# Quasi-Isometric Reductions Between Infinite Strings

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## Abstract

This paper studies the recursion-theoretic aspects of large-scale geometries of infinite strings, a subject initiated by Khoussainov and Takisaka (2017). We investigate several notions of quasi-isometric reductions between recursive infinite strings and prove various results on the equivalence classes of such reductions. The main result is the construction of two infinite recursive strings $\alpha$ and $\beta$ such that $\alpha$ is strictly quasi-isometrically reducible to $\beta$, but the reduction cannot be made recursive. This answers an open problem posed by Khoussainov and Takisaka.

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## 1 Introduction

Quasi-isometry is an important concept in geometric group theory that has been used to solve problems in group theory. Loosely speaking, two metric spaces are said to be quasi-isometric iff there is a mapping (called a *quasi-isometry*) from one metric space to the other that preserves the distance between any two points in the first metric space up to some multiplicative and additive constants. Thus, for example, while the Euclidean plane is not isometric to $\mathbb{R}^2$ equipped with the taxicab distance, the two spaces are quasi-isometric to each other since the Euclidean distance between any two points does not differ from the taxicab distance between them up to a multiplicative factor of $\sqrt{2}$. The study of group properties – where groups are represented by their Cayley graphs – that are invariant under quasi-isometries is quite a prominent theme in geometric group theory; examples of such group properties include hyperbolicity and growth rate [2].

This paper studies quasi-isometries of the ordered sets $(\mathbb{N}, <)$ with the objects being infinite strings, recursive functions from $\mathbb{N}$ to a finite alphabet or isomorphic copies of these structures defined with automatic functions in automata theory replacing recursive ones (the latter being delayed to the journal version of this paper). The notion of quasi-isometry for infinite strings was introduced by Khoussainov and Takisaka [6], enabling the study of global patterns on strings and linking the study of large-scale geometries with automata theory, computability theory, algorithmic randomness and model theory. Furthermore, quasi-isometries between hyperbolic metric spaces in general – an example of which is an infinite string when viewed as a colored metric space – are well-studied in geometric group theory. Isometries between computable metric spaces have also been studied by Melnikov [10].

Among the various questions investigated by Khoussainov and Takisaka was the computational complexity of the *quasi-isometry problem*: given any two infinite strings $\alpha$ and $\beta$, is there a quasi-isometry from $\alpha$ to $\beta$? They found that for any two quasi-isometric strings, a quasi-isometry that is recursive in the halting problem relative to $\alpha$ and $\beta$ always exists between them, and that the quasi-isometry problem between any two recursive strings is $\Sigma^0_2$-complete [7][1]. In comparison, the corresponding problem for isometry with respect to recursive strings is $\Pi^0_1$-complete [10]. Khoussainov and Takisaka also had the following open problem which was mentioned in many talks and discussions: *if a quasi-isometric reduction from $\alpha$ to $\beta$ exists, does there always exist a recursive quasi-isometric reduction?* This is a very natural question for computer science, specifically for computability theory, since it seeks to understand how complex such a reduction is. We answered this question in the negative, that is, there are cases where the reduction exists but cannot be made recursive. The fourth author's bachelor thesis [9] which contains this result was cited by Khoussainov and Takisaka in the journal version [7] of their paper [6].

To complete the picture, the present work examines, in more detail, the recursion-theoretic aspects of quasi-isometries between infinite strings. We study various natural restrictions on quasi-isometric reductions between strings: first, *many-one reductions*, where the quasi-isometric reduction is required to be *recursive* and *many-one*; second, *one-one reductions*, which are injective many-one reductions; third, *permutation quasi-isometric reductions*, which are surjective one-one reductions.

The main subjects of this work are the structural properties of the equivalence classes induced by the different types of reductions and the relationships between these reductions. In accordance with recursion-theoretic terminology, we call an equivalence class induced by a reduction type a *degree* of that reduction type. We show, for example, that within each many-one quasi-isometry degree, any pair of strings has a common upper bound as well as a common lower bound with respect to one-one reductions. Furthermore, there are two strings for which their many-one quasi-isometry degrees have a unique least common upper bound. The main result is the separation of quasi-isometry from *recursive* quasi-isometry, that is, we construct two recursive strings such that one is quasi-isometric reducible to the other but no recursive many-one quasi-isometry exists between them. This main result answers the above-mentioned open problem posed by Khoussainov and Takisaka.

[1] Note that [7] is the journal version of their paper [6], containing some corrections from the earlier paper.

## 2 Notation

Any unexplained recursion-theoretic notation may be found in [11, 13, 14]. The set of positive integers will be denoted by $\mathbb{N}$; $\mathbb{N} \cup \{0\}$ will be denoted by $\mathbb{N}_0$. The finite set $\Sigma$ will denote the alphabet used. We assume knowledge of elementary computability theory over different size alphabets [1]. An infinite string $\alpha \in \Sigma^\omega$ can also be viewed as a $\Sigma$-valued function defined on $\mathbb{N}$. The length of an interval $I$ is denoted by $|I|$. For $\alpha_i \in \Sigma^*$ and $i \in \mathbb{N}$, we write $(\alpha_i)_{i=1}^\infty$ to denote $\alpha_1\alpha_2\cdots$, a possibly infinite string.

## 3 Colored Metric Spaces and Infinite Strings

▶ **Definition 1** (Colored Metric Spaces, [6]). *A* colored metric space $(M; d_M, Cl)$ *consists of the* underlying metric space $(M; d_M)$ *with metric* $d_M$ *and the color function* $Cl : M \to \Sigma$, *where* $\Sigma$ *is a finite set of colors called an* alphabet. *We say that* $m \in M$ has color $\sigma \in \Sigma$ *if* $\sigma = Cl(m)$.

▶ **Definition 2** (Quasi-isometries Between Colored Metric Spaces, [6]). *For any* $A \geq 1$ *and* $B \geq 0$, *an* $(A, B)$-quasi-isometry *from a metric space* $\mathcal{M}_1 = (M_1; d_1)$ *to a metric space* $\mathcal{M}_2 = (M_2; d_2)$ *is a function* $f : M_1 \to M_2$ *such that for all* $x, y \in M_1$, $\frac{1}{A} \cdot d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A \cdot d_1(x, y) + B$, *and for all* $y \in M_2$, *there exists an* $x \in M_1$ *such that* $d_2(f(x), y) \leq A$.

*Given two colored metric spaces* $\mathcal{M}_1 = (M_1; d_1, Cl_1)$ *and* $\mathcal{M}_2 = (M_2; d_2, Cl_2)$, *a function* $f : M_1 \to M_2$ *is a* quasi-isometric reduction *from* $\mathcal{M}_1$ *to* $\mathcal{M}_2$ *iff for some* $A \geq 1$ *and* $B \geq 0$, $f$ *is an* $(A, B)$*-quasi-isometry from* $(M_1; d_1)$ *to* $(M_2; d_2)$ *and* $f$ *is* color-preserving, *that is, for all* $x \in M_1$, $Cl_1(x) = Cl_2(f(x))$.

An infinite string $\alpha$ can then be seen as a colored metric space $(\mathbb{N}; d, \alpha)$, where $d$ is the metric on $\mathbb{N}$ defined by $d(i, j) = |i - j|$ and $\alpha : \mathbb{N} \to \Sigma$ is the color function. For any two infinite strings $\alpha$ and $\beta$, we write $\alpha \leq_{qi} \beta$ to mean that there is a quasi-isometric reduction from $\alpha$ to $\beta$. The relation $\leq_{qi}$ is a preorder on $\Sigma^\omega$. For any pair of distinct letters $a_1, a_2 \in \Sigma$, $a_1^\omega$ and $a_2^\omega$ are incomparable with respect to $\leq_{qi}$, so this relation is not total.

The following proposition gives a useful simplification of the definition of quasi-isometry in the context of infinite strings.

▶ **Proposition 3.** *Given two infinite strings* $\alpha$ *and* $\beta$, *let* $f : \mathbb{N} \to \mathbb{N}$ *be a color-preserving function. Then* $f$ *is a quasi-isometric reduction from* $\alpha$ *to* $\beta$ *iff there exists a constant* $C \geq 1$ *such that for all* $x, y$ *in the domain of* $\alpha$, *the following conditions hold:*

**(a)** $d(f(x), f(x+1)) \leq C$;
**(b)** $x + C < y \Rightarrow f(x) < f(y)$.

**Proof.** First, suppose that $f : \mathbb{N} \to \mathbb{N}$ is a color-preserving quasi-isometric reduction from $\alpha$ to $\beta$. We show that there exists a constant $C \geq 1$ for which Conditions (a) and (b) hold for any $x, y \in \mathbb{N}$. By the definition of a quasi-isometric reduction, there exist constants $A \geq 1$ and $B \geq 0$ such that

$$\frac{1}{A} \cdot d(x, y) - B \leq d(f(x), f(y)) \leq A \cdot d(x, y) + B. \tag{1}$$

We first derive, for each of the two conditions, a choice of $C$ satisfying it.

