choice of units for x and k as continuous variables ranging over $\mathbb{R}$.

In analogy to standard practise in physics, we may use f to describe a "wave-function",[8]

$$|f\rangle\!\rangle \;:=\; \int\limits_{x\in\mathbb{Z}_D} f(x)\,|x\rangle\!\rangle\;\mathrm{d}\mu(x)\,. \tag{21}$$

A similar "wave function" for $\hat{f}$ would involve the measure μ', the measure on the argument of $\hat{f}$, which may in principle differ from μ:

$$|\hat{f}\,\rangle\!\rangle \;=\; \int\limits_{k\in\mathbb{Z}_D} \hat{f}(k)\,|k\rangle\!\rangle\;\mathrm{d}\mu'(k)\,, \tag{22}$$

integrating with respect to that different measure. Taking $\mu' \neq \mu$ would imply that the functions $f : (\mathbb{Z}_D, \mu) \to \mathbb{C}$, defined on $\mathbb{Z}_D$ considered as a space with measure μ, are strictly speaking not of the same type as their Fourier transforms $\hat{f} : (\mathbb{Z}_D, \mu') \to \mathbb{C}$ which are defined over a space with a different measure. String diagrams representing the transformations of such functions would then require wires of more than one type. While this is admissible in principle, we prefer to consider f and $\hat{f}$ to have the same measure space $(\mathbb{Z}_D, \mu)$ for their domains, so that we may treat them using string diagrams with wires of a single type, as we do in the ZX and ZH calculi. Identifying $\mu' = \mu$, we obtain

$$|\hat{f}\,\rangle\!\rangle \;=\; \int\limits_{k\in\mathbb{Z}_D} \hat{f}(k)\,|k\rangle\!\rangle\;\mathrm{d}\mu(k) \;=\; \iint\limits_{k,x\in\mathbb{Z}_D} \mathrm{e}^{-2\pi i k x/D} f(x)\;|k\rangle\!\rangle\;\mathrm{d}\mu(x)\;\mathrm{d}\mu(k)\,. \tag{23}$$

This motivates the definition of the discrete Fourier transform operator F over $\mathbb{Z}_D$, as

$$F \;=\; \iint\limits_{k,x\in\mathbb{Z}_D} \mathrm{e}^{-2\pi i k x/D}\;|k\rangle\!\rangle\langle\!\langle x|\;\mathrm{d}\mu(x)\;\mathrm{d}\mu(k)\;, \tag{24}$$

so that $|\hat{f}\rangle\!\rangle = F|f\rangle\!\rangle$; this is the interpretation given to the "Hadamard minus box" in Eqn. (28). We adopt the convention that F is unitary, to allow it to directly represent a possible transformation of state-vectors over $\mathcal{H}$. This has the benefit that the inverse Fourier transform can be expressed similarly (now suppressing the differentials $\mathrm{d}\mu$, for brevity):

$$f(x) \;=\; \langle\!\langle x|F^\dagger|\hat{f}\rangle\!\rangle \;=\; \iiint\limits_{x,h,k\in\mathbb{Z}_D} \mathrm{e}^{2\pi i k x/D}\;|x\rangle\!\rangle\langle\!\langle k|\Big(\hat{f}(h)\,|h\rangle\!\rangle\Big) \;=\; \int\limits_{k\in\mathbb{Z}_D} \mathrm{e}^{2\pi i k x/D}\;\hat{f}(x)\;. \tag{25}$$

The definition of F in Eqn. (24) and the constraint that it should be unitary, imposes a constraint on the measure μ on $\mathbb{Z}_D$. We first prove a routine Lemma (which is used often in the Appendices of Ref. [15] in simplifying iterated integrals):

▶ **Lemma 1.** *Let* $\omega = \mathrm{e}^{2\pi i/D}$ *and* $E \in \mathbf{D}$*. Then* $\displaystyle\int\limits_{k\in\mathbb{Z}_D} \omega^{Ek}\;\mathrm{d}\mu(k) \;=\; \langle\!\langle E|0\rangle\!\rangle\, D\nu^4$.

Proof. This holds by reduction to the usual exponential sum:

$$\int\limits_{k\in\mathbf{D}} \mathrm{e}^{2\pi i E k/D}\;\mathrm{d}\mu(k) \;=\; \nu^2 \sum_{k\in\mathbf{D}} \big(\omega^E\big)^k \;=\; \left\{\begin{array}{r} \nu^2\cdot\omega^{EL_D}\cdot\dfrac{(\omega^E)^D - 1}{\omega - 1}\,,\;\;\text{if } \omega^E \neq 1 \\ \nu^2\cdot D,\;\;\text{if } \omega^E = 1 \end{array}\right\}$$
$$=\; \delta_{E,0}\;D\nu^2 \;=\; \langle E|0\rangle\;D\nu^2 \;=\; \langle\!\langle E|0\rangle\!\rangle\;D\nu^4\;. \qquad ◀$$

[8] Note that $|f\rangle\!\rangle$ may not be a unit vector; whether this is the case depends on the values taken by f.

