

Preservation Theorems on Sparse Classes Revisited

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Abstract

We revisit the work studying homomorphism preservation for first-order logic in sparse classes of structures initiated in [Atserias et al., JACM 2006] and [Dawar, JCSS 2010]. These established that first-order logic has the homomorphism preservation property in any sparse class that is monotone and addable. It turns out that the assumption of addability is not strong enough for the proofs given. We demonstrate this by constructing classes of graphs of bounded treewidth which are monotone and addable but fail to have homomorphism preservation. We also show that homomorphism preservation fails on the class of planar graphs. On the other hand, the proofs of homomorphism preservation can be recovered by replacing addability by a stronger condition of amalgamation over bottlenecks. This is analogous to a similar condition formulated for extension preservation in [Atserias et al., SiCOMP 2008].

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1 Introduction

Preservation theorems have played an important role in the development of finite model theory. They provide a correspondence between the syntactic structure of first-order sentences and their semantic behaviour. In the early development of finite model theory it was noted that many classical preservation theorems fail when we limit ourselves to finite structures. An important case in point is the Łoś-Tarski or *extension* preservation theorem, which asserts that a first-order formula is preserved by embeddings between all structures if, and only if, it is equivalent to an existential formula. Interestingly, this was shown to fail on finite structures [15] much before the question attracted interest in finite model theory [12]. On the other hand, the *homomorphism* preservation theorem, asserting that formulas preserved by homomorphisms are precisely those equivalent to existential-positive ones, was remarkably shown to hold on finite structures by Rossman [14], spurring applications in constraint satisfaction and database theory.

However, even before Rossman's celebrated result, these preservation properties were investigated on subclasses of the class of finite structures. In the case of both the extension and homomorphism preservation theorems, the direction of the theorem stating that the syntactic restriction implies the semantic closure condition is easy and holds in restriction to any class of structures. It is the other direction that may fail and, restricting to a subclass weakens both the hypothesis and the conclusion, therefore leading to an entirely new question. Thus, while the class of all finite structures is combinatorially wild, it contains *tame* classes which are both algorithmically and model-theoretically better behaved [6]. A study of



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preservation properties for such restricted classes of finite structures was initiated in [3] and [2], which looked at homomorphism preservation and extension preservation respectively. The focus was on tame classes defined by *sparsity* conditions, which allows for methods based on the *locality* of first-order logic. In particular, the sparsity conditions were based on what have come to be called wideness conditions.

We recall the formal definition of wideness in Section 4 below but, informally, a class of structures \mathcal{C} is called *wide* if in any large enough structure in \mathcal{C} we can find a large set of elements that are pairwise far away from each other. The class \mathcal{C} is *almost wide* if there is a constant s so that in any large enough structure in \mathcal{C} , removing at most s elements gives a structure in which we can find a large set of elements that are pairwise far away from each other. Finally, \mathcal{C} is said to be *quasi-wide* if there is a function s so that in any large enough structure in \mathcal{C} , removing at most $s(d)$ elements gives a structure in which we can find a large set of elements that are pairwise at distance d from each other. In the latter two cases, we refer to a set of elements whose removal yields a large scattered set as a *bottleneck* set.

The main result asserted in [3] is that homomorphism preservation holds in any class \mathcal{C} which is almost wide and is *monotone* (i.e. closed under substructures) and *addable* (i.e. closed under disjoint unions). From this, it is concluded that homomorphism preservation holds for any class \mathcal{C} whose Gaifman graphs exclude some graph G as a minor, as long as \mathcal{C} is monotone and addable. The result was extended from almost wide to quasi-wide classes in [7], from which homomorphism preservation was deduced for classes that locally exclude minors and classes that have bounded expansion, again subject to the proviso that they are monotone and addable. Quasi-wide classes were later identified with *nowhere dense* classes, which are now central in structural and algorithmic graph theory [13].

The main technical construction in [3] is concerned with showing that classes of graphs which exclude a minor are indeed almost wide. The fact that homomorphism preservation holds in monotone and addable almost wide classes is deduced from a construction of Ajtai and Gurevich [1] which shows the “density” of minimal models of a first-order sentence preserved by homomorphisms, and the fact that in an almost wide class a collection of such dense models must necessarily be finite. While the Ajtai and Gurevich construction is carried out within the class of all finite structures, it is argued in [3] that it can be carried out in any monotone and addable class because of “the fact that disjoint union and taking a substructure are the only constructions used in the proof” [3, p. 216]. This argument is sketched in a bit more detail in [7, Lemma 7]. The starting point of the present paper is that this argument is flawed. The construction requires us to take not just disjoint unions, but unions that identify certain elements: in other words *amalgamations* over sets of points. On the other hand, we can relax the requirement of monotonicity to just hereditariness (i.e. closure under induced substructures). The conclusion is that homomorphism preservation holds in any class \mathcal{C} that is quasi-wide, hereditary and closed under amalgamation over bottleneck points. The precise statement is given in Theorem 9 below. We also show that the requirements formulated in [3] are insufficient by constructing a class that is almost wide (indeed, has bounded treewidth), is monotone and addable, but fails to have the homomorphism preservation property.

Interestingly, the requirement of amalgamations over bottlenecks is similar to that used to define classes on which the extension preservation property holds in [2], even though the construction uses rather different methods. The result there can be understood, in our terms, as showing that the extension preservation theorem holds in any almost wide, hereditary class with amalgamation over bottlenecks. As we observe below (in Corollary 3), this implies that homomorphism preservation holds in all such classes. Our Theorem 9 then strengthens this to quasi-wide classes, where we do not know if extension preservation holds. The class

of planar graphs is an interesting case as it is used in [2] as an example of a hereditary, addable class with excluded minors in which extension preservation fails. We show here that homomorphism preservation also fails in this class, strengthening the result of [2].

In the rest of this paper, we introduce notation and necessary background in Section 2. We construct a monotone, addable class of graphs of small treewidth in Section 3, providing the first counterexample to the claims of [3]. Section 4 states and proves the corrected version of the preservation theorems and Section 5 shows the failure of homomorphism preservation in the class of planar graphs.

2 Preliminaries

We assume familiarity with the standard notions of finite model theory and structural graph theory, and refer to [10] and [13] for reference. We henceforth fix a finite relational vocabulary τ ; by a structure we implicitly mean a τ -structure. We often abuse notation and do not distinguish between a structure and its domain. Given two structures A, B , a homomorphism $f : A \rightarrow B$ is a map such that for all relation symbols R and tuples \bar{a} from A we have $\bar{a} \in R^A \implies f(\bar{a}) \in R^B$. If moreover $f(\bar{a}) \in R^B \implies \bar{a} \in R^A$ then f is said to be *strong*. An injective strong homomorphism is called an *embedding*. We also call a homomorphism $f : A \rightarrow B$ *full* if it is surjective and for any relation symbol R and tuple \bar{b} from B we have $\bar{b} \in R^B \implies \exists \bar{a} \in R^A$ with $f(\bar{a}) = \bar{b}$.

A structure B is said to be a *weak substructure* of a structure A if $B \subseteq A$ and the inclusion map $\iota : B \hookrightarrow A$ is a homomorphism. Likewise, B is an *induced substructure* of A if the inclusion map is an embedding. Given a structure A and a subset $S \subseteq A$ we write $A[S]$ for the unique induced substructure of A with underlying set S . An induced substructure B of A is said to be *free in A* if there is some structure C such that A is the disjoint union $B + C$. Finally, a substructure B of A is said to be *proper* if the inclusion map is not full. We say that a class of structures is *monotone* if it is closed under weak substructures, and it is *hereditary* if it is closed under induced substructures. Moreover a class is called *addable* if it is closed under taking disjoint unions. We often consider classes of undirected graphs. Seen as a relational structure, this is a set with an irreflexive symmetric relation E on it. A weak substructure of such a graph need not be a graph. However, when we speak of a monotone class of undirected graphs, we mean it in the usual sense of a class of graphs closed under the operations of removing edges and vertices.