**(i)** Plugging $y = x + 1$ into the upper bound in (1) yields $d(f(x), f(x + 1)) \leq A + B$.

**(ii)** Assume for the sake of a contradiction that for all $C \geq 1$, there are $x \in \mathbb{N}$ and $C' > C$ such that $f(x + C') \leq f(x)$. We show that if $C$ is chosen so that $A + B \leq \frac{1}{A} \cdot C - B$, then the existence of some $C' > C$ with $f(x + C') \leq f(x)$ would lead to a contradiction. Fix such a $C$, and suppose there were indeed some $C'$ with $C' > C \geq 1$ and

$$f(x + C') \leq f(x) . \tag{2}$$

Then,

$$\begin{aligned} f(x + C' + 1) - f(x + C') &\leq d(f(x + C' + 1), f(x + C')) \\ &\leq A + B \quad \text{(by statement (i))} \\ &\leq \frac{1}{A} \cdot C - B \quad \text{(by the choice of } C\text{)} \\ &< \frac{1}{A} \cdot C' - B \quad \text{(since } C' > C\text{)} \\ &\leq f(x) - f(x + C') \quad \text{(by (1) and (2))} , \end{aligned}$$

giving $f(x + C' + 1) < f(x)$. One can repeat the preceding argument inductively, yielding the inequality $f(x + C' + k + 1) - f(x + C' + k) < f(x) - f(x + C' + k)$, or equivalently $f(x + C' + k + 1) < f(x)$, for each $k \geq 0$. But this is impossible since $f(x)$ is finite and $d(f(x + C' + k + 1), f(x + C' + k' + 1)) > 0$ whenever $|k - k'|$ is sufficiently large.

It follows from (i) and (ii) that Conditions (a) and (b) are satisfied for $C = A \cdot (A + 2B)$.

For a proof of the converse direction, fix a $C$ satisfying Conditions (a) and (b). Suppose $x \in \mathbb{N}$. Then by Condition (a), $d(f(x), f(x+1)) \leq C$. Inductively, assume that $d(f(x), f(x + n)) \leq n \cdot C$. Then by the inductive hypothesis and Condition (a), $d(f(x), f(x + n + 1)) \leq d(f(x), f(x+n)) + d(f(x+n), f(x+n+1)) \leq n \cdot C + C = (n+1) \cdot C$ where the first inequality follows from the triangle inequality. Consequently, for all $x, y \in \mathbb{N}$,

$$d(f(x), f(y)) \leq d(x, y) \cdot C . \tag{3}$$

Next, we establish a lower bound for $d(f(x), f(y))$. Without loss of generality, assume $x < y$. Write $y = x + i(C+1) + j$ for some $i \in \mathbb{N}_0$ and $0 \leq j \leq C$. By a simple induction, one can show that $f(x + i(C + 1)) \geq f(x) + i$ and thus $d(f(x), f(x + i(C + 1)) \geq i$. Furthermore, $d(f(x + i(C + 1)), f(y)) \leq C^2$. Thus $d(f(x), f(y)) \geq i - C^2$ and $i \geq d(x, y)/(C + 1) - 1$. It follows that $d(f(x), f(y)) \geq d(x, y)/(C + 1) - 1 - C^2$. Thus one can select $A = (C + 1)$ and $B = C^2 + 1$ to establish the required bounds for the quasi-isometric mapping.

To establish that for all $y \in M_2$, there exists an $x \in M_1$ such that $d_2(f(x), y) \leq A$, one can choose any $A \geq max(C, f(1))$, as the distance between $f(x)$ and $f(x + 1)$ is bounded by $C$. ◀

By Proposition 3, we can now redefine quasi-isometric reduction in terms of one constant $C$, instead of two constants $A$ and $B$ as in Definition 2, reducing the number of constants by 1.

▶ **Definition 4.** *Suppose $C \geq 1$. Given infinite strings $\alpha$ and $\beta$, a $C$-*quasi-isometry *from $\alpha$ to $\beta$ is a color-preserving function $f : \mathbb{N} \to \mathbb{N}$ such that for all $x, y$ in the domain of $\alpha$,*

**(a)** *$f(1) \leq C$ and $f(x) - C \leq f(x + 1) \leq f(x) + C$.*

**(b)** *$x + C < y \Rightarrow f(x) < f(y)$.*

*For the rest of the paper, we shall use "Condition (a)" and "Condition (b)" to refer to the above conditions respectively, without necessarily mentioning the definition number.*

A useful property of a $C$-quasi-isometry $f$ from $\alpha$ to $\beta$ is that any position of $\beta$ has at most $C+1$ pre-images under $f$.

▶ **Lemma 5** ([6, Corollary II.4]). *Given two infinite strings $\alpha$ and $\beta$, suppose that $f$ is a $C$-quasi-isometry from $\alpha$ to $\beta$. Then for all $y \in \mathbb{N}$, $|f^{-1}(y)| \leq C+1$.*

It was proven earlier that for any infinite strings $\alpha, \beta$ and any $C$-quasi-isometry $f$ from $\alpha$ to $\beta$, there is a constant $D$ such that each position of $\beta$ is at most $D$ positions away from some image of $f$. The next lemma states that each position of $\beta$ in the range of $f$ is at most $C$ positions away from a different image of $f$. The proof is omitted due to space restrictions.

▶ **Lemma 6.** *Given two infinite strings $\alpha$ and $\beta$, suppose that $f$ is a $C$-quasi-isometry from $\alpha$ to $\beta$. Then $\min\{f(x) : x \in \mathbb{N}\} \leq C$ and for each $y \in \mathbb{N}$, $min\{f(x) : x \in \mathbb{N}$ and $f(x) > f(y)\} \leq f(y) + C$. Hence for each $z \in \mathbb{N}$, there is some $x \in \mathbb{N}$ such that $d(f(x), z) \leq C$.*

▶ **Corollary 7.** *Let $\Sigma = \{a_1, \ldots, a_l\}$ and let $\alpha, \beta$ be two infinite strings. Let $f$ be a $C$-quasi-isometry from $\alpha$ to $\beta$. Suppose that there is a positive integer $K$ such that there is at least one occurrence of $a_i$ in any interval of positions of $\alpha$ of length $K$. Then there is at least one occurrence of $a_i$ in any interval of positions of $\beta$ of length $KC$.*

A quasi-isometry $f$ can fail to be order-preserving in that there are pairs $x, y \in \mathbb{N}$ with $x < y$ and $f(x) > f(y)$. Nonetheless, as Khoussainov and Takisaka noted [6, Lemma II.2], every quasi-isometry enforces a uniform upper bound on the size of a *cross-over* – the difference $f(x) - f(y)$ for such a pair $x, y \in \mathbb{N}$.

▶ **Lemma 8** (Small Cross-Over Lemma, [6, Lemma II.2]). *Given two infinite strings $\alpha$ and $\beta$, suppose that $f$ is a $C$-quasi-isometry from $\alpha$ to $\beta$. Then for all $n, m \in \mathbb{N}$ with $n < m$, we have $f(n) - f(m) \leq C^2$.*

## 4 Recursive Quasi-Isometric Reductions

Khoussainov and Takisaka [6] investigated the structure of the partial-order $\Sigma^{\omega}_{qi}$ of the quasi-isometry degrees over an alphabet $\Sigma = \{a_1, \ldots, a_l\}$. They proved that $\Sigma^{\omega}_{qi}$ has a greatest element, namely the degree of $(a_1 \cdots a_n)^{\omega}$, and that $\Sigma^{\omega}_{qi}$ contains uncountably many minimal elements. Furthermore, they showed that $\Sigma^{\omega}_{qi}$ includes a chain of the type of the integers, and that it includes an antichain. In connection with computability theory, in particular with the arithmetical hierarchy, they established that the quasi-isometry relation on recursive infinite strings is $\Sigma^0_2$-complete [7]. In this section, we continue research into the recursion-theoretic aspects of quasi-isometries on infinite strings. We consider the notions of many-one and one-one recursive reducibilities first introduced by Post [12] as relations between recursive functions, and apply them to quasi-isometric reductions. We also define a third type of quasi-isometric reducibility – permutation reducibility – which is bijective. We then prove a variety of results on the degrees of such reductions.

▶ **Definition 9** (Many-One Reducibility). *A string $\alpha$ is* many-one reducible*, or* mqi-reducible*, to a string $\beta$ iff there exists a quasi-isometric reduction $f$ from $\alpha$ to $\beta$ such that $f$ is* recursive*. We call such an $f$ a* many-one quasi-isometry (or mqi-reduction)*, and write $\alpha \leq_{mqi} \beta$ to mean that $\alpha$ is many-one reducible to $\beta$; if, in addition, $f$ is a $C$-quasi-isometry, then we call $f$ a* $C$-many-one quasi-isometry *(or* $C$-mqi-reduction*). We write $\alpha <_{mqi} \beta$ to mean that $\alpha \leq_{mqi} \beta$ and $\beta \not\leq_{mqi} \alpha$.*

▶ **Definition 10** (One-One Reducibility). *A string $\alpha$ is* one-one reducible*, or* 1qi-reducible*, to a string $\beta$ iff there exists a many-one quasi-isometry $f$ from $\alpha$ to $\beta$ such that $f$ is one-one. We call such an $f$ a* one-one quasi-isometry (or 1qi-reduction)*, and write $\alpha \leq_{1qi} \beta$ to mean that $\alpha$ is one-one reducible to $\beta$; if, in addition, $f$ is a $C$-quasi-isometry, then we call $f$ a* $C$-one-one quasi-isometry *(or* $C$-1qi-reduction*). We write $\alpha <_{1qi} \beta$ to mean that $\alpha \leq_{1qi} \beta$ and $\beta \not\leq_{1qi} \alpha$.*