We may apply this in the case of the Fourier transform as follows. If F as expressed in Eqn. (24) is unitary, we have

$$\begin{aligned}\mathbf{1} \;=\; F^\dagger F \;&=\; \Biggl[\;\iint\limits_{y,h\in\mathbf{D}} \mathrm{e}^{2\pi i h y/D}\;|y\rangle\!\rangle\langle\!\langle h|\;\mathrm{d}\mu(y)\;\mathrm{d}\mu(h)\Biggr]\Biggl[\;\iint\limits_{k,x\in\mathbf{D}} \mathrm{e}^{-2\pi i k x/D}\;|k\rangle\!\rangle\langle\!\langle x|\;\mathrm{d}\mu(k)\;\mathrm{d}\mu(x)\Biggr]\\
&=\; \iiiint\limits_{y,h,k,x\in\mathbf{D}} \mathrm{e}^{2\pi i(hy-kx)/D}\;|y\rangle\!\rangle\langle\!\langle h|k\rangle\!\rangle\langle\!\langle x|\;\mathrm{d}\mu(y)\;\mathrm{d}\mu(h)\;\mathrm{d}\mu(k)\;\mathrm{d}\mu(x)\\
&=\; \iiint\limits_{y,k,x\in\mathbf{D}} \mathrm{e}^{2\pi i k(y-x)/D}\;|y\rangle\!\rangle\langle\!\langle x|\;\mathrm{d}\mu(y)\;\mathrm{d}\mu(k)\;\mathrm{d}\mu(x)\\
&=\; \iint\limits_{y,x\in\mathbf{D}} \Biggl[\;\int\limits_{k\in\mathbf{D}} \mathrm{e}^{2\pi i k(y-x)/D}\;\mathrm{d}\mu(k)\Biggr]\;|y\rangle\!\rangle\langle\!\langle x|\;\mathrm{d}\mu(y)\;\mathrm{d}\mu(x)\\
&=\; \iint\limits_{y,x\in\mathbf{D}} \Bigl[D\nu^4\cdot\langle\!\langle y|x\rangle\!\rangle\Bigr]\;|y\rangle\!\rangle\langle\!\langle x|\;\mathrm{d}\mu(y)\;\mathrm{d}\mu(x)\\
&=\; D\nu^4\int\limits_{x\in\mathbf{D}} |x\rangle\!\rangle\langle\!\langle x|\;\mathrm{d}\mu(x) \;=\; D\nu^4\cdot\mathbf{1}\,. \end{aligned} \tag{26}$$

This implies that $\nu = D^{-1/4}$ (or equivalently, $N = \mu(\mathbb{Z}_D) = D\nu^2 = \sqrt{D}$).

As there are multiple conventions for representing the discrete Fourier transform, one might wish to consider how adopting a different convention to Eqn. (20) affects constraints on the measure μ; we consider this question in Ref. [15, Appendix B.5].

4 Semantics for ZX- and ZH-diagrams using discrete interals

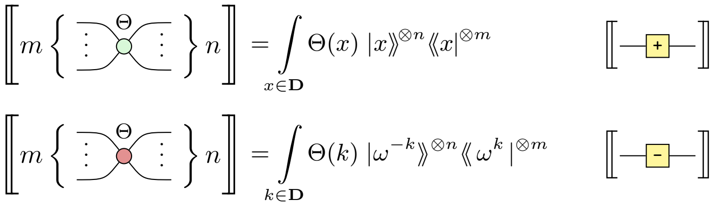
We present an approach to simply and systematically define semantic maps for ZX and ZH generators, which **(a)** yields simple diagrams for unitary transformations of interest, **(b)** admits scalar-exact diagrammatic rewrites involving few scalar gadgets, and **(c)** allows the two notational systems to be used seamlessly together.

Our approach is to subsume all considerations of normalising factors into the measure of a discrete integral, and its accompanying point-mass functions, as indicated on page 2. Our use of integrals and discrete measures in this way is standard, if somewhat uncommon in quantum information theory: see Ref. [35, 27] for comparable examples. Our intent is explicitly to draw attention to the freedom involved in the choice of measure, as a way forward to defining a semantic map $[\![\,\cdot\,]\!]$ for ZX and ZH diagrams that has desirable features.

