On Low Complexity Colorings of Grids

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Abstract

A d-dimensional configuration is a coloring of the infinite grid $\mathbb{Z}^d$ using a finite number of colors. For a finite subset $D \subseteq \mathbb{Z}^d$, the D-patterns of a configuration are the patterns of shape D that appear in the configuration. A configuration is said to be admitted by these patterns. The number of distinct D-patterns in a configuration is a natural measure of its complexity. We focus on low complexity configurations, where the number of distinct D-patterns is at most $|D|$, the size of the shape. This framework includes the notorious open Nivat's conjecture and the recently solved Periodic Tiling problem. We use algebraic tools to study the periodicity of low complexity configurations. In the two-dimensional case, if $D \subseteq \mathbb{Z}^2$ is a rectangle or any convex shape, we establish an algorithm to determine if a given collection of $|D|$ patterns admits any configuration. This is based on the fact that if the given patterns admit a configuration, then they admit a periodic configuration. We also demonstrate that a two-dimensional low complexity configuration must be periodic if it originates from the well-known Ledrappier subshift or from several other algebraically defined subshifts.

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1 Overview

In the *domino problem*, one is given a finite collection of allowed $m \times n$ arrays of colors, or patterns, and must determine whether it is possible to color the infinite grid $\mathbb{Z}^2$ such that only the allowed $m \times n$ patterns appear in the coloring. A classical result by Robert Berger states that the domino problem is undecidable: no algorithm can provide the correct answer to every instance of the problem. An underlying structural property is aperiodicity, where some choices of allowed patterns force the valid colorings to be non-periodic [1].

However, the situation is different in the low complexity setting where the number of allowed patterns is at most mn, the size of the array. In this case, there are no aperiodic systems, meaning that if a valid coloring of $\mathbb{Z}^2$ exists, then a periodic coloring also exists [5]. This fact leads to an algorithm to solve the domino problem in the low complexity setting. Similar results hold if, instead of $m \times n$ rectangles, one considers patterns of any convex shape $D \subseteq \mathbb{Z}^2$.

The low complexity setting also encompasses two interesting problems: Nivat's conjecture and the Periodic Tiling problem. These specific questions are of independent interest and have driven our research on this topic in recent years. Nivat's conjecture remains open, while the Periodic Tiling problem was recently solved positively by Siddhartha Bhattacharya [2] in the two-dimensional case and negatively in sufficiently high-dimensional grids by Rachel Greenfeld and Terrence Tao [3].

To address the low complexity setting, we initiated an algebraic approach in [6]. In this approach, we represent colors as integers and colorings of $\mathbb{Z}^d$ as formal Laurent power series with d variables. If the coloring has low complexity with respect to some finite shape $D \subseteq \mathbb{Z}^d$, a simple linear algebra argument shows that the corresponding power series has a non-trivial annihilator: a d-variate Laurent polynomial whose formal product with the power series vanishes. Moreover, periodicity can be simply formulated in terms of having a difference binomial annihilator. By studying the structure of the multivariate polynomial ideal of annihilators, we prove that a two-dimensional uniformly recurrent configuration with low complexity with respect to a convex shape must be periodic [5], implying the decidability of the domino problem in such a low complexity setting. We also show that low complexity elements of certain algebraic subshifts are all periodic [4].

References

1. R. Berger. *The Undecidability of the Domino Problem.* American Mathematical Society memoirs. American Mathematical Society, 1966.
2. S. Bhattacharya. Periodicity and decidability of tilings of $\mathbb{Z}^2$. *American Journal of Mathematics*, 142:255–266, 2016.
3. R. Greenfeld and T. Tao. *A counterexample to the periodic tiling conjecture.* arXiv preprint, 2022. `arXiv:2211.15847`.
4. J. Kari and E. Moutot. Nivat's conjecture and pattern complexity in algebraic subshifts. *Theoretical Computer Science*, 777:379–386, 2019.
5. J. Kari and E. Moutot. Decidability and periodicity of low complexity tilings. *Theory of Computing Systems*, 67(1):125–148, 2023.
6. J. Kari and M. Szabados. An algebraic geometric approach to Nivat's conjecture. In *Proceedings of ICALP 2015, part II*, volume 9135 of *Lecture Notes in Computer Science*, pages 273–285, 2015.