When Lawvere Meets Peirce: An Equational Presentation of Boolean Hyperdoctrines

Filippo Bonchi
University of Pisa, Italy

Alessandro Di Giorgio
University College London, UK

Davide Trotta
University of Padova, Italy

Abstract

Fo-bicategories are a categorification of Peirce's calculus of relations. Notably, their laws provide a proof system for first-order logic that is both purely equational and complete. This paper illustrates a correspondence between fo-bicategories and Lawvere's hyperdoctrines. To streamline our proof, we introduce peircean bicategories, which offer a more succinct characterization of fo-bicategories.

2012 ACM Subject Classification Theory of computation → Logic; Theory of computation → Categorical semantics

Keywords and phrases relational algebra, hyperdoctrines, cartesian bicategories, string diagrams

Digital Object Identifier 10.4230/LIPIcs.MFCS.2024.30

Related Version *Full Version*: `https://arxiv.org/abs/2404.18795` [5]

Funding *Filippo Bonchi*: Supported by MUR project PRIN 2022 PNRR No. P2022HXNSC – RAP (Resource Awareness in Programming) and University of Pisa project PRA_2022_99 – FM4HD.
Alessandro Di Giorgio: Supported by EPSRC project EP/V002376/1 – Nominal String Diagrams.

1 Introduction

The first appearances of the characteristic features of first-order logic can be traced back to the works of Peirce [54] and Frege [21]. Frege was mainly motivated by the pursuit of a rigorous foundation for mathematics: his work was inspired by real analysis, bringing the concept of functions and variables into the logical realm [18]. On the other hand Peirce, inspired by the work of De Morgan [16] on relational reasoning, introduced a calculus in which operations allow the combination of relations and adhere to a set of algebraic laws. Like Boole's algebra of classes [9], Peirce's calculus of relations does not feature variables nor quantifiers and its sole deduction rule is substituting equals by equals.

Despite several negative results [51, 28, 63, 22, 2, 60] regarding axiomatizations for the calculus, its lack of binder-related complexities, coupled with purely equational proofs, has rendered the calculus of relations highly influential in computer science, e.g., in the context of database theory [13], programming languages [61, 27, 38, 1, 37] and proof assistants [58, 59, 36]. In logic, the calculus played a secondary role for many years, likely because it is strictly less expressive than first-order logic [43]. This was until Tarski in [67] recognized its algebraic flavour and initiated a program of algebraizing first-order logic, including works such as [17, 26, 62]. Quoting Quine [62]:

> "Logic in his adolescent phase was algebraic. There was Boole's algebra of classes and Peirce's algebra of relations. But in 1879 logic come of age, with Frege's quantification theory. Here the bound variables, so characteristic of analysis rather than of algebra, became central to logic."

Such a perspective, which regarded algebraic aspects and those concerning quantifiers as separate entities, changed with the work of Lawvere.

Thanks to the recent development of a new branch of mathematics, namely category theory, Lawvere introduced in [40, 41, 42] *hyperdoctrines* which enabled the study of logic from a pure algebraic perspective. The crucial insights of Lawvere was to show that quantifiers, as well as many logical constructs, can be algebraically captured through the crucial notion of adjointness. Hyperdoctrines, along with many categorical structures related to logics, such as regular, Heyting, and boolean categories [32, 33], align with Frege's functional perspective: arrows represent functions (terms), and relations are derived through specific constructions.

In the last decade, the paradigm shift towards treating data as a physical resource has motivated many computer scientists to move from traditional term-based (cartesian) syntax toward a string diagrammatic (monoidal) syntax [34, 65] (see e.g., [66, 4, 6, 8, 14, 19, 20, 24, 52, 56]). This shift in syntax enables an extension of Peirce's calculus of relations that is as expressive as first-order logic, accompanied by an axiomatization that is purely equational and complete. The axioms are those of *first-order bicategories* [3]: see Figures 1, 3 and 4. In essence, a first-order bicategory, or fo-bicategory, encompasses a cartesian and a cocartesian bicategory [11], interacting as a linear bicategory [12], while additionally satisfying linear versions of Frobenius equations and adjointness conditions.

In this paper, we reconcile Lawvere's understanding of logic with Peirce's calculus of relations by illustrating a formal correspondence between boolean hyperdoctrines and first-order bicategories.

To reach such a correspondence, we found convenient to introduce *peircean bicategories*: these are cartesian bicategories with each homset carrying a boolean algebra where the negation behaves appropriately with *maps* – special arrows that intuitively generalize functions. Our first result (Theorem 27) states that peircean and fo-bicategories are equivalent.

While the definition of peircean bicategories is not purely equational, as in the case of fo-bicategories, it is notably more concise. Moreover, it allows us to reuse from [7] an adjunction between cartesian bicategories and *elementary and existential doctrines* [46, 45, 47], which are a generalisation of hyperdoctrines, corresponding to the $(\exists, =, \top, \wedge)$-fragment of first-order logic. Our main result (Theorem 32) reveals an adjunction between the category of first-order bicategories and the category of boolean hyperdoctrines.

It is essential to note that our theorem establishes an adjunction rather than an equivalence. The discrepancy can be intuitively explained by noting that, akin to first-order logic, terms and formulas are distinct entities in hyperdoctrines. Thus for two terms t_1 and t_2, the hyperdoctrine where the formula $t_1 = t_2$ is true differs from the hyperdoctrine where t_1 and t_2 are equated as terms, a distinction not present in fo-bicategories. These issues, related to the extensionality of equality, are thoroughly analyzed in the literature (see e.g. [45, 31]).

Leveraging another result from [7], we demonstrate (Theorem 37) that the adjunction in Theorem 32 becomes an equivalence when restricted to well-behaved hyperdoctrines (i.e., those whose equality is extensional and satisfying the rule of unique choice [44]).

Synopsis. In § 2, we provide a review of (co)cartesian, linear and fo-bicategories. § 3 covers a recap of elementary and existential doctrines and boolean hyperdoctrines. The key adjunction from [7] is recalled in §4. Our original contributions commence in § 5, where we introduce peircean bicategories and establish their equivalence with fo-bicategories. This result is used in § 6 to show the adjunction and in § 7 to establish the equivalence. Missing proofs can be found in [5].

Terminology and Notation. All bicategories considered in this paper are just poset-enriched symmetric monoidal categories. For a bicategory $\mathbf{C}$, we will write $\mathbf{C}^{\mathrm{op}}$ for the bicategory having the same objects as $\mathbf{C}$ but homsets $\mathbf{C}^{\mathrm{op}}[X,Y] \stackrel{\text{def}}{=} \mathbf{C}[Y,X]$. Similarly, we will write $\mathbf{C}^{\mathrm{co}}$ to denote the bicategory having the same objects and arrows of $\mathbf{C}$ but equipped with the reversed ordering $\geq$. The cartesian bicategories in this paper are called in [11] cartesian bicategories of relations. We refer the reader to [3, Rem. 2] for a comparison with the presentation of linear bicategories in [12]. In a category with finite products, we write $\langle f, g\rangle$ for the pairing of f and g and Δ_X for $\langle id^\circ_X, id^\circ_X\rangle$.

2 From (Co)Cartesian to First-Order Bicategories

In this section we recall the notion of *first-order bicategory* from [3]. To provide a preliminary intuition, it is convenient to consider **Rel**, the first-order bicategory of sets and relations.

It is well known that sets and relations form a symmetric monoidal category, hereafter denoted as $\mathbf{Rel}^\circ$, with composition, identities, monoidal product and symmetries defined as

$$\begin{array}{ll} a \mathbin{⨾} b \stackrel{\text{def}}{=} \{(x,z) \mid \exists y{\in}Y \,.\, (x,y)\in a \wedge (y,z)\in b\} \subseteq X\times Z & id^\circ_X \stackrel{\text{def}}{=} \{(x,y) \mid x=y\} \subseteq X\times X \\ a\otimes c \stackrel{\text{def}}{=} \{(\,(x,z),(y,v)\,) \mid (x,y)\in a \wedge (z,v)\in c\} \subseteq (X\times Z)\times(Y\times V) & \\ \sigma^\circ_{X,Y} \stackrel{\text{def}}{=} \{(\,(x,y),(y',x')\,) \mid x=x' \wedge y=y'\} \subseteq (X\times Y)\times(Y\times X) & \end{array} \tag{1}$$

for all sets X,Y,Z,V and relations $a \subseteq X\times Y$, $b\subseteq Y\times Z$ and $c\subseteq Z\times V$. As originally observed by Peirce in [55], beyond $⨾$ there exists another form of relational composition that enjoys noteworthy algebraic properties. This different composition gives rise to another symmetric monoidal category of sets and relations, hereafter denoted by $\mathbf{Rel}^\bullet$ and defined as follows.

$$\begin{array}{ll} a \mathbin{\bullet\!\!;} b \stackrel{\text{def}}{=} \{(x,z) \mid \forall y{\in}Y \,.\, (x,y)\in a \vee (y,z)\in b\} \subseteq X\times Z & id^\bullet_X \stackrel{\text{def}}{=} \{(x,y) \mid x\neq y\} \subseteq X\times X \\ a\mathbin{\otimes\!\!\!\!\bullet} c \stackrel{\text{def}}{=} \{(\,(x,z),(y,v)\,) \mid (x,y)\in a \vee (z,v)\in c\} \subseteq (X\times Z)\times(Y\times V) & \\ \sigma^\bullet_{X,Y} \stackrel{\text{def}}{=} \{(\,(x,y),(y',x')\,) \mid x\neq x' \vee y\neq y'\} \subseteq (X\times Y)\times(Y\times X) & \end{array} \tag{2}$$

Note that $\otimes$ and $\otimes\!\!\!\!\bullet$ are both defined on objects as the cartesian product of sets and have as unit the singleton set $I \stackrel{\text{def}}{=} \{\star\}$. Both $\mathbf{Rel}^\circ$ and $\mathbf{Rel}^\bullet$ are poset-enriched symmetric monoidal categories when taking as ordering the inclusion $\subseteq$ and the complement $\neg\colon (\mathbf{Rel}^\circ)^{\mathrm{co}} \to \mathbf{Rel}^\bullet$ is an isomorphism. As we will explain in § 2.1, the relations defined for all sets X as

$$\begin{array}{ll} \blacktriangleleft^\circ_X \stackrel{\text{def}}{=} \{(x,\ (y,z)) \mid x=y \wedge x=z\} \subseteq X\times(X\times X) & \blacktriangleleft^\bullet_X \stackrel{\text{def}}{=} \{(x,\ (y,z)) \mid x\neq y \vee x\neq z\} \subseteq X\times(X\times X) \\ \blacktriangleright^\circ_X \stackrel{\text{def}}{=} \{((y,z),\ x) \mid x=y \wedge x=z\} \subseteq (X\times X)\times X & \blacktriangleright^\bullet_X \stackrel{\text{def}}{=} \{((y,z),\ x) \mid x\neq y \vee x\neq z\} \subseteq (X\times X)\times X \\ !^\circ_X \stackrel{\text{def}}{=} \{(x,\star) \mid x\in X\} \subseteq X\times I & !^\bullet_X \stackrel{\text{def}}{=} \varnothing \subseteq X\times I \\ ¡^\circ_X \stackrel{\text{def}}{=} \{(\star,x) \mid x\in X\} \subseteq I\times X & ¡^\bullet_X \stackrel{\text{def}}{=} \varnothing \subseteq I\times X \end{array} \tag{3}$$

make $\mathbf{Rel}^\circ$ a cartesian bicategory, while $\mathbf{Rel}^\bullet$ a cocartesian one.