Defining a discrete integral on $\mathbf{D}$ with $\mu(\mathbf{D}) = \sqrt{D}$, as we do in the preceding Section, allows us to easily define semantic maps for ZX and ZH diagrams with a number of convenient properties. Let

$$|\omega^k\rangle\!\rangle \;=\; F|k\rangle\!\rangle \;=\; \int\limits_{x\in\mathbf{D}} \omega^{-kx}\,|x\rangle\!\rangle \tag{27}$$

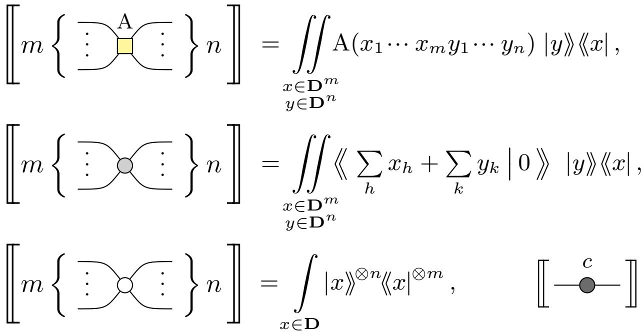
be the non-normalised point-mass distributions analogous to the Fourier basis states $|\omega^k\rangle$ introduced on page 4 (so that $|\omega^k\rangle\!\rangle$ is an ω^k-eigenvector of the cyclic shift operator X given by $X\,|a\rangle = |a{+}1\rangle$). We then define a semantic map $[\![\,\cdot\,]\!]$ on the ZX generators of Eqns. (6),



$$= \int_{x\in\mathbf{D}} \Theta(x)\, |x\rangle\!\rangle^{\otimes n} \langle\!\langle x|^{\otimes m} \qquad = F^\dagger = \iint_{x,k\in\mathbf{D}} \mathrm{e}^{2\pi i k x/D} |k\rangle\!\rangle\langle\!\langle x|$$

$$= \int_{k\in\mathbf{D}} \Theta(k)\, |\omega^{-k}\rangle\!\rangle^{\otimes n} \langle\!\langle\, \omega^{k}\, |^{\otimes m} \qquad = F = \iint_{x,k\in\mathbf{D}} \mathrm{e}^{-2\pi i k x/D} |k\rangle\!\rangle\langle\!\langle x| \tag{28}$$

and the ZH generators of Eqn. (9):



$$= \iint_{\substack{x\in\mathbf{D}^m\\ y\in\mathbf{D}^n}} \mathrm{A}(x_1\cdots x_m y_1\cdots y_n)\; |y\rangle\!\rangle\langle\!\langle x|\,,$$

$$= \iint_{\substack{x\in\mathbf{D}^m\\ y\in\mathbf{D}^n}} \langle\!\langle\, \sum_h x_h + \sum_k y_k \,\big|\, 0 \,\rangle\!\rangle\;\; |y\rangle\!\rangle\langle\!\langle x|\,, \tag{29}$$

$$= \int_{x\in\mathbf{D}} |x\rangle\!\rangle^{\otimes n}\langle\!\langle x|^{\otimes m}\,, \qquad = \int_{x\in\mathbf{D}} |{-c}{-x}\rangle\!\rangle\langle\!\langle x|\,,$$

These semantics are consistent with those set out in Eqns. (8) and (10), replacing the sums and the vectors $|x\rangle$ with discrete integrals and the corresponding point-mass distributions $|x\rangle\!\rangle$, and substitute proportionality relations with equalities. The discrete integrals (and point-mass functions) serve to specify specific scalar factors for the proportionalities.

These definitions are ones that we could choose to make, regardless of the measure μ that we consider for $\mathbf{D}$. Regardless of the choice made for ν, the above interpretations are certainly similar in their simplicity to the standard interpretations. By taking $\nu = D^{-1/4}$ as suggested in the preceding section, we not only obtain rewrite systems involving very few scalar gadgets – see Figs. 1 and 2 – but also, the most commonly considered states and unitary operations of qudit circuits admit simple presentations using these semantics. We may demonstrate this as follows.

4.1 The stabiliser sub-theory of ZX for arbitrary $D > 1$

We describe below a *stabiliser subtheory* of the ZX calculus, concerning ZX diagrams which suffice to represent stabiliser states [13] on systems of arbitrary dimension $D > 1$. These are characterised by ZX diagrams whose phase parameters are governed by restricted functions, for which arithmetic modulo D plays a central role.

We begin by describing the stabiliser sub-theory of quantum circuits. Following Ref. [13], define the complex unit $\tau = \mathrm{e}^{\pi i(D^2+1)/D}$, which is relevant to the analysis of stabiliser circuits on qudits of dimension D. The scalar τ is defined in such a way that $\tau^2 = \omega$, but also so that $\tau X^\dagger Z^\dagger$ is an operator of order D, where X and Z given by

$$X\,|t\rangle \;=\; |t+1\rangle\,, \qquad\qquad Z\,|t\rangle \;=\; \omega^{t}\,|t\rangle\,, \tag{30}$$

are the D-dimensional generalised Pauli operators. (As always, arithmetic performed in the kets are evaluated modulo D.) Choosing τ in this way makes it possible [13] to define a simple and uniform theory of unitary stabiliser circuits on qudits of dimension D, generated

