Generalized Completion Problems with Forbidden Tournaments

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Abstract

A recent result by Bodirsky and Guzmán-Pro gives a complexity dichotomy for the following class of computational problems, parametrized by a finite family $\mathcal{F}$ of finite tournaments: given an undirected graph, does there exist an orientation of the graph that avoids every tournament in $\mathcal{F}$? One can see the edges of the input graphs as constraints imposing to find an orientation. In this paper, we consider a more general version of this problem where the constraints in the input are not necessarily about pairs of variables and impose local constraints on the global oriented graph to be found. Our main result is a complexity dichotomy for such problems, as well as a classification of such problems where the yes-instances have bounded treewidth duality. As a consequence, we obtain a streamlined proof of the result by Bodirsky and Guzmán-Pro using the theory of smooth approximations due to Mottet and Pinsker.

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1 Introduction

1.1 Completion Problems

For every fixed finite family $\mathcal{F}$ of tournaments, the $\mathcal{F}$-free orientation problem consists in deciding whether an undirected graph G admits an orientation G^* such that no tournament from $\mathcal{F}$ is contained in G^*. The complexity of such problems was systematically studied recently in [5], where it is shown that every such problem is solvable in polynomial time or NP-complete.

We extend this result by considering the following variation of the problem. Fix an $r \geq 2$ and let R be a set of tournaments on r vertices, labelled with the numbers $1, \dots, r$. An input in our problem is a set V of vertices where some r-tuples are marked. The yes-instances to this problem are those where there exists an $\mathcal{F}$-free digraph on V such that whenever $(x_1, \dots, x_r)$ is marked, then the labelled subdigraph induced on $\{x_1, \dots, x_r\}$ is isomorphic to an element of R.

This is an extension of the $\mathcal{F}$-free orientation problem, as can be seen by taking $r = 2$ and R consisting of the two possible tournaments on 2 (labelled) vertices. We call this the $(\mathcal{F}, R)$-orientation problem, see Figure 1 for an example. One can view $\mathcal{F}$ as imposing *global* constraints (find a directed graph on V that globally avoids every tournament in $\mathcal{F}$), and R as imposing *local* constraints (find a directed graph on V that satisfies the local restrictions on the marked tuples). Our more general result is a P versus NP-complete complexity dichotomy for such problems.

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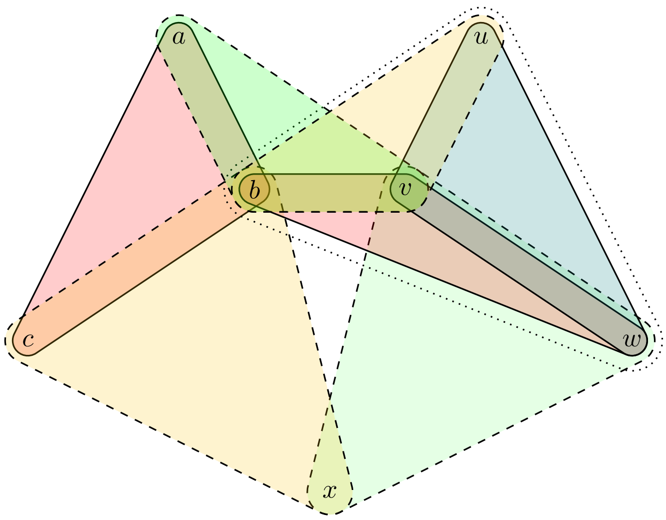

Figure 1 Example of an instance of $(\mathcal{F}, R)$-orientation where $\mathcal{F}$ consists of the transitive tournament on 4 vertices, and R contains the oriented graphs on 3 vertices inducing a transitive tournament. The markings are given by hyperedges. Note that due to the symmetry of R we do not need to specify the order of the vertices in the hyperedges. To solve such an instance, one needs to determine whether there exists an oriented graph on the given vertices, not inducing a transitive tournament on any four vertices, while all marked triples of vertices induce a transitive tournament.

▶ **Theorem 1.** *Let $\mathcal{F}$ be a finite set of finite tournaments and let R be a set of labelled r-vertex tournaments. Then the $(\mathcal{F}, R)$-orientation problem is solvable in polynomial time or NP-complete.*

1.2 Constraint Satisfaction Problems

Given a finite family $\mathcal{F}$ of tournaments, Fraïssé's theorem asserts that there exists a countably infinite digraph $D_{\mathcal{F}}$ whose finite subgraphs are exactly the oriented graphs avoiding every tournament in $\mathcal{F}$. Moreover, $D_{\mathcal{F}}$ satisfies a certain model-theoretic condition called *homogeneity*, which defines $D_{\mathcal{F}}$ uniquely up to isomorphism (see Section 2 for the detailed definitions). The following observation by Bodirsky and Guzmán-Pro in [5] is crucial for their complexity result. Consider the symmetric closure of $D_{\mathcal{F}}$, i.e., the undirected graph $H_{\mathcal{F}}$ obtained by forgetting about the directions of arcs in $D_{\mathcal{F}}$. Then it can be observed that given an undirected graph G, one has a homomorphism $G \to H_{\mathcal{F}}$ if, and only if, G admits an $\mathcal{F}$-free orientation. Thus, the $\mathcal{F}$-free orientation problem coincides with the *constraint satisfaction problem* for the graph $H_{\mathcal{F}}$, denoted by $\mathrm{CSP}(H_{\mathcal{F}})$. The complexity of this CSP can then be investigated using standard methods.

In order to obtain a generalization of [5] for the $(\mathcal{F}, R)$-orientation problem, we start with a similar observation. In $D_{\mathcal{F}}$, consider the subset $R' \subseteq V^r$ containing all tuples $(v_1, \ldots, v_r)$ inducing in $D_{\mathcal{F}}$ a tournament from R. The structure $\mathbb{A} = (V; R')$ is a so-called *first-order reduct* of $D_{\mathcal{F}}$. Note that if R consists of the two labelled tournaments on 2 vertices, then $\mathbb{A}$ thus defined coincides with $H_{\mathcal{F}}$. An input to the $(\mathcal{F}, R)$-orientation problem can then be seen as a structure in the same signature as $\mathbb{A}$. One obtains that an input to the $(\mathcal{F}, R)$-orientation problem admits a solution if, and only if, it admits a homomorphism to the structure $\mathbb{A}$. The structure $\mathbb{A}$ obtained in this way moreover satisfies two important properties:

- Provided R contains more than one tournament, the symmetric closure U of the edge relation of $D_{\mathcal{F}}$ is invariant under the higher-order symmetries of $\mathbb{A}$, called *polymorphisms*. In particular, the group of automorphisms of $\mathbb{A}$ is a subgroup of $\mathrm{Aut}(H_{\mathcal{F}})$.
- $\mathbb{A}$ has a particular type of binary injective polymorphism that we call an *injective projection*.

Theorem 1 then follows from the following technical result. The notion that "$\mathbb{A}$ pp-constructs every finite structure" appearing in the statement is defined in Section 2; for now, the reader can view this as a natural condition allowing to reduce the boolean satisfiability problem to the constraint satisfaction problem of $\mathbb{A}$, which is then NP-hard.

▶ **Theorem 2.** *Let $\mathcal{F}$ be a finite set of tournaments, and let $\mathbb{A}$ be a first-order reduct of $D_{\mathcal{F}}$ that admits injective projections and such that U is invariant under* $\mathrm{Pol}(\mathbb{A})$*. One of the following holds:*
- $\mathbb{A}$ *pp-constructs every finite structure and* $\mathrm{CSP}(\mathbb{A})$ *is NP-complete, or*
- $\mathrm{CSP}(\mathbb{A})$ *is solvable in polynomial time.*

The proof follows the *smooth approximations* approach introduced in [25] and uses the refinements thereof from [22, 26]. This provides a streamlined proof and an extension of the dichotomy result from [5].

Another consequence of our proof is the following. Say that the class of $\mathcal{F}$-free orientable graphs has *bounded treewidth duality* if there exists a set $\mathcal{G}$ of undirected graphs of bounded treewidth such that for every finite graph G, there exists an $\mathcal{F}$-free orientation of G if, and only if, no graph from $\mathcal{G}$ admits a homomorphism to G. This notion can be extended to structures with an $(\mathcal{F}, R)$-orientation by generalizing the notion of treewidth for relational structures in general, we refer the interested reader to [15] for precise definitions. Any class of structures corresponding to a CSP and having bounded treewidth duality can be recognized in polynomial time by a Datalog program, giving a particularly simple algorithm recognizing the class.

As a by-product of our proof, we obtain a characterization of the sets $\mathcal{F}, R$ for which the class of structures with an $(\mathcal{F}, R)$-orientation has bounded treewidth duality. As above, this characterization is phrased in terms of the algebraic properties of a first-order reduct of $D_{\mathcal{F}}$. Using the recent results from [22], this also allows us to obtain a bounded on the treewidth in a possible duality depending on the size of the tournaments in $\mathcal{F}$ and R.