Intuitively, a first-order bicategory $\mathbf{C}$ consists of a cartesian bicategory $\mathbf{C}^\circ$, called the "white structure", and a cocartesian bicategory $\mathbf{C}^\bullet$, called the "black structure", that interact by obeying the same laws of $\mathbf{Rel}^\circ$ and $\mathbf{Rel}^\bullet$. The name "first-order" is due to the fact that such laws provide a complete system of axioms for first-order logic.

The axioms can be conveniently given by means of a graphical representation inspired by string diagrams [34, 65]: composition is depicted as horizontal composition while the monoidal product by vertically "stacking" diagrams. However, since there are two compositions $⨾$ and

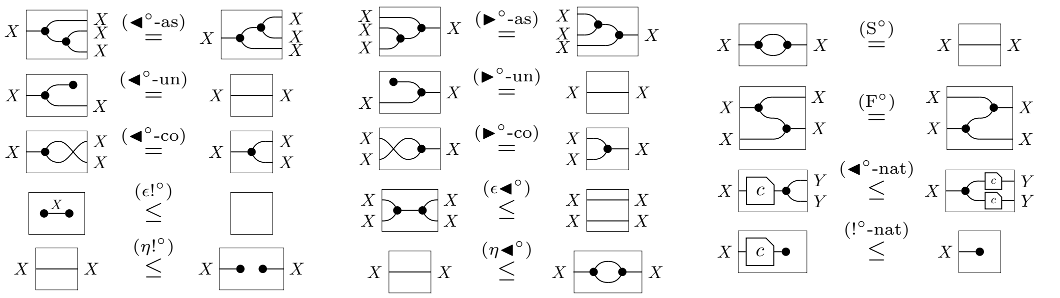
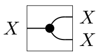
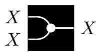


Figure 1 Axioms of cartesian bicategories.

$\fatsemi$ and two monoidal products $\otimes$ and $\circledast$, to distinguish them we use different colors. All white constants have white background, mutatis mutandis for the black ones: for instance $◀^{\circ}_{X}$ and $▶^{\bullet}_{X}$ are drawn  and , while for some arrows a, b, c, d of the appropriate type, $(a \otimes c) \fatsemi (b \circledast d)$ is drawn as on the right of (ν_l°) in Figure 3.

2.1 (Co)Cartesian Bicategories

We commence with the notion of cartesian bicategories by Carboni and Walters [11].

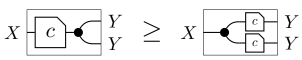
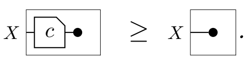
▶ **Definition 1.** *A* cartesian bicategory $(\mathbf{C}, \otimes, I, ◀^{\circ}, !^{\circ}, ▶^{\circ}, ¡^{\circ})$, *shorthand* $(\mathbf{C}, ◀^{\circ}, ▶^{\circ})$, *is a poset-enriched symmetric monoidal category* $(\mathbf{C}, \otimes, I)$ *and, for every object* X *in* $\mathbf{C}$, *arrows* $◀^{\circ}_{X} : X \to X \otimes X$, $!^{\circ}_{X} : X \to I$, $▶^{\circ}_{X} : X \otimes X \to X$, $¡^{\circ}_{X} : I \to X$ *such that*

1. $(◀^{\circ}_{X}, !^{\circ}_{X})$ *is a comonoid and* $(▶^{\circ}_{X}, ¡^{\circ}_{X})$ *a monoid, i.e., the equalities* (◀°-as), (◀°-un), (◀°-co) *and* (▶°-as), (▶°-un), (▶°-co) *in Figure 1 hold;*
2. *every arrow* $c : X \to Y$ *is a lax comonoid homomorphism, i.e.,* (◀°-nat) *and* (!°-nat) *hold;*
3. *comonoids are left adjoints to the monoids, i.e.,* $(\eta\, ◀^{\circ})$, $(\epsilon\, ◀^{\circ})$, $(\eta !^{\circ})$ *and* $(\epsilon !^{\circ})$ *hold;*
4. *monoids and comonoids form special Frobenius bimonoids, i.e.,* (F°) *and* (S°) *hold;*
5. *monoids and comonoids satisfy the expected coherence conditions (see e.g. [7]).*

$\mathbf{C}$ *is a* cocartesian bicategory *if* $\mathbf{C}^{\mathrm{co}}$ *is a cartesian bicategory. A* morphism of (co)cartesian bicategories *is a poset-enriched strong symmetric monoidal functor preserving monoids and comonoids. We denote by* $\mathbb{CB}$ *the category of cartesian bicategories and their morphisms.*

As already mentioned, $\mathbf{Rel}^{\circ}$ with $◀^{\circ}_{X}$, $!^{\circ}_{X}$, $▶^{\circ}_{X}$ and $¡^{\circ}_{X}$ defined in (3) form a cartesian bicategory: the reader can easily check, using the definitions in (1) and (3), that all the laws in Figure 1 are satisfied. Similarly, one can observe that the opposite inequality of (◀°-nat) holds iff the relation $c \subseteq X \times Y$ is single-valued (i.e., deterministic), while the opposite of (!°-nat) iff c is total. In other words, c is a function iff both (◀°-nat) and (!°-nat) hold as equalities.

▶ **Definition 2.** *Let* $c : X \to Y$ *be an arrow of a cartesian bicategory* $\mathbf{C}$. *It is a* map *if*

 *and*  . (4)

Maps form a monoidal subcategory of $\mathbf{C}$, denoted by $\mathsf{Map}(\mathbf{C})$, that has finite products [11].

In a cartesian bicategory $\mathbf{C}$, each homset $\mathbf{C}[X, Y]$ carries the structure of inf-semilattice, defined for all $c, d : X \to Y$ as in (5) below. Furthermore, the equation (6) defines an identity-on-objects isomorphism of cartesian bicategories $(\cdot)^{\dagger} : \mathbf{C} \to \mathbf{C}^{\mathrm{op}}$.

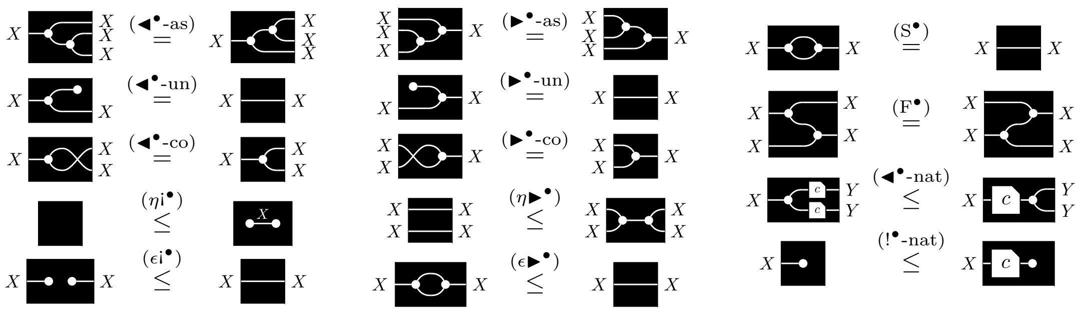
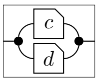
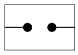
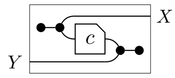


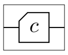
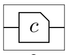
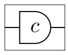
Figure 2 Axioms of cocartesian bicategories.

$c \wedge d \stackrel{\text{def}}{=}$ X  Y $\quad \top \stackrel{\text{def}}{=}$ X  Y (5) $\quad c^\dagger \stackrel{\text{def}}{=}$  (6)

The reader can check, using (1) and (3) that in $\mathbf{Rel}^\circ$, $c^\dagger \colon Y \to X$ is the opposite of the relation c, namely $\{(y,x) \mid (x,y) \in c\}$. It is well known that a relation c is a function iff it is left adjoint to $c^\dagger$. More generally in a cartesian bicategory c is a map iff it is left adjoint to $c^\dagger$. Summarising:

▶ **Proposition 3.** *Let* $\mathbf{C}$ *be a cartesian bicategory and* $c \colon X \to Y$ *an arrow of* $\mathbf{C}$. *The following hold:*

1. *every homset carries the inf-semilattice structure, defined as in* (5)*;*
2. *there is an isomorphism of cartesian bicategories* $(\cdot)^\dagger \colon \mathbf{C} \to \mathbf{C}^{\text{op}}$*, defined as in* (6)*;*
3. c *is a map iff* c *is left adjoint to* $c^\dagger$*;*
4. $\mathsf{Map}(\mathbf{C})$ *is a category with finite products; moreover, a morphism of cartesian bicategories* $F \colon \mathbf{C} \to \mathbf{D}$ *restricts to a functor* $\tilde{F} \colon \mathsf{Map}(\mathbf{C}) \to \mathsf{Map}(\mathbf{D})$ *preserving finite products.*

Hereafter, we draw Y  X for (X  Y $)^\dagger$ and X  Y for a map $c \colon X \to Y$.

We mentioned that $\mathbf{Rel}^\bullet$ with $◀^\bullet_X$, $!^\bullet_X$, $▶^\bullet_X$ and $¡^\bullet_X$ defined in (3) forms a cocartesian bicategory. To prove this, it is enough to observe that the complement $\neg$ is a poset-enriched symmetric monoidal isomorphism $\neg \colon (\mathbf{Rel}^\circ)^{\text{co}} \to \mathbf{Rel}^\bullet$ preserving (co)monoids.