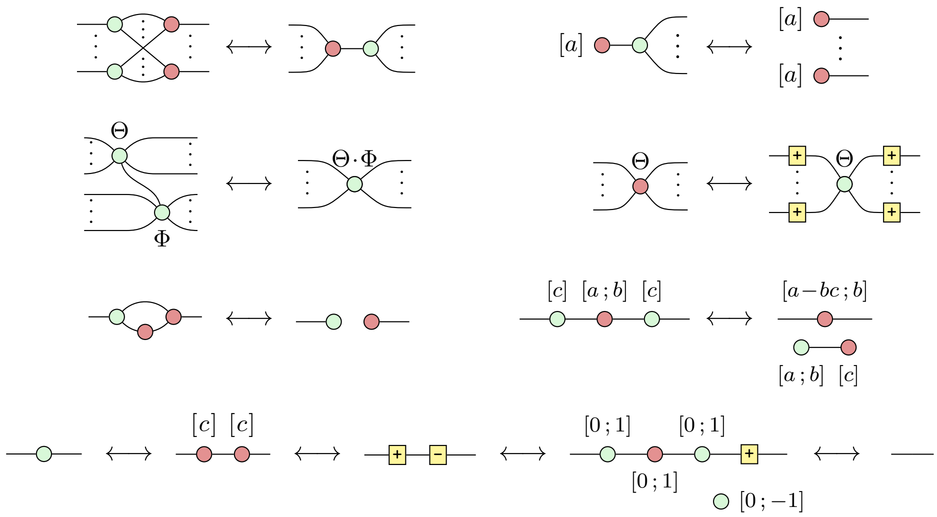


Figure 1 A sample of the (scalar-exact) rewrites which are sound for ZX diagrams with semantics as in Eqn. (28), when $\nu = D^{-1/4}$. Throughout, we have $\Theta, \Phi : \mathbb{Z} \to \mathbb{C}$, and $a, b, c \in \mathbb{Z}$. Node labels of the form $[a]$ or $[a\,;b]$ stand respectively for the amplitude functions $x \mapsto \tau^{2ax}$ and $x \mapsto \tau^{2ax+bx^2}$, where $\tau = \exp(\pi i(D^2{+}1)/D)$. A more complete list of rewrites, and proofs of their soundness, may be found in Ref. [15, Appendix A.3.3].

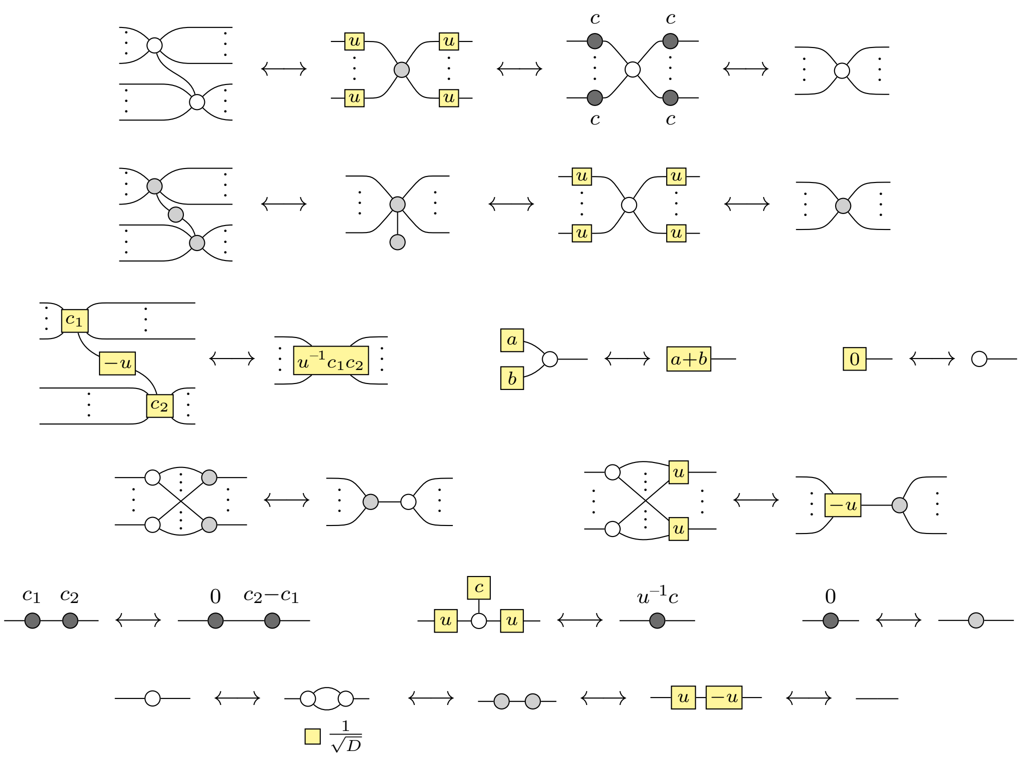


Figure 2 A sample of the (scalar-exact) rewrites which are sound for ZH diagrams with semantics as in Eqn. (29) when $\nu = D^{-1/4}$. H-boxes which are labeled inside with an integer parameter such as $c \in \mathbb{Z}$, indicate an amplitude of $\omega^c = \mathrm{e}^{2\pi i c/D}$; H-boxes labelled with "+" or "−" indicate $c = \pm 1$. Throughout, we have$a, b, c, c_1, c_2, u, v \in \mathbb{Z}$ (which may be evaluated modulo D), where in particular u and v are coprime to D. A more complete list of rewrites, and proofs of their soundness, may be found in Ref. [15, Appendix A.3.1].