Given a structure A and an equivalence relation $E \subseteq A \times A$ we define the quotient structure A/E as the structure whose domain $A/E = \{[a]_E : a \in A\}$ is the set of E -equivalence classes and such that for every relation symbol R of arity n we have $R^{A/E} = \{([a_1], \dots, [a_n]) \in A/E : (a_1, \dots, a_n) \in R^A\}$. Clearly, the quotient map $\pi_E : A \rightarrow A/E$ is a full homomorphism which we call the *quotient homomorphism*. Given structures A, B and a set $S \subseteq A \cap B$ such that $A[S] = B[S]$, we write $A \oplus_S B$ for the *free amalgam of A and B over S* . This can be defined as the quotient of the disjoint union $A + B$ by the equivalence relation generated by $\{(\iota_A(s), \iota_B(s)) : s \in S\}$, where $\iota_A : S \rightarrow A$ and $\iota_B : S \rightarrow B$ are the inclusion maps. Evidently, there is an injective homomorphism $A \rightarrow A \oplus_S B$ given by composing the inclusion $A \rightarrow A + B$ with the quotient $A + B \rightarrow A \oplus_S B$, and a full homomorphism $A + A \rightarrow A$ which descends to a full homomorphism $A \oplus_S A \rightarrow A$.

By the *Gaifman graph* of a structure A we mean the undirected graph $\text{Gaif}(A)$ with vertex set A such that two elements are adjacent if, and only if, they appear together in some tuple of a relation of A . Given a structure A , $r \in \mathbb{N}$, and $a \in A$, we write $B_A^r(a)$ for the *ball of radius r around a in A* , that is, the set of elements of M whose distance from a

in $\text{Gaif}(A)$ is at most r ; we shall often abuse notation and write $B_A^r(a)$ to mean the induced substructure $A[B_A^r(a)]$ of A , possibly with a constant for the element a . A set $S \subseteq A$ is said to be r -independent if $b \notin B_A^r(a)$ for any $a, b \in S$.

For $r \in \mathbb{N}$, let $\text{dist}(x, y) \leq r$ be the first-order formula expressing that the distance between x and y in the Gaifman graph is at most r , and $\text{dist}(x, y) > r$ its negation. Evidently, the quantifier rank of $\text{dist}(x, y) \leq r$ less than r . A *basic local sentence* is a sentence

$$\exists x_1, \dots, x_n \left(\bigwedge_{i \neq j} \text{dist}(x_i, x_j) > 2r \wedge \bigwedge_{i \in [n]} \psi^{B^r(x_i)}(x_i) \right),$$

where $\psi^{B^r(x_i)}(x_i)$ denotes the relativisation of ψ to the r -ball around x_i , i.e. the formula obtained from ψ by replacing every quantifier $\exists x \theta$ with $\exists x (\text{dist}(x_i, x) \leq r \wedge \theta)$, and likewise every quantifier $\forall x \theta$ with $\forall x (\text{dist}(x_i, x) \leq r \rightarrow \theta)$. We call r the *locality radius*, n the *width*, and ψ the *local condition* of ϕ . Recall the Gaifman locality theorem [10, Theorem 2.5.1].

► **Theorem 1** (Gaifman Locality). *Every first-order sentence of quantifier rank q is equivalent to a Boolean combination of basic local sentences of locality radius 7^q .*

We say that a formula ϕ is preserved by homomorphisms (resp. extensions) over a class of structures \mathcal{C} if for all $A, B \in \mathcal{C}$ such that there is a homomorphism (resp. embedding) from A to B , $A \models \phi$ implies that $B \models \phi$. We say that a class of structures \mathcal{C} has the *homomorphism preservation property* (resp. *extension preservation property*) if for every formula ϕ preserved by homomorphisms (resp. extensions) over \mathcal{C} there is an existential-positive (resp. existential) formula ψ such that $M \models \phi \iff M \models \psi$ for all $M \in \mathcal{C}$.

Given a formula ϕ and a class of structures \mathcal{C} , we say that $M \in \mathcal{C}$ is a *minimal induced model* of ϕ in \mathcal{C} if $M \models \phi$ and for any proper induced substructure N of M with $N \in \mathcal{C}$ we have $N \not\models \phi$. The relationship between minimal models and preservation theorems is highlighted by the following lemma, which is standard, and combines [3, Theorem 3.1] and [2, Lemma 2.1]

► **Lemma 2.** *Let \mathcal{C} be a hereditary class of finite structures. Then a sentence preserved by homomorphisms (resp. extensions) in \mathcal{C} is equivalent to an existential-positive (resp. existential) sentence over \mathcal{C} if and only if it has finitely many minimal induced models in \mathcal{C} .*

► **Corollary 3.** *Let \mathcal{C} be a hereditary class of finite structures. If \mathcal{C} has the extension preservation property, then \mathcal{C} has the homomorphism preservation property.*

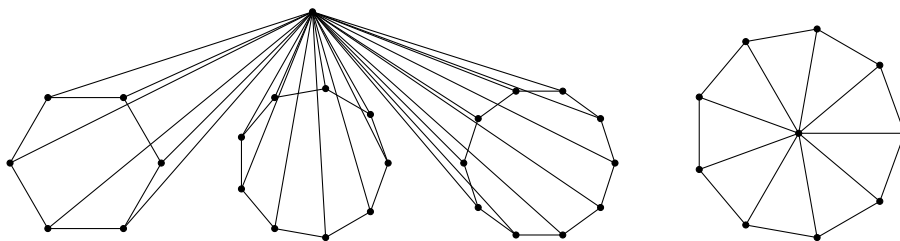
Proof. If a formula ϕ is preserved by homomorphisms then, in particular, it is preserved by extensions. It follows that ϕ is equivalent to an existential sentence over \mathcal{C} , and so by Lemma 2 it has finitely many minimal induced models in \mathcal{C} . Consequently, the same lemma implies that ϕ is equivalent to an existential-positive sentence over \mathcal{C} as required. ◀

Another immediate consequence of the above is that both preservation properties hold over any class \mathcal{C} that is *well-quasi-ordered* by the induced substructure relation, i.e. for every infinite subclass $\{M_i : i \in I\} \subseteq \mathcal{C}$ there are $i \neq j \in I$ such that either M_i is an induced substructure of M_j or vice versa. In fact, any property (i.e. not necessarily definable by a first-order formula) preserved by homomorphisms (resp. extensions) is equivalent to an existential positive (resp. existential) formula over a well-quasi-ordered class. In particular, this applies to classes of cliques or more generally classes of bounded shrub-depth [11].

3 Preservation can fail on classes of small treewidth

Theorem 4.4 of [3] can be paraphrased in the language of this paper as saying that *homomorphism preservation holds over any monotone and addable class of bounded treewidth*. In this section we provide a simple counterexample to this, exhibiting a monotone and addable class of graphs of treewidth 3 where homomorphism preservation fails. More generally, this contradicts Corollary 3.3 of [3] and Theorem 9 of [7]. To witness failure of preservation, we must exhibit the relevant class, a formula preserved by homomorphisms over this class, and an infinite collection of minimal induced models in the class. We then conclude by Lemma 2.

► **Definition 4.** Fix $k \in \mathbb{N}$ and $n_i \geq 3$ for every $i \in [k]$. We define the *bouquet of cycles of type (n_1, \dots, n_k)* , denoted by W_{n_1, \dots, n_k} , as the graph obtained by taking the disjoint union of k cycles of length n_1, \dots, n_k respectively, and adding an apex vertex, i.e. a vertex adjacent to every vertex in these cycles. When $k = 1$, we refer to the graph W_n as the *wheel of order n* .



■ **Figure 1** The bouquet of cycles of type $(6, 9, 10)$ and the wheel of order 9 respectively.

► **Lemma 5.** Fix $n, m \in \mathbb{N}$ odd. Then the wheel W_n has chromatic number 4, while its proper subgraphs have chromatic number 3. Consequently, any homomorphism $W_n \rightarrow W_m$ is full.

