Half-Space Separation in Monophonic Convexity

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Abstract

We study half-space separation in the convexity of chordless paths of a graph, i.e., monophonic convexity. In this problem, one is given a graph and two (disjoint) subsets of vertices and asks whether these two sets can be separated by complementary convex sets, called half-spaces. While it is known this problem is NP-complete for geodesic convexity – the convexity of shortest paths – we show that it can be solved in polynomial time for monophonic convexity.

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1 Introduction

A (finite) convexity space is a pair $(V, \mathcal{C})$ where V is a (finite) groundset and $\mathcal{C}$ a collection of subsets of V, called *convex sets*, containing $\emptyset$, V and closed under taking intersections. Graphs provide a wide variety of different convexity notions, known as graph convexities. These are usually defined based on paths and include for instance the geodesic convexity [20], the monophonic convexity [10, 12, 15], the m^3-convexity [11], the triangle-path convexity [5], the toll convexity [1], or the weakly toll convexity [9].

In this paper, we are interested in the *half-space separation* problem: with an implicitly given convexity space $(V, \mathcal{C})$ and two (convex) subsets A, B of V, are there complementary convex sets $H, \overline{H}$ – the so-called *half-spaces* – such that $A \subseteq H$ and $B \subseteq \overline{H}$? This problem is a generalization to abstract convexity of the half-space separation problem in $\mathbb{R}^d$, being well-studied in machine learning [3, 16, 25]. Half-space separation has motivated the study of structural separation properties of convexity spaces. Among these properties, two have received particular attention, notably within graph convexities (see e.g. [2, 7, 14, 18, 23]): the S_3 property stating that any convex set C can be separated from any element of V not in C; and the S_4 or Kakutani property stating that any pair of disjoint convex sets can be separated. Besides, the study of the half-space separation problem on its own has

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recently been brought to the context of graph convexities [21, 22]. In particular, Seiffarth et al. [21] show that half-space separation is NP-complete for geodesic convexity, the convexity induced by the shortest paths of a graph. To our knowledge though, the complexity status of half-space separation for the other aforementioned graph convexities is still unknown.

In our contribution, we follow this latter line of research and study half-space separation for the monophonic convexity. Given a graph G with vertices $V(G)$, a set $C \subseteq V(G)$ is monophonically convex if for any two vertices u, v of C, all the vertices on a chordless path u and v lie in C. We prove that half-space separation can be decided in polynomial time for monophonic convexity. More formally, our main theorem reads as follows:

▶ **Theorem 1.** *Given a graph G and two subsets A, B of $V(G)$, whether A, B are separated by monophonic half-spaces can be decided in polynomial time.*

Theorem 1 contrasts with the NP-completeness of half-space separation for geodesic convexity [21] and suggests to study separation in further graph convexities. Besides, half-space separation also relates to the p-partition problem (in graph convexities). In the p-partition problem, one is given a graph G and has to decide whether $V(G)$ can be partitioned into p convex sets, where the meaning of convex depends on the convexity at hand. For monophonic convexity, Gonzalez et al. [17] show that p-partition is NP-complete for $p \geq 3$, but they leave open the case $p = 2$. Since 2-partition is possible if and only if there exists two separable vertices, Theorem 1 proves that 2-partition can be decided in polynomial time.

▶ Remark 2. In very recent contributions, Chepoi [8] and Bressan et al. [4] also showed that half-space separation can be decided in polynomial time for monophonic convexity. Their results have been obtained independently and using different approaches, even though they share some common points with the technique used in this paper.

Organization of the paper

In Section 2 we provide definitions, notations and we formally define the problem we investigate in the paper. In Section 3 we prove Theorem 1 by giving an algorithm which decides whether two sets can be separated by half-spaces. We conclude in Section 4.

▶ Remark 3. Due to space limitations, most proofs are omitted. All of them can found in the `arXiv` version of the present contribution [13].

2 Preliminaries

All the objects considered in this paper are finite. Let V be a set. The powerset of V is denoted $\mathbf{2}^V$. Given $X \subseteq V$, we write $\overline{X}$ the complement of X in V, i.e., $\overline{X} = V \setminus X$. Sometimes, we shall write a set X as the concatenation of its elements, e.g., uv instead of $\{u, v\}$. As a result, $X \cup uv$ and $X \cup v$ stands for $X \cup \{u, v\}$ and $X \cup \{v\}$ respectively.