by the single-qudit operators [9]

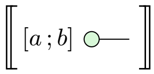
$$S = \int_{x\in\mathbf{D}} \tau^{x^2} \, |x\rangle\!\rangle\langle\!\langle x| \; ; \qquad F = \iint_{k,x\in\mathbf{D}} \tau^{-2kx} \, |k\rangle\!\rangle\langle\!\langle x| \; ; \qquad M_u = \int_{x\in\mathbf{D}} |ux\rangle\!\rangle\langle\!\langle x| \; , \tag{31}$$

where in the case of M_u we restrict to $u \in \mathbb{Z}$ which is relatively prime to D; and either one of the two-qudit operators

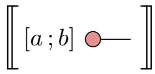
$$\mathrm{CX} = \iint_{x,y\in\mathbf{D}} |x\rangle\!\rangle\langle\!\langle x| \otimes |x{+}y\rangle\!\rangle\langle\!\langle y| \; ; \qquad\qquad \mathrm{CZ} = \iint_{x,y\in\mathbf{D}} \tau^{2xy} \, |x,y\rangle\!\rangle\langle\!\langle x,y| \; . \tag{32}$$

Finally, the full range of stabiliser circuits also admit measurements in the standard basis (and bases which may be related to the standard basis by the above unitaries).

Booth and Carette [6] describe a version of the ZX calculus which is complete for this subtheory, for the special case of D an odd prime. Following them, we may describe how the semantics of Eqn. (28) allows a simplification of these rewrites, extending them in most cases to arbitrary $D > 1$. To this end, it will be helpful to use a slightly different notational convention to Booth and Carette [6], we may easily denote these with ZX diagrams using the semantics of Eqn. (28). For $a, b \in \mathbb{Z}$, when parameterising a green or red dot, let $[a\,;b]$ stand for the amplitude function $\Theta(x) = \tau^{2ax+bx^2}$, so that

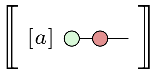


$$= \int_{x\in\mathbf{D}} \tau^{\,2ax\,+\,bx^2} \, |x\rangle\!\rangle \; ;$$

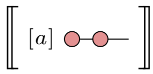


$$= \int_{k\in\mathbf{D}} \tau^{\,2ak\,+\,bk^2} \, |\omega^{-k}\rangle\!\rangle \; ; \tag{33}$$

generalising these to dots with multiple edges (or with none) similarly to Ref. [6]. When $b = 0$, we may abbreviate this function simply by $[a]$, so that we may represent the states $|a\rangle\!\rangle$ and $|\omega^a\rangle\!\rangle$ straightforwardly (albeit with the use of auxiliary red dots to represent an antipode operator, mapping $|\omega^a\rangle\!\rangle \mapsto |\omega^{-a}\rangle\!\rangle$ and $|a\rangle\!\rangle \mapsto |{-a}\rangle\!\rangle$ for $a \in \mathbb{Z}_D$):

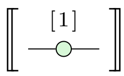


$$= \iint_{k,x\in\mathbf{D}} \tau^{2ax} \, |\omega^{-k}\rangle\!\rangle\langle\!\langle \omega^k | x\rangle\!\rangle \;=\; |\omega^a\rangle\!\rangle \; ; \tag{34a}$$

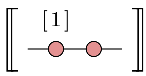


$$= \iint_{h,k\in\mathbf{D}} \tau^{2ah} \, |\omega^{-k}\rangle\!\rangle\langle\!\langle \omega^k | \omega^{-h}\rangle\!\rangle \;=\; |a\rangle\!\rangle \, . \tag{34b}$$

We may also easily represent the operators Z, and X as $1 \to 1$ dots:

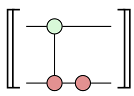
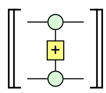
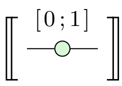


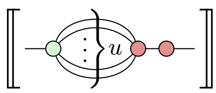
$$= \int_{x\in\mathbf{D}} \tau^{2x} \, |x\rangle\!\rangle\langle\!\langle x| \;=\; Z \, ;$$



$$= \int_{h\in\mathbf{D}} \tau^{2h} \, |\omega^h\rangle\!\rangle\langle\!\langle \omega^h| \;=\; X \, . \tag{35}$$

Regarding the unitary stabiliser operators on qudits, we may express them without any phases, using multi-edges between green and red dots, or using Hadamard boxes:

 $= \mathrm{CX}$,  $= \mathrm{CZ}$, 

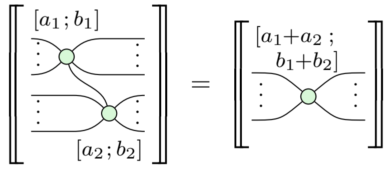
$= S$, 

$= M_u \, .$ (36)

(The diagram shown for M_u also generalises to operators $M_u = \int_x |ux\rangle\!\rangle\langle\!\langle x|$ for u not a multiplicative unit modulo D, though in that case the operator will not be invertible.)