Proof. Fix $n, m \in \mathbb{N}$ odd. Clearly, any proper colouring of W_n must use a unique colour for the apex as it is adjacent to every other node in W_n . Moreover, we require an additional three colours to colour the vertices in the odd-length cycle of W_n . It follows that $\chi(W_n) = 4$. Now, let W be a proper subgraph of W_n . It follows that there is at least one edge (u, v) present in W_n which is not in W . If u and v are both in the cycle of odd length then we may define a proper 3-colouring of W by giving u and v the same colour, alternating between this and a second colour along the cycle, and using a final third colour for the apex. If one of u or v is the apex of W_n , then we once again colour u and v with the same colour and use an additional two colours to alternate between along the cycle. In particular, it follows that the chromatic number of any homomorphic image of W_n is at least 4, and so it cannot be a proper substructure of W_m . This implies that any homomorphism $f : W_n \rightarrow W_m$ is full. ◀

The advantage of working with bouquets of cycles is that, unlike single cycles, there is a formula that asserts the existence of such a structure as a free induced subgraph. Indeed, let

$$\phi := \exists x \exists y [E(x, y) \wedge \forall z (z \neq x \wedge \text{dist}(x, z) \leq 2 \rightarrow E(x, z) \wedge \psi(x, z))], \text{ where}$$

$$\psi(x, z) := \exists u \exists v [u \neq v \wedge u \neq x \wedge v \neq x \wedge E(z, u) \wedge E(z, v) \wedge \forall w (E(w, z) \rightarrow w = u \vee w = v \vee w = x)]$$

Intuitively, ϕ asserts the following: “there is a vertex x of degree at least one such that every other vertex reachable from x by a path of length two is adjacent to x and has exactly two distinct neighbours which are not x ”.

► **Lemma 6.** *Let G be an arbitrary finite graph. Then $G \models \phi$ if, and only if, it contains a bouquet of cycles as a free induced subgraph.*

Proof. Suppose that G contains a bouquet W as a free induced subgraph. Then the apex of the bouquet is a vertex of degree at least one, while every vertex reachable from the apex by a path of length two must be in one of the cycles, since W is free in G . Since all vertices in the cycles have degree exactly two, not considering the apex itself, it follows that $G \models \phi$.

Conversely, suppose that $G \models \phi$, and let x be the vertex that is guaranteed to exist by ϕ . Let $S \subseteq V(G)$ be the vertices that are adjacent to x . Since x has degree at least one it follows that S is non-empty. Partition S into k classes S_1, \dots, S_k , by putting two vertices in the same class if, and only if, there is a path between them in $G \setminus \{x\}$. We argue that for each $i \in [k]$, S_i is a free induced cycle in $G \setminus \{x\}$. First, notice that since every vertex reachable from x by a path of length two has degree exactly two in $G \setminus \{x\}$, it follows that S_i induces a cycle in $G \setminus \{x\}$. Moreover, there is no $y \in G \setminus (\{x\} \cup S_i)$ which is adjacent to S_i . Indeed, if $y \neq x$ is adjacent to some $v \in S_i$ then y is reachable from x by a path of length two. It follows that y is itself adjacent to x , and therefore y is in S ; in particular, y and v are in the same class and so $y \in S_i$. Consequently each S_i is a free induced cycle in $G \setminus \{x\}$, and so the connected component of x is a free induced bouquet of cycles in G . ◀

It is evident that ϕ is not preserved by homomorphisms in general as every W_n maps homomorphically to the structure $W_n \cup \{c\}$ with an additional vertex adjacent to the apex, and while the latter contains a bouquet as an induced subgraph, it does not contain a bouquet as a free induced subgraph. However, when restricting to subgraphs of disjoint unions of wheels we no longer have non-free-occurring bouquets. This is the core of the next argument.

► **Theorem 7.** *Let \mathcal{C} be the monotone and addable closure of $\{W_{2n+1} : n \in \mathbb{N}\}$. Then homomorphism preservation fails over \mathcal{C} .*

Proof. Let ϕ be as above; we argue that it is preserved by homomorphisms over \mathcal{C} . Indeed, if some $G \in \mathcal{C}$ is such that $G \models \phi$ then by Lemma 6 it contains a bouquet of cycles as a free induced subgraph. By the choice of \mathcal{C} , this necessarily implies that G contains W_n as a free induced subgraph for some odd n . Let $H \in \mathcal{C}$ and $f : G \rightarrow H$ be a homomorphism. Then f restricts onto a homomorphism $W_n \rightarrow H$ which, by the connectivity of W_n and the fact that $H \in \mathcal{C}$, descends to a homomorphism $\hat{f} : W_n \rightarrow W_m$ for some odd $m \in \mathbb{N}$. It follows by Lemma 5 that \hat{f} is full, and therefore H contains W_m as a subgraph. The choice of \mathcal{C} once again ensures that W_m is a free induced subgraph of H , and so Lemma 6 implies that $H \models \phi$ as required. Finally, Lemma 6 implies that every W_n is a model of ϕ , while every proper subgraph of W_n cannot possibly contain a bouquet of cycles as a free induced subgraph, and so it cannot model ϕ . Consequently, each W_n is a minimal model of ϕ in \mathcal{C} , and so we conclude by Lemma 2 that ϕ is not equivalent to an existential-positive formula over \mathcal{C} . ◀

Finally, observe that each graph W_n has treewidth 3 (in fact even pathwidth 3). Indeed, taking a tree decomposition of the cycle C_n of width 2, and adding the apex to every bag in the decomposition gives the required tree decomposition of W_n .

4 Preservation under bottleneck amalgamation

The results of [3] were later extended to classes that are *quasi-wide*. Recall that a class is called quasi-wide if for every $r \in \mathbb{N}$ there exist $k_r \in \mathbb{N}$ and $f_r : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $m \in \mathbb{N}$ and $M \in \mathcal{C}$ of size at least $f_r(m)$ there are disjoint sets $A, S \subseteq M$ with $|A| \geq m$, $|S| \leq k_r$ and such that A is r -independent in $M \setminus S$. Intuitively, quasi-wideness ensures that, for every

choice of distance, we may find in suitably large structures a large set of elements that are pairwise far away after removing a small set of *bottleneck points*. This notion was introduced for the purpose of extending the arguments of [3] to more general sparse classes, such as classes of bounded expansion, or classes that locally exclude a minor. Paraphrased into the language of this paper, Theorem 9 of [7] asserts that: *homomorphism preservation holds over any monotone and addable quasi-wide class*. Evidently, this is violated by Theorem 7 above. Nonetheless, we may salvage the proof by replacing additivity by the stronger assumption of closure under amalgamation over the bottleneck points that witness quasi-wideness.

The proof proceeds by arguing that any suitably large model M of a sentence ϕ preserved by homomorphisms over a class \mathcal{C} satisfying our assumptions, has a proper induced substructure N which also models ϕ . We thus obtain a concrete bound on the size of minimal models of ϕ , and conclude by Lemma 2. The existence of this bound is guaranteed by quasi-wideness, as any large enough structure contains a large scattered set after removing a small number of bottleneck points. To isolate the bottleneck points \bar{p} of M we consider a structure $\bar{p}M$ in an expanded language which is bi-interpretable with M , and work with the corresponding interpretation ϕ^k of ϕ ; in particular $\bar{p}M$ contains a large scattered set itself and it models ϕ^k . Then, by removing a carefully chosen point from the scattered set of $\bar{p}M$, we obtain a proper induced substructure $\bar{p}N$ of $\bar{p}M$ such that $N \in \mathcal{C}$ by hereditariness. To argue that this still models ϕ^k , we use a relativisation of the locality argument of Ajtai and Gurevich from [1]. While in its original version the argument only considers disjoint copies of M , working with the interpretation $\bar{p}M$ of M corresponds to taking free amalgams of M over the set of bottleneck points; this is precisely the subtlety that was missed in [3] and [7].

We now define the structure $\bar{p}M$; in the following we only consider the case of undirected graphs for simplicity. For arbitrary relational structures the idea is analogous, in that we isolate the tuple \bar{p} by forgetting any relation that contains some p_i , and introduce new relation symbols of smaller arities to recover the forgotten relations.