▶ **Theorem 3.** *Let $\mathcal{F}$ be a finite set of tournaments, and let $\mathbb{A}$ be a first-order reduct of $D_{\mathcal{F}}$ that admits injective projections and such that U is invariant under* $\mathrm{Pol}(\mathbb{A})$*. Let r be the maximal arity of a relation of $\mathbb{A}$ and ℓ be the maximal size of a tournament in $\mathcal{F}$. Then, the following are equivalent:*
1. $\mathrm{Pol}(\mathbb{A})$ *contains an* $\mathrm{Aut}(D_{\mathcal{F}})$*-canonical pseudo-majority polymorphism modulo* $\overline{\mathrm{Aut}(D_{\mathcal{F}})}$*,*
2. *The class of finite structures admitting a homomorphism to $\mathbb{A}$ has a duality of treewidth at most* $\max(6, r, \ell)$*.*

In particular, when the class of $\mathcal{F}$-free orientable graphs has a bounded treewidth duality, then Theorem 3 implies that the duality $\mathcal{G}$ can be chosen to consist of graphs of treewidth at most $\max(6, \ell)$.

1.3 Related Work

The approach we follow here shows an equivalence between $(\mathcal{F}, R)$-completion problems and a subclass of constraint satisfaction problems (CSPs). A P/NP-complete complexity dichotomy is known for CSPs with a finite template [14, 29], while for CSPs with infinite templates in general it is known that the complexity varies greatly [4, 17, 18]. For a subclass of infinite structures, so-called *first-order reducts of finitely bounded homogeneous structures*, Bodirsky and Pinsker have conjectured that a dichotomy similar to that of finite-domain CSPs exists. The templates $D_{\mathcal{F}}$ studied here are in the scope of that conjecture. The conjecture has been proved for a variety of finitely bounded homogeneous structures (e.g., all homogeneous undirected graphs [8], certain homogeneous hypergraphs [26], the universal homogeneous tournament [25], $(\mathbb{Q}; <)$ [7], and the universal homogeneous poset [20]). There exist natural subclasses of the Bodirsky-Pinsker class for which the conjecture is still open; this is the case for example for first-order reducts of finitely bounded homogeneous directed graphs, or for first-order reducts of *homomorphically bounded* homogeneous structures. It can be shown that if D is a homogeneous homomorphically bounded homogeneous directed graph, then $D = D_{\mathcal{F}}$ for some finite family $\mathcal{F}$ of finite tournaments. Thus, our result can be seen as making progress on the Bodirsky-Pinsker conjecture for both mentioned subclasses.

1.4 Organization of the paper

We recall some elementary notions from graph theory and the universal-algebraic approach to the complexity of infinite-domain CSPs in Section 2. In Section 3, we prove some elementary properties of the templates $\mathbb{A}$ arising from $(\mathcal{F}, R)$-completion problems. Due to space restrictions, we only describe a high-level proof strategy for Theorems 2 and 3 in Section 4, focusing on making the presentation accessible to a non-expert.

2 Definitions and Notations

2.1 Elementary model-theoretic notions

For the purposes of this paper, a *structure* is a tuple $\mathbb{A} = (A; R_1^{\mathbb{A}}, \dots, R_k^{\mathbb{A}})$ consisting of a set A (the *domain*) together with finitely many relations $R_i^{\mathbb{A}} \subseteq A^{r_i}$ on A. The *signature* of $\mathbb{A}$ is the list $(r_1, \dots, r_k)$ containing the arities of the relations of $\mathbb{A}$. We assume the reader is familiar with the standard notions of homomorphisms, embeddings, and isomorphisms between structures. As is standard in model theory, all substructures and subgraphs in this paper are *induced* substructures and subgraphs.

An oriented graph is a directed graph $G = (V, E)$ where at most one of $(u, v) \in E$ or $(v, u) \in E$ holds for all $u, v \in V$, and where $(u, u) \notin E$ for all $u \in V$. A tournament is an oriented graph such that the symmetric closure of its edge relation induces a complete graph. For a (finite) set $\mathcal{F}$ of finite tournaments we say that an oriented graph is $\mathcal{F}$-free if it contains no $F \in \mathcal{F}$ as an induced subgraph.

Let $\mathcal{F}$ be a finite set of finite tournaments. Let $\mathcal{C}_{\mathcal{F}}$ be the class of all $\mathcal{F}$-free oriented graphs. It can be seen that this class has the so-called *amalgamation property*: given two $\mathcal{F}$-free oriented graphs, their union is also $\mathcal{F}$-free due to $\mathcal{F}$ consisting of tournaments only. By the classical result of Fraïssé [16], there exists an oriented graph $D_{\mathcal{F}} = (V, E)$ on a countable set whose finite subgraphs are exactly the graphs isomorphic to a graph in $\mathcal{C}_{\mathcal{F}}$. Moreover, this graph is *homogeneous*, in the sense that for every finite subset $S \subseteq V$ and every partial isomorphism $f\colon S \to V$, there exists an automorphism α of $D_{\mathcal{F}}$ such that $f|_S = \alpha|_S$. These two properties describe $D_{\mathcal{F}}$ uniquely up to isomorphism. We write $H_{\mathcal{F}} = (V, U)$ for the undirected graph whose edge set is the symmetric closure of E. We note that $H_{\mathcal{F}}$ is in general not homogeneous, and in fact we will focus in this paper on the case where $H_{\mathcal{F}}$ is *not* homogeneous for reasons that are made clear below.

A *first-order reduct* of $\mathbb{A}$ is a structure $\mathbb{B}$ with the same domain as $\mathbb{A}$ and whose relations all have a first-order definition in $\mathbb{A}$. For example, $H_{\mathcal{F}}$ is a first-order reduct of $D_{\mathcal{F}}$, because the symmetric closure U of E is definable by $(v,w) \in E \vee (w,v) \in E$, which is a first-order definition of U in $D_{\mathcal{F}}$. An *expansion* of $\mathbb{A}$ is a structure $\mathbb{B}$ obtained from $\mathbb{A}$ by adding new relations.

An example of a natural expansion of $D_{\mathcal{F}}$ is obtained as follows. Consider the class $\mathcal{C}_{\mathcal{F}}^{<}$ of all structures $(V', E', \prec)$ where $(V', E') \in \mathcal{C}_{\mathcal{F}}$ and where $\prec$ is an arbitrary linear order on V'. This class also has the amalgamation property as described above, and its Fraïssé limit can be taken to be $(V, E, <)$, an expansion of $D_{\mathcal{F}}$ by a linear order.

Let $\mathscr{G}$ be a group of permutations on a set A. Let $\equiv_k$ be the equivalence relation on A^k containing the pairs (a,b) of tuples such that there exists $\alpha \in \mathscr{G}$ such that $\alpha(a_i) = b_i$ for all $i \in \{1,\dots,k\}$. We call the equivalence classes of $\equiv_k$ the *orbits* of $\mathscr{G}$. The orbit of a pair $(a,b) \in V^2$ under $\mathrm{Aut}(D_{\mathcal{F}})$ is given, by the homogeneity of $D_{\mathcal{F}}$, by the isomorphism type of the labelled graph induced by $\{a,b\}$. There are therefore 4 orbits, corresponding to whether $a = b$, whether $a \neq b$ are connected by an edge, and if they are whether $(a,b) \in E$ or $(b,a) \in E$. We denote the orbit containing the pairs (a,b) such that $a \neq b$ and $(a,b) \notin U$ by N, the orbit of all $(a,b) \in E$ by $\rightarrow$, and the orbit of all (a,b) such that $(b,a) \in E$ by $\leftarrow$.

Let $\mathbb{A}, \mathbb{B}$ be structures with $B = A^n$. We say $\mathbb{B}$ is a *pp-power* of A if every relation in $\mathbb{B}$ is definable by a primitive positive formula over $\mathbb{A}$, that is a formula only using $\exists$ and $\wedge$. For arbitrary structures $\mathbb{A}, \mathbb{B}$ we say $\mathbb{A}$ *pp-constructs* $\mathbb{B}$ if there is a pp-power $\mathbb{C}$ of $\mathbb{A}$ such that $\mathbb{B}$ and $\mathbb{C}$ are homomorphically equivalent.

2.2 Clones, naked and affine sets

A relation $R \subseteq A^m$ is said to be *invariant* under a function $f\colon A^n \to A$ if for every $a^1,\dots,a^n \in R$, then $f(a^1,\dots,a^n)$, the tuple obtained by applying f componentwise to the tuples $a^1,\dots,a^n$, is also in R. A function $f\colon A^n \to A$ is a *polymorphism* of a structure $\mathbb{A}$ if all the relations of $\mathbb{A}$ are invariant under f. In particular, every automorphism and endomorphism of $\mathbb{A}$ is a polymorphism of $\mathbb{A}$. The set $\mathrm{Pol}(\mathbb{A})$ of all polymorphisms of $\mathbb{A}$ forms a *clone*: it contains all the projections $p_i^k\colon (a_1,\dots,a_k) \mapsto a_i$ for $1 \le i \le k$ and it is closed under composition. We write $\mathscr{P}$ for the clone consisting of only the projections on the set $\{0,1\}$. This clone is relevant in the theory of constraint satisfaction because it is exactly the clone of polymorphisms of a structure $\mathbb{S}$ with domain $\{0,1\}$ and having all ternary relations on $\{0,1\}$ as its relations, whose CSP corresponds to the problem CNF-3SAT.