▶ **Definition 11** (Permutation Reducibility). *A string $\alpha$ is* permutation reducible*, or* pqi-reducible*, to a string $\beta$ iff there exists a one-one quasi-isometry $f$ from $\alpha$ to $\beta$ such that $f$ is surjective. We call such an $f$ a* permutation quasi-isometry (or pqi-reduction)*, and write $\alpha \leq_{pqi} \beta$ to mean that $\alpha$ is permutation reducible to $\beta$; if, in addition, $f$ is a $C$-quasi-isometry, then we call $f$ a* $C$-permutation quasi-isometry *(or* $C$-pqi-reduction*). We write $\alpha <_{pqi} \beta$ to mean that $\alpha \leq_{pqi} \beta$ and $\beta \not\leq_{pqi} \alpha$. Here, note that it can be shown that $\alpha \leq_{pqi} \beta$ implies $\beta \leq_{pqi} \alpha$.*

Given an alphabet $\Sigma$, the relations $\leq_{mqi}$, $\leq_{1qi}$, $\leq_{pqi}$ and $\leq_{qi}$ are preorders on the class of infinite strings over $\Sigma$. Let $\equiv_{mqi}$ be the relation on $\Sigma^\omega$ such that $\alpha \equiv_{mqi} \beta$ iff $\alpha \leq_{mqi} \beta$ and $\beta \leq_{mqi} \alpha$. Then $\equiv_{mqi}$ is an equivalence relation on $\Sigma^\omega$. We call an equivalence class on $\Sigma^\omega$ induced by $\equiv_{mqi}$ a *many-one quasi-isometry degree* (or *mqi-degree*), and denote the mqi-degree of an infinite string $\alpha$ by $[\alpha]_{mqi}$. Analogous definitions apply to $\equiv_{1qi}$, $[\alpha]_{1qi}$, $\equiv_{pqi}$, $[\alpha]_{pqi}$, $\equiv_{qi}$ and $[\alpha]_{qi}$.

We denote the partial orders induced by $\leq_{pqi}, \leq_{1qi}, \leq_{mqi}$ and $\leq_{qi}$ on the pqi-degrees, 1qi-degrees, mqi-degrees and qi-degrees by $\Sigma^\omega_{pqi}, \Sigma^\omega_{1qi}, \Sigma^\omega_{mqi}$ and $\Sigma^\omega_{qi}$ respectively.

By definition, $\Sigma^\omega_{pqi}$ is a refinement of $\Sigma^\omega_{1qi}$ in the sense that for all infinite strings $\alpha$ and $\beta$, $[\alpha]_{pqi} \leq_{pqi} [\beta]_{pqi} \Rightarrow [\alpha]_{1qi} \leq_{1qi} [\beta]_{1qi}$. In a similar manner, $\Sigma^\omega_{1qi}$ is a refinement of $\Sigma^\omega_{mqi}$, which is in turn a refinement of $\Sigma^\omega_{qi}$. The first subsection deals with the mqi-degrees, starting with the inner structure of each mqi-degree.

## 4.1 Structure of the mqi-Degrees

Fix any two distinct infinite strings $\beta$ and $\gamma$ belonging to $[\alpha]_{mqi}$. It can be shown that $\beta$ and $\gamma$ have a common upper bound as well as a common lower bound in $[\alpha]_{mqi}$ such that these bounds are witnessed by 1qi-reductions.

▶ **Proposition 12.** *For any two distinct infinite strings $\beta, \gamma \in [\alpha]_{mqi}$, there exists a $\delta \in [\alpha]_{mqi}$ such that $\beta \leq_{1qi} \delta$ and $\gamma \leq_{1qi} \delta$.*

**Proof.** Let $f$ be a $C$-mqi-reduction from $\beta$ to $\gamma$. Let $\delta$ be the infinite string obtained from $\gamma$ by repeating $C+1$ times each letter of $\gamma$. Then $\gamma \leq_{1qi} \delta$ via a $(C+1)$-1qi-reduction $g$ defined by $g(n) = (n-1) \cdot (C+1) + 1$ for each $n \in \mathbb{N}$. Furthermore, $\delta \leq_{mqi} \gamma$ via a $C$-mqi-reduction $g'$ defined by $g'(n) = \lceil \frac{n}{C+1} \rceil$. Thus $\delta \in [\alpha]_{mqi}$.

Next, one constructs a $(C^2 + 2C)$-1qi-reduction $f'$ from $\beta$ to $\delta$ using the function $f$. For each $y$ in the range of $f$, map the pre-image of $y$ under $f$, which by Lemma 5 has at most $C+1$ elements, to the set of positions of $\delta$ corresponding to the $C+1$ copies of the letter at position $y$. Formally, define

$$f'(n) = \begin{cases} g(f(n))\,, & \text{if } f(n) \neq f(n') \text{ for all } n' < n; \\ g(f(n)) + C'\,, & \text{otherwise; where } 1 \leq C' < C+1 \text{ is minimum such that} \\ & g(f(n)) + C' \neq f'(n') \text{ for all } n' < n. \end{cases}$$

We verify that $f'$ is an injective $(C^2+2C)$-quasi-isometry. Injectiveness follows from the definition of $f'$: in the first case, the injectiveness of $g$ ensures that $f'(x) \neq f'(x')$ for all $x' < x$; in the second case, it is directly enforced that $f'(x) \neq f(x')$ for all $x' < x$. Since $f$ is a $C$-reduction, $x + C < y \Rightarrow f(x) < f(y) \Rightarrow g(f(x)) < g(f(y)) \Rightarrow f'(x) < f'(y)$, and so $f'$ satisfies Condition (b) with constant $C$. Now we show that $f'$ satisfies Condition (a) with constant $C^2+2C$. By Condition (a), $d(f(x), f(x+1)) \leq C$. Without loss of generality, assume that $f(x) \leq f(x+1)$. By the definition of $f'$, $f'(x) \geq g(f(x))$ and $f'(x+1) \leq g(f(x+1)) + C$. Since $f(x) \leq f(x+1)$, it follows that $f'(x) \leq f'(x+1)$ and so

$$\begin{aligned} d(f'(x+1), f'(x)) &\leq g(f(x+1)) + C - g(f(x)) \\ &= (C+1)\cdot(f(x+1)-1) + 1 + C - (C+1)\cdot(f(x)-1) - 1 \\ &= (C+1)\cdot(f(x+1)-f(x)) + C \\ &\leq C\cdot(C+1) + C \\ &= C^2 + 2C\,. \end{aligned}$$

This completes the proof. ◀

Next, we prove a lower bound counterpart of Proposition 12.

▶ **Proposition 13.** *For any two distinct infinite strings $\beta, \gamma \in [\alpha]_{mqi}$, there exists a $\delta \in [\alpha]_{mqi}$ such that $\delta \leq_{1qi} \beta$ and $\delta \leq_{1qi} \gamma$.*

**Proof.** Suppose $\beta = \beta_1\beta_2\ldots$, where $\beta_i \in \Sigma$. Let $f : \mathbb{N} \to \mathbb{N}$ be a $C$-mqi-reduction from $\beta$ to $\gamma$. Now define $\delta = \beta_{i_1}\beta_{i_2}\ldots$, where $i_k$ is the minimum index such that $i_k \neq i_l$ for all $l < k$ and for all $j < i_k$, $f(j) \neq f(i_k)$. By Condition (b), the range of $f$ is infinite and thus each $i_k$ is well-defined. We verify that $\delta \leq_{1qi} \beta$ and $\delta \leq_{1qi} \gamma$.

Define $f'(n) = i_n$ for all $n \in \mathbb{N}$. We show that $f'$ is a 1qi-reduction from $\delta$ to $\beta$. By the choice of the $i_n$'s, $f'(n) > f'(m)$ whenever $n > m$; in particular, $f'$ is injective and Condition (b) holds for $f'$. Furthermore, given any $n$, by applying Condition (b) to $f$ and all $n' \leq n$, it follows that $f'(n+1) \leq f'(n) + C + 1$. Hence $f'$ also satisfies Condition (a).

Next, define a 1qi-reduction $f''$ from $\delta$ to $\gamma$ by $f''(n) = f(i_n)$. The injectiveness of $f''$ follows from the choice of the $i_n$'s (though $f''$ is not necessarily strictly monotone increasing). Using the fact that $i_{n+1} \leq i_n + C + 1$, as well as applying Condition (a) $i_{n+1} - i_n$ times, $d(f''(n+1), f''(n)) = d(f(i_{n+1}), f(i_n)) \leq C \cdot d(i_{n+1}, i_n) \leq C\cdot(C+1)$. Hence $f''$ satisfies Condition (a) with constant $C\cdot(C+1)$. Since the $i_n$'s are strictly increasing, $m + C < n \Rightarrow i_m + C < i_n \Rightarrow f(i_m) < f(i_n)$. Thus $f''$ is a $C\cdot(C+1)$-1qi-reduction.