Graphs

We assume the reader is familiar with standard graph terminology. We consider loopless undirected graphs. Let G be a graph with vertices $V(G)$ and edge set $E(G)$. A subgraph of G is any graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The *(open) neighborhood* of a vertex v in G is denoted $N(v)$ and is defined as $N(v) = \{u \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of v in G is $N[v] = N(v) \cup v$. For $X \subseteq V(G)$, we put similarly $N(X) = \{u \in V(G) \setminus X : xu \in E(G) \text{ for some } x \in X\}$ and $N[X] = N(X) \cup X$. The subgraph of G *induced* by X is $G[X] = (X, E(G[X]))$, where $E(G[X]) = \{uv \in E(G) : u, v \in X\}$. If this is clear from the context, we identify X with $G[X]$, and we use $G - X$ to denote $G[V(G) - X]$. A *path* in G is a subgraph P of G with $V(P) = \{v_1, \ldots, v_k\}$ and such that $v_i v_{i+1} \in E(P)$ for each $1 \leq i < k$. Putting $u = v_1$ and $v = v_k$, P is an *uv-path* of G. An *induced uv-path* or *chordless uv-path* of G is an induced subgraph of G being an uv-path. A shortest path is an induced path with the least possible number of vertices. For simplicity we will identify a path P with the sequence $v_1, \ldots, v_k$ of its vertices. Let $A, B \subseteq V(G)$ be non-empty. The *(inner) frontier* of A with respect to B is $F(A, B) = A \cap N[B]$. We note that if A, B are disjoint, we obtain $F(A, B) = A \cap N(B)$. Remark that for every $X \subseteq V(G)$, $F(\overline{X}, X) = N(X)$.

Convexity spaces

We refer the reader to [24] for a thorough introduction to convexity theory. A *convexity space* is a pair $(V, \mathcal{C})$, with $\mathcal{C} \subseteq \mathbf{2}^V$, such that $\emptyset, V \in \mathcal{C}$ and for every $C_1, C_2 \in \mathcal{C}$, $C_1 \cap C_2 \in \mathcal{C}$. The sets in $\mathcal{C}$ are *convex* sets. A convexity space $(V, \mathcal{C})$ induces a *(convex) hull* operator $h \colon \mathbf{2}^V \to \mathbf{2}^V$ defined for all $X \subseteq V$ by:

$$h(X) = \bigcap \{C \in \mathcal{C} : X \subseteq C\}$$

The operator h satisfies, for all $X, Y \subseteq V$: $X \subseteq h(X)$; $h(X) \subseteq h(Y)$ if $X \subseteq Y$; and $h(h(X)) = h(X)$. The *Carathéodory number* of $(V, \mathcal{C})$ is the least integer d such that for every $X \subseteq V$ and $v \in V$, if $v \in h(X)$, there exists a subset Y of X such that $v \in h(Y)$ and $|Y| \leq d$. A *half-space* of $(V, \mathcal{C})$ is a convex set H whose set complement $\overline{H}$ is convex, that is, $H, \overline{H} \in \mathcal{C}$. Let A, B be two subsets of V. We say that A and B are *(half-space) separable* if there exists half-spaces $H, \overline{H}$ satisfying $A \subseteq H$ and $B \subseteq \overline{H}$. This is equivalent to $h(A) \subseteq H$ and $h(B) \subseteq \overline{H}$. The *shadow of A with respect to B* [6, 7] is the set $A/B = \{v \in V : h(B \cup v) \cap A \neq \emptyset\}$. Observe that $A \subseteq A/B$ and $B \subseteq B/A$.

▶ Remark 4. Usually, A/B is defined for disjoint sets. Here, it is more convenient to extend this definition to sets that may intersect. If $A \cap B \neq \emptyset$, then $A/B = V$ vacuously.