If $S \subseteq A^k$ is invariant under a clone $\mathscr{C}$, and θ is an equivalence relation on S that is also invariant under $\mathscr{C}$, then the operations in $\mathscr{C}$ naturally induce a clone of functions on the set S/θ, where $f \in \mathscr{C}$ of arity n induces the function $([s_1],\dots,[s_n]) \mapsto [f(s_1,\dots,s_n)]$. We use the notation $\mathscr{C} \curvearrowright S/\theta$ to denote this clone and if θ has at least two equivalence classes we call (S,θ) a *subfactor* of $\mathscr{C}$. A subfactor is *minimal* if for all $a,b \in S$ that are not θ-equivalent, the smallest $\mathscr{C}$-invariant set containing a,b is equal to S.

A subfactor (S,θ) of $\mathscr{C}$ is called a *naked set* if $\mathscr{C} \curvearrowright S/\theta$ only consists of projections. Similarly, if S/θ is finite and can be endowed with a structure of a module over a ring R, in a way that $\mathscr{C} \curvearrowright S/\theta$ consists of *affine* functions, of the form $(x_1,\dots,x_n) \mapsto \sum \lambda_i \cdot x_i$ for $\lambda_1,\dots,\lambda_n \in R$ such that $\sum \lambda_i = 1$, then we call (S,θ) an *affine set* for $\mathscr{C}$. Note that every naked set is a particular example of an affine set: if (S,θ) is a naked set of $\mathscr{C}$, then the maps induced by $\mathscr{C}$ on S/θ have exactly one non-zero λ_i, which is equal to 1.

For a clone $\mathscr{C}$ on a 2-element set (in this paper, the 2-element set is $\{\leftarrow,\rightarrow\}$), the notions of having a naked set or an affine set can be rephrased using Post's classification of such clones [28]:

- $\mathscr{C}$ has a naked set if, and only if, every operation in $\mathscr{C}$ is essentially unary, of the form $(x_1, \ldots, x_n) \mapsto e(x_i)$ for some permutation e of $\{\leftarrow, \rightarrow\}$ and $i \in \{1, \ldots, n\}$. If $\mathscr{C}$ does not have a naked set, then it contains an operation f satisfying the equation

$$f(x, y, z) = f(y, z, x)$$

for all $x, y, z \in \{\leftarrow, \rightarrow\}$. Such an f is called a *ternary cyclic operation*.
- $\mathscr{C}$ has an affine set if, and only if, every operation in $\mathscr{C}$ is of the form

$$(x_1, \ldots, x_n) \mapsto \sum \lambda_i \cdot x_i + \mu$$

where $\lambda_1, \ldots, \lambda_n, \mu \in \{0, 1\}$ are such that $\sum \lambda_i = 1$, addition and multiplication are understood modulo 2, and after choosing an arbitrary bijection between $\{\leftarrow, \rightarrow\}$ and $\{0, 1\}$. If $\mathscr{C}$ does not have an affine set, then it contains an operation m satisfying the identities

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x$$

for all $x, y \in \{\leftarrow, \rightarrow\}$ or a binary operation s that is associative, commutative, and satisfies $s(x, x) = x$ for all $x \in \{\leftarrow, \rightarrow\}$. Such an m is called a *majority* operation, while s is called a *semilattice*.

For an ω-categorical structure $\mathbb{A}$ it is known that the complexity of CSP($\mathbb{A}$) is captured by the polymorphisms of $\mathbb{A}$. For our purposes, we need the following two special cases:

▶ **Theorem 4** ([21]). *Let $\mathbb{A}$ be an ω-categorical structure. The following hold:*
- *If* Pol($\mathbb{A}$) *has a naked set, then $\mathbb{A}$ pp-constructs every finite structure and* CSP($\mathbb{A}$) *is NP-hard.*
- *If* Pol($\mathbb{A}$) *has an affine set, then the class of structures admitting a homomorphism to $\mathbb{A}$ does not have bounded treewidth duality.*

2.3 Canonical Functions

Let $S \subseteq A^k$ be invariant under a function $f\colon A^n \to A$. We say that $f\colon A^n \to A$ is *canonical on S with respect to a permutation group $\mathscr{G}$* if whenever $a^1 \equiv_k b^1, \ldots, a^n \equiv_k b^n$ for $a^1, b^1, \ldots, a^n, b^n \in S$, then $f(a^1, \ldots, a^n) \equiv_k f(b^1, \ldots, b^n)$. In other words, f is canonical on S if the restriction of $\equiv_k$ to S is invariant under f; in particular, the set of functions f that are canonical on S with respect to $\mathscr{G}$ forms a clone $\mathscr{C}$, and this clone has an action $\mathscr{C} \curvearrowright S/{\equiv_k}$. If $f(a^1, \ldots, a^n) \equiv_k f(\alpha a^1, \ldots, \alpha a^n)$ holds for all $\alpha \in \mathrm{Aut}(\mathbb{A})$ and all $a^1, \ldots, a^n$, we call f *diagonally* canonical. In order to simplify notation and to make the dependence on the group clearer, we write $S/\mathscr{G}$ for the set of orbits of S induced by $\mathscr{G}$.

Let $k \geq 1$. We write $I_k \subseteq V^k$ for the set of tuples with pairwise distinct entries, and $K \subseteq I_k$ for the set of tuples whose components induce a clique in $H_{\mathcal{F}}$. We will be considering the following clones:
- $\mathscr{C}^K_{(D_{\mathcal{F}},<)}$ is the clone of polymorphisms of $\mathbb{A}$ that are canonical on K with respect to $\mathrm{Aut}(D_{\mathcal{F}}, <)$.
- $\mathscr{C}^K_{D_{\mathcal{F}}}$ the clone of polymorphisms of $\mathbb{A}$ that are canonical on K with respect to $\mathrm{Aut}(D_{\mathcal{F}})$,
- $\mathscr{C}^I_{\mathbb{A}}$ the clone of polymorphisms of $\mathbb{A}$ that are, for all $k \geq 1$, canonical on I_k with respect to $\mathrm{Aut}(\mathbb{A})$.

Note that since canonicity is considered with respect to different groups, the clones above are not necessarily comparable with respect to inclusion.

2.4 Smooth approximations

Central to our proof of Theorem 2 is the theory of smooth approximations, further developed in [23]. It relies on comparing two clones $\mathscr{C} \subseteq \mathscr{D}$ and whether a naked set (resp. affine set) for $\mathscr{C}$ can be lifted to a naked set (resp. affine set) for $\mathscr{D}$. The lifting is formalized by smooth approximations.

▶ **Definition 5** (Smooth Approximations). *Let A be a set, $k \geq 1$, and $\sim$ be an equivalence relation on $S \subseteq A^k$. An equivalence relation η on some set S' with $S \subseteq S' \subseteq A^k$ approximates $\sim$ if the restriction of η to S is a possible (non-proper) refinement of $\sim$; in that case, η is an* approximation *of $\sim$.*

Suppose that the $\sim$-equivalence classes as well as η are invariant under a group $\mathscr{G}$ of permutations on A. We say that the approximation is

- presmooth *with respect to a group $\mathscr{G}$ if each equivalence class C of $\sim$ intersects some equivalence class C' of η such that $C \cap C'$ contains two disjoint tuples in the same $\mathscr{G}$-orbit;*
- very smooth *with respect to a group $\mathscr{G}$ if $\equiv_k$ is a (possibly non-proper) refinement of η; in other words, if any two k-tuples in the same orbit must be η-equivalent.*

The equivalence relations $\sim$ for which we want to find approximations come from subfactors of $\mathscr{C}$. If $\mathscr{D}$ contains $\mathscr{C}$, it might not act at all on $S/{\sim}$ (if S or $\sim$ is not invariant under $\mathscr{D}$), and even in the case that it does, its action might contain operations that are not from $\mathscr{C} \curvearrowright S/{\sim}$. However, the theory of smooth approximations gives us that we can (under certain conditions) find a $\mathscr{D}$-invariant set $S' \supseteq S$ and an equivalence relation η on S' that approximates $\sim$, and such that $\mathscr{D} \curvearrowright S'/\eta$ is not "richer" than $\mathscr{C} \curvearrowright S/{\sim}$.

One of the central results from [25] is the so-called *loop lemma of smooth approximations*. We do not give the general formulation of the loop lemma here and rather phrase it directly the way we apply it in our proof.