Lastly, define a mqi-reduction $g$ from $\beta$ to $\delta$ by $g(n) = k$ where $k$ is the minimum integer with $f(n) = f(i_k)$. As the $i_n$'s cover the whole range of $f$, $g$ is well-defined. For any given $n$, suppose $g(n) = k_1$ and $g(n+1) = k_2$, so that $f(n) = f(i_{k_1})$ and $f(n+1) = f(i_{k_2})$. By Condition (b), $d(n, i_{k_1}) \leq C$ and $d(n+1, i_{k_2}) \leq C$, and so

$$\begin{aligned} d(g(n), g(n+1)) &= d(k_1, k_2) \\ &\leq d(i_{k_1}, i_{k_2}) \\ &\leq d(n, i_{k_1}) + d(n, n+1) + d(n+1, i_{k_2}) \\ &\leq 2C + 1\,. \end{aligned}$$

Hence $g$ satisfies Condition (a) with constant $2C+1$. To verify that $g$ satisfies Condition (b) for some constant, fix any $n$ and apply Condition (b) $C\cdot(C+1)$ times to $f$, giving $f(n) + C\cdot(C+1) \leq f(n + C\cdot(C+1)^2)$. Suppose $g(n) = i_{k_1}$ and $g(n + C\cdot(C+1)^2) = i_{k_2}$, so that $f(n) = f(i_{k_1})$ and $f(n + C\cdot(C+1)^2) = f(i_{k_2})$. Then $d(f(i_{k_1}), f(i_{k_2})) = d(f(n), f(n +$

$C \cdot (C+1)^2)) \geq C \cdot (C+1)$. So by applying Condition (a) $d(i_{k_1}, i_{k_2})$ times to $f$, we have $C \cdot d(i_{k_1}, i_{k_2}) \geq d(f(i_{k_1}), f(i_{k_2})) \geq C \cdot (C+1)$. Dividing both sides of the inequality by $C$ yields $d(i_{k_1}, i_{k_2}) \geq C+1$. Applying the contrapositive of Condition (b) to $f$ then gives $f(i_{k_2}) \geq f(i_{k_1}) \Rightarrow i_{k_2} + C \geq i_{k_1}$. Since $d(i_{k_1}, i_{k_2}) \geq C+1$, this implies that $g(n + C \cdot (C+1)^2) = i_{k_2} > i_{k_1} = g(n)$. Thus $g$ satisfies Condition (b) with constant $C \cdot (C+1)^2 - 1$. ◀

## 4.2 1qi-Degrees Within mqi-Degrees

We now investigate the structural properties of 1qi-degrees within individual mqi-degrees. As will be seen shortly, these properties can vary quite a bit depending on the choice of the mqi-degree.

▶ **Proposition 14.** *There exists an infinite string $\alpha$ such that $[\alpha]_{mqi}$ is the union of an infinite ascending chain of 1qi-degrees.*

**Proof.** Let $\Sigma = \{0, 1\}$ and let $\alpha = 10^\omega$. Then $[\alpha]_{mqi}$ consists of all infinite strings with a finite, positive number of occurrences of 1. Given any infinite string $\beta$ with $k \geq 1$ occurrences of 1, $\beta$ is 1qi-equivalent to a string $\gamma$ in $[\alpha]_{mqi}$ iff $\gamma$ has exactly $k$ occurrences of 1. If $1 \leq k < k'$, then each string $\beta \in [\alpha]_{mqi}$ with exactly $k$ occurrences of 1 is 1qi-reducible to any string $\beta' \in [\alpha]_{mqi}$ with exactly $k'$ occurrences of 1. Thus $[\alpha]_{mqi}$ is the union of an ascending chain $[\alpha]_{1qi} < [110^\omega]_{1qi} < [1110^\omega]_{1qi} < \ldots$, where the $i$-th term of this chain is $1^i0^\omega$. ◀

▶ **Proposition 15.** *There exists an infinite string $\alpha$ such that the poset of 1qi-degrees within $[\alpha]_{mqi}$ is isomorphic to $\mathbb{N}^2$ with the componentwise ordering. That is, $[\alpha]_{mqi}$ is the union of infinitely many disjoint infinite ascending chains of 1qi-degrees such that every pair of these ascending chains has incomparable elements. Also, $[\alpha]_{mqi}$ does not contain infinite anti-chains of 1qi-degrees.*

**Proof.** Let $\Sigma = \{0, 1, 2\}$ and let $\alpha = 120^\omega$. Then $[\alpha]_{mqi}$ consists of all infinite strings with a finite, positive number of 1's and a finite, positive number of 2's. Furthermore, $[\alpha]_{1qi}$ consists of all infinite strings with exactly one occurrence of 1 and exactly one occurrence of 2.

Based on the proof of Proposition 14, $[\alpha]_{mqi}$ is the union, over all $k \geq 1$, of chains of the form $[12^k0^\omega]_{1qi} < [1^22^k0^\omega]_{1qi} < \ldots$, where the $i$-th term of each chain is $[1^i2^k0^\omega]_{1qi}$. Given any two chains $\Gamma_j = \{[1^i2^j0^\omega]_{1qi} : i \in \mathbb{N}\}$ and $\Gamma_k = \{[1^i2^k0^\omega]_{1qi} : i \in \mathbb{N}\}$, where $j < k$, the classes $[1^22^j0^\omega]_{1qi} \in \Gamma_j$ and $[12^k0^\omega]_{1qi} \in \Gamma_k$ are incomparable with respect to $\leq_{1qi}$.

It remains to show that any anti-chain of 1qi-degrees contained in $[\alpha]_{mqi}$ must be finite. Consider any anti-chain of 1qi-degrees containing the class $[1^i2^j0^\omega]_{1qi} \subseteq [\alpha]_{mqi}$. Every element of this anti-chain that is different from $[1^i2^j0^\omega]_{1qi}$ is of the form $[1^{i'}2^{j'}0^\omega]_{1qi}$, where either $i < i'$ and $j > j'$, or $i > i'$ and $j < j'$. Thus, if the anti-chain were infinite, then it would contain at least 2 1qi-degrees, $[\beta]_{1qi}$ and $[\gamma]_{1qi}$, such that either $\beta$ has the same number of occurrences of 1 as $\gamma$, or $\beta$ has the same number of occurrences of 2 as $\gamma$. This is a contradiction as it would imply that either $\beta \leq_{1qi} \gamma$ or $\gamma \leq_{1qi} \beta$. ◀

## 4.3 pqi-Reductions

We now discuss pqi-reductions, which are the most stringent kind of quasi-isometric reductions considered in the present work. Pqi-reductions are 1qi-reductions that are surjective; an example of such a reduction is the mapping $2m-1 \mapsto 2m$, $2m \mapsto 2m-1$ from $(01)^\omega$ to $(10)^\omega$. We record a few elementary properties of pqi-reductions.

▶ **Lemma 16.** *If $f$ is a pqi-reduction and if $x + D = f(x)$ for some $D \geq 1$ and some $x \in \mathbb{N}$, then there are at least $D$ positions $y > x$ such that $f(y) < f(x)$.*

**Proof.** If $x + D = f(x)$ for some $D \geq 1$, then $\{1, \ldots, x + D - 1\} \setminus \{f(1), \ldots, f(x-1)\}$ must contain at least $D$ elements as the former set contains $D$ more elements than the latter. Thus, for $f$ to be a bijection, there must exist at least $D$ positions $y > x$ that are mapped by $f$ into $\{1, \ldots, x + D - 1\} \setminus \{f(1), \ldots, f(x-1)\}$. ◀

We next observe that for any pqi-reduction $f$, there is a uniform upper bound on the difference $x - f(x)$.

▶ **Proposition 17.** *If $f$ is a $C$-pqi-reduction, then for all $x \in \mathbb{N}$, $x - f(x) < 2C^2 + 1$.*

**Proof.** Assume, by way of contradiction, that there is some $x \in \mathbb{N}$ such that $x - f(x) \geq 2C^2 + 1$. First, suppose that there are at least $C^2 + 1$ numbers $z$ such that $z > x$ and $f(z) \in \{f(x) + 1, f(x) + 2, \ldots, x - 1\}$. Then there are at least $C^2 + 1$ numbers $z'$ such that $z' < x$ and $f(z') > x > f(x)$, among which there is at least one $z'_0$ with $f(z'_0) \geq x + C^2 + 1$. This would contradict the fact that by the Small Cross-Over Lemma (Lemma 8), $z'_0 < x \Rightarrow f(z'_0) \leq f(x) + C^2 < x + C^2$.

Second, suppose that $f$ maps at most $C^2$ numbers greater than $x$ into $\{f(x) + 1, f(x) + 2, \ldots, x - 1\}$. Then there are at least $C^2 + 1$ numbers less than $x$ that are mapped into $\{f(x) + 1, f(x) + 2, \ldots, x - 1\}$ and in particular, there is at least one number $y < x$ such that $f(y) \geq f(x) + C^2 + 1$, contradicting the Small Cross-over Lemma. Thus for all $x \in \mathbb{N}$, $x - f(x) < 2C^2 + 1$. ◀

Lemma 16 and Proposition 17 together give a uniform upper bound on the absolute difference between any position number and its image under a $C$-pqi-reduction.