2.2 Linear Bicategories

We have seen that $\mathbf{Rel}^\circ$ forms a cartesian bicategory, and $\mathbf{Rel}^\bullet$ a cocartesian bicategory. The next step consists of merging them into one entity and studying their algebraic interactions. However, the coexistence of two different compositions $\mathbin{⨟^\circ}$ and $\mathbin{⨟^\bullet}$ on the same class of objects and arrows brings us out of the realm of ordinary categories. The appropriate setting is provided by *linear bicategories* [12] by Cockett, Koslowski and Seely.

▶ **Definition 4.** *A* linear bicategory $(\mathbf{C}, ⨟^\circ, id^\circ, ⨟^\bullet, id^\bullet)$ *consists of two poset-enriched categories* $(\mathbf{C}, ⨟^\circ, id^\circ)$ *and* $(\mathbf{C}, ⨟^\bullet, id^\bullet)$ *with the same class of objects, arrows and orderings (but possibly different identities and compositions) such that* $⨟^\circ$ *linearly distributes over* $⨟^\bullet$*, i.e.,* (δ_l) *and* (δ_r) *in Figure 3 hold.*

A symmetric monoidal linear bicategory $(\mathbf{C}, ⨟^\circ, id^\circ, ⨟^\bullet, id^\bullet, \otimes, \sigma^\circ, \varotimes, \sigma^\bullet, I)$, *shortly* $(\mathbf{C}, \otimes, \varotimes, I)$, *consists of a linear bicategory* $(\mathbf{C}, ⨟^\circ, id^\bullet, ⨟^\bullet, id^\bullet)$ *and two poset-enriched symmetric monoidal categories* $(\mathbf{C}, \otimes, I)$ *and* $(\mathbf{C}, \varotimes, I)$ *s.t.* $\otimes$ *and* $\varotimes$ *agree on objects, i.e.,* $X \otimes Y = X \varotimes Y$*, share the same unit* I *and*

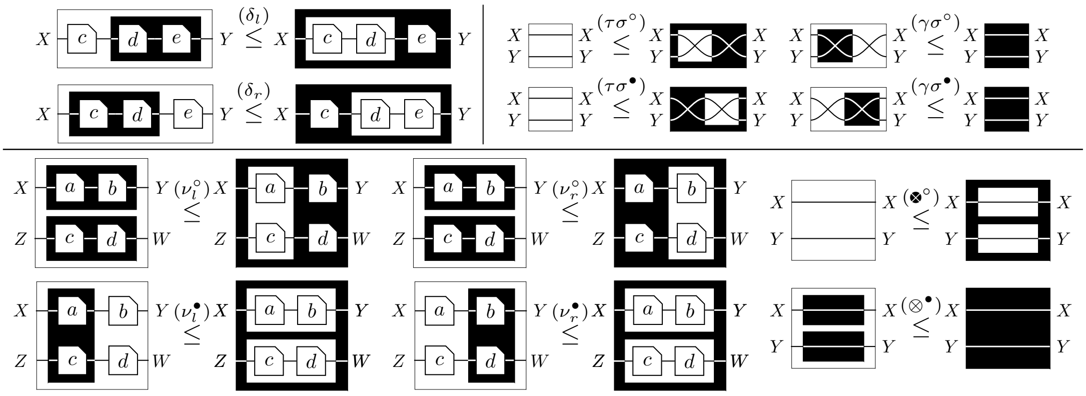


Figure 3 Axioms of closed symmetric monoidal linear bicategories.

2. *there are linear strengths for* $(\otimes, \otimes\!\!\!\bullet)$, *i.e., the inequalities* (ν_l°), (ν_r°), $(\nu_l^\bullet)$ *and* $(\nu_r^\bullet)$ *hold;*
3. $\otimes\!\!\!\bullet$ *preserves* id° *colaxly and* $\otimes$ *preserves* $id^\bullet$ *laxly, i.e.,* $(\otimes^\bullet)$ *and* $(\otimes\!\!\!\bullet^\circ)$ *hold.*

A morphism of symmetric monoidal linear bicategories $F\colon (\mathbf{C_1}, \otimes, \otimes\!\!\!\bullet, I) \to (\mathbf{C_2}, \otimes, \otimes\!\!\!\bullet, I)$ *consists of two poset-enriched symmetric monoidal functors* $F^\circ\colon (\mathbf{C_1}, \otimes, I) \to (\mathbf{C_2}, \otimes, I)$ *and* $F^\bullet\colon (\mathbf{C_1}, \otimes\!\!\!\bullet, I) \to (\mathbf{C_2}, \otimes\!\!\!\bullet, I)$ *that agree on objects and arrows:* $F^\circ(X) = F^\bullet(X)$ *and* $F^\circ(c) = F^\bullet(c)$.

We omit the adjective *symmetric monoidal*, since all linear bicategories in this paper are such. In linear bicategories one can define *linear* adjoints: for $a\colon X \to Y$ and $b\colon Y \to X$, a is *left linear adjoint* to b, or b is *right linear adjoint* to a, written $b \Vdash a$, if $id_X^\circ \leq a \,;\!\bullet\, b$ and $b \,;\!\circ\, a \leq id_Y^\bullet$.

▶ **Definition 5.** *A linear bicategory* $(\mathbf{C}, \otimes, \otimes\!\!\!\bullet, I)$ *is said to be* closed *if every* $a\colon X \to Y$ *has both a left and a right linear adjoint and, in particular, the white symmetry* σ° *is both left and right linear adjoint to the black symmetry* $\sigma^\bullet$ $(\sigma^\bullet \Vdash \sigma^\circ \Vdash \sigma^\bullet)$, *i.e.* $(\tau\sigma^\circ)$, $(\gamma\sigma^\circ)$, $(\tau\sigma^\bullet)$ *and* $(\gamma\sigma^\bullet)$ *in Figure 3 hold.*

Our main example is the closed linear bicategory **Rel** of sets and relations. The white structure is the symmetric monoidal category $\mathbf{Rel}^\circ$ and the black structure is $\mathbf{Rel}^\bullet$. Observe that the two have the same objects, arrows and ordering. The white and black monoidal products $\otimes$ and $\otimes\!\!\!\bullet$ agree on objects (they are the cartesian product of sets) and have common unit object (the singleton set I). By (1) and (2), one can easily check all the inequalities in Figure 3. Both left and right linear adjoints of a relation $c \subseteq X \times Y$ are given by $\neg c^\dagger$.

2.3 First-Order Bicategories

After (co)cartesian and linear bicategories, we can recall first-order bicategories from [3].

▶ **Definition 6.** *A* first-order bicategory **C** *consists of a closed linear bicategory* $(\mathbf{C}, \otimes, \otimes\!\!\!\bullet, I)$, *a cartesian bicategory* $(\mathbf{C}, \blacktriangleleft^\circ, \blacktriangleright^\circ)$ *and a cocartesian bicategory* $(\mathbf{C}, \blacktriangleleft^\bullet, \blacktriangleright^\bullet)$, *such that*

1. *the white comonoid* $(\blacktriangleleft^\circ, !^\circ)$ *is left and right linear adjoint to black monoid* $(\blacktriangleright^\bullet, ¡^\bullet)$ *and* $(\blacktriangleright^\circ, ¡^\circ)$ *is left and right linear adjoint to* $(\blacktriangleleft^\bullet, !^\bullet)$ *i.e., the 16 inequalities in the top of Figure 4 hold;*
2. *white and black (co)monoids satisfy the linear Frobenius laws, i.e.* $(\mathrm{F}^\bullet_\circ)$, $(\mathrm{F}^\circ_\bullet)$, $(\mathrm{F}_\bullet^\circ)$, $(\mathrm{F}_\circ^\bullet)$ *hold.*

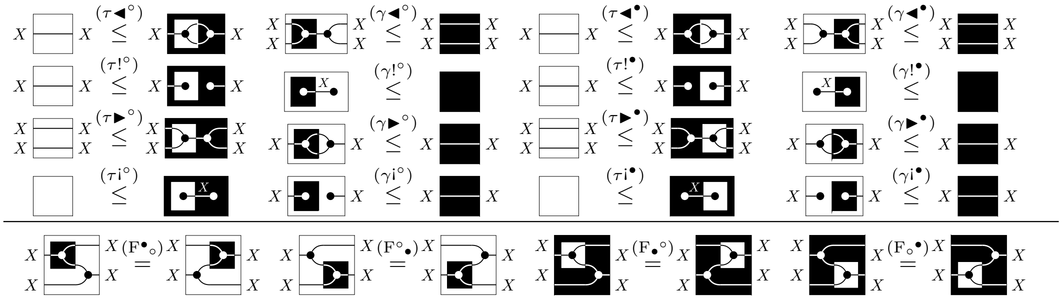


Figure 4 Additional axioms for fo-bicategories.

A morphism of fo-bicategories *is a morphism of linear bicategories* and *of (co)cartesian bicategories. We denote by* $\mathbb{FOB}$ *the category of fo-bicategories and their morphisms.*

We have seen that $\mathbf{Rel}$ is a closed linear bicategory, $\mathbf{Rel}^\circ$ a cartesian bicategory and $\mathbf{Rel}^\bullet$ a cocartesian bicategory. Given (3), it is easy to check the inequalities in Figure 4.

If $\mathbf{C}$ is a fo-bicategory, then $\mathbf{C}^{\text{co}}$ is a fo-bicategory when swapping white and black structures. Similarly, $\mathbf{C}^{\text{op}}$ is a fo-bicategory when swapping monoids and comonoids.

In a fo-bicategory $\mathbf{C}$, left and right linear adjoints of an arrow c coincide and are denoted by $c^\perp$. The assignment $c \mapsto c^\perp$ gives rise to an identity-on-objects isomorphism of fo-bicategories $(\cdot)^\perp \colon \mathbf{C} \to (\mathbf{C}^{\text{co}})^{\text{op}}$. Similarly, $(\cdot)^\dagger \colon \mathbf{C} \to \mathbf{C}^{\text{op}}$ in (6) is also an isomorphism of fo-bicategories.

Since the following diagram commutes, one can define the complement as the diagonal of the square, namely $\neg(\cdot) \stackrel{\text{def}}{=} ((\cdot)^\perp)^\dagger$.