▶ **Theorem 6** (Consequence of Theorem 11 and Lemma 14 in [25]). *Let $k \geq 1$, and suppose that $\mathscr{C}^I_{\mathbb{A}} \curvearrowright I_k/\mathrm{Aut}(\mathbb{A})$ has a naked (resp. affine) set. Then there exists a naked (resp. affine) set $(S, \sim)$ of $\mathscr{C}^I_{\mathbb{A}}$ with $S \subseteq I_k$ and $\mathrm{Aut}(\mathbb{A})$-invariant $\sim$-classes such that one of the following holds:*

1. *$\sim$ is approximated by a $\mathrm{Pol}(\mathbb{A})$-invariant equivalence relation that is presmooth with respect to $\mathrm{Aut}(D_{\mathcal{F}})$;*
2. *there exists $f \in \mathrm{Pol}(\mathbb{A})$ such that $f(a,b) \sim f(b,a)$ holds for all disjoint injective tuples $a, b \in V^k$ such that $f(a,b), f(b,a) \in S$.*

▶ **Lemma 7.** *Let $f\colon V^n \to V$ be an arbitrary operation defined on V. There exists $g\colon V^n \to V$ that is canonical with respect to $\mathrm{Aut}(D_{\mathcal{F}}, <)$ and that is* locally interpolated *by f, i.e., for every finite $S \subseteq V$ there exist $\alpha_1, \ldots, \alpha_n, \beta \in \mathrm{Aut}(D_{\mathcal{F}}, <)$ such that $g(a_1, \ldots, a_n) = \beta f(\alpha_1 a_1, \ldots, \alpha_n a_n)$ holds for all $a_1, \ldots, a_n \in S$.*

Proof. By the theorem of Nešetřil and Rödl [27], the class of all $\mathcal{F}$-free oriented graphs endowed with a linear order is a so-called *Ramsey class*. The conclusion is then obtained by applying [13, Lemma 14], see also [12, Theorem 5] for an alternative presentation. ◀

In particular, if $\mathbb{A}$ is a first-order reduct of $D_{\mathcal{F}}$, then for every $f \in \mathrm{Pol}(\mathbb{A})$ there exists $g \in \mathscr{C}^K_{(D_{\mathcal{F}},<)}$ that is locally interpolated by f. Similarly, for every $f\colon V^n \to V$, there exists $g\colon V^n \to V$ that is diagonally canonical with respect to $\mathrm{Aut}(D_{\mathcal{F}}, <)$ and that is *diagonally interpolated* by f, that is, for every finite $S \subseteq V$ there exist $\alpha, \beta \in \mathrm{Aut}(D_{\mathcal{F}}, <)$ such that $g(a_1, \ldots, a_n) = \beta f(\alpha a_1, \ldots, \alpha a_n)$ holds for all $a_1, \ldots, a_n \in S$.

3 Injective Projections and Preliminary Results

We define here the notion of injective projections appearing in Theorem 2.

▶ **Definition 8** (Injective projections). *An* injective projection *is a function* $q_i^n \colon V^n \to V$ *such that for all* $a_1, \ldots, a_n \in V^2$*, at least one of which is not a constant pair, the following hold:*
- $q_i^n(a_1, \ldots, a_n) \in \mathrm{N}$ *if* $a_j \notin U$ *for some* $j \in \{1, \ldots, n\}$*,*
- $q_i^n(a_1, \ldots, a_n)$ *and* a_i *are in the same orbit under* $\mathrm{Aut}(D_{\mathcal{F}})$ *otherwise.*

If $\mathbb{A}$ *is a reduct of* $D_{\mathcal{F}}$ *and* $q_i^n \in \mathscr{C}_{\mathbb{A}}^I$ *for all* $1 \leq i \leq n$ *then we say* $\mathbb{A}$ *has injective projections.*

Note that injective projections are canonical with respect to $D_{\mathcal{F}}$. Indeed, suppose that $a_1, b_1, \ldots, a_n, b_n$ are tuples of the same length m such that a_j and b_j induce the same graph in $D_{\mathcal{F}}$, for all $j \in \{1, \ldots, n\}$. Then by definition, $q_i^n(a_1, \ldots, a_n)$ and $q_i^n(b_1, \ldots, b_n)$ induce the same directed graph in $D_{\mathcal{F}}$, and thus they are in the same orbit under $\mathrm{Aut}(D_{\mathcal{F}})$ by homogeneity of $D_{\mathcal{F}}$.

▶ **Proposition 9.** *Let* $\mathcal{F}$ *be a finite set of finite tournaments, and let* $\mathbb{A}$ *be a first-order reduct of* $D_{\mathcal{F}}$ *whose relations only contain tuples inducing tournaments in* $D_{\mathcal{F}}$*. Then* $q_i^n \in \mathrm{Pol}(\mathbb{A})$ *for all* $1 \leq i \leq n$*. If one of the relations of* $\mathbb{A}$ *contains tuples inducing different tournaments, then* U *is invariant under* $\mathrm{Pol}(\mathbb{A})$*.*

Proof. Let R be a relation of $\mathbb{A}$ that contains two tuples inducing different tournaments in $D_{\mathcal{F}}$. In particular the arity n of R must be at least 2. Since R contains tuples a, b inducing different tournaments, there exist distinct $i, j \in \{1, \ldots, n\}$ such that (a_i, a_j) and (b_i, b_j) induce different tournaments, i.e., they form edges in opposite directions. Thus, the projection of R onto the coordinates i, j is equal to U, and it follows that U is invariant under $\mathrm{Pol}(\mathbb{A})$.

Let $1 \leq i \leq n$. We define a directed graph $D = (V^n, E')$ by $(x, y) \in E'$ if, and only if, $(x_j, y_j) \in U$ for all $j \in \{1, \ldots, n\}$ and $(x_i, y_i) \in E$.

We prove that D is $\mathcal{F}$-free. Assume for contradiction otherwise. Then there is a finite $V' \subseteq V^n$ inducing a tournament from $\mathcal{F}$ in D. By definition, the projection

$$\{v \in V \mid \exists x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in V : (x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_n) \in V'\}$$

of V' onto its ith coordinate induces the same tournament in $D_{\mathcal{F}}$. But this contradicts the fact that $D_{\mathcal{F}}$ is $\mathcal{F}$-free. Thus, since $D_{\mathcal{F}}$ is universal for the class of $\mathcal{F}$-free digraphs, there exists an embedding $q_i^n \colon D \hookrightarrow D_{\mathcal{F}}$.

We can then view q_i^n as an n-ary function on V. We prove that it is a polymorphism of $\mathbb{A}$. Let R be a relation of $\mathbb{A}$, and let $r_1, \ldots, r_n \in R$. By assumption on R, all $r_1, \ldots, r_n$ induce tournaments in $D_{\mathcal{F}}$. Thus, $q_i^n(r_1, \ldots, r_n)$ induce the same tournament as r_i, and therefore $q_i^n(r_1, \ldots, r_n) \in R$. ◀

As a consequence of Proposition 9, we obtain that all templates $\mathbb{A}$ arising from a $(\mathcal{F}, R)$-completion problem in the way outlined in the introduction satisfy the assumptions of Theorem 2.

We now prove that the injective projections are canonical with respect to $\mathrm{Aut}(\mathbb{A})$, for any first-order reduct $\mathbb{A}$ of $D_{\mathcal{F}}$ such that $\mathrm{Aut}(\mathbb{A}) \subseteq \mathrm{Aut}(H_{\mathcal{F}})$. The proof of this fact is the only place where the classification of first-order reducts of $D_{\mathcal{F}}$ due to Agarwal and Kompatscher [1] is needed.

▶ **Lemma 10.** *Let* $\mathbb{A}$ *be a first-order reduct of* $D_{\mathcal{F}}$ *with* $\mathrm{Aut}(\mathbb{A}) \subseteq \mathrm{Aut}(H_{\mathcal{F}})$*. The injective projections are canonical with respect to* $\mathrm{Aut}(\mathbb{A})$ *and are therefore elements of* $\mathscr{C}_{\mathbb{A}}^I$*.*

Proof. Let $1 \leq k \leq n$ and let $a^1, b^1, \ldots, a^n, b^n$ be such that a^i and b^i are injective tuples that are in the same orbit under $\mathrm{Aut}(\mathbb{A})$, for all $i \in \{1, \ldots, n\}$. We need to show that $q_k^n(a^1, \ldots, a^n)$ and $q_k^n(b^1, \ldots, b^n)$ are in the same orbit under $\mathrm{Aut}(\mathbb{A})$.

The $\mathrm{Aut}(D_\mathcal{F})$-orbit of $q_k^n(a^1, \ldots, a^n)$ is the orbit of n-tuples c such that for all $i \neq j$, either $(c_i, c_j) \in \mathrm{N}$ whenever for some ℓ, $(a_i^\ell, a_j^\ell) \in \mathrm{N}$, and otherwise (c_i, c_j) is in the same $\mathrm{Aut}(D_\mathcal{F})$-orbit as (a_i^k, a_j^k). Since a^ℓ and b^ℓ are in the same orbit under $\mathrm{Aut}(\mathbb{A})$ for all $\ell \in \{1, \ldots, n\}$, for all $i \neq j$ we have $(a_i^\ell, a_j^\ell) \in \mathrm{N}$ if, and only if, $(b_i^\ell, b_j^\ell) \in \mathrm{N}$. Therefore, the indices i, j where the restrictions of the tuples $q_k^n(a^1, \ldots, a^n)$ and $q_k^n(b^1, \ldots, b^n)$ are in N coincide.