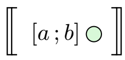
[9] Despite the different convention we adopt for the labeling of the standard basis, the definitions below are equivalent to those of Ref. [13]: the relative phases τ^{2ax+bx^2} remain well-defined on substitution of values $x < 0$ with $D + x$, as $\tau^{2a(D+x)+b(D+x)^2} = \tau^{2aD+2ax+bD^2+2bD+bx^2} = \tau^{2ax+bx^2}$ (using the fact that $\tau^{D^2} = \tau^{2D} = 1$ for both even and odd D).

The stabiliser subtheory of ZX may produce green or red dots *of degree zero* with phase parameter $[a\,;\,b]$ for some $a, b \in \mathbb{Z}$. These may occur when evaluating the probability of measurement outcomes (e.g., in the standard basis) arising from a stabiliser qudit circuit. As we show in Ref. [15, Appendix A.3.3] (as a simple corollary of a more general fusion rule), we have



(37)

where each instance of "⋮" denotes some number (zero or more) of incident wires. In the case where there are *no* other dots connected to two green dots as above, the right-hand side would be an isolated dot denoting a scalar, for which we define the notation $\Gamma(a, b, D)$:



$$= \int_{x\in\mathbf{D}} \tau^{2ax+bx^2} \;=:\; \Gamma(a,b,D)\;. \tag{38}$$

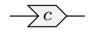
Evaluating such a discrete integral is connected with the subject of quadratic Gaussian sums, which is addressed in some detail in Ref. [15, Appendix C]. As a result of the normalisation convention for our discrete integrals, it is possible to show (see Ref. [15, Appendix C, Eqn. (103)]) that

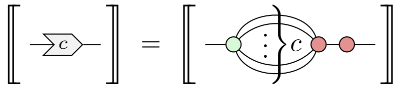
$$\Gamma(a,b,D) \;:=\; \int_{x\in\mathbf{D}} \tau^{2ax+bx^2} \;=\; \begin{cases} \sqrt{t}\cdot \mathrm{e}^{i\gamma}, & \text{if } t=\gcd(b,D) \text{ and } a \text{ is divisible by } t; \\ 0, & \text{otherwise,} \end{cases} \tag{39}$$

where γ is a phase parameter described in more detail in Ref. [15, Appendix C, Eqn. (103)]. In particular, if b is a multiplicative unit modulo D, this represents a global phase factor. (If we also have $a = 0$, then $\Gamma(a, b, D)$ is in fact a power of $\mathrm{e}^{\pi i/4}$.) More generally, $\Gamma(a, b, D)$ will either be 0, or have magnitude $\sqrt{t}$, where $t = \gcd(b, D)$.

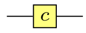
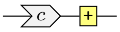
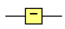
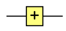
In this way, we obtain a diagrammatic language which is capable of expressing the rewrites similar to those described by Ref. [6], while involving fewer scalar factors (see Ref. [15, Appendix A.3.3, Fig. 6] for a more complete list of sound rewrites).

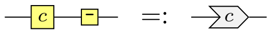
4.2 Multipliers and multicharacters in qudit ZH

It would be cumbersome to reason about stabiliser multiplication operators M_u or iterated CX or CZ gates using parallel edges between dots. Booth and Carette [6] describe how these may be denoted using gadgets called "multipliers", denoted  for $c \in \mathbb{N}$, which represent a limited form of scalable ZX notation [9, 8]. Using discrete integrals and the semantics described in Eqn. (28), we would simply write



$$= \int_{x\in\mathbf{D}} |cx\rangle\!\rangle\langle\!\langle x|\;. \tag{40}$$

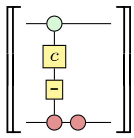
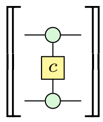
Using these multipliers, Booth and Carette [6, p. 24] then define “Fourier boxes”  :=  (using our notation for Hadamard boxes), whose interpretation coincides with the ones we assign using Eqn. (29) to an H-box with an amplitude parameter ω^c. Using this as a primitive, and composing this with the inverse  of the positive Hadamard box , we may directly describe multipliers instead as a ZH gadget, loosely following Roy [33]:

 (41)

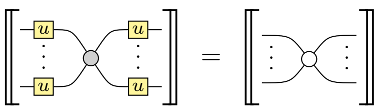
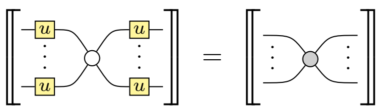
On the left, we employ a short-hand for H-boxes with an amplitude parameter ω^c. This is short-hand for a character function $\chi_c : \mathbb{Z} \to \mathbb{C}$ given by $\chi_c(x) = \omega^{cx}$, which is well-defined modulo D, and which we may then regard as a character on $\mathbb{Z}_D$. The function $\mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ given by $(x, y) \mapsto \chi_c(xy)$ is a *bicharacter*, which is also well-defined modulo D on each of its arguments; and more generally we may consider *multicharacters*, which are functions $\mathbb{Z}_D \times \cdots \times \mathbb{Z}_D \to \mathbb{C}$ given by $(x_1, \ldots, x_n) \mapsto \omega^{cx_1 \cdots x_n}$. We may call H-boxes with any number of edges, and with amplitude parameter ω^c for some $c \in \mathbb{Z}_D$, a ($\mathbb{Z}_D$-)*multicharacter box*.