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{(\cdot)^\dagger} & \mathbf{C}^{\text{op}} \\ {\scriptstyle (\cdot)^\perp}\downarrow & & \downarrow{\scriptstyle (\cdot)^\perp} \\ (\mathbf{C}^{\text{co}})^{\text{op}} & \xrightarrow{(\cdot)^\dagger} & \mathbf{C}^{\text{co}} \end{array}$$

Clearly $\neg \colon \mathbf{C} \to \mathbf{C}^{\text{co}}$ is an isomorphism of fo-bicategories. Moreover, it induces a boolean algebra on each homset of $\mathbf{C}$.

▶ **Proposition 7.** *Let* $\mathbf{C}$ *be a fo-bicategory. Then, every homset of* $\mathbf{C}$ *is a boolean algebra.*

▶ **Proposition 8.** *Let* $F \colon \mathbf{C} \to \mathbf{D}$ *be a morphism of fo-bicategories. For all arrows* c*,* $\neg F(c) = F(\neg c)$*.*

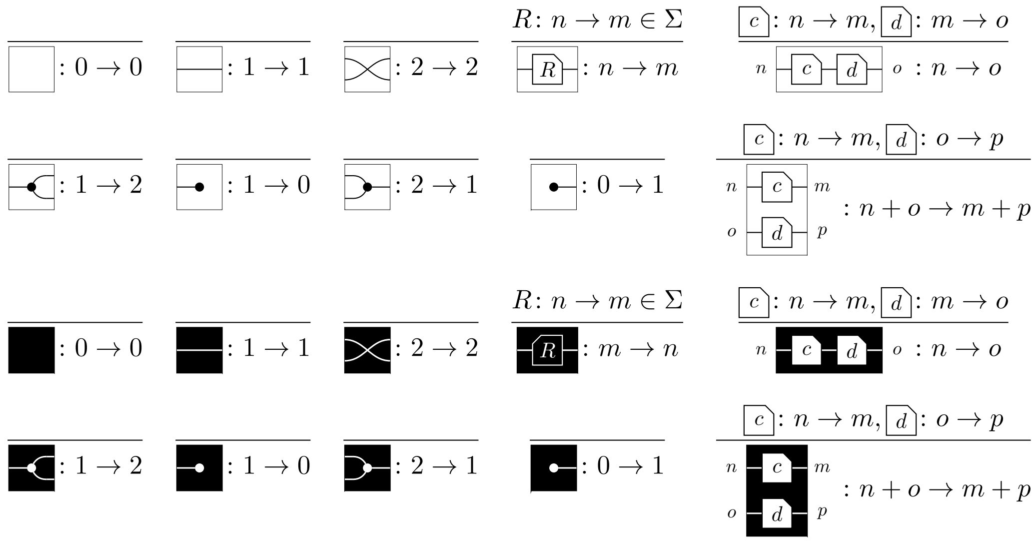
The next property of maps (Definition 2) plays a key role in our work.

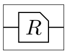
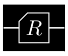
▶ **Proposition 9.** *For all maps* $f \colon X \to Y$ *and arrows* $c \colon Y \to Z$*, it holds that* $f \fatsemi \neg c = \neg(f \fatsemi c)$*.*

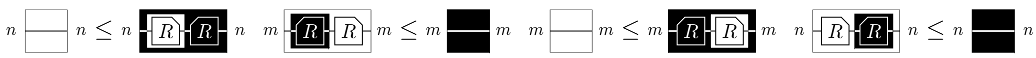
2.4 Freely Generated First-Order Bicategories

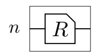
We conclude this section by giving to the reader a taste of how fo-bicategories relate to first-order theories. First, we recall from [3] the freely generated fo-bicategory $\mathbf{FOB}_\Sigma$.

Given a monoidal signature Σ, namely a set of symbols $R \colon n \to m$ with arity n and coarity m, $\mathbf{FOB}_\Sigma$ is the fo-bicategory whose objects are natural numbers and arrows $c \colon n \to m$ are string diagrams generated by the following rules:



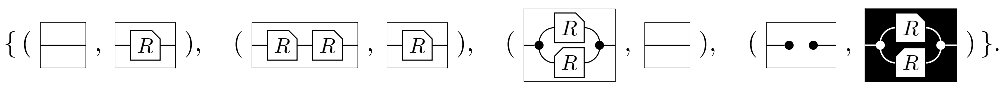
More precisely, arrows are equivalence classes of string diagrams w.r.t $\lesssim \cap \gtrsim$, where $\lesssim$ is the precongruence (w.r.t. ⨟, $\otimes$, ⨟ and ⊗) generated by the axioms in Figures 1,2,3,4 (with X, Y, Z, W replaced by natural numbers, and a, b, c, d by diagrams of the appropriate type) and the axioms forcing  and  to be linear adjoints:



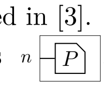
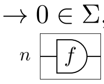
To give semantics to these diagrams we need *interpretations*, i.e. pairs $\mathcal{I} = (X, \rho)$, where X is a set and ρ is a function assigning to each $R\colon n \to m \in \Sigma$ a relation $\rho(R)\colon X^n \to X^m$. Since $\mathbf{FOB}_\Sigma$ is the free fo-bicategory, for any interpretation $\mathcal{I}$ there exists a unique morphism of fo-bicategories $\mathcal{I}^\sharp\colon \mathbf{FOB}_\Sigma \to \mathbf{Rel}$ such that $\mathcal{I}^\sharp(1) = X$ and $\mathcal{I}^\sharp($  $) = \rho(R) \subseteq X^n \times X^m$. Intuitively, $\mathcal{I}^\sharp$ is defined inductively by (1), (2) and (3) with the free cases provided by $\mathcal{I}$.

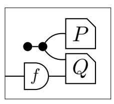
A *diagrammatic first-order theory* is a pair $\mathbb{T} = (\Sigma, \mathbb{I})$ where Σ is a monoidal signature and $\mathbb{I}$ is a set of *axioms*: pairs (c, d) for $c, d\colon n \to m$ in $\mathbf{FOB}_\Sigma$, standing for $c \leq d$. An interpretation $\mathcal{I}$ is a *model* of $\mathbb{T}$ if and only if, for all $(c, d) \in \mathbb{I}$, $\mathcal{I}^\sharp(c) \subseteq \mathcal{I}^\sharp(d)$. As illustrated in [3], one can generate the fo-bicategory $\mathbf{FOB}_\mathbb{T}$ and, in the spirit of Lawvere's functorial semantics [39], models of $\mathbb{T}$ are in one-to-one correspondence with morphisms $F\colon \mathbf{FOB}_\mathbb{T} \to \mathbf{Rel}$.

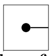
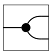
▶ **Example 10.** Consider the theory $\mathbb{T} = (\Sigma, \mathbb{I})$, where $\Sigma = \{R\colon 1 \to 1\}$ and $\mathbb{I}$ be as follows:



An interpretation is a set X and a relation $R \subseteq X \times X$. It is a model iff R is an order, i.e., reflexive ($id^\circ_X \subseteq R$), transitive ($R \,⨟\, R \subseteq R$), antisymmetric ($R \cap R^\dagger \subseteq id^\circ$) and total ($\top \subseteq R \cup R^\dagger$).

▶ Remark 11. A direct encoding of traditional first-order theories into diagrammatic ones is illustrated in [3]. Shortly, a predicate symbol P of arity n becomes a symbol $P\colon n \to 0 \in \Sigma$, drawn as , and a n-ary function symbol f becomes $f\colon n \to 1 \in \Sigma$, drawn as . For instance, the formula $\exists x.P(x) \wedge Q(x, f(y))$ is rendered as follows



where  plays the role of $\exists$ and  that of $\wedge$. Note that both predicate and function symbols of traditional first-order theories are regarded as symbols of the monoidal signature Σ. Function symbols are constrained to represent functions by adding to $\mathbb{I}$ the axioms of maps, i.e., the inequalities in (4).

3 From Elementary-Existential Doctrines to Boolean Hyperdoctrines

The notion of hyperdoctrine was introduced by Lawvere in a series of seminal papers [40, 42]. Over the years, various generalizations and specializations of this concept have been formulated and applied across multiple domains in the fields of logic and computer science. In this work, we employ a generalization of the notion of hyperdoctrine introduced by Maietti and Rosolini in [46, 45, 47], namely that of an *elementary and existential doctrine.*

▶ **Definition 12.** *An* elementary and existential doctrine *is a functor* $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *from the opposite of a category* $\mathbf{C}$ *with finite products to the category of inf-semilattices such that:*

- *for every* Y *in* $\mathbf{C}$ *there exists an element* δ_Y *in* $P(Y \times Y)$*, called* equality predicate*, such that for a morphism* $id^{\circ}_X \times \Delta_Y \colon X \times Y \to X \times Y \times Y$ *in* $\mathbf{C}$ *and every element* α *in* $P(X \times Y)$*, the assignment*

$$\exists_{id^{\circ}_X \times \Delta_Y}(\alpha) \overset{def}{=} P_{\langle \pi_1, \pi_2 \rangle}(\alpha) \wedge P_{\langle \pi_2, \pi_3 \rangle}(\delta_Y)$$

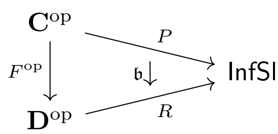
  determines a left adjoint to the functor $P_{id^{\circ}_X \times \Delta_Y} \colon P(X \times Y \times Y) \to P(X \times Y)$*;*
- *for any projection* $\pi_X \colon X \times Y \to X$*, the functor* $P_{\pi_X} \colon P(X) \to P(X \times Y)$ *has a left adjoint* $\exists_{\pi_X}$*, and these satisfy the* Beck-Chevalley condition *and* Frobenius reciprocity*, see [46, Sec. 2].*

▶ Remark 13. In an elementary and existential doctrine, for every $f\colon X \to Y$ of $\mathbf{C}$ the functor P_f has a left adjoint $\exists_f$ that can be computed as $\exists_{\pi_Y}(P_{f \times id^{\circ}_{X\,Y}}(\delta_Y) \wedge P_{\pi_X}(\alpha))$ for α in $P(X)$, where π_X and π_Y are the projections from $X \times Y$. These left ajoints satisfy the Frobenius reciprocity but not necessarily the Beck-Chevalley condition. See [48, Rem. 6.4].