► **Definition 8.** Fix $k \in \mathbb{N}$, and let $\sigma = \{E, P_1, \dots, P_k, Q_1, \dots, Q_k\}$ be the expansion of the language of graphs with $2k$ unary predicates. Given a graph $G = (V, E)$ and a tuple $\bar{p} \in V^k$, define the σ -structure $\bar{p}G$ on the same domain V such that for all $i \in [k]$:

- $E^{\bar{p}G} = \{(u, v) \in E : u, v \notin \{p_1, \dots, p_k\}\};$
- $P_i^{\bar{p}G} = \{p_i\};$
- $Q_i^{\bar{p}G} = \{v \in V : (p_i, v) \in E\}.$

Consider the formula $\epsilon(x, y) := \bigvee_{i \in [k]} (P_i(x) \wedge Q_i(y)) \vee E(x, y)$. Given a sentence ϕ , write ϕ^k for the σ -sentence obtained by ϕ by replacing every atom $E(x, y)$ by $(\epsilon(x, y) \vee \epsilon(y, x))$. It is then clear that for every $G = (V, E)$ and $\bar{p} \in V^k$ as above $G \models \phi \iff \bar{p}G \models \phi^k$.

With this, we turn to our main theorem in this section. Instead of proving a base case and invoking that for the interpretation step as in [2], [3], and [7], we opt for a direct proof to illustrate the relevance of our assumptions on the class.

► **Theorem 9.** Let \mathcal{C} be hereditary such that for every $r \in \mathbb{N}$ there exist $k_r \in \mathbb{N}$ and $f_r : \mathbb{N} \rightarrow \mathbb{N}$ so that for every $m \in \mathbb{N}$ and $M \in \mathcal{C}$ of size $\geq f_r(m)$ there are disjoint sets $A, S \subseteq M$ with:

- $|A| \geq m$ and $|S| \leq k_r$;
- A being r -independent in $M \setminus S$;
- $\bigoplus_S^n M := \underbrace{M \oplus_S M \oplus_S \dots \oplus_S M}_{n \text{ times}} \in \mathcal{C}$ for every $n \in \mathbb{N}$.

Then homomorphism preservation holds over \mathcal{C} .

Proof. Let \mathcal{C} be as above, and fix ϕ which is preserved by homomorphisms over \mathcal{C} . Denote the quantifier rank of ϕ by q , and let $r = 2 \cdot 7^q$. It follows that there is some $k \in \mathbb{N}$ and some $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $m \in \mathbb{N}$ and every M in \mathcal{C} of size at least $f(m)$, there are disjoint sets $A, S \subseteq M$ such that $|A| \geq m$, $|S| \leq k$, A is r -independent in $M \setminus S$, and $\bigoplus_S^n M \in \mathcal{C}$ for every $n \in \mathbb{N}$. We consider the formula ϕ^k : by Gaifman locality, there is a set $\{\phi_1, \dots, \phi_s\}$ of basic local sentences such that ϕ^k is equivalent to a Boolean combination of these. For $i \in [s]$, let r_i and n_i be the radius and width of locality respectively of ϕ_i , and ψ_i its local condition. Observe that ϕ and ϕ^k have the same quantifier rank, and so $2 \cdot \max_{i \in [s]} r_i \leq r$. Set $n := \max_{i \in [s]} n_i$ and $m := 2^s + 1$. We argue that every minimal model $M \in \mathcal{C}$ of ϕ has size $< f(m)$.

Let M be a minimal model of ϕ , and assume for a contradiction that $|M| \geq f(m)$. It follows that there is a set $S \subseteq M$ of size k such that $M \setminus S$ contains an r -independent set A of size m . Let $\bar{p} \in M^k$ be an enumeration of S ; this implies that $\bar{p}M \models \phi^k$ and A is an r -independent set in $\bar{p}M$. For each $i \in [s]$ define

$$\Psi_i(x) := \exists y (\text{dist}(x, y) \leq r_i \wedge \psi_i^{B_{r_i}(y)}(y)).$$

Since $|A| \geq m = 2^s + 1$, it follows that there are at least two vertices $u, v \in A$ satisfying

$$B_{\bar{p}M}^{2r_i}(u) \models \Psi_i(u) \iff B_{\bar{p}M}^{2r_i}(v) \models \Psi_i(v)$$

for all $i \in [s]$. Let N' be the substructure of $\bar{p}M$ induced on $\bar{p}M \setminus \{u\}$. Since A does not intersect the vertices in S , the substructure N of M induced on $M \setminus \{u\}$ satisfies that $N' = \bar{p}N$. We shall argue that $\bar{p}N \models \phi^k$ and so $N \models \phi$, contradicting that M is a minimal induced model of ϕ .

By our closure assumptions on \mathcal{C} , $N_n := \bigoplus_S^n N$ and $M_n := M \oplus_S (\bigoplus_S^n N)$ are both in \mathcal{C} , as the latter is an induced substructure of $\bigoplus_S^{n+1} M$. Since there is a homomorphism $M \rightarrow M_n$ we obtain that $M_n \models \phi$ and thus $\bar{p}M_n \models \phi^k$. We shall argue that

$$\bar{p}M_n \models \phi_i \iff \bar{p}N_n \models \phi_i$$

for all basic local sentences ϕ_i of ϕ^k . In particular, this implies that $\bar{p}N_n \models \phi^k$, and so $N_n \models \phi$. Since there is a homomorphism $N_n \rightarrow N$, the preservation of ϕ implies that $N \models \phi$ as claimed.

Clearly, if $\bar{p}N_n \models \phi_i$ then $\bar{p}M_n \models \phi_i$ by the fact that ϕ_i is a local sentence and $\bar{p}N_n$ a free induced substructure of $\bar{p}M_n$. Conversely, if $\bar{p}M_n \models \phi_i$ then there is a $2r_i$ -independent subset X of size n_i such that $B_{\bar{p}M_n}^{r_i}(x) \models \psi_i(x)$ for every $x \in X$. Observe that if $X \subseteq S$ then clearly $\bar{p}N_n \models \phi_i$. So, since S is isolated in $\bar{p}M_n$ and $\bar{p}N_n$, we focus on elements of $X \setminus S$, which we may assume to be non-empty. We therefore distinguish two cases.

If $|X \setminus S| > 1$ then, by the $2r_i$ -independence of X , there is at least one $x \in X \setminus S$ such that $u \notin B_{\bar{p}M_n}^{r_i}(x)$. It follows that the r_i -ball centered at x is isomorphic to an r_i -ball centered at an element in a disjoint copy of $N \setminus S$. Since $\bar{p}N_n$ contains $n \geq n_i$ such copies, it follows that there is a $2r_i$ -independent subset Y of N_n of size n_i such that $B_{\bar{p}N_n}^{r_i}(y) \models \psi_i(y)$ for every $y \in Y$, i.e. $\bar{p}N_n \models \phi_i$.

On the other hand if $|X \setminus S| = 1$, let x be the unique element of $X \setminus S$. Clearly if $u \notin B_{\bar{p}M_n}^{r_i}(x)$ then the r_i -ball centered at x is isomorphic to the r_i -ball centered at an element in $\bar{p}N_n$, and so $\bar{p}N_n \models \phi_i$. If $u \in B_{\bar{p}M_n}^{r_i}(x)$, then $\bar{p}M \models \Psi_i(u)$, and so by the choice of u and v , $\bar{p}M \models \Psi_i(v)$. Consequently, there is some $y \in B_{\bar{p}M}^{r_i}(v)$ such that $B_{\bar{p}M}^{r_i}(y) \models \psi_i(y)$. Observe that because v and u are $2r_i$ -independent, $u \notin B_{\bar{p}M_n}^{r_i}(y)$. As before, this implies that $\bar{p}(B_n) \models \phi_i$.

The above implies that there are finitely many minimal induced models of ϕ in \mathcal{C} , and so we conclude that ϕ is equivalent to an existential-positive formula over \mathcal{C} by Lemma 2. \blacktriangleleft

Going back to bouquets of cycles, it is easy to see that if a bouquet has more than $m^2 \cdot (r + 1)$ vertices then after removing the apex it either contains m disjoint cycles or it contains a cycle of size at least $m \cdot (r + 1)$; in either case it contains an r -independent set of size m . In this case the apex is the only bottleneck point, and so amalgamating over this corresponds to adding more cycles to the bouquet. Consequently, homomorphism preservation holds for the hereditary closure of the class of bouquets of cycles by Theorem 9 above.