Monophonic convexity

We introduce monophonic convexity. We redirect the reader to [24, 20] for further details on graph and interval convexities. Let G be a graph, and let $u, v \in V(G)$. The *monophonic closed interval* of u, v is the set of all vertices that lie on a chordless uv-path, denoted by $J[u, v]$. For $X \subseteq V(G)$, we put $J[X] = \bigcup_{u,v \in X} J[u, v]$. A set C is *monophonically convex* if $J[C] = C$. Throughout the paper, if there is no ambiguity, we use the term convex sets as a shortening for monophonically convex sets. With $\mathcal{C} = \{C \subseteq V(G) : C \text{ is monophonically convex}\}$, the pair $(V(G), \mathcal{C})$ is a convexity space, the *monophonic convexity* of G. Its convex hull operator h is defined for all $X \subseteq V(G)$ by:

$$h(X) = \bigcup_{k=0}^{\infty} X_k$$

where $X_0 = X$ and $X_i = J[X_{i-1}]$ for $i \geq 1$. We now gather existing results regarding monophonic convexity that will be useful throughout the paper. These statements are rephrased to match our terminology.

▶ **Observation 5** (see also [12]). *In a connected graph G, every convex set is connected.*

▶ **Theorem 6** ([12], Theorem 5.1). *The monophonic convexity of a connected graph has Carathéodory number is* 1 *if the graph is a clique and* 2 *otherwise.*

▶ **Theorem 7** ([10], Theorem 4.1). *Let G be a graph and let $X \subseteq V(G)$. Then, $h(X)$ can be computed in polynomial time in the size of G.*

▶ **Theorem 8** ([10], Theorem 2.1). *Let G be a connected graph and let $C \subseteq V$. The set C is convex if and only if for every connected component S of $G - C$, $F_G(C, S)$ is a clique.*

▶ **Lemma 9** ([17], Lemma 14). *Let G be a connected graph, K a clique separator of G, and X the union of some of the connected components of $G - K$. Then $X \cup K$ is convex.*

We end these preliminaries by stating the problem we investigate in this paper. It is the problem of separating two sets of vertices by half-spaces. Its decision version is:

Half-space separation in monophonic convexity
Input: A graph G and two (non-empty and disjoint) subsets A, B of $V(G)$.
Question: Are A and B half-space separable?

Since h can be computed in polynomial time by Theorem 7 and A, B are separable if and only if $h(A)$, $h(B)$ are separable, we can assume w.l.o.g. that the sets A and B are convex.

3 Half-space separation

In this section, we prove Theorem 1, which we first restate.

▶ **Theorem 1.** *Given a graph G and two subsets A, B of $V(G)$, whether A, B are separated by monophonic half-spaces can be decided in polynomial time.*

Remark that if the input graph is not connected, one just has to solve the problem for each connected component. Thus, we can consider without loss of generality that the graphs we consider are connected. Hence, for the section, we fix a connected graph G. Let A, B be two (disjoint) convex sets of G. To prove Theorem 1, we give a polynomial time algorithm that decides whether A, B are separable. The algorithm first computes a shortest path $a = v_1, \ldots, v_k = b$ for some $a \in A$ and $b \in B$ in polynomial time. We show in Lemma 11 that A and B are separable if and only if there exists $1 \leq i < k$ such that $A_i := h(A \cup \{v_1, \ldots, v_i\})$ and $B_i := h(B \cup \{v_{i+1}, \ldots, v_k\})$ are separable. This step is the *linkage* of A and B. Then, for each i, the algorithm does the subsequent operations:

(1) It computes the *saturation* $A'_i := S(A_i, B_i)$, $B'_i := S(B_i, A_i)$ of A_i, B_i (resp.) with respect to h (see Subsection 3.2). Informally, the saturation step extends A_i and B_i with vertices that are forced on one of the two sides by the hull operator h. Lemma 13 shows that A_i, B_i are separable if and only if A'_i, B'_i are separable. Corollary 16 proves that computing saturation takes polynomial time.

(2) From A'_i and B'_i, it builds an equivalence relation $\equiv_{A'_i B'_i}$ on $\overline{A'_i \cup B'_i}$ and an associated graph $G_{A'_i B'_i}$. Theorem 31 states that A'_i and B'_i are separable if and only if $G_{A'_i B'_i}$ is bipartite and no equivalence class of $\equiv_{A'_i B'_i}$ contains a so-called *forbidden pair* of vertices. Finally, Theorem 32 proves that the conditions of Theorem 31 can be tested in polynomial time.

The algorithm outputs that A and B are separable if there is an integer i for which step (2) succeeds. Otherwise, A and B are not separable. The correctness of the algorithm follows from Lemma 11, Lemma 13 and Theorem 31. The fact that it runs in polynomial time is a consequence of Corollary 16 and Theorem 32. This proves Theorem 1.

The rest of the section is dedicated to the proof of the aforementioned statements.