▶ **Definition 14.** *Let* $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *and* $R\colon \mathbf{D}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *be two elementary and existential doctrines. A* morphism of elementary and existential doctrines *is given by a pair* $(F, \mathfrak{b})$ *where*

- $F\colon \mathbf{C} \to \mathbf{D}$ *is a finite product preserving functor;*
- $\mathfrak{b}\colon P \to F^{\mathrm{op}} \mathbin{;} R$ *is a natural transformation;*

satisfying the following conditions:



1. *for every object* X *of* $\mathbf{C}$, $\mathfrak{b}_{X \times X}(\delta_X) = \delta_{FX \times FX}$*;*
2. *for every* $\pi_X \colon X \times Y \to X$ *of* $\mathbf{C}$ *and for every* α *in* $P(X \times Y)$, $\exists_{F(\pi_X)} \mathfrak{b}_{X \times Y}(\alpha) = \mathfrak{b}_X(\exists_{\pi_X}(\alpha))$.

We write $\mathbb{EED}$ for the category of elementary and existential doctrines and morphisms.

▶ **Example 15.** The powerset functor $\mathcal{P}\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ is the archetypal example of an elementary and existential doctrine. More generally, for any regular category $\mathbf{C}$, the subobjects functor $\mathsf{Sub}_{\mathbf{C}}\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ is an elementary and existential doctrine, see [45, 46]. This assignment extends to an inclusion of the category $\mathbb{REG}$ of regular categories into $\mathbb{EED}$.

▶ **Example 16.** For a cartesian bicategory $\mathbf{C}$, the functor $\mathbf{C}[-, I]\colon \mathsf{Map}(\mathbf{C})^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ is an elementary and existential doctrine, where the actions of left adjoints is given $\exists_g(f) := f \mathbin{;} g^{\dagger}$ [7, Thm. 20]. As we will see in §4, this assignment extends to an inclusion of $\mathbb{CB}$ into $\mathbb{EED}$.

Similarly to cartesian bicategories, elementary and existential doctrines have enough structure to deal with the notion of *functional* (or single-valued) and *entire* (total) predicates.

▶ **Definition 17** (From [44]). *Let $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be an elementary and existential doctrine. An element $\alpha \in P(X \times Y)$ is said to be* functional from X *to* Y *if* $P_{\langle \pi_1,\pi_2\rangle}(\alpha) \wedge P_{\langle \pi_1,\pi_3\rangle}(\alpha) \leq P_{\langle \pi_2,\pi_3\rangle}(\delta_Y)$ *in* $P(X \times Y \times Y)$*. Also, α is said to be* entire from X to Y *if* $\top_X \leq \exists_{\pi_X}(\alpha)$ *in* $P(X)$*.*

▶ Remark 18. By definition, a morphism of elementary and existential doctrines preserves both $\exists_{\pi_X}$ and δ_Y. Therefore it preserves functional and entire elements.

▶ **Example 19.** In $\mathcal{P}\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ from Example 15, an $\alpha \in \mathcal{P}(X \times Y)$ is functional iff it defines a partial function from X to Y, while it is entire iff it is a total relation from X to Y.

▶ **Example 20.** In the doctrine $\mathbf{C}[-, I]\colon \mathsf{Map}(\mathbf{C})^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ from Example 16, functional and entire elements are precisely maps of $\mathbf{C}$. A detailed proof is in [5, Appendix E].

We can now recall the definition of *boolean hyperdoctrine.*

▶ **Definition 21** (boolean hyperdoctrine). *Let $\mathbf{C}$ be a category with finite products. A functor $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathsf{Bool}$ is a* boolean hyperdoctrine *if it is an elementary and existential doctrine.*

A morphism $(F, \mathfrak{b}) : P \to R$ of boolean hyperdoctrines is a morphism of elementary and existential doctrines such that $\mathfrak{b}_X$ is a morphism of boolean algebras for all objects X of $\mathbf{C}$. We denote by $\mathbb{BHD}$ the category of boolean hyperdoctrines and their morphisms.

It is well-known that in first-order logic the universal quantifier can be derived by the existential quantifier and the negation. The same happens in boolean hyperdoctrines: for all arrows $f\colon X \to Y$, the functor $\forall_f(-) \stackrel{\text{def}}{=} \neg\exists_f\neg(-)$ is a right adjoint to P_f (see [5, Appendix B.1]).

▶ **Example 22.** The powerset functor $\mathcal{P}\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Bool}$ provides an example of a boolean hyperdoctrine. This can be generalized to an arbitrary *boolean category* $\mathbf{B}$, namely a coherent category such that every subobject has a complement, see [33, Sec. A1.4, p. 38]. The subobjects functor on $\mathbf{B}$ is a boolean hyperdoctrine $\mathsf{Sub}_{\mathbf{B}}\colon \mathbf{B}^{\mathrm{op}} \longrightarrow \mathsf{Bool}$.

▶ **Example 23.** Given a standard first-order theory $\textsc{Th}$ in a first-order language $\mathcal{L}$ (for simplicity single sorted), one can consider the functor $\mathcal{L}^{\textsc{Th}}\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathsf{Bool}$. The base category $\mathcal{V}$ is the *syntactic* category of $\mathcal{L}$, i.e. the category where objects are natural numbers and morphisms are lists of terms, while the predicates of $\mathcal{L}^{\textsc{Th}}(n)$ are given by equivalence classes (with respect to provable reciprocal consequence $\dashv\vdash$) of well-formed formulae with free variables in $\{x_1, \dots, x_n\}$, and the partial order is given by the provable consequences, according to the fixed theory $\textsc{Th}$. In this case, the left adjoint to the weakening functor $\mathcal{L}^{\textsc{Th}}_{\pi}$ is computed by existentially quantifying the variables that are not involved in the substitution induced by the projection π. Dually, the right adjoint is computed by quantifying universally. The equality predicate is give by the formula $x_1 = x_2$.

▶ **Example 24.** Let A be a boolean algebra. The representable functor $\mathsf{A}^{(-)}\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Bool}$ assigning to a set X the poset A^X of functions from X to A with the point-wise order is a boolean hyperdoctrine.

We conclude this section with a result that, intuitively, is the analogous of Proposition 9.

▶ **Lemma 25.** *Let $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathsf{Bool}$ be a boolean hyperdoctrine and $\phi \in P(X \times Y)$ a functional and entire element from X toY. For all $\psi \in P(Y \times Z)$, it holds that*

$$\exists_{\pi_{X\times Z}}(P_{\pi_{X\times Y}}(\phi) \wedge P_{\pi_{Y\times Z}}(\neg\psi)) = \neg(\,\exists_{\pi_{X\times Z}}(P_{\pi_{X\times Y}}(\phi) \wedge P_{\pi_{Y\times Z}}(\psi))\,).$$

4 An Adjunction and an Equivalence

In [7], cartesian bicategories and elementary existential doctrines are compared. The main results of [7, Thm. 28] states that there exists the following adjunction.

$$\mathbb{CB} \underset{\mathsf{Hml}}{\overset{\mathsf{Rel}}{\underset{\bot}{\leftrightarrows}}} \mathbb{EED} \tag{7}$$

The embedding $\mathsf{Hml}\colon \mathbb{CB} \to \mathbb{EED}$ maps a cartesian bicategory $\mathbf{C}$ into the hom-functor $\mathbf{C}[-,I]\colon \mathsf{Map}(\mathbf{C})^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ that, as explained in Example 16, is an elementary existential doctrine. The functor $\mathsf{Rel}\colon \mathbb{EED} \to \mathbb{CB}$ is a generalisation to elementary and existential doctrines of the construction of bicategory relations associated with a regular category (see [11, Ex. 1.4]). For $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$, the cartesian bicategory $\mathsf{Rel}(P)$ is defined as follows:

- objects are those of $\mathbf{C}$; for objects X, Y, the homsets $\mathsf{Rel}(P)[X,Y]$ are the posets $P(X \times Y)$;
- the identity for an object X is the equality predicate δ_X in $P(X \times X)$;
- composition of $\phi\colon X \to Y$ and $\psi\colon Y \to Z$ is given by $\exists_{\pi_{X\times Z}}(P_{\pi_{X\times Y}}(\phi) \wedge P_{\pi_{Y\times Z}}(\psi))$.

The reader is referred to [7] or to [5, Appendix C] for further details on the adjunction in (7).

Another result in [7, Thm. 35] shows that the adjunction in (7) restricts to an equivalence

$$\mathbb{CB} \equiv \overline{\mathbb{EED}} \tag{8}$$

where $\overline{\mathbb{EED}}$ is a full subcategory of $\mathbb{EED}$ whose objects are particularly well-behaved doctrines. For the sake of readability, we will make clear in §7 what these doctrines are.

5 Peircean Bicategories

We now introduce *peircean bicategories*, an alternative presentation of fo-bicategories. The name peircean is due to the fact that, like in Peirce's algebra of relations [55], and differently from fo-bicategories, the structure of boolean algebra is taken as a primitive.

▶ **Definition 26.** *A* peircean bicategory *consists of a cartesian bicategory* $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$ *such that*

1. *every homset* $\mathbf{C}[X,Y]$ *carries a Boolean algebra* $(\mathbf{C}[X,Y], \vee, \bot, \wedge, \top, \neg)$*;*
2. *for all maps* $f\colon X \to Y$ *and arrows* $c\colon Y \to Z$*,*

$$f ⨾ \neg c = \neg(f ⨾ c). \tag{$\neg\mathcal{M}$}$$

A morphism of peircean bicategories *is a morphism of cartesian bicategories* $F\colon \mathbf{C} \to \mathbf{D}$ *such that* $F(\neg c) = \neg F(c)$*. We write* $\mathbb{PB}$ *for the category of peircean bicategories and their morphisms.*

By Propositions 7 and 9 every fo-bicategory is a peircean bicategory. By Proposition 8 every morphism of fo-bicategories is a morphism of peircean bicategories.