▶ **Corollary 18.** *If $f$ is a $C$-pqi-reduction, then for all $x \in \mathbb{N}$, $|x - f(x)| < 2C^2 + 1$.*

**Proof.** By Condition (b), there cannot be more than $C$ numbers $y$ such that $y > x$ and $f(y) < f(x)$. Lemma 16 thus implies that there cannot exist any $D > C$ such that $x + D = f(x)$, and so $f(x) - x \leq C$. Combining the latter inequality with that in Proposition 17 yields $|x - f(x)| < \max\{C + 1, 2C^2 + 1\} = 2C^2 + 1$. ◀

Given any infinite string $\alpha$, it was observed earlier that by the definitions of pqi, 1qi and mqi-reductions, $[\alpha]_{pqi} \subseteq [\alpha]_{1qi} \subseteq [\alpha]_{mqi}$. In the following example, we give instances of strings $\alpha$ where each of the two subset relations is proper or can be replaced with the equals relation.

▶ **Example 19.**

**(a)** $[\alpha]_{pqi} = [\alpha]_{1qi} = [\alpha]_{mqi}$. Set $\alpha = 0^\omega$. For any infinite string $\gamma$ such that $\gamma \leq_{mqi} 0^\omega$, $\gamma$ can only contain occurrences of 0, and therefore $[0^\omega]_{pqi} = [0^\omega]_{1qi} = [0^\omega]_{mqi} = \{0^\omega\}$.

**(b)** $[\alpha]_{1qi} = [\alpha]_{mqi}$ and $[\alpha]_{pqi} \subset [\alpha]_{1qi}$. Set $\alpha = (01)^\omega$. First, $(001)^\omega \leq_{1qi} (01)^\omega$, as witnessed by the 1qi-reduction $3n - 2 \mapsto 4n - 3, 3n - 1 \mapsto 4n - 1, 3n \mapsto 4n$ for $n \in \mathbb{N}$. We also have $(01)^\omega \leq_{1qi} (001)^\omega$ via the 1qi-reduction $2n - 1 \mapsto 3n - 2, 2n \mapsto 3n$ for $n \in \mathbb{N}$. However, $(001)^\omega \notin [(01)^\omega]_{pqi}$ because the density of 0's and 1's in the two strings are different, making it impossible to construct a permutation reduction between them. More formally, if there were a pqi-reduction from $(001)^\omega$ to $(01)^\omega$, then by Corollary 18, there would be a constant $D$ such that for each $n$, the first $3n$ positions of $(001)^\omega$ are mapped into the first $3n + D$ positions of $(01)^\omega$. But the first $3n$ positions of $(001)^\omega$ contain $2n$ occurrences of 0 while the first $3n + D$ positions of $(01)^\omega$ contain at most $\lceil 1.5n + \frac{D}{2} \rceil$ occurrences of 0, and for large enough $n$, one has $2n > \lceil 1.5n + \frac{D}{2} \rceil$. Hence no pqi-reduction from $(001)^\omega$ to $(01)^\omega$ can exist, and so $[\alpha]_{pqi} \subset [\alpha]_{1qi}$.

To see that $[(01)^\omega]_{mqi} \subseteq [(01)^\omega]_{1qi}$, we first note that any string that is mqi-reducible to $(01)^\omega$ (or to any other recursive string) must be recursive. Thus if $\beta \leq_{mqi} (01)^\omega$, then a 1qi-reduction from $\beta$ to $(01)^\omega$ can be constructed by mapping the $n$-th position of $\beta$ to the position of the matching letter in the $n$-th occurrence of 01 in $(01)^\omega$. Next, suppose that $f$ is a $C$-mqi-reduction from $(01)^\omega$ to $\beta$. By Corollary 7, $f$ maps the positions of $(01)^\omega$ to a sequence of positions of $\beta$ that contains 0 and 1 every $2C$ positions. Thus a 1qi-reduction can be constructed from $(01)^\omega$ to $\beta$ by mapping, for each $n$, the $(2n-1)$-st and $(2n)$-th positions of $(01)^\omega$ to the positions of the first occurrence of 0 and first occurrence of 1 respectively in the interval $[2C(n-1)+1, 2Cn]$ of positions of $\beta$. Therefore $\beta \in [(01)^\omega]_{1qi}$.

**(c)** $[\alpha]_{1qi} \subset [\alpha]_{mqi}$ and $[\alpha]_{pqi} = [\alpha]_{1qi}$. Set $\alpha = 10^\omega$. We recall from the proof of Proposition 14 that $[10^\omega]_{pqi}$ and $[10^\omega]_{1qi}$ consist of all binary strings with a single occurrence of 1, while $[10^\omega]_{mqi}$ consists of all binary strings with a finite, positive number of occurrences of 1. Thus $[10^\omega]_{pqi} = [10^\omega]_{1qi}$ and $[10^\omega]_{pqi} \neq [10^\omega]_{mqi}$.

**(d)** $[\alpha]_{pqi} \subset [\alpha]_{1qi} \subset [\alpha]_{mqi}$. Set $\alpha = (0^n 1)_{n=1}^\infty$, the concatenation of all strings $0^n1$ where $n \in \mathbb{N}$. Then $\beta = (0^n 11)_{n=1}^\infty \in [\alpha]_{mqi}$; however, $\beta \notin [\alpha]_{1qi}$ as each pair of adjacent positions of 1's in $\beta$ must be mapped to distinct positions of 1's in $\alpha$, but the distance between the $n$-th and $(n+1)$-st occurrences of 1 in $\alpha$ increases linearly with $n$, meaning that Condition (a) cannot be satisfied.

To construct an mqi-reduction from $\beta$ to $\alpha$, map the positions of the substring $0^n 11$ of $\beta$ to the positions of the substring $0^n 1$ of $\alpha$ as follows: for $k \in \{1, \ldots, n\}$, the position of the $k$-th occurrence of 0 in $0^n 11$ is mapped to that of the $k$-th occurrence of 0 in $0^n 1$, while the two positions of 1's in $0^n 11$ are mapped to the position of the single 1 in $0^n 1$. For an mqi-reduction from $\alpha$ to $\beta$, for each substring $0^n 1$ of $\alpha$ and each substring $0^n 11$ of $\beta$, the positions of $0^n$ in $0^n 1$ are mapped to the corresponding positions of $0^n$ in $0^n 11$, while the position of 1 in $0^n 1$ is mapped to the position of the first occurrence of 1 in $0^n 11$. Thus $\beta \in [\alpha]_{mqi}$.

Furthermore, $\gamma = 1(0^n 1)_{n=1}^\infty \in [\alpha]_{1qi}$ but $\gamma \notin [\alpha]_{pqi}$. The reason for $\gamma$ not being pqi-reducible to $\alpha$ is similar to that given in Example (b). If such a pqi-reduction did exist, then by Corollary 18, there would exist a constant $D$ such that for all $n$, the first $1 + \sum_{k=1}^{n} (k+1) = 1 + \frac{n(n+3)}{2}$ positions of $\gamma$ are mapped into the first $1 + \frac{n(n+3)}{2} + D$ positions of $\alpha$. But the first $1 + \frac{n(n+3)}{2}$ positions of $\gamma$ contain $n+1$ occurrences of 1 and for large enough $n$, the first $1 + \frac{n(n+3)}{2} + D$ positions of $\alpha$ contain at most $n$ occurrences of 1. Hence no pqi-reduction from $\gamma$ to $\alpha$ is possible.

For a 1qi-reduction from $\gamma$ to $\alpha$, map the starting position of $\gamma$, where the letter 1 occurs, to the first occurrence of 1 in $\alpha$. For subsequent positions of $\gamma$, for each $n \geq 1$, the set of positions of $\gamma$ where the substring $0^n 1$ occurs can be mapped in a one-to-one fashion into the set of positions of $\alpha$ where the substring $0^{n+1}1$ occurs. To see that $\alpha$ is 1qi-reducible to $\gamma$, it suffices to observe that $\alpha$ is a suffix of $\gamma$, so one can map the positions of $\alpha$ in a one-to-one fashion to the positions of the suffix of $\gamma$ corresponding to $\alpha$. The 1qi-reduction from $\alpha$ to $\gamma$ is trivial.

Proposition 21 extends the first example in Example 19 by characterising all recursive strings whose pqi, 1qi and mqi-degrees all coincide. In fact, there are only $|\Sigma|$ many such strings: those of the form $a_i^\omega$, where $a_i \in \Sigma$. We call the pqi, 1qi and mqi-degrees of such strings *trivial*. Due to space constraint, the proof of Proposition 21 is omitted.

▶ **Definition 20.** *The pqi, 1qi and mqi-degrees of each string $a_i^\omega$, where $a_i \in \Sigma$, will be called* trivial *pqi, 1qi and mqi-degrees respectively.*

▶ **Proposition 21.** *If, for some recursive string $\alpha$, $[\alpha]_{pqi} = [\alpha]_{1qi} = [\alpha]_{mqi}$, then all three degree classes are trivial.*

We observe next that every non-trivial pqi degree must be infinite.

▶ **Proposition 22.** *All non-trivial pqi-degrees are infinite.*

**Proof.** Suppose that at least two distinct letters occur in $\alpha$. Fix a letter, say $a_1$, that occurs infinitely often in $\alpha$. Let $a_2$ be a letter different from $a_1$ that occurs in $\alpha$. For each $n \in \mathbb{N}$, let $\beta_n = a_1^n a_2 \alpha^{(n+1)}$, where $\alpha^{(n+1)}$ is obtained from $\alpha$ by removing the first occurrence of $a_2$ as well as the first $n$ occurrences of $a_1$. Since $\beta_n$ is built from $\alpha$ by permuting the letters occurring at a finite set of positions of $\alpha$, $\beta_n \in [\alpha]_{pqi}$. As the $\beta_n$'s are all distinct, it follows that $[\alpha]_{pqi}$ is indeed infinite. ◀

We close this subsection by illustrating an application of Proposition 22, showing that if the mqi-degree of $\alpha$ contains at least two distinct strings such that one is 1qi-reducible to the other, then the first string is 1qi-reducible to infinitely many strings in $[\alpha]_{mqi}$.