Closure under amalgamation over bottlenecks is a technical condition and one might consider if it could be replaced by more natural conditions. For example, we could strengthen it by considering closure under arbitrary amalgamation. However, this is a condition that does not sit well with sparsity requirements. Indeed, any hereditary class of undirected graphs that is also closed under arbitrary amalgamation contains arbitrarily large 1-subdivided cliques, and hence, cannot be quasi-wide. Nonetheless, there are naturally defined sparse families of structures that satisfy the conditions of Theorem 9. One such class is known to exist by [2], that is, the class \mathcal{T}_k of all graphs of treewidth bounded by k , for any value of $k \in \mathbb{N}$. Indeed, for any suitably large graph of bounded treewidth we may pick a set of bottleneck points that comes from the same bag in a tree decomposition of the graph, and so amalgamating over this set of points does not increase the treewidth. Another naturally defined such class is the class of outerplanar graphs. For our purposes, we may define outerplanar graphs as those omitting K_4 and $K_{2,3}$ as minors [4]. The quasi-wideness of this class follows by the following fact, which moreover permits some control over the bottleneck points.

► **Theorem 10** ([3]). *For every $k, r, m \in \mathbb{N}$ there is an $N = N(k, r, m) \in \mathbb{N}$ such that if G is a graph of size at least N excluding K_k as a minor, then there are disjoint sets $A, S \subseteq V(G)$ with $|A| \geq m$ and $|S| \leq k - 2$ such that A is $2r$ -independent in $G \setminus S$. Moreover, the bipartite graph $K_{A,S}$ with parts A and S defined by putting an edge between $a \in A$ and $s \in S$ if and only if there is some $u \in B_{G \setminus S}^r(a)$ such that $(u, s) \in E(G)$ is complete.*

► **Theorem 11.** *Homomorphism preservation holds for the class of outerplanar graphs.*

Proof. Since outerplanar graphs are K_4 -minor-free, it follows by Theorem 10 that for every $r, m \in \mathbb{N}$ there exists an $N = N(r, m) \in \mathbb{N}$ such that if G is an outerplanar graph of size at least N , then there are disjoint sets $A, S \subseteq V(G)$ with $|A| \geq m, |S| \leq 2$, A r -independent in $G \setminus S$, and $K_{A,S}$ complete. Since outerplanar graphs also forbid $K_{2,3}$ as a minor, this implies that $|S| \leq 1$. It is then clear that for any such G and $S \subseteq V(G)$, the graph $\oplus_S^n G$ is still outerplanar for all $n \in \mathbb{N}$, as no 1-point free amalgams can create K_4 or $K_{2,3}$ minors. We thus conclude by Theorem 9. ◀

Interestingly, the examples exhibited above are in fact *almost-wide*, that is, the number of the bottleneck points does not depend on the radius of independence. It would be interesting to find natural quasi-wide classes which are not almost-wide, and which are closed under bottleneck amalgamation. One potential candidate might be the class of all graphs whose local treewidth is bounded by the same constant.

5 Homomorphism preservation fails on planar graphs

In this section we witness that homomorphism preservation fails on the class of planar graphs. Previously, it was established [2] that the extension preservation property fails on planar graphs. Since extension preservation implies homomorphism preservation on

47:10 Preservation Theorems on Sparse Classes Revisited

hereditary classes by Corollary 3, our result strengthens the above. Recall that by Wagner’s theorem [16] a graph is planar if and only if it omits $K_{3,3}$ and K_5 as minors. Our construction, in fact, also reveals that homomorphism preservation fails on the class of K_5 -minor-free graphs.

► **Definition 12.** Fix $n \in \mathbb{N}$. Define G_n as the undirected graph on vertex set $V(G_n) = \{v_1, v_2\} \cup \{a_i : i \in [n]\} \cup \{b_i : i \in [n]\}$ and edge set

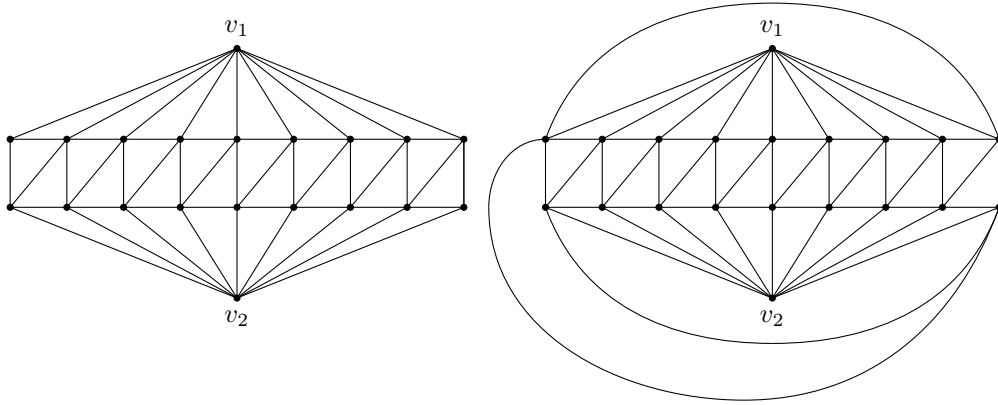
$$E(G_n) = \{(v_1, a_i) : i \in [n]\} \cup \{(v_2, b_i) : i \in [n]\} \cup \{(a_i, b_i) : i \in [n]\}$$

$$\cup \{(a_i, a_{i+1}) : i \in [n-1]\} \cup \{(b_i, b_{i+1}) : i \in [n-1]\} \cup \{(a_{i+1}, b_i) : i \in [n-1]\}.$$

We define D_n as the extension of G_n on the same vertex set, with

$$E(D_n) = E(G_n) \cup \{(a_1, a_n), (b_1, b_n), (a_1, b_n)\}.$$

We also define A_n as the graph obtained from G_n by taking the quotient over the equivalence relation generated by (a_1, a_n) , and we write $\alpha_n : G_n \rightarrow A_n$ for the corresponding quotient homomorphism. Likewise, we let $B_n := G_n / (a_1, b_n)$ and $C_n := G_n / (b_1, b_n)$, and write $\beta_n : G_n \rightarrow B_n$ and $\gamma_n : G_n \rightarrow C_n$ for the respective quotient homomorphisms.



■ **Figure 2** Planar embeddings of G_9 and D_9 respectively.

Consider the following observations. First, for every $n \geq 3$ the graphs G_n, D_n, A_n, B_n, C_n are all planar and 4-chromatic. In particular D_n, B_n, C_n are maximal planar. For $n = 3$ the graphs D_n and B_n contain a copy of K_4 , while for $n \geq 4$ they are K_4 -free. Likewise, for $n \in \{3, 4\}$ the graphs A_n and C_n contain a copy of K_4 , while for $n \geq 5$ they are K_4 -free. Finally, for $3 \leq m \leq n$ there is a homomorphism $\delta_{n,m} : G_n \rightarrow D_m$ that “wraps” G_n around D_m . Labelling their vertices as above, this satisfies

$$\delta_{n,m}(v_1) = v_1, \delta_{n,m}(v_2) = v_2, \delta_{n,m}(a_i) = a_{i \bmod m} \text{ and } \delta_{n,m}(b_i) = b_{i \bmod m},$$

for all $i \in [n]$.

We proceed to characterise the K_5 -minor-free homomorphic images of D_n . We argue that these either contain K_4 , or an induced copy of D_m for some m with $m|n$. The proof proceeds by first characterising the K_5 -minor-free homomorphic images of G_n by induction, and then using that G_n is a subgraph of D_n . While this requires a fair amount of book-keeping, it is not conceptually difficult. The base case of the induction is the following lemma.