We may use these ideas to define a *multicharacter subtheory* of ZH, consisting of the subtheory in which the H-boxes are indexed by paramters $c \in \mathbb{Z}_D$ in this way. Roy [33] has substantially investigated this fragment of ZH, in odd prime dimension. Our choice of semantic map allows us [15, Appendix A.3.1, Fig. 4] to reproduce many of the rewrites considered by Roy, while making minor simplifications and extending them to arbitrary dimensions $D > 1$.

We may use multiplier gadgets and multicharacter boxes to usefully describe unitary transformations, such as exponentiations of the qudit controlled-X and controlled-Z gates:

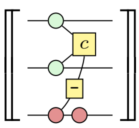
 $= \mathrm{CX}^c = \displaystyle\iint_{x,y\in\mathbf{D}} |x, y{+}cx\rangle\!\rangle\langle\!\langle x, y|$,  $= \mathrm{CZ}^c = \displaystyle\iint_{x,y\in\mathbf{D}} \omega^{cxy}\, |x, y\rangle\!\rangle\langle\!\langle x, y|$. (42)

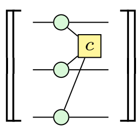
These degree-2 multicharacter boxes are effectively a Fourier transform over an isomorphic presentation of $\mathbb{Z}_D$ in some cases. This occurs in particular when $c = u \in \mathbb{Z}_D^\times$ is a multiplicative unit modulo D. We can witness this by rewrites which are valid for H-boxes parameterised by units $u \in \mathbb{Z}_D^\times$, such as ones which relate the white dots and the gray dots among the ZH generators (similar remarks apply for the green and red ZX generators):

 and  (43)

While there cannot be any perfect symmetry between the white and gray dots in general in the ZH calculus (as it involves the standard basis as a preferred basis), in this case a symmetry is recovered which one does not normally expect of presentations of the ZH calculusx.

We may also easily describe multi-qudit analogues of the qudit controlled-X and controlled-Z gates, using the fact that the H-boxes denote multi-characters. For example:

 $= \mathrm{CCX}^c = \displaystyle\iiint_{x,y,z\in\mathbf{D}} |x, y, z{+}cxy\rangle\!\rangle\langle\!\langle x, y, z|$, (44a)

 $= \mathrm{CCZ}^c = \displaystyle\iiint_{x,y,z\in\mathbf{D}} \omega^{cxyz}\, |x, y, z\rangle\!\rangle\langle\!\langle x, y, z|$. (44b)

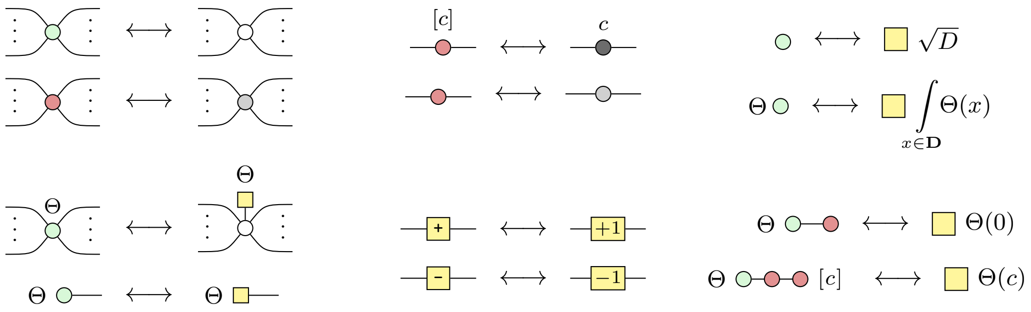
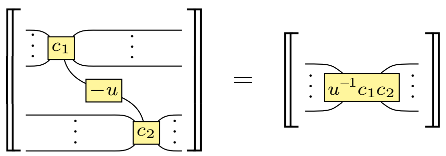


Figure 3 Sound rewrites between the ZX generators and the ZH generators, subject to the semantics of Eqns. (28) and (29) in the case $\nu = D^{-1/4}$. The proofs of the soundness of these rewrites are shown in Ref. [15, Appendix A.3.2].