Vice versa, every peircean bicategory $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$ gives rise to a fo-bicategory. The black structure $(\mathbf{C}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$ is defined as expected from the white one and $\neg$. Namely:

$$\begin{array}{cccc} c \mathbin{\bullet\!\!⨾} d \stackrel{\mathrm{def}}{=} \neg(\neg c ⨾ \neg d) & id^{\bullet}_X \stackrel{\mathrm{def}}{=} \neg id^{\circ}_X & c \mathbin{\otimes\!\!\!\bullet} d \stackrel{\mathrm{def}}{=} \neg(\neg c \otimes \neg d) & \sigma^{\bullet}_{X,Y} \stackrel{\mathrm{def}}{=} \neg\sigma^{\circ}_{X,Y} \\ \blacktriangleleft^{\bullet}_X \stackrel{\mathrm{def}}{=} \neg\blacktriangleleft^{\circ}_X & !^{\bullet}_X \stackrel{\mathrm{def}}{=} \neg !^{\circ}_X & \blacktriangleright^{\bullet}_X \stackrel{\mathrm{def}}{=} \neg\blacktriangleright^{\circ}_X & ¡^{\bullet}_X \stackrel{\mathrm{def}}{=} \neg ¡^{\circ}_X \end{array} \tag{9}$$

With this definition, it is immediate to see that $\neg\colon (\mathbf{C}^{\mathrm{co}}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}) \to (\mathbf{C}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$ is an isomorphism and thus to conclude that $(\mathbf{C}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$ is a cocartesian bicategory. Proving that $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$ and $(\mathbf{C}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$ give rise to a fo-bicategory is the main technical effort of this paper.

▶ **Theorem 27.** *There is an isomorphism of categories* $\mathbb{FOB} \cong \mathbb{PB}$.

Proof (sketch). As mentioned above, most of the proof is devoted to showing that every peircean bicategory is a fo-bicategory.

This is achieved by proving first that the relational operations on the homsets, like $(\cdot)^\dagger$ defined in (6), are preserved by negation, e.g. $\neg(c^\dagger) = (\neg c)^\dagger$. This is also where the property on maps ($\neg\mathcal{M}$) mostly comes into play.

Then, to prove that the axioms of fo-bicategories are satisfied, one crucially exploits the laws of boolean algebras.

Finally, to show that every morphism F of peircean bicategories is a morphism of fo-bicategories, it suffices to observe that F preserves the strucure in (9), as it preserves negation by definition.

The full diagrammatic proof can be found in [5, Appendix D]. ◀

Note that, differently from Definition 6, Definition 26 is not purely axiomatic, since ($\neg\mathcal{M}$) requires f to be a map. However, the notion of a peircean bicategory is notably more succinct than that of a fo-bicategory, making it more convenient for our purposes.

6 An Equational Presentation of Boolean Hyperdoctrines

The main purpose of this section is to establish a formal link between fo-bicategories and boolean hyperdoctrines. In particular, we are going to show that the adjunction presented in (7) restricts to an adjunction between $\mathbb{FOB}$ and $\mathbb{BHD}$. Theorem 27 allows us to conveniently work with peircean bicategories. We commence with the following result.

▶ **Proposition 28.** *Let* $\mathbf{C}$ *be a peircean bicategory. Then* $\mathsf{Hml}(\mathbf{C})$ *is a boolean hyperdoctrine.*

Proof. By (7), $\mathbf{C}[-, I] \colon \mathsf{Map}(\mathbf{C})^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ is an elementary and existential doctrine and, by definition of peircean bicategories, $\mathbf{C}[X, I]$ is a boolean algebra for all objects X. To conclude that $\mathbf{C}[-, I] \colon \mathsf{Map}(\mathbf{C})^{\mathrm{op}} \longrightarrow \mathsf{Bool}$, one has only to show that, for all maps $f \colon X \to Y$, $\mathbf{C}[f, I] \colon \mathbf{C}[Y, I] \to \mathbf{C}[X, I]$ is a morphism of boolean algebras. Since, by (7), $\mathbf{C}[f, I]$ is a morphism of inf-semilattices, it is enough to show that it preserves negation: for all $c \in \mathbf{C}[Y, I]$

$$\begin{aligned} \mathbf{C}[f, I](\neg c) &= f \fatsemi \neg c && \text{(Definition of } \mathbf{C}[-, I]) \\ &= \neg(f \fatsemi c) && (\neg\mathcal{M}) \\ &= \neg\mathbf{C}[f, I](c) && \text{(Definition of } \mathbf{C}[-, I]) \end{aligned}$$

◀

The above proposition allows us to characterize peircean bicategories as follows:

▶ **Corollary 29.** *Let* $\mathbf{C}$ *be a cartesian bicategory. Then it is a peircean bicategory if and only if* $\mathsf{Hml}(\mathbf{C})$ *is a boolean hyperdoctrine.*

To prove that, for any boolean hyperdoctrine P, $\mathsf{Rel}(P)$ is a peircean bicategory, we need to establish a formal correspondence between Definition 2 and Definition 17.

▶ **Proposition 30.** *Let* $P \colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *be an elementary and existential doctrine. Then the maps of* $\mathsf{Rel}(P)$ *are precisely the functional and entire elements of* P.

▶ **Proposition 31.** *Let* P *be a boolean hyperdoctrine. Then* $\mathsf{Rel}(P)$ *is a peircean bicategory.*

Proof. By (7), $\mathsf{Rel}(P)$ is a cartesian bicategory. Since $P(X)$ is a boolean algebra for all objects X, then each hom-set $\mathsf{Rel}(P)[X,Y]$ – by definition $P(X \times Y)$ – is a boolean algebra. To conclude that $\mathsf{Rel}(P)$ is a peircean bicategory, it is enough to show that $(\neg\mathcal{M})$ holds, that is

$$\phi \mathbin{;} \neg\psi = \neg(\phi \mathbin{;} \psi)$$

for all maps $\phi \in \mathsf{Rel}(P)[X,Y]$ and arrows $\psi \in \mathsf{Rel}(P)[Y,Z]$. By Proposition 30, ϕ is a functional and entire element of P. Thus, one can rely on Lemma 25 to conclude that

$$\begin{aligned} \phi \mathbin{;} \neg\psi &= \exists_{\pi_{X\times Z}}(P_{\pi_{X\times Y}}(\phi) \wedge P_{\pi_{Y\times Z}}(\neg\psi)) && \text{(Defintion of } \mathsf{Rel}(P)\text{)} \\ &= \neg(\, \exists_{\pi_{X\times Z}}(P_{\pi_{X\times Y}}(\phi) \wedge P_{\pi_{Y\times Z}}(\psi))\,) && \text{(Lemma 25)} \\ &= \neg(\phi \mathbin{;} \psi) && \text{(Defintion of } \mathsf{Rel}(P)\text{)} \end{aligned}$$

◀

By Propositions 28 and 31 proving the following result amounts to a few routine checks.

▶ **Theorem 32.** *The adjunction in* (7)*, restricts to the adjunction below on the left.*

$$\mathbb{PB} \underset{\mathsf{Hml}}{\overset{\mathsf{Rel}}{\leftrightarrows}} \mathbb{BHD} \quad (\bot)$$

Thus, by Theorem 27, there is an adjunction $\mathbb{FOB} \leftrightarrows \mathbb{BHD}$ $(\bot)$.

Proof. First, we want to prove that the inclusion $\mathsf{Hml}\colon \mathbb{CB} \hookrightarrow \mathbb{EED}$ in (7) restricts to an inclusion of categories $\mathbb{PB} \hookrightarrow \mathbb{BHD}$. By Proposition 28, one only needs to check for morphisms in $\mathbb{PB}$. Given a morphism of peircean bicategories $F\colon \mathbf{C} \to \mathbf{D}$, $\mathsf{Hml}(F)$ is the morphism of elementary and existential doctrines $(\tilde{F}, \mathfrak{b}^F)$ defined in Section 4. In order to conclude that it is a morphism of boolean doctrines, it is enough to show that $\mathfrak{b}^F_X$ is a morphism of boolean algebras for all objects X. Since $(\tilde{F}, \mathfrak{b}^F)$ is a morphism of doctrines, $\mathfrak{b}^F_X$ is a morphism of inf-semilattices. Thus it is enough to show that $\mathfrak{b}^F_X$ preserve negation. But this is trivial since, for all $c \in \mathbf{C}[X, I]$,

$$\begin{aligned} \mathfrak{b}^F_X(\neg c) &= F(\neg c) && \text{(Def. } \mathfrak{b}^F\text{)} \\ &= \neg F(c) && \text{(morphism of Peircean, Definition 26)} \\ &= \neg \mathfrak{b}^F_X(c) && \text{(Def. } \mathfrak{b}^F\text{)} \end{aligned}$$

Now, to prove that Rel restrict to a functor $\mathsf{Rel}\colon \mathbb{BHD} \to \mathbb{PB}$, by Proposition 31, one only needs to check that for all morphisms of boolean hyperdoctrines $(F, \mathfrak{b})\colon P \to Q$, $\mathsf{Rel}(F,\mathfrak{b})\colon \mathsf{Rel}(P) \to \mathsf{Rel}(Q)$ is a morphism of peicean bicategories. Since by (7), $\mathsf{Rel}(F,\mathfrak{b})$ is a morphism of cartesian bicategories, one only needs to check that it preserves the negation. But this is obvious since for all arrows $\phi \in \mathsf{Rel}(P)[X,Y]$, $\mathsf{Rel}(F,\mathfrak{b})(\phi)$ is – by definition – $\mathfrak{b}_{X\times Y}(\phi)$ and $\mathfrak{b}_{X\times Y}$ is a morphism of boolean algebras.

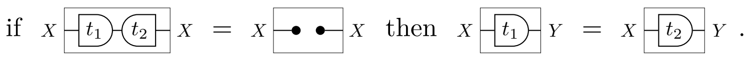
To conclude, one only needs to check the unit and the counit of the adjunction in (7). The counit is an isomorphism of cartesian bicategories (see Equation (9) in [7]), and then it provides an isomorphism of peircean bicategories $\mathbf{C} \cong \mathsf{Rel}(\mathbf{C}[-, I])$ whenever $\mathbf{C}$ is a peircean bicategory. The unit of the adjunction $\eta_P\colon P \to \mathsf{Rel}(P)[-, I]$ is the morphism of elementary and existential doctrines (Γ_P, ρ) illustrated in [7, Section 7] or [5, Appendix C]. To conclude that η_P is a morphism of boolean hyperdoctrine whenever P is a boolean hyperdoctrine, one has only to prove that ρ is a morphism of boolean algebras, but this is trivial since ρ is always an isomorphism of inf-semilattices. ◀

7 Boolean Hyperdoctrines Representing First-Order Bicategories

As anticipated in §4, the adjunction in (7) becomes an equivalence for certain well-behaved doctrines. Definitions 33 and 34 state the conditions that such doctrines must satisfy.