By the classification of all first-order reducts $\mathbb{A}$ of $D_\mathcal{F}$ such that $\mathrm{Aut}(D_\mathcal{F}) \leq \mathrm{Aut}(\mathbb{A}) \leq \mathrm{Aut}(H_\mathcal{F})$ due to Agarwal and Kompatscher [1], the tuple b^k can be obtained from a^k by a sequence of *switching* steps and *reversing* steps defined as follows. Given a directed graph, a switching step consists in choosing a vertex of the graph and reversing the direction of every edge incident to that vertex; a reversing step consists in reversing the direction of all edges. Note that if the directed graph induced by b can be obtained by finitely many such operations starting from the directed graph induced by a, then the same is true if one removes in a and b edges at the same position. It follows that $q_k^n(a^1, \ldots, a^k)$ and $q_k^n(b^1, \ldots, b^k)$ are in the same orbit under $\mathrm{Aut}(\mathbb{A})$. ◀

4 Description of the proof strategy

We describe here the strategy for the proof of Theorem 2 on a relatively high level.

4.1 Preprocessing of the Reducts of $D_\mathcal{F}$

A structure $\mathbb{A}$ is a *model-complete core* if for every finite $S \subseteq A$ and every endomorphism $f\colon \mathbb{A} \to \mathbb{A}$, there exists an automorphism $\alpha \in \mathrm{Aut}(\mathbb{A})$ such that $f|_S = \alpha|_S$. It is often very convenient for studying the complexity of $\mathrm{CSP}(\mathbb{A})$ and the polymorphisms of $\mathbb{A}$ to work with a structure that is a model-complete core; as an example of an important application for us, if $\mathbb{A}$ is a model-complete core, then for all $n \geq 1$ and $a \in A^n$, the orbit of a under $\mathrm{Aut}(\mathbb{A})$ is invariant under all the polymorphisms of $\mathbb{A}$.

While not every structure is a core, it is known that every ω-categorical structure, and in particular every first-order reduct $\mathbb{A}$ of $D_\mathcal{F}$, is homomorphically equivalent to a structure $\mathbb{B}$ that is a model-complete core, i.e., such that there exist homomorphisms $\mathbb{A} \to \mathbb{B}$ and $\mathbb{B} \to \mathbb{A}$ [3, 2]. Moreover, this structure is unique up to isomorphism and is called the *model-complete core of* $\mathbb{A}$.

If $\mathbb{A}$ is a first-order reduct of $D_\mathcal{F}$, it is *a priori* not guaranteed that the model-complete core of $\mathbb{A}$ is a first-order reduct of $D_\mathcal{F}$. The following statement that we prove first establishes this property and relies on a recent result by Mottet and Pinsker [24].

▶ **Lemma 11.** *Let $\mathbb{A}$ be a first-order reduct of $D_\mathcal{F}$, and let $\mathbb{A}'$ be the model-complete core of $\mathbb{A}$. Then $\mathbb{A}'$ is either isomorphic to $\mathbb{A}$, or a 1-element structure, or a first-order reduct of a homogeneous undirected graph or the universal homogeneous tournament.*

In the following statement, a function $g\colon A \to A$ with respect to a group $\mathscr{G}$ of permutations on A is *range-rigid* if for every $\alpha \in \mathscr{G}$ and every finite $S \subseteq A$, there exists $\beta \in \mathscr{G}$ such that $g|_S = \beta \circ g \circ \alpha \circ g|_S$. In words, this means that g essentially behaves like a retraction (modulo elements of $\mathscr{G}$) on every orbit of $\mathscr{G}$ that intersects the range of g.

While the first outcome in Theorem 6 only yields a presmooth approximation, it is a very smooth approximation that is needed in Proposition 21. As in [25], the step from presmooth to very smooth is achieved by leveraging a certain primitivity property of the automorphism group of the base structure under consideration, in this case $D_{\mathcal{F}}$.

For $n \geq 1$, we call a permutation group $\mathscr{G}$ acting on a set A *n-primitive* if for every orbit $O \subseteq A^n$ of $\mathscr{G}$, every $\mathscr{G}$-invariant equivalence relation on O containing a pair (a, b) where a, b are disjoint is full. We say an ω-categorical structure $\mathbb{A}$ has *no algebraicity* if none of its elements are first-order definable using other elements as parameters.

▶ **Lemma 12.** *For all $n \geq 1$, $\mathrm{Aut}(D_{\mathcal{F}})$ is n-primitive and has no algebraicity.*

Proof. Let $n \geq 1$ and O an orbit of n-tuples under $\mathrm{Aut}(D_{\mathcal{F}})$ and $a \sim b$ for some equivalence relation $\sim$ on O with a, b disjoint, and c, d arbitrary tuples in O. We define a digraph $\mathbb{X}$ on $5n$ vertices, partitioned into five n-tuples $x, y, z, u, v \in V^n$ such that the entries of x, u and v, z induce the same graph as the entries of a, b in $D_{\mathcal{F}}$. Similar let u, y and y, v induce b, a. Finally let x, z induce c, d. Then $\mathbb{X}$ is $\mathcal{F}$-free as all induced tournaments in $\mathbb{X}$ contain only vertices belonging to at most two tuples which are neighbors in the cycle x, u, y, v, z, x. By definition these two tuples induce a graph isomorphic to the graph induced by a, b or c, d in $D_{\mathcal{F}}$ respectively, so $\mathbb{X}$ is $\mathcal{F}$-free. Then there is an embedding $f \colon \mathbb{X} \to D_{\mathcal{F}}$ with $f(x) = c$ and $f(z) = d$ by homogeneity. Also by transitivity of $\sim$ we have $c = f(x) \sim f(z) = d$.

The class $\mathcal{C}$ of $\mathcal{F}$-free oriented graphs has the *free amalgamation property*, i.e., given any two such oriented graphs $D = (W, F), D' = (W', F')$ inducing the same directed graph on $W \cap W'$, then the union $(W \cup W', F \cup F')$ is an $\mathcal{F}$-free oriented graph. This implies (see, e.g., [19]) that the Fraïssé limit of $\mathcal{C}$, which is exactly $D_{\mathcal{F}}$, has no algebraicity. ◀

▶ **Lemma 13.** *Let $\mathbb{A}$ be a first-order reduct of $D_{\mathcal{F}}$ that is a model-complete core and such that $\mathrm{Aut}(\mathbb{A}) \subseteq \mathrm{Aut}(H_{\mathcal{F}})$. Then $\neq$ and N are invariant under $\mathrm{Pol}(\mathbb{A})$.*

Proof. Since $\mathrm{Aut}(\mathbb{A}) \subseteq \mathrm{Aut}(H_{\mathcal{F}})$, N is a single orbit under $\mathrm{Aut}(\mathbb{A})$ and is therefore invariant under $\mathrm{Pol}(\mathbb{A})$ since $\mathbb{A}$ is a model-complete core. Now, let O be another orbit of injective pairs. Then every pair (a, b) with $a \neq b$ satisfied the formula $\varphi(x, y) := \exists z((x, z) \in O \wedge (y, z) \in \mathrm{N})$. This is a primitive positive formula, and since both O and N are invariant under $\mathrm{Pol}(\mathbb{A})$, then so is the relation defined by φ. ◀

Let us call a first-order reduct $\mathbb{A}$ of $D_{\mathcal{F}}$ a *proper reduct* if the following conditions are satisfied:

- $\mathbb{A}$ is a model-complete core, i.e., if every homomorphism $\mathbb{A} \to \mathbb{A}$ locally resembles an automorphism an automorphism of $\mathbb{A}$ (the precise definition is given in Section 4.1),
- $\mathrm{Aut}(\mathbb{A}) \subseteq \mathrm{Aut}(H_{\mathcal{F}})$,
- if $\mathrm{Aut}(\mathbb{A}) = \mathrm{Aut}(H_{\mathcal{F}})$, then $H_{\mathcal{F}}$ is not homogeneous.