▶ **Proposition 23.** *If there exist distinct $\beta \in [\alpha]_{mqi}$ and $\gamma \in [\alpha]_{mqi}$ such that $\beta \leq_{1qi} \gamma$, then $\beta$ is 1qi-reducible to infinitely many strings in $[\alpha]_{mqi}$.*

**Proof.** Suppose that $\beta \leq_{1qi} \gamma$ and $\beta \neq \gamma$ for some $\beta \in [\alpha]_{mqi}$ and $\gamma \in [\alpha]_{mqi}$. Then $[\alpha]_{mqi}$ is non-trivial, so by Proposition 22, $[\gamma]_{pqi}$ is infinite. Since $[\gamma]_{pqi} \subseteq [\gamma]_{1qi}$, $[\gamma]_{1qi}$ is also infinite. Thus $\beta$ is 1qi-reducible to each of the infinitely many strings in $[\gamma]_{1qi}$. ◀

## 4.4 The Partial Order of All mqi-Degrees

As discussed earlier, Khoussainov and Takisaka [6] observed that for any alphabet $\Sigma = \{a_1, \ldots, a_l\}$, the partial order $\Sigma^\omega_{qi}$ has a greatest element equal to $[(a_1 \cdots a_l)^\omega]_{qi}$. Their proof also extends to the partial order of all recursive mqi-degrees, showing that for each recursive string $\alpha$, $[\alpha]_{mqi} \leq_{mqi} [(a_1 \cdots a_l)^\omega]_{mqi}$. We next prove that there is a pair of recursive mqi-degrees whose join is precisely the maximum recursive mqi-degree $[(a_1 \cdots a_l)^\omega]_{mqi}$.

▶ **Proposition 24.** *Suppose that $\Sigma = \{a_1, \ldots, a_l\}$. Then there exist two distinct infinite strings $\alpha$ and $\beta$ such that $[(a_1 \cdots a_l)^\omega]_{mqi}$ is the unique recursive common upper bound of $[\alpha]_{mqi}$ and of $[\beta]_{mqi}$ under $\leq_{mqi}$.*

**Proof.** Let $\alpha = (a_1)^\omega$ and $\beta = (a_2 a_3 \cdots a_l)^\omega$. Suppose that for some recursive string $\gamma$, $\alpha \leq_{mqi} \gamma$ via a $C$-mqi-reduction. Since $a_1$ is the only letter occurring in $\alpha$, Condition (a) implies that there must be at least one occurrence of $a_1$ in $\gamma$ every $C$ positions. Similarly, if $\beta \leq_{mqi} \gamma$ via a $C'$-mqi-reduction, then for each $a_i$ with $i \geq 2$, since $a_i$ occurs every $l-1$ positions, it must also occur in $\gamma$ every $C' \cdot (l-1)$ positions. Hence there exists a constant $C''$ such that every substring of $\gamma$ of length $C''$ contains at least one occurrence of $a_i$ for every $i \in \{1, \ldots, l\}$, and therefore $(a_1 \cdots a_l)^\omega \leq_{mqi} \gamma$. Since $\gamma \leq_{mqi} (a_1 \cdots a_l)^\omega$ follows from the proof of [6, Proposition II.1], one has $\gamma \in [(a_1 \cdots a_l)^\omega]_{mqi}$, as required. ◀

Khoussainov and Takisaka [6] showed that the partial order $\Sigma^\omega_{qi}$ is not dense. In particular, given any distinct $a_i, a_j \in \Sigma$, there is no element $[\beta]_{qi}$ that is strictly between the minimal element $[(a_j)^\omega]_{qi}$ and the "atom" $[a_i(a_j)^\omega]_{qi}$ [6, Proposition II.1]. The next theorem shows similarly that the partial order $\Sigma^\omega_{mqi}$ is non-dense with respect to pairs of mqi-degrees. The high-level proof of the theorem is given below; the proofs of claims used are not shown due to space limitations.

▶ **Theorem 25.** *There exist two pairs $(\alpha, \beta)$ and $(\gamma, \delta)$ of recursive strings such that both $\alpha$ and $\beta$ are mqi-reducible to $\gamma$ as well as mqi-reducible to $\delta$, but there is no string $\xi$ such that $\alpha \leq_{mqi} \xi, \beta \leq_{mqi} \xi, \xi \leq_{mqi} \gamma$ and $\xi \leq_{mqi} \delta$.*

**Proof.** Let $\Sigma = \{0, 1\}$. Define the strings

$$\alpha = \sigma_1\sigma_2 \ldots, \quad \text{where } \sigma_n = (01)^{2^{2^n}} 0^n 1^n ;$$

$$\beta = \tau_1\tau_2 \ldots, \quad \text{where } \tau_n = (01)^{2^{2^n}} 1^n 0^n ;$$

$$\gamma = \mu_1\mu_2 \ldots, \quad \text{where } \mu_n = (01)^{2^{2^n}} 0^n ;$$

$$\delta = \nu_1\nu_2 \ldots, \quad \text{where } \nu_n = (01)^{2^{2^n}} 1^n .$$

We first show that $\alpha \leq_{mqi} \gamma$. For each $i \in \mathbb{N}$, define the following intervals of positions.

- $K_i = [k_i, k_i + 2^{2^i+1} - 1]$ is the interval of positions of the substring $(01)^{2^{2^i}}$ of $\sigma_i$ in $\alpha$.
- $R_i = [r_i, r_i + 2i - 1]$ is the interval of positions of the substring $0^i1^i$ of $\sigma_i$ in $\alpha$.
- $L_i = [l_i, l_i + 2^{2^i+1} - 1]$ is the interval of positions of the substring $(01)^{2^{2^i}}$ of $\mu_i$ in $\gamma$.
- $L'_i = [l_i + 2^{2^i+1}, l_i + 2^{2^i+1} + i - 1]$ is the interval of positions of the substring $0^i$ of $\mu_i$ in $\gamma$.

Define an mqi-reduction $g$ from $\alpha$ to $\gamma$ as follows. For $i \in \mathbb{N}$,

$$g(k_i + 4w + 2u + x) = l_i + 2(i-1) + 2w + x\,, \quad 0 \leq w \leq i-2, u, x \in \{0,1\}\,;$$

$$g(k_i + m) = l_i + m\,, \quad 4i - 4 \leq m \leq 2^{2^i+1} - 1\,;$$

$$g(r_i + m) = l_i + 2^{2^i+1} + m\,, \quad 0 \leq m \leq i-1\,;$$

$$g(r_i + i + m) = l_{i+1} + 2m + 1\,, \quad 0 \leq m \leq i-1.$$

The mqi-reduction $g$ maps the interval $K_i$ to the suffix of the interval $L_i$ starting at its $(2i-1)$-st position such that each of the first $i-1$ pairs of positions of this suffix is the image of two consecutive pairs of positions of $K_i$, while the remaining $|K_i| - 2(i-1)$ positions of $K_i$ is mapped by $g$ to the remaining positions of the suffix of $L_i$ in a one-to-one fashion. Thus $g$ is a 4-mqi-reduction from $\alpha$ to $\gamma$. A similar mqi-reduction can be constructed from $\beta$ to $\gamma$, from $\alpha$ to $\delta$, as well from $\beta$ to $\delta$.

Assume, by way of contradiction, that there is a string $\xi$ and there are mqi-reductions $f_1$ from $\alpha$ to $\xi$, $f_2$ from $\beta$ to $\xi$, $f_3$ from $\xi$ to $\gamma$ and $f_4$ from $\xi$ to $\delta$ with constants $C_1, C_2, C_3$ and $C_4$ respectively. Set $C = \max\{C_1, C_2, C_3, C_4\}$ and fix some $n > 2C^7 + 1$.

For $i \in \mathbb{N}$, let $K'_i = [k_i + C^2 + 2, k_i + 2^{2^i+1} - 3 - C^2]$ be the interval obtained from $K_i$ by removing the first and last $C^2 + 2$ positions. We make the following observation. The proof is omitted due to space constraint.

▷ Claim 26. For all positions $m \in K'_n$, for $i \in \{1, 2\}$ and $j \in \{3, 4\}$, $f_j(f_i(m)) \in L_n$.

Define the sets $H_i = f_1(K'_i) \cup f_2(K'_i)$ for $i \in \mathbb{N}$. We show that the sets $H_n$ and $H_{n+1}$ are non-overlapping by proving $\max(H_n) < \min(H_{n+1})$. By Claim 26, for all $m \in H_n$ and $j \in \{3, 4\}$ we have $f_j(m) \in L_n$. Then, we have $f(\min(H_{n+1})) - f(\max(H_n)) \geq \min(L_{n+1}) - \max(L_n) = n + 1 > 2C^7 + 2 > C^2$. So by the Small Cross-Over Lemma, $\min(H_{n+1}) > \max(H_n)$.