While it would quickly become cumberson to represent each of the integrals in such an operation – this being a motivation for diagrammatic calculi in general – this demonstrates the genericity of the representation for these unitary transformations, and the relative lack of minor details to attend to in using them. Finally, we note the quasi-spider property that H-boxes are known for in the qubit case and in (and also shown in a more complicated form for odd prime D by Roy [33]), which can also be shown for a pair of multicharacter boxes connected to a common H-box with parameter $u \in \mathbb{Z}_D^\times$:



(45)

We do not claim to have a complete multicharacter subtheory for ZH over arbitrary qudits, but many of the rewrites which one may show in this case [15, Appendix A.3.1, Fig. 4] can be specialised in a useful way to the multicharacter case.

4.3 Compatibility and universality

In addition to the semantics of Eqns. (28) and (29) yielding ZX and ZH calculi which are each convenient in their own right, it also assigns the same semantics to certain ZX generators and certain ZH generators. This is illustrated in Fig. 3. This allows us to relate the two calculi to each other, to describe a "ZXH calculus" which has the features of both.

It is not necessary to consider such a united calculus to be able to denote arbitrary operators: see Ref. [15, Appendix A.2] for a sketch of a proof of universality of the ZH diagrams, in terms of a "normal form"-like diagram which mirrors the construction of Ref. [3]. However, while using both sets of generators may be redundant in principle, it should be expected to facilitate analysis, as the rewrite rules of each system effectively represents at least an important Lemma or Theorem of the other system.

We do not demonstrate any completeness results for these calculi; more rewrites may be necessary to prove completeness for arbitrary $D > 1$, for each of these two calculi, even in the stabiliser and multicharacter fragments described above.

5 Discussion

As well as providing an approach to define ZX and ZH calculi with simple rewrite systems on qudits, this approach is a simpler, and apparently independent, way to reproduce[10] the "well-tempered" semantic map $[\![\cdot]\!]_\nu$ described in Ref. [14] for $D = 2$. In this way, Eqns. (28) and (29) provide a more intuitive definition of those semantics, and extend them to arbitrary $D > 1$. It is possible to show that this is essentially down to the constraints imposed on the representation of the discrete Fourier transform in Section 3.2. Ref. [15, Appendix B] describes **(a)** the way that Eqns. (28) and (29) fail to constrain a generic "Ockhamic" interpretation of ZH diagrams while fixing a specific "Ockhamic" interpretations of ZX diagrams; and **(b)** to what extent this approach to fixing semantics actually differs from the approach of Ref. [14].

As well as discrete integrals, and amplitude functions $\Theta, \mathrm{A} : \mathbb{Z} \to \mathbb{C}$ in place of (vectors of) phases or amplitudes, we consider an index set for $\mathcal{H}$ which is not simply $\{0, 1, \ldots, D{-}1\}$, but instead $\{L, L+1, \ldots, U-1, U\}$ for some integers such that $U - L + 1 = D$. One conventional choice is $L = 0$ and $U = D - 1$, but most of out results (in particular: all those to do with the stabiliser / multicharacter fragments of ZX or ZH) hold equally well with any such set of labels for the standard basis. This less committal choice of index set demonstrates the flexibility of this system, which may prove useful for future applications (e.g., problems in physics where it may prove useful to consider negative index values).

We conclude with a highly speculative thought regarding discrete measures. One constraint which we imposed on the measure μ on $\mathbf{D}$ – interpreted as a measure on $\mathbb{Z}_D$ – was that the Fourier transform should be interpretable as an involution $\mathbb{C}^{(\mathbb{Z}_D,\mu)} \to \mathbb{C}^{(\mathbb{Z}_D,\mu)}$ on functions on the measure space $(\mathbb{Z}_D, \mu)$, rather than a bijection $\mathbb{C}^{(\mathbb{Z}_D,\mu)} \to \mathbb{C}^{(\mathbb{Z}_D,\mu')}$ between functions on distinct measure spaces $(\mathbb{Z}_D, \mu)$ and $(\mathbb{Z}_D, \mu')$. This may seem like a technical but necessary step; for a conventional presentation of ZX diagrams, it *is* necessary, if all of the wires are to have the same type. However, many quantum algorithms have a structure in which some classical operation with a distinguished control register, where that control operates on a state which is conceived as being in the Fourier basis. This structure is consistent with the control register having different "datatypes" at different stages of the algorithm. Could it be more appropriate to make a distinction on logical qudits of each dimension D, between a "standard" type and a "Fourier" type (possibly among others), than to have just a single "type" for each D? It would be interesting to consider what insights into the structure of quantum algorithms might arise by investigating along these lines; it is conceivable that this could give rise to new insights into structured quantum programming.

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[10] Strictly speaking, the calculi of Ref. [14] involve red and green dots with parameters $\theta \in \mathbb{R}$, H-boxes with parameters $\alpha \in \mathbb{C}$, only one type of Hadamard box instead of two, and a "nu box" which is missing from the calculus presented here. We may bridge these differences using the short-hand described for phases / amplitudes to parameterise these nodes, identifying both Hadamard plus and minus boxes with the single Hadamard box of Ref. [14], and replacing the nu-boxes with some suitable scalar gadgets (such as H-boxes parameterised by powers of $\nu = D^{-1/4}$).

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