Our next step is to show that $\mathbb{A}$ can without loss of generality be assumed to be proper. Indeed, if $\mathbb{A}$ is not a model-complete core, then by Lemma 11 the model-complete core of $\mathbb{A}$ is a 1-element structure, or is a first-order reduct of a homogeneous graph, or is a first-order reduct of the universal homogeneous tournament; Theorem 2 is known to hold for all such structures [8, 25]. Since replacing $\mathbb{A}$ by its model-complete core does not change the outcome of Theorem 2, we are immediately done if $\mathbb{A}$ is not a model-complete core. If $\mathrm{Aut}(\mathbb{A}) \not\subseteq \mathrm{Aut}(H_{\mathcal{F}})$, then by Theorem 2.2(i) of [1] we have $\mathrm{Aut}(H_{\mathcal{F}}) \subsetneq \mathrm{Aut}(\mathbb{A})$, in which case either $H_{\mathcal{F}}$ is a homogeneous undirected graph, or $H_{\mathcal{F}}$ is not homogeneous and then

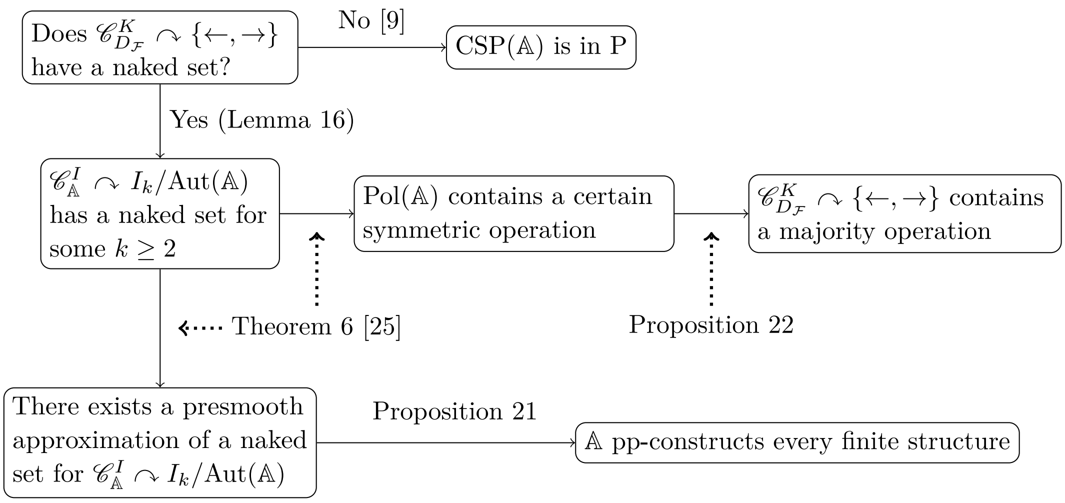

Figure 2 A simplified overview of the proof strategy of Theorem 2 after the preprocessing step.

Aut($\mathbb{A}$) is the full symmetric group by Theorem 2.2(iii) of [1]. Both of these cases can be handled by [25].[1] Finally, if $\mathrm{Aut}(\mathbb{A}) = \mathrm{Aut}(H_\mathcal{F})$ and $H_\mathcal{F}$ is homogeneous, then we are again done by [25].

4.2 An Algebraic Dichotomy for Proper Reducts

After this “preprocessing” step, the main technical result is the following.

▶ **Theorem 14.** *Let $\mathcal{F}$ be a finite set of finite tournaments. Let $\mathbb{A}$ be a proper reduct of $D_\mathcal{F}$ that admits injective projections and such that U is invariant under* Pol($\mathbb{A}$)*. Then exactly one of the following holds:*

1. Pol($\mathbb{A}$) *has a naked set,*
2. Pol($\mathbb{A}$) *contains a ternary operation f that is canonical with respect to* Aut($D_\mathcal{F}$) *and $u, v \in \overline{\mathrm{Aut}(D_\mathcal{F})}$ such that $u \circ f(x, y, z) = v \circ f(y, z, x)$ holds for all $x, y, z \in V$.*

Note that Theorem 14 is indeed a refined version of Theorem 2: the first item of Theorem 14 implies the first item of Theorem 2 by [11], and the second item of Theorem 14 implies the second item of Theorem 2 by [9].

The proof strategy is represented in Figure 2 and is based on distinguish upon whether $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow, \rightarrow\}$, which is a clone of functions on the two-element set $\{\leftarrow, \rightarrow\}$, has a naked set. If $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow, \rightarrow\}$ does not have a naked set, then we show that Pol($\mathbb{A}$) contains an operation f as in the second item of Theorem 14.

▶ **Proposition 15.** *Let $\mathbb{A}$ be a proper reduct of $D_\mathcal{F}$ that admits injective projections. The following hold:*

1. *If $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow, \rightarrow\}$ does not have a naked set, then there exists $f \in \mathrm{Pol}(\mathbb{A})$ and $u, v \in \overline{\mathrm{Aut}(D_\mathcal{F})}$, canonical with respect to* Aut($D_\mathcal{F}$)*, and such that the identity*

$$u \circ f(x, y, z) = v \circ f(y, z, x)$$

holds for all $x, y, z \in V$;

[1] The first proof in the first case was given in [8], while the first proof in the case of the full symmetric group is due to [6].

2. *If* $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow,\rightarrow\}$ *does not have an affine set, then there exists* $f \in \mathrm{Pol}(\mathbb{A})$ *such that for all* $n \geq 1$ *and* $a, b \in V^n$*, the tuples* $a, f(a,a,b), f(a,b,a), f(b,a,a)$ *are all in the same orbit under* $\mathrm{Aut}(D_\mathcal{F})$*.*

Proof. If $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow,\rightarrow\}$ does not have a naked set, then it contains a ternary function f' that acts as a cyclic operation on $\{\leftarrow,\rightarrow\}$. Then $f(x,y,z) := f'(q^3_1(x,y,z), q^3_1(y,z,x), q^3_1(z,x,y))$ is a canonical polymorphism of $\mathbb{A}$ which is pseudo-cyclic modulo $\overline{\mathrm{Aut}(D_\mathcal{F})}$. If $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow,\rightarrow\}$ does not have an affine set, then it contains a ternary function m that induces a majority operation on $\{\leftarrow,\rightarrow\}$ (note that any binary operation $f \in \mathscr{C}^K_{D_\mathcal{F}}$ induces on $\{\leftarrow,\rightarrow\}$ a function such that $f(\rightarrow,\leftarrow) \neq f(\leftarrow,\rightarrow)$, so that $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow,\rightarrow\}$ cannot contain a semilattice operation). Then $f(x,y,z) := m(q^3_1(x,y,z), q^3_1(y,z,x), q^3_1(z,x,y))$ is a canonical polymorphism of $\mathbb{A}$ that satisfies the statement. ◀

Then $\mathrm{CSP}(\mathbb{A})$ can be solved in polynomial time by reducing it to a tractable finite-domain CSP [10]. Otherwise, $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow,\rightarrow\}$ has a naked set and one can prove in this case that $\mathscr{C}^I_{\mathbb{A}} \curvearrowright I_k/\mathrm{Aut}(\mathbb{A})$ also has a naked set for some $k \geq 1$.

▶ **Lemma 16.** *Let* $\mathbb{A}$ *be a proper reduct of* $D_\mathcal{F}$ *that admits injective projections and such that* U *is invariant under* $\mathrm{Pol}(\mathbb{A})$*. Assume* $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow,\rightarrow\}$ *has a naked (resp. affine) set. Then* $\mathscr{C}^I_{\mathbb{A}} \curvearrowright I_k/\mathrm{Aut}(\mathbb{A})$ *has a naked (resp. affine) set for some* $k \geq 2$*.*

In other words, there exist $S \subseteq I_k$ invariant under $\mathscr{C}^I_{\mathbb{A}}$ and a $\mathscr{C}^I_{\mathbb{A}}$-invariant equivalence relation $\sim$ on S with $\mathrm{Aut}(\mathbb{A})$-invariant equivalence classes such that $\mathscr{C}^I_{\mathbb{A}} \curvearrowright S/{\sim}$ only contains projections. The loop lemma of smooth approximations applies (Theorem 6), giving us two possible outcomes.

4.2.1 First Case: Presmooth Approximation

In the first case, there exists a presmooth approximation for a naked set of $\mathscr{C}^I_{\mathbb{A}} \curvearrowright I/\mathrm{Aut}(\mathbb{A})$. We first show how to "upgrade" this approximation into a very smooth approximation, applying general principles from the theory of smooth approximations.

▶ **Proposition 17.** *Let* $\mathbb{A}$ *be a first-order reduct of* $D_\mathcal{F}$ *that is a model-complete core and such that* $\mathrm{Aut}(\mathbb{A}) \subseteq \mathrm{Aut}(H_\mathcal{F})$*. If* $(S,\sim)$ *is a minimal subfactor of* $\mathscr{C}^I_{\mathbb{A}}$ *such that* $\sim$ *has* $\mathrm{Aut}(\mathbb{A})$*-invariant classes, and* η *is a presmooth approximation of* $\sim$ *with respect to* $\mathrm{Aut}(\mathbb{A})$*, then* η *is very smooth with respect to* $\mathrm{Aut}(D_\mathcal{F})$*.*

Proof. We show that η is presmooth with respect to $\mathrm{Aut}(D_\mathcal{F})$. Let C be an equivalence class of $\sim$. By assumption, there exists an equivalence class C' of η and $a, b \in C \cap C'$ that are disjoint. By Lemma 12, $\mathrm{Aut}(D_\mathcal{F})$ has no algebraicity. Thus, there exists an automorphism $\alpha \in \mathrm{Aut}(D_\mathcal{F}, a)$ such that $\alpha(b)$ and b are disjoint. Note that b and $\alpha(b)$ are $\sim$-equivalent, since the equivalence classes of $\sim$ are $\mathrm{Aut}(\mathbb{A})$-invariant. Moreover, b and $\alpha(b)$ are η-equivalent, since $(a,b) \in \eta$ and η is $\mathrm{Aut}(\mathbb{A})$-invariant. Thus, we have disjoint elements $b, \alpha(b)$ in $C' \cap C$ and in the same orbit under $\mathrm{Aut}(D_\mathcal{F})$, i.e., η is presmooth with respect to $\mathrm{Aut}(D_\mathcal{F})$.