▶ **Definition 33.** *An elementary and existential doctrine* $P\colon \mathbf{C}^{op} \longrightarrow \mathsf{InfSl}$ *has* comprehensive diagonals *if for the equality predicate* $\delta_X \in P(X)$ *it holds that* $P_{\Delta_X}(\delta_X) = \top_X$ *and every arrow* $f\colon Y \to X \times X$ *such that* $P_f(\delta_X) = \top_Y$ *factors (uniquely) through* Δ_X *.*

Intuitively, a doctrine has comprehensive diagonals if its equality is *extensional*, namely if a formula $t_1 = t_2$ is true, then the terms t_1 and t_2 are syntactically equal. In the language of cartesian bicategories, for two maps t_1, t_2, this can be stated by means of diagrams as



(10)

While it is sometimes meaningful to consider syntactic doctrines (e.g. Example 23) in which the equality is not extensional, in several semantical doctrines this condition is satisfied.

▶ **Definition 34.** *Let* $P\colon \mathbf{C}^{op} \longrightarrow \mathsf{InfSl}$ *be an elementary existential doctrine. We say that* P *satisfies the* Rule of Unique Choice *(RUC) if for every entire functional element* ϕ *in* $P(X \times Y)$ *there exists an arrow* $f\colon X \to Y$ *such that* $\top_X \leq P_{\langle id_X^\circ, f\rangle}(\phi)$.

The reader can think that a doctrine has (RUC) if for every element (intuitively formula) that is entire and functional, there exists an arrow in $\mathbf{C}$ (intuitively a term) that represents it.

▶ **Example 35.** The doctrine $\mathcal{P}\colon \mathsf{Set}^{op} \longrightarrow \mathsf{InfSl}$ has comprehensive diagonals, and it satisfies the (RUC) (since every functional and total relation can be represented by a function). More generally, every subobject doctrine $\mathsf{Sub}_{\mathbf{C}}\colon \mathbf{C}^{op} \longrightarrow \mathsf{InfSl}$ on a regular category, as presented in Example 15 satisfies the (RUC) and it has comprehensive diagonals, as observed in [44].

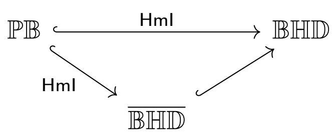
▶ **Example 36.** The doctrine $\mathbf{C}[-, I]\colon \mathsf{Map}(\mathbf{C})^{op} \longrightarrow \mathsf{InfSl}$ presented in Example 16 satisfies the (RUC) and it has comprehensive diagonals, as proved in [7]. The reader can find a diagrammatic proof of (10) in [5, Appendix A].

Hereafter – and in the equivalence in (8) – $\overline{\mathbb{EED}}$ is the full subcategory of $\mathbb{EED}$ whose objects are doctrines satisfying (RUC) and with comprehensive diagonals. Similarly $\overline{\mathbb{BHD}}$ is the full subcategory of $\mathbb{BHD}$ whose objects are boolean hyperdoctrines satisfying (RUC) and with comprehensive diagonals.

By means of Theorem 32, it is easy to prove that the equivalence in (8) restricts as follows.

▶ **Theorem 37.** $\mathbb{PB} \equiv \overline{\mathbb{BHD}}$ *and thus, by Theorem 27,* $\mathbb{FOB} \equiv \overline{\mathbb{BHD}}$*.*

Proof. By Equation (8) we have that the Hml and Rel functors provide an equivalence between the categories $\mathbb{CB}$ and $\overline{\mathbb{EED}}$. Now, since every peircean category is in particular a cartesian bicategory, we have that every boolean hyperdoctrine arising from a peircean bicategory satisfies (RUC) and it has comprehensive diagonals. Then, we have that the functor $\mathsf{Hml}\colon \mathbb{PB} \hookrightarrow \mathbb{BHD}$ factors through the canonical inclusion $\overline{\mathbb{BHD}} \hookrightarrow \mathbb{BHD}$:



By Theorem 32, we have that $\mathsf{Hml}\colon \mathbb{PB} \hookrightarrow \mathbb{BHD}$ is fully and faithful (since the counit of the adjunction is an iso), so it remains to prove that it is essentially surjective (with respect to the objects of $\overline{\mathbb{BHD}}$). By the equivalence presented in Equation (8), we know that every boolean hyperdoctrine (that is in particular an elementary and existential doctrine) satisfying (RUC) and having comprehensive diagonals, is isomorphic to an elementary and existential doctrine $\mathbf{C}[-,I]\colon \mathsf{Map}(\mathbf{C})^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ for some cartesian bicategory $\mathbf{C}$. Thus, we can conclude that $\mathbf{C}[-,I]\colon \mathsf{Map}(\mathbf{C})^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ is a boolean hyperdoctrine and, by Corollary 29, that $\mathbf{C}$ is a peircean bicategory. This concludes the proof that $\mathbb{PB} \equiv \overline{\mathbb{BHD}}$. ◀

8 Related Work

There exists many structures that are closely related to fo-bicategories, as discussed in [3]. The introduction of peircean bicategory in § 5 provide clearer correspondences with such structures. Here we discuss some of them.

Boolean hyperdoctrines are used in [10] as a categorical treatment of another work of Peirce: *existential graphs* [64]. While the latter share some similarities with the graphical language of fo-bicategories there is one notable difference: negation is a primitive operator rather than a derived one, as it happens for instance also in [25] and Definition 26. In [3] and in §5, it is emphasised how this choice makes the resulting calculus less algebraic in flavour, having to deal with convoluted rules such as the one for (de)iteration or properties which are not purely equational, such as $(\neg\mathcal{M})$.

Inspired by [10], another graphical language [50] akin to Peirce's graphs is based on a decomposition of a hyperdoctrine into a bifibration. In this work, the categorical treatment revolves around the notion of monoidal *chiralities* [49], which are much more closer in spirit to fo-bicategories. We believe that our results might set an initial step towards a connection between fo-bicategories and chiralities.

A recent work [15] proposes a relational understanding of doctrines. However, these corresponds to the regular fragment of first-order logic, and thus it might by intriguing to understand the role of the additional black structure of first-order bicategories in this setting. Another route, suggested by the equivalence in Theorem 37, might be to understand the role of $(\neg\mathcal{M})$ in relational doctrines with boolean fibres.

Finally, it is also worth remarking that peircean bicategories, as well as fo-bicategories, are poset-enriched categories. Such categorical treatements of first-order logic are also found in works such as [30, 23], along with the references therein. Their primary focus, though, is on the categorical approach to classical proof theory instead of semantics.

9 Conclusions and Future Work

Theorems 32 and 37 provide a solid bridge between functional and relational approaches to classical logic. The former rely on categorical structures that are usually defined by means of exactness properties; the latter on fo-bicategories which enjoy a purely equational presentation, much in the spirit of Boole's algebra and Peirce's calulus.

To achieve our result, we found it extremely convenient to introduce the notion of peircean bicategories that, by Theorem 27, provide a far handier characterisation of fo-bicategories.

Theorem 27 might also be useful to establish a correspondence with *allegories* [22]: since cartesian bicategories are equivalent to unitary pretabular allegories [35], we expect that such allegories where, additionally, homsets carry boolean algebras satisfying $(\neg\mathcal{M})$ are equivalent to fo-bicategories. Despite searching the literature, we did not find analogous structures. Interestingly, the property $(\neg\mathcal{M})$ can be proven in any Peirce allegories, as shown in Proposition 4.6.1 in [53].

Finally, as future work we aim to investigate how our characterizations can be extended to higher-order classical logic, which is categorically represented through the notion of *tripos* [29, 57]. Indeed, we believe that the constructions and results presented in this work, together with the notion of tripos, can serve as a guide for defining a variant of fo-bicategories – hopefully, purely equational – capable of representing higher-order classical logic.

References

1 Richard Bird and Oege De Moor. The algebra of programming. *NATO ASI DPD*, 152:167–203, 1996.

2 Benedikt Bollig, Alain Finkel, and Amrita Suresh. Bounded Reachability Problems Are Decidable in FIFO Machines. In Igor Konnov and Laura Kovács, editors, *31st International Conference on Concurrency Theory (CONCUR 2020)*, volume 171 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 49:1–49:17, Dagstuhl, Germany, 2020. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.CONCUR.2020.49.

3 Filippo Bonchi, Alessandro Di Giorgio, Nathan Haydon, and Pawel Sobocinski. Diagrammatic algebra of first order logic. In *Proceedings of the 39th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '24, New York, NY, USA, 2024. Association for Computing Machinery. doi:10.1145/3661814.3662078.

4 Filippo Bonchi, Fabio Gadducci, Aleks Kissinger, Pawel Sobocinski, and Fabio Zanasi. String diagram rewrite theory I: rewriting with frobenius structure. *J. ACM*, 69(2):14:1–14:58, 2022. doi:10.1145/3502719.

5 Filippo Bonchi, Alessandro Di Giorgio, and Davide Trotta. When lawvere meets peirce: an equational presentation of boolean hyperdoctrines, 2024. arXiv:2404.18795.

6 Filippo Bonchi, Joshua Holland, Robin Piedeleu, Paweł Sobociński, and Fabio Zanasi. Diagrammatic algebra: From linear to concurrent systems. *Proceedings of the ACM on Programming Languages*, 3(POPL):25:1–25:28, January 2019. doi:10.1145/3290338.

7 Filippo Bonchi, Alessio Santamaria, Jens Seeber, and Paweł Sobociński. On doctrines and cartesian bicategories. In *9th Conference on Algebra and Coalgebra in Computer Science (CALCO 2021)*. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021.

8 Filippo Bonchi, Pawel Sobocinski, and Fabio Zanasi. Full Abstraction for Signal Flow Graphs. In *Proceedings of the 42nd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, POPL '15, pages 515–526, New York, NY, USA, January 2015. Association for Computing Machinery. doi:10.1145/2676726.2676993.

9 George Boole. *The mathematical analysis of logic*. Philosophical Library, 1847.

10 Geraldine Brady and Todd Trimble. A categorical interpretation of c.s. peirce's propositional logic alpha. *Journal of Pure and Applied Algebra - J PURE APPL ALG*, 149:213–239, June 2000. doi:10.1016/S0022-4049(98)00179-0.