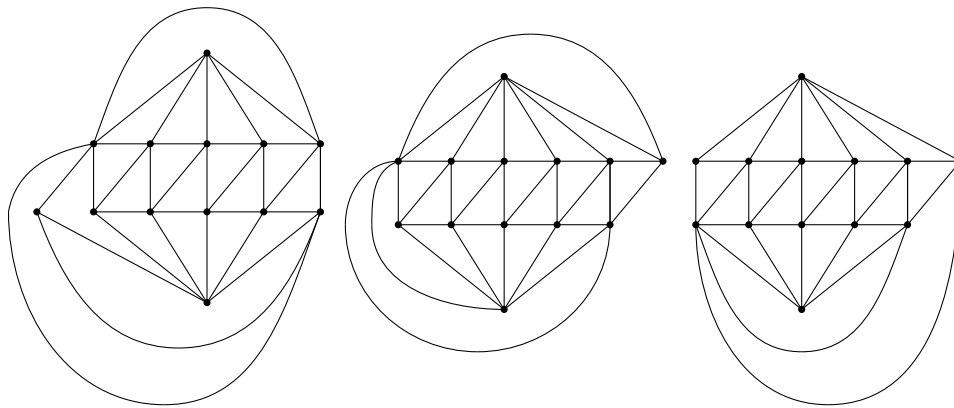
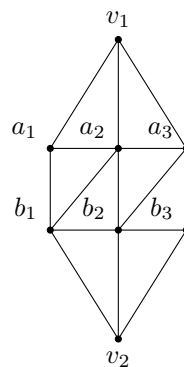


Figure 3 Planar embeddings of $A_6, B_6,$ and C_6 respectively.

► **Lemma 13.** *Let $f : G_3 \rightarrow H$ be a homomorphism. If H is K_4 -free then f is injective.*

Proof. Let H be a K_4 -free graph, and $f : G_3 \rightarrow H$ a homomorphism. Label the vertices of G_3 as in the picture below; we shall argue that f is injective.



First, notice that f is injective on $\{v_2, b_1, b_2\}$ since they form a triangle in G_3 . If $f(a_2) = f(v_2)$ then the set $\{f(v_2), f(b_2), f(a_3), f(b_3)\}$ induces K_4 in H ; it follows that $f(a_2) \neq f(v_2)$, and so f is injective on $\{v_2, b_1, b_2, a_2\}$. Likewise, $f(b_3) \neq f(a_2)$ as otherwise $\{f(b_1), f(b_2), f(b_3), f(v_2)\}$ induce K_4 in H . Moreover, $f(b_3) \neq f(b_1)$ as $\{f(b_1), f(b_2), f(a_2), f(a_3)\}$ would otherwise induce K_4 in H . It follows that f is injective on $\{v_2, b_1, b_2, b_3, a_2\}$. From this we deduce that $f(a_3) \neq f(b_1)$ as otherwise $\{f(v_2), f(b_1), f(b_2), f(b_3)\}$ would induce K_4 in H , and that $f(a_3) \neq f(v_2)$ as otherwise $\{f(v_2), f(b_1), f(b_2), f(a_2)\}$ would also induce K_4 . Hence, f is injective on $\{v_2, b_1, b_2, b_3, a_2, a_3\}$. By symmetry, it follows that f is also injective on $\{v_1, a_1, a_2, a_3, b_1, b_2\}$. Notice that $f(a_1) \neq f(b_3)$ and $f(v_1) \neq f(b_3)$ since otherwise $\{f(a_2), f(a_3), f(b_2), f(b_3)\}$ would induce K_4 in H . Finally, $f(a_1) \neq f(v_2)$ and $f(v_1) \neq f(v_2)$ as otherwise $\{f(a_2), f(b_1), f(b_2), f(v_2)\}$ would induce K_4 in H . Putting all the above together, we conclude that f is injective on all of G_3 as required. ◀

We now proceed to the general case.

► **Lemma 14.** Fix $n \geq 3$. Let $f : G_n \rightarrow H$ be a homomorphism, where H is K_4 -free and K_5 -minor-free. Then one of the following is true:

1. f is injective;
2. there is some $m \in [4, n-1]$ and an embedding $\hat{f} : D_m \rightarrow H$ such that $f = \hat{f} \circ \delta_{n,m}$;
3. $n \geq 5$ and there is an injective homomorphism $\hat{f} : A_n \rightarrow H$ such that $f = \hat{f} \circ \alpha_n$;
4. $n \geq 4$ and there is an embedding $\hat{f} : B_n \rightarrow H$ such that $f = \hat{f} \circ \beta_n$;
5. $n \geq 5$ and there is an embedding $\hat{f} : C_n \rightarrow H$ such that $f = \hat{f} \circ \gamma_n$;

Proof. We prove the claim by induction on n . The base case $n = 3$ follows by Lemma 13. So, fix a K_4 -free and K_5 -minor-free graph H and consider a homomorphism $f : G_{n+1} \rightarrow H$. Evidently, this restricts to a homomorphism $f' : G_n \rightarrow H$. By the induction hypothesis, we may assume that f' satisfies one of the five conditions of this proposition.

Assume that f' satisfies (1), i.e. f is injective on $G_n = G_{n+1} \setminus \{a_{n+1}, b_{n+1}\}$. We consider the images of the vertices a_{n+1} and b_{n+1} under f . Observe that $(a_{n+1}, b_{n+1}) \in E(G_{n+1})$ so $f(a_{n+1}) \neq f(b_{n+1})$. Clearly, if $f(a_{n+1})$ and $f(b_{n+1})$ are not any of the vertices in $f[G_n]$, then f is itself injective and (1) holds. We hence distinguish three cases.

First, suppose that $f(a_{n+1}) \in f[G_n]$ and $f(b_{n+1}) \notin f[G_n]$. Since there are edges (v_1, a_{n+1}) , (a_n, a_{n+1}) , and (b_n, a_{n+1}) , it follows that $f(a_{n+1}) \notin \{f(v_1), f(a_n), f(b_n)\}$. Moreover, $f(a_{n+1}) \neq f(v_2)$ as otherwise $\{f(b_{n-1}), f(b_n), f(a_n), f(v_2)\}$ would induce K_4 in H . Similarly, $f(a_{n+1}) \neq f(a_{n-1})$, as otherwise $\{f(a_{n-1}), f(b_{n-1}), f(a_n), f(b_n)\}$ would induce K_4 , and $f(a_{n+1}) \neq f(b_{n-1})$, as otherwise $\{f(v_1), f(a_{n-1}), f(a_n), f(b_{n-1})\}$ would induce K_4 . In addition, $f(a_{n+1}) \notin \{f(a_i) : 2 \leq i \leq n-2\}$ as otherwise an edge $(f(a_i), f(a_n))$ for some $i \in [2, n-2]$ would produce a K_5 -minor in H , namely the minor arising from $S_1 = \{f(v_1)\}, S_2 = \{f(a_i)\}, S_3 = \{f(a_n)\}, S_4 = \{f(a_j) : i+1 \leq j \leq n-1\}, S_5 = \{f(v_2), f(b_1), f(a_1), f(b_i), f(b_{i+1}), f(b_n)\}$. A similar argument reveals that $f(a_{n+1}) \notin \{f(b_i) : 2 \leq i \leq n-2\}$. It follows that $f(a_{n+1}) = f(b_1)$ or $f(a_{n+1}) = f(a_1)$. The former case would produce a copy of K_4 , namely $\{f(v_1), f(a_1), f(a_2), f(b_1)\}$, leading to a contradiction. Hence $f(a_{n+1}) = f(a_1)$. Since $f(b_{n+1}) \notin f[G_n]$, it follows that f factors through the quotient homomorphism α_n , i.e. case (3) is true.