Consider the interval $[\min(H_n), \max(H_n)]$ in the domain of $\xi$. By Claim 26, $f_3$ (resp. $f_4$) maps each element of $H_n$ into $L_n$. Fix any other position $z$ in the interval. Then $f_3$ cannot map $z$ into $L'_n$, which is the set of positions in $\gamma$ of the string $0^n$. To see this, we note that if $\ell$ and $\ell + 1$ are the two largest values of $K_n$, then $\ell$ is at least $C^2 + 1$ more than the value $x$ such that $f_i(x) = \max(H_n)$ for some $i \in \{1, 2\}$, and so by Condition (b), $z + C < \max(H_n) + C < f_k(\ell + 1)$ for $k \in \{1, 2\}$. Thus $f_3(z) < f_3(f_k(\ell + 1))$ for

$k \in \{1,2\}$. Furthermore, by applying Condition (a) repeatedly to $f_3$ and then to $f_k$, we have $d(f_3(f_k(\ell+1)), f_3(f_k(\max(K'_n)))) \leq C \cdot d(f_k(\ell+1), f_k(\max(K'_n))) \leq C^2 \cdot (C^2+2) = C^4+2C^2$. Since $f_3(f_k(\max(K'_n))) \in L_n$ and we fixed $n > 2C^7 + 1$, then $f_3(f_k(\ell+1)) \notin L_{n+1}$. Furthermore, the letter at position $f_3(f_k(\ell+1))$ of $\gamma$ is 1. Thus $f_3(z)$ cannot lie in $L'_n$ as there is no occurrence of 1 in $L'_n$. A similar argument, using position $\ell$ rather than position $\ell+1$, shows that $f_4(z)$ cannot lie in $L'_n$. One can also prove similarly that none of the positions in the interval $[\min(H_{n+1}), \max(H_{n+1})]$ is mapped by $f_3$ or $f_4$ into the interval $L'_n$.

Next, we consider the positions of $\xi$ between $\max(H_n)$ and $\min(H_{n+1})$. Since none of the positions of $\xi$ in the union $[\min(H_n), \max(H_n)] \cup [\min(H_{n+1}), \max(H_{n+1})]$ is mapped by $f_3$ into $L'_n$ and $L'_n$ is an interval of length $n > 2C^7$, Lemma 6 implies that there are at least $\lfloor \frac{n}{C_3} \rfloor$ positions of $\xi$ between $H_n$ and $H_{n+1}$ which are mapped into $L'_n$. Then, we can make the following observations – the proofs of which are omitted due to space constraint.

▷ Claim 27. The string $\xi$ contains a substring of 0's (resp. 1's) of length $\Omega(C^4)$ between $H_n$ and $H_{n+1}$ such that all positions of this substring are mapped by $f_3$ (resp. $f_4$) into $L'_n$.

▷ Claim 28. There cannot exist between $H_n$ and $H_{n+1}$ two $\Omega(C^4)$-long substrings of 0's (resp. 1's) such that an $\Omega(C^4)$-long substring of 1's (resp. 0's) lies between them.

Based on these two claims, there are exactly two maximal intervals $J_1$ and $J_2$, each of length $\Omega(C^4)$, such that the substrings of $\xi$ occupied by $J_1$ and $J_2$ belong to $\{0\}^*$ and $\{1\}^*$ respectively. Then $f_1$ maps $\Omega(C^3)$ positions of $[r_n, r_n+n-1]$ into $J_1$ and $\Omega(C^3)$ positions of $[r_n+n, r_n+2n-1]$ into $J_2$; further, there are two positions that are $\Omega(C^3)$ positions apart, one in $[r_n, r_n+n-1]$ and the other in $[r_n+n, r_n+2n-1]$, such that $f_1$ maps the first position into $J_1$ and the second position into $J_2$. This implies that $J_1$ must precede $J_2$, for otherwise Condition (b) would be violated. Arguing similarly with $f_2$ in place of $f_1$ (that is, the mapping from $\beta$ to $\xi$), it follows that $J_2$ must precede $J_1$, a contradiction. We conclude that the string $\xi$ cannot exist. ◀

Example 19 established separations between various notions of recursive quasi-reducibility: pqi, 1qi and mqi-reducibilities. It remains to separate general quasi-isometry from its recursive counterpart. Due to space constraint, we only give a proof sketch of Theorem 29.

▶ **Theorem 29.** *There exist two recursive strings $\alpha$ and $\beta$ such that $\alpha \leq_{qi} \beta$ but $\alpha \not\leq_{mqi} \beta$.*

**Proof sketch.** We begin with an overview of the construction of $\alpha$ and $\beta$. To ensure that only non-recursive quasi-isometries between $\alpha$ and $\beta$ exist, we use a tool from computability theory, which is a Kleene tree [8] – an infinite uniformly recursive binary tree with no infinite recursive branches (see, for example, [11, §V.5]). The idea of the proof is to encode a fixed Kleene tree into $\beta$, and construct $\alpha$ such that for any quasi-isometry $f$ from $\alpha$ to $\beta$, an infinite branch of the encoded Kleene tree can be computed recursively from $f$. Hence, $f$ cannot be recursive, as otherwise the chosen infinite branch of the Kleene tree must be recursive, contradicting the definition of a Kleene tree.

We now describe the construction of $\alpha$ and $\beta$ based on some fixed Kleene tree $T \subseteq \{0,1\}^*$. The building blocks for $\alpha$ and $\beta$ are called *$n$-blocks*, which are strings of the following form for some $n \in \mathbb{N}$:

- $\lambda_{(n,0)} = 0^n 1^n$,
- $\lambda_{(n,i)} = 0^{\lfloor \frac{n+1}{2} \rfloor} 1^i 0^{\lceil \frac{n+1}{2} \rceil} 1^n$, for $1 \leq i \leq n-1$,
- $\lambda'_n = (01)^n 1^n$,

We may also call an $n$-block as simply a *block*.

To construct $\alpha$ and $\beta$ we concatenate the blocks in stages, where in stage $n$ we arrange $n$-blocks to form $\theta_n$ and $\zeta_n$. Then, $\theta_n$ and $\zeta_n$ for $n \in \mathbb{N}$ are concatenated to form $\alpha$ and $\beta$ respectively. That is, $\alpha = \theta_1\theta_2 \dots$ and $\beta = \zeta_1\zeta_2 \dots$ where $\theta_n$ and $\zeta_n$ are defined as follows:

- $\theta_1 = \zeta_1 = \lambda_{(1,0)}$.
- For $n \geq 2$, $\theta_n = v_{n,1}\ s_{n,1}\ v_{n,2}\ s_{n,2}\ v_{n,3}\ t_n\ v_{n,4}\ u_n$.
- For $n \geq 2$, $\zeta_n = v'_{n,1}\ s'_{n,1}\ v'_{n,2}\ s'_{n,2}\ v'_{n,3}\ t'_n\ v'_{n,4}\ u'_n$.

We now state the definitions of each variables $v_{n,1}$ and so on, and explain their purpose later. For $n \geq 2$, we define the following:

- Let $B^n_1, B^n_2, B^n_3, B^n_4, B^n_5, B^n_6, B^n_7, B^n_8$ be the number of blocks in $\alpha$ before the start of $v_{n,1},\ s_{n,1},\ v_{n,2},\ s_{n,2},\ v_{n,3},\ t_n,\ v_{n,4},\ u_n$ respectively.
- *Join segments* $v_{n,i} = v'_{n,i} = (\lambda_{(n,0)})^{3nB^n_{2i-1}}$.
- *Scaling segments* $s_{n,1}v_{n,2}s_{n,2}$ and $s'_{n,1}v'_{n,2}s'_{n,2}$, each containing two of the *scaling parts* $s_{n,i} = s'_{n,i} = (\lambda_{(n,1)})^{nB}(\lambda_{(n,2)})^{nB} \dots (\lambda_{(n,n-1)})^{nB}(\lambda_{(n,0)})^{2nB}$ where $B$ means $B^n_{2i}$.
- *Branching segments* $t_n = (\lambda_{(n,0)})^{2nB^n_6+1}$ and $t'_n = (\lambda_{(n,0)})^{2nB^n_6}\lambda'_n$.
- *Selection segments* $u_n$ and $u'_n$ defined as follows. Let

$$S_n = \left\{ \sum_{m=1}^{n-1} b_m 4^{n-1-m} : b_1 \cdots b_{n-1} \in T \cap \{0,1\}^{n-1} \right\}.$$

  Then, each element $\sum_{m=1}^{n-1} b_m 4^{n-1-m}$ of $S_n$ corresponds to the binary string $b_1 \cdots b_{n-1} \in T$. Define $u_n = \lambda_{(n,1)}(\lambda_{(n,0)})^{\max(S_n)}$. For $1 \leq i \leq \max(S_n) + 1$, let the $i$-th block of $u'_n$ be $\lambda_{(n,0)}$ if $i - 1 \notin S_n$ and be $\lambda_{(n,1)}$ if $i - 1 \in S_n$.

Note that the number of blocks in the respective segments of $\alpha$ and $\beta$ are the same. So, for example, the number of blocks in $\alpha$ before $v_{n,i}$ is the same as that of $\beta$ before $v'_{n,i}$.