11 A. Carboni and R. F. C. Walters. Cartesian bicategories I. *Journal of Pure and Applied Algebra*, 49(1):11–32, November 1987. doi:10.1016/0022-4049(87)90121-6.

12 J. Robin B. Cockett, Jürgen Koslowski, and Robert AG Seely. Introduction to linear bicategories. *Mathematical Structures in Computer Science*, 10(2):165–203, 2000.

13 Edgar Frank Codd. A relational model of data for large shared data banks. *Communications of the ACM*, 26(1):64–69, 1983.

14 Bob Coecke and Ross Duncan. Interacting quantum observables: Categorical algebra and diagrammatics. *New Journal of Physics*, 13(4):043016, April 2011. doi:10.1088/1367-2630/13/4/043016.

15 Francesco Dagnino and Fabio Pasquali. Quotients and Extensionality in Relational Doctrines. In Marco Gaboardi and Femke van Raamsdonk, editors, *8th International Conference on Formal Structures for Computation and Deduction (FSCD 2023)*, volume 260 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 25:1–25:23, Dagstuhl, Germany, 2023. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.FSCD.2023.25.

16 Augustus De Morgan. On the syllogism, no. iv. and on the logic of relations. *Printed by C.J. Clay at the University Press*, 1860.

17 Charles J Everett and Stanislaw Ulam. Projective algebra i. *American Journal of Mathematics*, 68(1):77–88, 1946.

18 William Ewald. The emergence of first-order logic. *Stanford Encyclopedia of Philosophy*, 2018.

19 Brendan Fong, Paweł Sobociński, and Paolo Rapisarda. A categorical approach to open and interconnected dynamical systems. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '16, pages 495–504, New York, NY, USA, July 2016. Association for Computing Machinery. doi:10.1145/2933575.2934556.

20 Brendan Fong and David Spivak. String diagrams for regular logic (extended abstract). In John Baez and Bob Coecke, editors, *Applied Category Theory 2019*, volume 323 of *Electronic Proceedings in Theoretical Computer Science*, pages 196–229. Open Publishing Association, September 2020. doi:10.4204/eptcs.323.14.

21 GOTTLOB FREGE. *Begriffsschrift und andere Aufsätze.* Georg Olms Verlag, 1977.

22 Peter Freyd and Andre Scedrov. *Categories, Allegories*, volume 39 of *North-Holland Mathematical Library.* Elsevier B.V, 1990.

23 CARSTEN FÜHRMANN and DAVID PYM. On categorical models of classical logic and the geometry of interaction. *Mathematical Structures in Computer Science*, 17(5):957–1027, 2007. doi:10.1017/S0960129507006287.

24 Dan R. Ghica and Achim Jung. Categorical semantics of digital circuits. In *2016 Formal Methods in Computer-Aided Design (FMCAD)*, pages 41–48, 2016. doi:10.1109/FMCAD.2016.7886659.

25 Nathan Haydon and Paweł Sobociński. Compositional diagrammatic first-order logic. In *11th International Conference on the Theory and Application of Diagrams (DIAGRAMS 2020)*, 2020.

26 Leon Henkin. Cylindric algebras, 1971.

27 CAR Hoare and He Jifeng. The weakest prespecification, part i. *Fundamenta Informaticae*, 9(1):51–84, 1986.

28 Ian Hodkinson and Szabolcs Mikulás. Axiomatizability of reducts of algebras of relations. *Algebra Universalis*, 43(2):127–156, August 2000. doi:10.1007/s000120050150.

29 J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. Tripos theory. *Math. Proc. Camb. Phil. Soc.*, 88:205–232, 1980.

30 Martin Hyland. Abstract interpretation of proofs: Classical propositional calculus. In Jerzy Marcinkowski and Andrzej Tarlecki, editors, *Computer Science Logic*, pages 6–21, Berlin, Heidelberg, 2004. Springer Berlin Heidelberg.

31 B. Jacobs. *Categorical Logic and Type Theory*, volume 141 of *Studies in Logic and the foundations of mathematics.* North Holland Publishing Company, 1999.

32 P.T. Johnstone. *Topos Theory.* cademic Press, 1977.

33 P.T. Johnstone. *Sketches of an elephant: a topos theory compendium*, volume 2 of *Studies in Logic and the foundations of mathematics.* Oxford Univ. Press, 2002.

34 André Joyal and Ross Street. The geometry of tensor calculus, I. *Advances in Mathematics*, 88(1):55–112, July 1991. doi:10.1016/0001-8708(91)90003-P.

35 Petrus Marinus Waltherus Knijnenburg and Frank Nordemann. *Two Categories of Relations.* Citeseer, 1994.

36 Alexander Krauss and Tobias Nipkow. Proof pearl: Regular expression equivalence and relation algebra. *Journal of Automated Reasoning*, 49(1):95–106, 2012. doi:10.1007/s10817-011-9223-4.

37 Ugo Dal Lago and Francesco Gavazzo. A relational theory of effects and coeffects. *Proc. ACM Program. Lang.*, 6(POPL):1–28, 2022. doi:10.1145/3498692.

38 Søren B Lassen. Relational reasoning about contexts. *Higher order operational techniques in semantics*, 91, 1998.

39 F. W. Lawvere. *Functorial Semantics of Algebraic Theories.* PhD thesis, Columbia University, New York, NY, USA, 1963.

40 F.W. Lawvere. Adjointness in foundations. *Dialectica*, 23:281–296, 1969.

41 F.W. Lawvere. Diagonal arguments and cartesian closed categories. In *Category Theory, Homology Theory and their Applications*, volume 2, pages 134–145. Springer, 1969.

42 F.W. Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. In A. Heller, editor, *New York Symposium on Application of Categorical Algebra*, volume 2, pages 1–14. American Mathematical Society, 1970.

43 Leopold Löwenheim. Über Möglichkeiten im Relativkalkül. *Mathematische Annalen*, 76(4):447–470, 1915.

44 M.E. Maietti, F. Pasquali, and G. Rosolini. Triposes, exact completions, and hilbert's ε-operator. *Tbilisi Mathematica journal*, 10(3):141–166, 2017.

45 M.E. Maietti and G. Rosolini. Elementary quotient completion. *Theory App. Categ.*, 27(17):445–463, 2013.

46 M.E. Maietti and G. Rosolini. Quotient completion for the foundation of constructive mathematics. *Log. Univers.*, 7(3):371–402, 2013.

47 M.E. Maietti and G. Rosolini. Unifying exact completions. *Appl. Categ. Structures*, 23:43–52, 2013.

48 M.E. Maietti and D. Trotta. A characterization of generalized existential completions. *Annals of Pure and Applied Logic*, 174(4):103234, 2023.

49 Paul-André Melliès. Dialogue categories and chiralities. *Publications of the Research Institute for Mathematical Sciences*, 52(4):359–412, 2016.

50 Paul-André Melliès and Noam Zeilberger. A bifibrational reconstruction of lawvere's presheaf hyperdoctrine. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 555–564, 2016.

51 Donald Monk. On representable relation algebras. *Michigan Mathematical Journal*, 11(3):207–210, 1964. doi:10.1307/mmj/1028999131.

52 Koko Muroya, Steven W. T. Cheung, and Dan R. Ghica. The geometry of computation-graph abstraction. In Anuj Dawar and Erich Grädel, editors, *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018*, pages 749–758. ACM, 2018. doi:10.1145/3209108.3209127.

53 Jean-Pierre Olivier and Dany Serrato. Peirce allegories. identities involving transitive elements and symmetrical ones. *Journal of Pure and Applied Algebra*, 116(1-3):249–271, 1997.

54 Charles S. Peirce. The logic of relatives. *The Monist*, 7(2):161–217, 1897. URL: http://www.jstor.org/stable/27897407.

55 Charles Sanders Peirce. *Studies in logic. By members of the Johns Hopkins university.* Little, Brown, and Company, 1883.

56 Robin Piedeleu and Fabio Zanasi. A String Diagrammatic Axiomatisation of Finite-State Automata. In Stefan Kiefer and Christine Tasson, editors, *Foundations of Software Science and Computation Structures*, Lecture Notes in Computer Science, pages 469–489, Cham, 2021. Springer International Publishing. doi:10.1007/978-3-030-71995-1_24.

57 A.M. Pitts. Tripos theory in retrospect. *Math. Struct. in Comp. Science*, 12:265–279, 2002.

58 Damien Pous. Kleene algebra with tests and coq tools for while programs. In *Interactive Theorem Proving: 4th International Conference, ITP 2013, Rennes, France, July 22-26, 2013. Proceedings 4*, pages 180–196. Springer, 2013.

59 Damien Pous. *Automata for relation algebra and formal proofs.* PhD thesis, ENS Lyon, 2016.

60 Damien Pous. On the positive calculus of relations with transitive closure. In Rolf Niedermeier and Brigitte Vallée, editors, *35th Symposium on Theoretical Aspects of Computer Science, STACS 2018, February 28 to March 3, 2018, Caen, France*, volume 96 of *LIPIcs*, pages 3:1–3:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018. doi:10.4230/LIPIcs.STACS.2018.3.

61 Vaughan R Pratt. Semantical considerations on floyd-hoare logic. In *17th Annual Symposium on Foundations of Computer Science (sfcs 1976)*, pages 109–121. IEEE, 1976.

62 W.V. Quine. Predicate-functor logics. In J.E. Fenstad, editor, *Proceedings of the Second Scandinavian Logic Symposium*, volume 63 of *Studies in Logic and the Foundations of Mathematics*, pages 309–315. Elsevier, 1971. doi:10.1016/S0049-237X(08)70850-4.

63 Valentin N Redko. On defining relations for the algebra of regular events. *Ukrainskii Matematicheskii Zhurnal*, 16:120–126, 1964.

64 Don D. Roberts. *The Existential Graphs of Charles S. Peirce.* De Gruyter Mouton, 1973.

65 P. Selinger. A Survey of Graphical Languages for Monoidal Categories. In B. Coecke, editor, *New Structures for Physics*, volume 813 of *Lecture Notes in Physics*, pages 289–355. Springer, Berlin, Heidelberg, 2010. doi:10.1007/978-3-642-12821-9_4.

66 Dario Stein and Sam Staton. Probabilistic programming with exact conditions. *Journal of the ACM*, 2023.

67 Alfred Tarski. On the calculus of relations. *The Journal of Symbolic Logic*, 6(3):73–89, September 1941. doi:10.2307/2268577.