Next, suppose that $f(a_{n+1}) \in f[G_n]$ and $f(b_{n+1}) \in f[G_n]$. As before, we deduce from the first assumption that $f(a_{n+1}) = f(a_1)$. This implies that there are edges $(f(a_1), f(a_n))$ and $(f(a_1), f(b_n))$ in H . Since there are edges $(a_{n+1}, b_{n+1}), (v_2, b_{n+1}), (b_n, b_{n+1})$ in G_{n+1} we deduce that $f(b_{n+1}) \notin \{f(a_1), f(b_n), f(v_2)\}$. Moreover, $f(b_{n+1}) \neq f(v_1)$ as otherwise the edge $(f(v_1), f(v_2))$ would produce a K_5 -minor in H , namely $S_1 = \{f(v_1)\}, S_2 = \{f(a_1)\}, S_3 = \{f(a_n)\}, S_4 = \{f(a_j) : j \in [2, n-1]\}, S_5 = \{f(v_2), f(b_1), f(b_2), f(b_n)\}$. Similarly, a K_5 -minor arises in H if $f(b_{n+1}) \in \{f(a_i), f(b_i) : i \in [2, n-1]\}$. We thus deduce that $f(b_{n+1}) = f(b_1)$, from which we conclude that $f[G_{n+1}]$ induces a copy of D_n in H , and more precisely, case (2) holds.

So, suppose that $f(a_{n+1}) \notin f[G_n]$ and $f(b_{n+1}) \in f[G_n]$. We consider the possible images of b_{n+1} under f . Since there are edges $(a_{n+1}, b_{n+1}), (v_2, b_{n+1}), (b_n, b_{n+1})$ in G_{n+1} we deduce that $f(b_{n+1}) \notin \{f(a_{n+1}), f(v_2), f(b_n)\}$. Moreover, $f(b_{n+1}) \neq f(v_1)$ as otherwise $\{f(v_1), f(a_n), f(a_{n+1}), f(b_n)\}$ would induce K_4 in H . Likewise, $f(b_{n+1}) \neq f(a_n)$ as otherwise $\{f(v_2), f(a_{n-1}), f(b_n), f(a_n)\}$ would induce K_4 in H . Moreover $f(b_{n+1}) \notin \{f(a_i) : i \in [2, n-1]\}$ as otherwise the edge $(f(a_i), f(a_{n+1}))$ would produce a K_5 -minor in H , namely $S_1 = \{f(v_1)\}, S_2 = \{f(a_i)\}, S_3 = \{f(a_{n+1})\}, S_4 = \{f(a_j) : j \in [i+1, n]\}, S_5 = \{f(v_2), f(b_1), f(a_1), f(b_i), f(b_{i+1}), f(b_n)\}$. Very similarly, we deduce that $f(b_{n+1}) \notin \{f(b_i) : i \in [2, n-1]\}$. It follows that $f(b_{n+1}) = f(a_1)$ or $f(b_{n+1}) = f(b_1)$. In the former case, it follows that f factors through the quotient homomorphism β_n , and so (4) is true, while in the latter case, it follows that f factors through γ_n , and so (5) is true.

Next, assume that f' satisfies (2), i.e. there is some $m \in [4, n-1]$ and an embedding $\hat{f} : D_m \rightarrow H$ such that $f' = \hat{f} \circ \delta_{n,m}$. Arguing as before, it is easy to see that the assumptions on H force $f(a_{n+1})$ to be equal to $\hat{f}(a_{n+1 \bmod m})$, and likewise $f(b_{n+1}) = \hat{f}(b_{n+1 \bmod m})$, implying that $f = \hat{f} \circ \delta_{n+1,m}$. Hence f also satisfies (2).

Finally, we argue that f' cannot satisfy any of (3), (4), and (5). Indeed, assume for a contradiction that f' satisfies (3) and write f' as $\hat{f} \circ \alpha_n$ for some injective homomorphism $\hat{f} : A_n \rightarrow H$. In particular, we know that $n \geq 5$. Consider the image of a_{n+1} under f ; this is some vertex adjacent to $f(a_n) = f(a_1), f(v_1)$, and $f(b_n)$. If $f(a_{n+1}) \notin f[G_n]$, then we obtain a K_5 -minor in H , namely $S_1 = \{f(v_1)\}, S_2 = \{f(a_1)\}, S_3 = \{f(a_{n-1})\}, S_4 = \{f(a_i) : i \in [2, n-1]\}, S_5 = \{f(v_2), f(b_1), f(b_2), f(b_{n-1}), f(b_n), f(a_{n+1})\}$. So $f(a_{n+1}) \in f[G_n]$. If $f(a_{n+1}) = f(a_i)$ for some $i \in [2, n-2]$ then we obtain a K_5 -minor in H by picking some $j \in [2, n-2] \setminus \{i\}$ and letting $S_1 = \{f(v_1), f(a_j)\}, S_2 = \{f(a_1)\}, S_3 = \{f(a_{n-1})\}, S_4 = \{f(b_i) : i \in [n-1]\}, S_5 = \{f(v_2), f(b_n), f(a_i)\}$. Likewise, if $f(a_{n+1}) = f(a_{n-1})$ then we obtain the K_5 -minor $S_1 = \{f(v_1)\}, S_2 = \{f(a_1)\}, S_3 = \{f(b_{n-1}), f(b_{n-2}), f(a_{n-2})\}, S_4 = \{f(a_i) : i \in [2, n-3]\}, S_5 = \{f(v_2), f(b_n), f(a_{n-1})\}$. Consequently, $f(a_{n+1}) = f(b_i)$ for some $i \in [1, n-1]$. This produces an edge $(f(v_1), f(b_i))$ and thus gives rise to the K_5 -minor $S_1 = \{f(v_1)\}, S_2 = \{f(a_j) : j \in [1, i-1]\}, S_3 = \{f(a_i)\}, S_4 = \{f(a_j) : j \in [i+1, n-1]\}, S_5 = \{f(v_2), f(b_1), f(b_i), f(b_{n-1})\}$. It follows that f' cannot satisfy (3); via very similar reasoning, we exclude cases (4) and (5). \blacktriangleleft

Having established the above, our characterisation of the K_5 -minor-free homomorphic images of D_n follows easily.

► **Proposition 15.** *Fix $n \geq 4$. Then any K_4 -free and K_5 -minor-free homomorphic image of D_n contains an induced copy of D_m for some $m \geq 4$ such that $m|n$.*

Proof. Consider a homomorphism $f : D_n \rightarrow H$ where H is K_4 -free and K_5 -minor-free. Then f descends to a homomorphism $f' : G_n \rightarrow H$. It follows that one of the five cases of Lemma 14 holds. If f' is injective, then in particular f is injective; since the addition of any edge in D_n creates a K_5 -minor, it follows that f is in fact an embedding as required. Suppose that case (2) is true, and let $m \in [4, n-1]$ and $\hat{f} : D_m \rightarrow H$ be such that $f' = \hat{f} \circ \delta_{n,m}$. In particular, D_m is an induced subgraph of H and $m|n$. Finally, case (3) leads to a contradiction as the edge (a_1, a_n) in D_n implies that $f(a_1) \neq f(a_n)$, case (4) leads to a contradiction as the edge (a_1, b_n) implies that $f(a_1) \neq f(b_n)$, and likewise, case (5) leads to a contradiction as the edge (b_1, b_n) implies that $f(b_1) \neq f(b_n)$. \blacktriangleleft

We also define G_∞ as the countably infinite analogue of G_n , i.e. the graph on the vertex set $V(G_\infty) = \{v_1, v_2\} \cup \{a_i : i \in \mathbb{N}_{>0}\} \cup \{b_i : i \in \mathbb{N}_{>0}\}$ and edge set

$$E(G_n) = \{(v_1, a_i) : i \in \mathbb{N}_{>0}\} \cup \{(v_2, b_i) : i \in \mathbb{N}_{>0}\} \cup \{(a_i, b_i) : i \in \mathbb{N}_{>0}\}$$

$$\cup \{(a_i, a_{i+1}) : i \in \mathbb{N}_{>0}\} \cup \{(b_i, b_{i+1}) : i \in \mathbb{N}_{>0}\} \cup \{(a_{i+1}, b_i) : i \in \mathbb{N}_{>0}\}.$$

Likewise, we define the homomorphism $\delta_{\infty,m} : G_\infty \rightarrow D_m$ in analogy to the homomorphisms $\delta_{n,m} : G_n \rightarrow D_m$. Lemma 14 allows us to also characterise the finite K_5 -minor-free homomorphic images of G_∞ .