By Lemma 13, $\neq$ is invariant under $\mathrm{Pol}(\mathbb{A})$, and by Lemma 12, $\mathrm{Aut}(D_\mathcal{F})$ is n-primitive for all n. By [25, Lemma 8], we obtain that η is very smooth with respect to $\mathrm{Aut}(D_\mathcal{F})$. ◀

We show that this can be used to obtain a naked set for $\mathrm{Pol}(\mathbb{A})$, which implies by Theorem 4 that $\mathbb{A}$ pp-constructs every finite structure. We are in the situation where the original theorem from [25, Theorem 13] used to extend a naked set (or affine set) does not apply. Indeed, this result would require that $\mathscr{C}^I_{\mathbb{A}}$ be locally interpolated by $\mathrm{Pol}(\mathbb{A})$, which we do not have.

However, we do have that $\mathscr{C}^K_{D_\mathcal{F}}$ *is* locally interpolated by $\mathrm{Pol}(\mathbb{A})$. Indeed, we already know that any $f \in \mathrm{Pol}(\mathbb{A})$ locally interpolates an operation g that is canonical with respect to $\mathrm{Aut}(D_\mathcal{F}, <)$. If g is not an element of $\mathscr{C}^K_{D_\mathcal{F}}$, we show that $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow, \rightarrow\}$ must contain a majority operation, a contradiction to our assumption that $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow, \rightarrow\}$ has a naked set. The proof of the following lemma is similar to the proof of Lemma 34 in [25].

▶ **Lemma 18.** *Let $\mathbb{A}$ be a proper reduct of $D_\mathcal{F}$ such that U is invariant under* $\mathrm{Pol}(\mathbb{A})$*. Suppose that:*

- $\mathrm{Pol}(\mathbb{A})$ *contains a binary injection acting like a projection on $\{\leftarrow, \rightarrow\}$,*
- *there is a function in $\mathscr{C}^K_{(D_\mathcal{F},<)}$ that is not in $\mathscr{C}^K_{D_\mathcal{F}}$.*

Then $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow, \rightarrow\}$ contains a majority operation and in particular does not have an affine set.

We can then proceed as in [26] and use Lemma 18 and the injective projections to circumvent the original limitation from [25, Theorem 13]. In the following, the "lifting" operation, which gives us that $\mathrm{Pol}(\mathbb{A})$ has a naked set, is performed by exhibiting a *uniformly continuous clone homomorphism* from $\mathrm{Pol}(\mathbb{A})$ to $\mathscr{C}^I_\mathbb{A} \curvearrowright S/{\sim}$, i.e., an arity-preserving map ξ such that $\xi(f \circ (g_1, \ldots, g_k)) = \xi(f) \circ (\xi(g_1), \ldots, \xi(g_k))$ for all $f, g_1, \ldots, g_k \in \mathrm{Pol}(\mathbb{A})$ of appropriate arities, and where the value of $\xi(f)$ for an n-ary $f \in \mathrm{Pol}(\mathbb{A})$ is only determined by the restriction $f|_S$ of f onto a finite subset S of A that does not depend on f. The existence of such a map shows that $\mathrm{Pol}(\mathbb{A})$ is not "richer" than $\mathscr{C}^I_\mathbb{A} \curvearrowright S/{\sim}$; in particular, if $\mathscr{C}^I_\mathbb{A} \curvearrowright S/{\sim}$ has a naked (resp. affine) set, then so does $\mathrm{Pol}(\mathbb{A})$. We refer the reader to [11] and in particular to Theorem 1 therein for more details.

▶ **Corollary 19.** *Let $\mathbb{A}$ be a proper reduct of $D_\mathcal{F}$ that admits injective projections and such that U is invariant under* $\mathrm{Pol}(\mathbb{A})$*. If $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow, \rightarrow\}$ has an affine set, we have $\mathscr{C}^K_{(D_\mathcal{F},<)} \subseteq \mathscr{C}^K_{D_\mathcal{F}}$. In particular, every $f \in$* $\mathrm{Pol}(\mathbb{A})$ *locally interpolates a function in $\mathscr{C}^K_{D_\mathcal{F}}$.*

Proof. Lemma 18 applies since $\mathbb{A}$ admits injective projections and U is invariant under $\mathrm{Pol}(\mathbb{A})$. For the second part, we know by Lemma 7 that every $f \in \mathrm{Pol}(\mathbb{A})$ locally interpolates an operation that is canonical with respect to $(D_\mathcal{F}, <)$. Such operations are in particular in $\mathscr{C}^K_{(D_\mathcal{F},<)}$. Since $\mathscr{C}^K_{(D_\mathcal{F},<)} \subseteq \mathscr{C}^K_{D_\mathcal{F}}$, we are done. ◀

▶ **Lemma 20.** *Let $\mathbb{A}$ be a proper reduct of $D_\mathcal{F}$ such that U is invariant under $\mathscr{C}^I_\mathbb{A}$. Let $(S, \sim)$ be an affine set of $\mathscr{C}^I_\mathbb{A}$ such that $S \subseteq I_k$ and where the $\sim$-equivalence classes are* $\mathrm{Aut}(\mathbb{A})$*-invariant. Then for all $a, b \in S$, a and b induce the same undirected graph in $H_\mathcal{F}$.*

Proof. Suppose first that there exist $a, b \in S$ with $a \not\sim b$ and such that a, b induce the same undirected graph in $H_\mathcal{F}$. Since both N and U are invariant under $\mathscr{C}^I_\mathbb{A}$, the set generated by $\{a, b\}$ under $\mathscr{C}^I_\mathbb{A}$ only consists of tuples all inducing the same undirected graph as a and b in $H_\mathcal{F}$. By minimality of $(S, \sim)$, such a set must be equal to S itself, so we are done.

Otherwise, whenever $a \not\sim b$, then a, b induce different undirected graphs. In other words, any tuples in S inducing the same undirected graph are in the same $\sim$-equivalence class. By assumption, $(S, \sim)$ is an affine set for $\mathscr{C}^I_\mathbb{A}$, and therefore there exists a ring R and a finite R-module M on $S/{\sim}$ such that all operations in $\mathscr{C}^I_\mathbb{A} \curvearrowright S/{\sim}$ are of the form $(x_1, \ldots, x_n) \mapsto \sum \lambda_i \cdot x_i$, where $\lambda_1, \ldots, \lambda_n \in R$ are such that $\sum \lambda_i = 1$. Let $a \in S$ be a tuple whose $\sim$-equivalence class is an arbitrary non-zero element of the module M, and let $b \in S$ be a tuple whose $\sim$-equivalence class is the zero element of M. Since M is finite, a (more precisely, its $\sim$-equivalence class $[a]$), has a finite order $n \geq 2$, that is, $n[a] = [b]$. Note that the tuples $q^n_1(a, b, \ldots, b), q^n_1(b, a, b, \ldots, b), \ldots, q^n_1(b, \ldots, b, a)$ all induce the same undirected graph, and they are all in S since $q^n_1 \in \mathscr{C}^I_\mathbb{A}$ by Lemma 10. By our assumption, all these tuples are in the same $\sim$-equivalence class. Since $\mathscr{C}^I_{\mathbb{A}} \curvearrowright S/{\sim}$ is affine, the action of q^n_1 on $S/{\sim}$ is an affine function of the form $(x_1,\ldots,x_n) \mapsto \sum \lambda_i \cdot x_i$, where $\lambda_1,\ldots,\lambda_n$ are elements of R and such that $\sum \lambda_i = 1$. If a is in the ith position, then $q^n_1(b,\ldots,b,a,b,\ldots,b)$ is $\sim$-equivalent to $\lambda_i \cdot [a]$. Thus, we get that $\lambda_i \cdot [a] = \lambda_j \cdot [a]$ for all $i,j \in \{1,\ldots,n\}$. We call this element $\lambda \cdot [a]$. Thus, the equivalence class of $q^n_1(a,\ldots,a)$ is $\sum_{i=1}^n \lambda \cdot [a] = n(\lambda \cdot [a]) = \lambda \cdot (n \cdot [a]) = \lambda \cdot [b] = [b]$. However, q^n_1 being affine, we also know that $q^n_1(a,\ldots,a)$ is $\sim$-equivalent to a, contradicting our assumption that $a \not\sim b$. ◀

▶ **Proposition 21.** *Let $\mathbb{A}$ be a proper reduct of $D_{\mathcal{F}}$ that admits injective projections and such that U is invariant under* Pol($\mathbb{A}$)*. Assume that $\mathscr{C}^K_{D_{\mathcal{F}}} \curvearrowright \{\leftarrow,\rightarrow\}$ has an affine set. Suppose that there exists an affine set $(S,\sim)$ of $\mathscr{C}^I_{\mathbb{A}}$ with $S \subseteq I_k$ and $\sim$ having* Aut($\mathbb{A}$)*-invariant classes. Suppose further that $\sim$ admits a* Pol($\mathbb{A}$)*-invariant very smooth approximation with respect to* Aut($D_{\mathcal{F}}$)*. Then* Pol($\mathbb{A}$) *admits a uniformly continuous clone homomorphism to $\mathscr{C}^I_{\mathbb{A}} \curvearrowright S/{\sim}$.*

Proof. Without loss of generality, we can assume that S is minimal with the property of intersecting two equivalence classes of $\sim$. Let η be a presmooth approximation of $\sim$ with respect to Aut($D_{\mathcal{F}}$). By [25, Lemma 8] and Lemmas 12 and 13, every Pol($\mathbb{A}$)-invariant presmooth approximation of this naked set must be very smooth with respect to Aut($D_{\mathcal{F}}$).