3.1 Linkage along a shortest path

Let A, B be two non-empty disjoint convex subsets of $V(G)$. Assume that A and B are separable and let $H, \overline{H}$ be half-spaces separating A and B. Then for each $a \in A$, and each $b \in B$, all the vertices on the shortests ab-paths are distributed among H and $\overline{H}$. We show that, in fact, for each shortest ab-path, there is a vertex before which all vertices are assigned one half-space and all vertices after which are assigned the other half-space.

▶ **Proposition 10.** *Let $a \in A$, $b \in B$ and let $a = v_1, \ldots, v_k = b$ be a shortest ab-path. For every half-space separation $H, \overline{H}$ of A and B with $A \subseteq H$ and $B \subseteq \overline{H}$, there exists $1 \leq i < k$ such that $\{v_1, \ldots, v_i\} \subseteq H$ and $\{v_{i+1}, \ldots, v_k\} \subseteq \overline{H}$.*

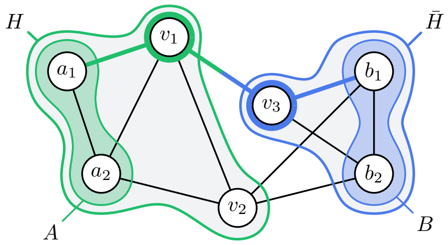

Figure 1 A graph G with two disjoint convex sets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ (circled in green and blue resp.). A and B are not linked, but they can be linked along the path a_1, v_1, v_3, b_1 (in bold green / bold blue). Namely, $A \cup v_1$ and $B \cup v_3$ are linked and convex. Two half-spaces $H, \overline{H}$ separating $A \cup v_1$ and $B \cup v_3$ (hence A and B) are drawn.

Following Proposition 10, we say that A and B are *linked* if there exists $a \in A$, $b \in B$ such that $ab \in E(G)$. Linked sets and Proposition 10 are illustrated in Figure 1. The next lemma is a direct consequence of Proposition 10.

▶ **Lemma 11.** *Let G be a connected graph and let A, B be two non-empty disjoint convex subsets of $V(G)$. Let $a \in A$, $b \in B$ and let $a = v_1, \ldots, v_k = b$ be a shortest ab-path. Then, A and B are separable if and only if there exists $1 \leq i < k$ such that $h(A \cup \{v_1, \ldots, v_i\})$ and $h(B \cup \{v_{i+1}, \ldots, v_k\})$ are separable.*

Given $a \in A$ and $b \in B$, finding a shortest ab-path can be done in polynomial time. Hence, making A and B linked can be done efficiently. Moreover, if A and B are linked, then for any disjoint $A', B' \subseteq V$ such that $A \subseteq A'$ and $B \subseteq B'$, A' and B' must be linked too. In what follows, we will thus consider disjoint, convex and linked subsets of $V(G)$.

3.2 Saturation with the hull operator

Let A, B be two disjoint, linked and convex subsets of $V(G)$. In this part, we use the hull operator h to define two sets $S(A, B)$ and $S(B, A)$ – the saturation of A and B (see below) – with $A \subseteq S(A, B)$, $B \subseteq S(B, A)$ and such that A, B are separable if and only if their saturation is separable. Informally, we use h to identify vertices that will appear in the same half-space as A in any possible half-space separation of A (and similarly with B), if any. We use two properties built on h:

(1) **Shadow-closing.** Remind that A/B, the shadow of A with respect to B, is defined by $A/B = \{v \in V(G) : h(B \cup v) \cap A \neq \emptyset\}$. In particular, $A \subseteq A/B$.

(2) **Forbidden sets.** Let $X \subseteq \overline{A \cup B}$ and assume that $h(X) \cap A \neq \emptyset$ and $h(X) \cap B \neq \emptyset$. Since $h(v) = \{v\}$ for all $v \in V$, we have $|X| \geq 2$. Thus, separating A, B implies to split the vertices of X. We say that X is a *forbidden set* of A and B with respect to G. A set X is forbidden if and only if it includes an inclusion-wise minimal forbidden set as a subset. Henceforth, in order to use forbidden sets, we need only consider the family of inclusion-wise minimal forbidden sets, denoted $\mathrm{MFS}(A, B)$. Formally,

$$\mathrm{MFS}(A, B) = \min_{\subseteq}\{X \subseteq \overline{A \cup B} : h(X) \cap A \neq \emptyset \text{ and } h(X) \cap B \neq \emptyset\}.$$