► **Lemma 16.** *Let $f : G_\infty \rightarrow H$ be a homomorphism where H is finite, K_4 -free, and K_5 -minor-free. Then there is some $m \geq 4$ and an embedding $\hat{f} : D_m \rightarrow H$ such that $f = \hat{f} \circ \delta_{\infty,m}$.*

47:14 Preservation Theorems on Sparse Classes Revisited

Proof. Fix f as above and let $n := |H|$. It follows that f restricts to a homomorphism $f' : G_n \rightarrow H$; this satisfies one of the five cases of Lemma 14. Clearly, cases (1),(3),(4), and (5) are ruled out due to size restrictions. Consequently, case (2) holds and the claim follows. ◀

With the above, we show that the existence of the graphs D_n as induced subgraphs is definable among K_4 -free K_5 -minor-free graphs by a simple first-order formula. Indeed, let

$$\begin{aligned} \chi(x_1, x_2, y_1, z_1, y_2, z_2) &= E(x_1, y_2) \wedge E(y_1, y_2) \wedge E(z_1, y_2) \wedge E(z_1, z_2) \wedge E(y_2, z_2) \wedge E(z_2, x_2), \text{ and} \\ \phi &= \exists x_1, x_2, y, z [E(x_1, y) \wedge E(y, z) \wedge E(z, x_2) \wedge \forall a, b (E(x_1, a) \wedge E(a, b) \wedge E(b, x_2)) \\ &\rightarrow \exists c, d \chi(x_1, x_2, a, b, c, d)] \end{aligned}$$

► **Lemma 17.** *Let H be a finite K_4 -free and K_5 -minor free graph. If $H \models \phi$ then there is some $n \geq 4$ such that H contains D_n as an induced subgraph.*

Proof. Fix a graph H as above, and suppose that $G \models \phi$. We inductively define a chain of partial homomorphisms $f_1 \subseteq f_2 \subseteq f_3 \subseteq \dots$ from $G_\infty \rightarrow H$ such that $\text{dom}(f_n) = G_n$. Then the map $f = \bigcup_{n=1}^\infty f_n$ is a homomorphism $G_\infty \rightarrow H$, and hence Lemma 16 implies that H contains some D_n as an induced subgraph.

Since $H \models \phi$ it follows that there are $x_1, x_2, y, z \in V(H)$ such that

$$H \models E(x_1, y) \wedge E(y, z) \wedge E(z, x_2).$$

Consequently, the map $f_1 : G_1 \rightarrow H$ given by $f(v_1) = x_1, f(v_2) = x_2, f(a_1) = y, f(b_1) = z$ is a homomorphism as required. So, suppose that f_n has been defined. Since

$$H \models E(x_1, f(a_n)) \wedge E(f(a_n), f(b_n)) \wedge E(f(b_n), x_2)$$

it follows that

$$H \models \exists c, d \chi(x_1, x_2, f(a_n), f(b_n), c, d).$$

We consequently extend $f_n : G_n \rightarrow H$ to $f_{n+1} : G_{n+1} \rightarrow H$ by letting $f_{n+1}(a_{n+1}) = c$ and $f_{n+1}(b_{n+1}) = d$; this is easily seen to be a valid homomorphism by the choice of χ . ◀

► **Lemma 18.** *Let H be a K_5 -minor-free graph. If H contains some D_n for $n \geq 3$ as an induced subgraph then $H \models \phi$.*

Proof. Let $D_n \leq H$ be as above. Clearly, $H \models E(v_1, a_1) \wedge E(a_1, b_1) \wedge E(b_1, v_2)$. So, let $a, b \in V(H)$ be arbitrary vertices such that $H \models E(v_1, a) \wedge E(a, b) \wedge E(b, v_2)$; we first argue that $a \in \{a_i : i \in [n]\}$ and $b \in \{b_i : i \in [n]\}$. Towards this, observe first that $b \notin \{a_i : i \in [n]\} \cup \{v_1\}$ as otherwise the edge (v_2, a_i) or (v_2, v_1) would contradict that D_n is induced in H . So, assume for a contradiction that $a \notin \{a_i : i \in [n]\}$. Since there is an edge (v_1, a) it follows that $a \neq v_1$, and so in particular the sets $S_1 = \{v_1\}, S_2 = \{a_1\}, S_3 = \{a_n\}, S_4 = \{a_j : j \in [2, n-1]\}, S_5 = \{v_2, b_1, b_n, a, b\}$ produce a K_5 -minor in H . With a symmetric argument we obtain that $b \in \{b_i : i \in [n]\}$. Now, since there is an edge (a, b) it follows that there is some $i \in [n]$ such that $a = a_i$ and $b = b_i$ or $b = b_{i-1 \bmod n}$. In these two respective cases we have that

$$H \models \chi(v_1, v_2, a, b, a_{i+1 \bmod n}, b_{i+1 \bmod n}), \text{ or } H \models \chi(v_1, v_2, a, b, a_{i-1 \bmod n}, b_{i-2 \bmod n}).$$

In either case, $H \models \exists c, d \chi(v_1, v_2, a, b, c, d)$, and since the choice of $a, b \in V(H)$ was arbitrary we obtain that $H \models \phi$ as required. ◀

Putting all the above together, we deduce the main theorem of this section.

► **Theorem 19.** *The class of planar graphs does not have the homomorphism preservation property.*

Proof. Let $\hat{\phi}$ be the disjunction of ϕ with the formula that induces a copy of K_4 , i.e.

$$\hat{\phi} := \phi \vee \exists x_1, x_2, x_3, x_4 \bigwedge_{i \neq j} E(x_i, x_j).$$

We argue that $\hat{\phi}$ is preserved by homomorphisms over the class of planar graphs. Indeed, let $f : G \rightarrow H$ be a homomorphism with G, H planar such that $G \models \hat{\phi}$. Clearly, if H contains a copy of K_4 then $H \models \hat{\phi}$. So, without loss of generality we may assume that $G \models \phi$ and G, H are K_4 -free. It follows by Lemma 17 that there exists some $n \geq 4$ such that G contains D_n as a subgraph. Consequently, Proposition 15 implies that there is some $m \geq 4$ such that H contains D_m as a subgraph. Lemma 18 then implies that $H \models \phi$, and thus $H \models \hat{\phi}$ as required. To conclude, observe that the minimal models of $\hat{\phi}$ over the class of planar graphs are K_4 and the graphs D_n for $n \geq 4$; since these are infinitely many Lemma 2 implies that $\hat{\phi}$ is not equivalent to an existential-positive formula over the class of planar graphs. ◀

Since we only use exclusion of K_5 -minors in the above, we additionally obtain the following.

► **Theorem 20.** *The class of all K_5 -minor-free graphs does not have the homomorphism preservation property.*

Finally, while we have not referred to topological minors to simplify our arguments, an easy check reveals that the above are still valid when considering graphs that forbid K_5 as a topological minor, implying that homomorphism preservation also fails on the class of K_5 -topological-minor-free graphs.

6 Conclusion

Much work in finite model theory explores *tame* classes of finite structures. In [6], two related notions of tameness are identified: algorithmic tameness and model-theoretic tameness. The former is centred around the tractability of model-checking for first-order logic while the latter is illustrated by preservation theorems and it was argued that these occurred together in sparse classes of structures. More recently, algorithmic tameness has been explored extensively for dense classes as well (see [8] for example). On the other hand, the results here show that the status of preservation theorems on sparse classes is more subtle and relies on closure properties that are not always present in natural classes such as planar graphs. Nonetheless, it is an interesting question whether the recent understanding of tame dense classes, such as *monadically stable* and more generally *monadically dependent* classes, can also cast light on preservation theorems. The arguments for homomorphism and extension preservation rely on quasi-wideness and almost-wideness respectively. Similar wideness phenomena occur for dense classes, by replacing deletion of bottleneck points with performing *flips*, that is, edge-complementations within subsets of the domain (see Table 1 in [9]). A question inspired by this is whether, for every $k \in \mathbb{N}$, the class of all graphs of cliquewidth at most k has the extension preservation property. For $k \leq 2$ this follows from the fact that cographs are well-quasi-ordered by the induced subgraph relation [5].

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