Let $f \in$ Pol($\mathbb{A}$) an n-ary function. Let $f' \in \mathscr{C}^K_{D_{\mathcal{F}}}$ be locally interpolated by f; such an operation exists by Corollary 19. Let q^n_i be the i-th injective projection. By Lemma 10, $q^n_i \in \mathscr{C}^I_{\mathbb{A}}$. We define $f''(x_1,\ldots,x_n) = f'(q^n_1(x_1,\ldots,x_n),\ldots,q^n_n(x_1,\ldots,x_n))$ and show that $f'' \in \mathscr{C}^I_{\mathbb{A}}$. For this let $a_1,a'_1,\ldots,a_n,a'_n$ be injective tuples of an arbitrary length such that a_i,a'_i are in the same Aut($\mathbb{A}$)-orbit for all $1 \le i \le n$. Since Aut($\mathbb{A}$) $\subseteq$ Aut($H_{\mathcal{F}}$), every pair a_i,a'_i induce the same undirected graph in $H_{\mathcal{F}}$. As $q^n_i \in \mathscr{C}^I_{\mathbb{A}}$ we know that $b_i := q^n_i(a_1,\ldots,a_n)$ and $b'_i := q^n_i(a'_1,\ldots,a'_n)$ are in the same Aut($\mathbb{A}$)-orbit, too, and for the same reason as above, each pair b_i,b'_i induce the same undirected graph in $H_{\mathcal{F}}$. Moreover, the pairs of coordinates where the projection of b_i is in N are exactly the pair of coordinates where the projection of b'_i is in N. Now let $c := f''(a_1,\ldots,a_n) = f'(b_1,\ldots,b_n)$ and $c' := f''(a'_1,\ldots,a'_n) = f'(b'_1,\ldots,b'_n)$. Since $f' \in \mathscr{C}^K_{D_{\mathcal{F}}}$, since Pol($\mathbb{A}$) preserves N, and since the operation induced by f' on $\{\leftarrow,\rightarrow\}$ is an affine map, we know that c and c' are in the same orbit under Aut($\mathbb{A}$). Thus, $f'' \in \mathscr{C}^I_{\mathbb{A}}$ and therefore it acts on $I_k/\mathrm{Aut}(\mathbb{A})$.

We define $\xi(f)$ as the action of f'' on $S/{\sim}$. As in [25, Theorem 13], for all $a_1,\ldots,a_n \in S$, and any f' that is locally interpolated by f, we have $f(a_1,\ldots,a_n)(\eta \circ \sim)f'(a_1,\ldots,a_n)$. It follows that

$$f(q^n_1(a_1,\ldots,a_n),\ldots,q^n_n(a_1,\ldots,a_n))(\eta \circ \sim)f''(a_1,\ldots,a_n)$$

holds for all $a_1,\ldots,a_n \in S$. In particular, the definition of $\xi(f)$ does not depend on the choice of f' in the construction. Moreover, if $a_1,\ldots,a_n$ induce the same undirected graph in $H_{\mathcal{F}}$, then $q^n_i(a_1,\ldots,a_n)$ and a_i are in the same orbit under Aut($D_{\mathcal{F}}$), for all $i \in \{1,\ldots,n\}$. It follows that $f''(a_1,\ldots,a_n)$ and $f'(a_1,\ldots,a_n)$ are in the same orbit with respect to Aut($D_{\mathcal{F}}$), as f' is canonical with respect to Aut($D_{\mathcal{F}}$). Since Aut($D_{\mathcal{F}}$) $\subseteq$ Aut($\mathbb{A}$), they are in the same orbit with respect to Aut($\mathbb{A}$), and therefore there are $\sim$-equivalent. Finally, this implies that $f(a_1,\ldots,a_n)(\eta \circ \sim)f''(a_1,\ldots,a_n)$ holds, whenever all $a_1,\ldots,a_n$ induce the same undirected graph in $H_{\mathcal{F}}$, which is the case for all $a_1,\ldots,a_n \in S$ by Lemma 20.

Now we show that ξ is a uniformly continuous clone homomorphism. It clearly preserves arities so we need to show it also preserves compositions. Let $f \in$ Pol($\mathbb{A}$) be n-ary, $g_1,\ldots,g_n \in$ Pol($\mathbb{A}$) be m-ary. Let $u_1,\ldots,u_m \in S$. Since $g_i(u_1,\ldots,u_m)(\eta \circ \sim)g''_i(u_1,\ldots,u_m)$ for all i, there exists $v_i \in S$ such that $g_i(u_1,\ldots,u_m)\ \eta\ v_i$. Then

$$f(g_1(u_1,\dots,u_m),\dots,g_n(u_1,\dots,u_m))\ \eta\ f(v_1,\dots,v_n)$$
$$(\eta\circ\sim) f''(v_1,\dots,v_n)$$
$$(\eta\circ\sim) f''(g_1(u_1,\dots,u_m),\dots,g_n(u_1,\dots,u_m)).$$

from which we conclude that

$$\xi(f(g_1,\dots,g_n))(u_1,\dots,u_m) = \xi(f)(\xi(g_1)(u_1,\dots,u_m),\dots,\xi(g_n)(u_1,\dots,u_m)).$$

To prove that ξ is uniformly continuous fix $n\in\mathbb{N}$ and let $a\not\sim b$ be any two k-ary tuples in S and define $F=\bigcup_{0\leq i\leq k}\{a_i,b_i\}^n$. Observe that if f,g agree on all $x\in\{a_i,b_i\}^n$ for each $i\leq k$, then they also agree on all $y\in\{a,b\}^n$ as $f(y)_i$ and $g(y)_i$ are fully determined by the values of f,g on a certain subset of $\{a_i,b_i\}^n$. Therefore they induce the same action on $S/{\sim}$ as in that case $f(x)\sim g(x)$ for all $x\in\{a,b\}^n$. Then it is clear that if f,g agree on F we also have $\xi(f)=\xi(g)$. This completes the proof. ◀

4.2.2 Second Case: Weakly Commutative Polymorphism

In the second case of Theorem 6, $\mathrm{Pol}(\mathbb{A})$ contains a well-behaved binary operation, which implies that $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow,\rightarrow\}$ contains a majority operation; this contradicts our assumption that $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow,\rightarrow\}$ has a naked set. The proof of Proposition 22 below is similar to that of Lemma 39 in [25] and is omitted due to space restrictions.

▶ **Proposition 22.** *Let $\mathbb{A}$ be a proper reduct of $D_\mathcal{F}$ with injective projections and such that U is invariant under $\mathrm{Pol}(\mathbb{A})$. Let $(S,\sim)$ be a minimal affine set of $\mathscr{C}^I_{\mathbb{A}}$ with $S\subseteq I_k$ and where $\sim$ has $\mathrm{Aut}(\mathbb{A})$-invariant equivalence classes. Suppose that there exists a binary $f\in\mathrm{Pol}(\mathbb{A})$ such that $f(a,b)\sim f(b,a)$ holds for all disjoint injective tuples $a,b\in V^k$ with $f(a,b),f(b,a)\in S$. Then $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow,\rightarrow\}$ contains a majority operation.*

4.2.3 Classifying Problems with Bounded Treewidth Dualities

Finally, we briefly describe the proof strategy for Theorem 3. It is very similar to the one outlined above and shares all the intermediate steps. Only the starting distinction changes, where we distinguish between whether $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow,\rightarrow\}$ has an affine set or not.

If $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow,\rightarrow\}$ does not have an affine set, then the second item in Proposition 15 states that $\mathrm{Pol}(\mathbb{A})$ contains a so-called *canonical pseudo-majority operation*, and $\mathbb{A}$ has bounded relational width by [9]. Moreover, since $D_\mathcal{F}$ is homogeneous in a binary language, [22, Theorem 2] entails that $\mathbb{A}$ has relational width at most $(4,\max(6,\ell))$, where ℓ is the maximal size of a tournament in $\mathcal{F}$. It follows from general principles [15] that there exists a duality for the class of structures admitting a homomorphism to $\mathbb{A}$ that has treewidth bounded by $\max(6,\ell,r)$, where r is the maximal arity of a relation of $\mathbb{A}$.

If $\mathscr{C}^I_{\mathbb{A}} \curvearrowright I_k/\mathrm{Aut}(\mathbb{A})$ has an affine set for some $k\geq 1$ (by Lemma 16), either we get an affine set for $\mathrm{Pol}(\mathbb{A})$ (by Proposition 21) or $\mathscr{C}^K_{D_\mathcal{F}} \curvearrowright \{\leftarrow,\rightarrow\}$ contains a majority operation (by Proposition 22), again a contradiction since a majority operation cannot be represented as an affine map over any module.

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