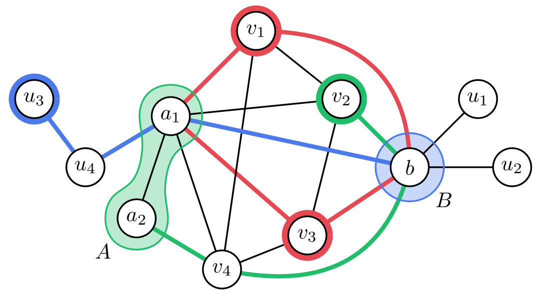

Figure 2 A graph G in which we seek to separate A and B (in circled green/blue resp.). The vertex a_1 is on a chordless u_3b-path (bold blue), so that $u_3 \in A/B$. Dually, b is on a chordless v_2a_2-path (bold green), i.e., $v_2 \in B/A$. Besides, $h(v_1v_3)$ intersects both A and B (bold red). Thus, v_1, v_3 must be separated to separate A and B, and $v_1v_3 \in \mathrm{MFS}(A, B)$ holds.

We illustrate shadows and forbidden sets in Figure 2. Now, we use A/B (resp. B/A) and $\mathrm{MFS}(A, B)$ in view of separating A and B. On the one hand, A/B cannot be separated from A by definition. On the other hand, for each $X \in \mathrm{MFS}(A, B)$ and every half-spaces $H, \bar{H}$ separating A and B with $A \subseteq H$, there exists at least one $x \in X$ such that $x \in H$, so that, $\bigcap_{x \in X} h(A \cup x) \subseteq H$ always hold. Based on the previous arguments, we define the *pre-saturation* of A with respect to B in G, denoted by $\sigma(A, B)$, by:

$$\sigma(A, B) = h\left(A/B \cup \bigcup\left\{\bigcap_{x \in X} h(A \cup x) : X \in \mathrm{MFS}(A, B)\right\}\right)$$

Observe that if $A \cap B \neq \emptyset$, then $\sigma(A, B) = \sigma(B, A) = V(G)$ as $A/B = B/A = V(G)$. In this case though, A and B cannot be separated. We prove in the next statement that $\sigma(A, B)$ preserves separation. Remark that it holds regardless of the disjointness of A and B.

▶ **Lemma 12.** *Let G be a connected graph, and let A, B be linked and convex subsets of $V(G)$. Then, A, B are separable if and only if $\sigma(A, B)$ and $\sigma(B, A)$ are separable.*

Proof. The if part follows from $A \subseteq A/B \subseteq \sigma(A, B)$ and $B \subseteq B/A \subseteq \sigma(B, A)$. We show the only if part. Suppose that A and B are separable and let $H, \overline{H}$ be half-spaces such that $A \subseteq H$, $B \subseteq \bar{H}$. Let $v \in A/B$. By definition, $h(B \cup v) \cap A \neq \emptyset$, hence $H \cap \overline{H} = \emptyset$ entails $v \in H$. Now let $X \in \mathrm{MFS}(A, B)$. By definition of forbidden sets, $X \cap H \neq \emptyset$ and $X \nsubseteq H$. Thus, there exists $x \in X$ such that $x \in H$, which entails $h(A \cup x) \subseteq H$ as H is convex. Since $\bigcap_{x' \in X} h(A \cup x') \subseteq h(A \cup x)$ for each $x \in X$, we deduce

$$A/B \cup \bigcup\left\{\bigcap_{x \in X} h(A \cup x) : X \in \mathrm{MFS}(A, B)\right\} \subseteq H.$$

As H is convex, we get $\sigma(A, B) \subseteq H$. Applying the symmetric reasoning on $\sigma(B, A)$ yields $\sigma(B, A) \subseteq \overline{H}$. This concludes the proof. ◀

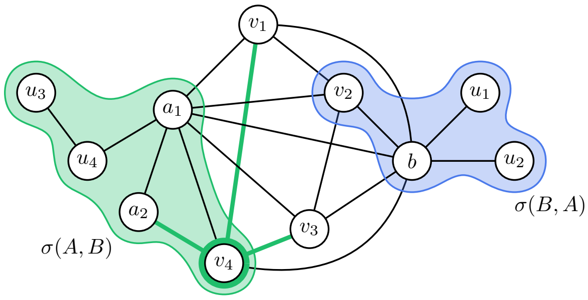

Figure 3 Pre-saturation applied to the sets A and B of Figure 2. For $\sigma(B, A)$, $u_1, u_2 \in B/A$ are added. For $\sigma(A, B)$, we have $u_3, u_4 \in A/B$ and $v_4 \in h(A \cup v_1) \cap h(A \cup v_3)$ (paths in bold green) with $v_1v_3 \in \mathrm{MFS}(A, B)$.