Before we explain the purpose of each segment, we first describe some useful properties of the $n$-blocks. The proofs are omitted due to space restrictions. Let $f$ be any $C$-quasi-isometric reduction from $\alpha$ to $\beta$. For all large enough $n$:

- No position of the $i$-th occurrence of an $n$-block in $\alpha$ is mapped by $f$ to the position of an $m$-block in $\beta$ with $m < n$.
- No position of a block $\lambda_{(n,1)}$ is mapped by $f$ to a position of a block $\lambda_{(n,0)}$ occurring in $\beta$.
- For $i \leq n-2$, $f$ maps the sequence of positions of each $\lambda_{(n,i)}$ block in a scaling segment of $\theta_n$ into either the sequence of positions of a $\lambda_{(n,i)}$ block in a scaling segment of $\zeta_n$ or the sequence of positions of a $\lambda_{(n,i+1)}$ block in a scaling segment of $\zeta_n$.
- $f$ can map the sequence of positions of a $\lambda_{(n,n-1)}$ block into the sequence of positions of exactly $k$ blocks $\lambda_{(n,0)}$ iff $k = 2$.
- Up to $C + 1$ blocks $\lambda_{(n,0)}$ can be mapped to a single $\lambda'_n$ block.
- A single $\lambda_{(n,0)}$ block can be mapped across $\lambda_{(n,0)}\lambda'_n$.

We can now describe the purpose of each segment. As above, the proofs are omitted due to space restrictions. Let $f$ be a $C$-quasi-isometric reduction from $\alpha$ to $\beta$. For sufficiently large $n \geq 2$:

- Each join segment $v_{n,i}$ has a non-negative *lead* $\ell$ such that for all $nB^n_{2i-1}+1 \leq j \leq 2nB^n_{2i-1}$, $f$ maps the $j$-th $\lambda_{(n,0)}$ block of $v_{n,i}$ to the $(j+\ell)$-th $\lambda_{(n,0)}$ block of $v'_{(n,i)}$.
- The scaling part doubles the lead in the previous join segment. Since a scaling segment contains two scaling parts, the scaling segment multiplies the lead by four. Hence, if the lead of $v_{n,1}$ is $\ell$, then the lead of $v_{n,3}$ is $4\ell$.
- The branching segment ensures that the lead is decreased by at most $C$ or increased by 1. So, if the lead of $v_{n,3}$ is $4\ell$, then the lead of $v_{n,4}$ is between $4\ell - C$ and $4\ell + 1$ inclusive.
- The selection segment ensures that the lead in the previous join segment $v_{n,4}$ is in $S_n$.

Then, one can show that there is some constant $c$ such that for large enough $n$, the leads $\ell_n$ and $\ell_{n+1}$ of the join segments $v_{n,4}$ and $v_{n+1,4}$ are contained in $S_n$ and $S_{n+1}$ respectively, and the binary strings $\sigma_n \in T \cap \{0,1\}^{n-1}$ and $\sigma_{n+1} \in T \cap \{0,1\}^n$, corresponding to $\ell_n$ and $\ell_{n+1}$ respectively have a common prefix of length $n-c$. Let $\tau_n$ be the prefix of $\sigma_n$ of length $n-c$. Then, for some large enough $n$, $\tau_n \prec \tau_{n+1} \prec \tau_{n+2} \prec \dots$ give an infinite branch of the Kleene tree $T$, which is non-recursive by definition of Kleene trees. Moreover, this infinite branch can be computed recursively from $f$. Hence, the quasi-isometric reduction $f$ from $\alpha$ to $\beta$ must be non-recursive. Hence, $\alpha \not\leq_{mqi} \beta$.

We now describe how to construct a quasi-isometric reduction $f$ from $\alpha$ to $\beta$, using some fixed infinite branch $\mathcal{B}(1)\mathcal{B}(2)\cdots$ of the Kleene tree. Since $\theta_1 = \zeta_1 = \lambda_{(n,0)}$, we can map $\theta_1$ to $\zeta_1$ in a strictly increasing manner and the lead in the next segment is 0. We can now describe the mappings for each segment in $\theta_n$ for $n \geq 2$. Observe that each block $\lambda_{(n,i)}$ can be mapped to a block $\lambda_{(n,j)}$ in a strictly increasing manner if $j = i$ or $i+1$. For each join segment $v_{(n,i)}$ with lead $\ell_1$ and $3nB^n_{2i-1}$ blocks, map the first $3nB^n_{2i-1} - \ell_1$ blocks of $v_{n,i}$ to the last $3nB^n_{2i-1} - \ell_1$ blocks of $v'_{(n,i)}$. Then, map the last $\ell_1$ blocks of $v_{n,i}$ to the first $\ell_1$ blocks of the following segment in $\zeta_n$. The lead in the next segment is $\ell_1$.

Next we describe the mapping for scaling part $s_{n,i}$ with lead $\ell_2$. For each $1 \leq j \leq (n-1)nB^n_{2i} - \ell_2$, map the $j$-th block of $s_{n,i}$ to the $(j+\ell_2)$-th block of $s'_{n,i}$. Map each of the last $\ell_2$ blocks $\lambda_{(n,n-1)}$ of $s_{n,i}$ to 2 blocks $\lambda_{(n,0)}$ of $s'_{n,i}$. Map the first $2nB^n_{2i} - 2\ell_2$ blocks $\lambda_{(n,0)}$ of $s_{n,i}$ to the remaining $2nB^n_{2i} - 2\ell_2$ blocks $\lambda_{(n,0)}$ of $s'_{n,i}$. Map the last $2\ell_2$ blocks $\lambda_{(n,0)}$ of $s_{n,i}$ to the first $2\ell_2$ blocks of the following join segment in $\zeta_n$. The lead of the next join segment is $2\ell_2$.

For the branching segment, suppose that the current lead is $\ell_3$. If $\mathcal{B}(n-1) = 1$, map the $(2nB^n_6 - \ell_3)$-th $\lambda_{(n,0)}$ block to the concatenation $\lambda_{(n,0)}\lambda'_n$ of two blocks in $t'_n v_{n,4}$. Otherwise, map the $(2nB^n_6 - \ell_3 + 1)$-st $\lambda_{(n,0)}$ block to the $\lambda'_n$ block in $t'_n$. Map the rest of the $\lambda_{(n,0)}$ blocks such that $f$ is strictly increasing. Then, the lead of the next join segment is $\ell_3 + \mathcal{B}(n-1)$.

For the selection segment, suppose that the current lead is $\ell_4$. Map the $\lambda_{(n,1)}$ block to the $(\ell_4+1)$-st block. By induction, $\ell_4 \in S_n$ and so the $(\ell_4+1)$-st block in the selection segment of $\zeta_n$ is $\lambda_{(n,1)}$. Map the $\lambda_{(n,0)}$ blocks in a strictly increasing manner. ◀

## 5 Conclusions and Future Investigations

The present paper introduced finer-grained notions of quasi-isometries between infinite strings, in particular requiring the reductions to be recursive. We showed that permutation quasi-isometric reductions are provably more restrictive than one-one quasi-isometric reductions, which are in turn provably more restrictive than many-one quasi-isometric reductions. One result was that general many-one quasi-isometries are strictly more powerful than recursive many-one quasi-isometries, which answers Khoussainov and Takisaka's open problem.

This work also presented some results on the structures of the permutation, one-one and many-one quasi-isometric degrees. It was shown, for example, that there are two infinite strings whose many-one quasi-isometric degrees have a unique common upper bound. It was also proven that the partial order $\Sigma^\omega_{mqi}$ is non-dense with respect to pairs of mqi-degrees. We conclude with the simple observation that the class of mqi-degrees does not form a lattice; in particular, the mqi-degrees $[0^\omega]_{mqi}$ and $[1^\omega]_{mqi}$ do not have a common lower bound.

For future work, we consider an automata-theoretic version of quasi-isometric reduction. Note that Definition 4 defines quasi-isometric reduction based only on the ordering of the natural numbers, as well as adding and subtracting constants. Then we can define an automatic version of Definition 4 for any automatically-ordered set $(A, \leq_A)$ which is order-isomorphic to $(\mathbb{N}, \leq)$: we replace natural number $i$ with the $i$-th smallest element of $A$, and study automatic functions $\mu, \nu : A \to \Sigma$ instead of infinite recursive strings $\alpha, \beta : \mathbb{N} \to \Sigma$. (The complete definitions of automatic functions and relations can be found in [3, 4, 5].) Note that the successor function is first-order definable in $(A, \leq_A)$ as follows: $succ(x) = x + 1 \coloneqq \min\{y : x <_A y\}$. So, adding and subtracting constants are automatic as well. Then this definition corresponds to the quasi-isometric reduction between colored metric spaces $(A; d_A, \mu)$ and $(A; d_A, \nu)$ where the metric function $d_A$ is defined in terms of the order-isomorphism $g$ from $(A, \leq_A)$ to $(\mathbb{N}, \leq)$; that is, $d_A(x, y) = |g(x) - g(y)|$.

Now one can ask, for which automatic functions $\mu, \nu : A \to \Sigma$, is there a quasi-isometric reduction from $\mu$ to $\nu$? And can every quasi-isometric reduction between some given $\mu, \nu$ be replaced by an automatic quasi-isometric reduction? The journal version of this paper will also study this automatic setting and show that the answer to the second question depends on $(A, \leq_A)$. Moreover, the expressibility, that is, which mappings from $\mathbb{N}$ to $\Sigma$ correspond to automatic functions from $A$ to $\Sigma$ in $(A, \leq_A)$, also depends on the choice of $(A, \leq_A)$.

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