We illustrate pre-saturation in Figure 3, where the operation is applied to the set A and B of Figure 2. In this example, once pre-saturation has been applied, no further vertices can be assigned by applying pre-saturation once more. There are cases however where applying pre-saturation twice yields new vertices to assign. Figure 4 illustrates this situation.

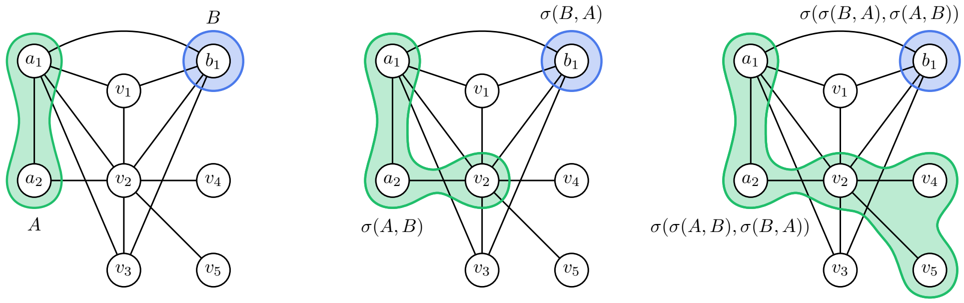

Figure 4 An example where pre-saturation can be applied twice. For $\sigma(A, B)$, v_2 is obtained from the forbidden pair v_1v_3. Once v_2 is added, v_4, v_5 become part of $\sigma(A, B)/\sigma(B, A)$. Observe that $B = \sigma(B, A) = \sigma(\sigma(B, A), \sigma(A, B))$. The remaining vertices v_1, v_3 can be separated in any way.

This suggests to iteratively apply the pre-saturation operator until no more vertices are added. For $A, B \subseteq V$, the *saturation* of A with respect to B, denoted by $S(A, B)$ is defined as follows:

$$S(A, B) = \bigcup_{i=0}^{\infty} \sigma(A_i, B_i)$$

where $A_0 = A$, $B_0 = B$ and for all $1 \leq i$, $A_i = \sigma(A_{i-1}, B_{i-1})$ and $B_i = \sigma(B_{i-1}, A_{i-1})$. Given $A, B \subseteq V(G)$, we say that A and B are *saturated* if $A = S(A, B)$ and $B = S(B, A)$. Since σ is increasing, the procedure for computing $S(A, B)$ terminates after $|V(G)|$ steps at most. Applying Lemma 12 inductively on $1 \leq i$, we get:

▶ **Lemma 13.** *Let G be a connected graph, and let A, B be two linked and convex subsets of $V(G)$. Then, A, B are separable if and only if $S(A, B)$, $S(B, A)$ are separable.*

▶ Remark 14. If $A_i \cap B_i \neq \emptyset$ for some i, then $S(A, B) = S(B, A) = V(G)$, and no separation can distinguish $S(A, B)$ and $S(B, A)$. In particular, A, B are thus not separable.

To conclude this paragraph, we argue that $S(A, B)$ can be computed in polynomial time in the size of G. Since S is at most $|V(G)|$ applications of σ on subsets of $V(G)$, it is sufficient to show that σ can be computed in polynomial time. The bottleneck of computing σ lies in finding $\mathrm{MFS}(A, B)$. However, the fact that the Carathéodory number of monophonic convexity is 2 by Theorem 6 makes the sets in $\mathrm{MFS}(A, B)$ of constant size.

▶ **Proposition 15.** *Given $A, B \subseteq V(G)$, $\sigma(A, B)$ can be computed in polynomial time in the size of G.*

▶ **Corollary 16.** *Given $A, B \subseteq V(G)$, $S(A, B)$ can be computed in polynomial time in the size of G.*

We note that saturation is not sufficient to decide separability, as suggested by Figure 5. This motivates the last step of the algorithm.

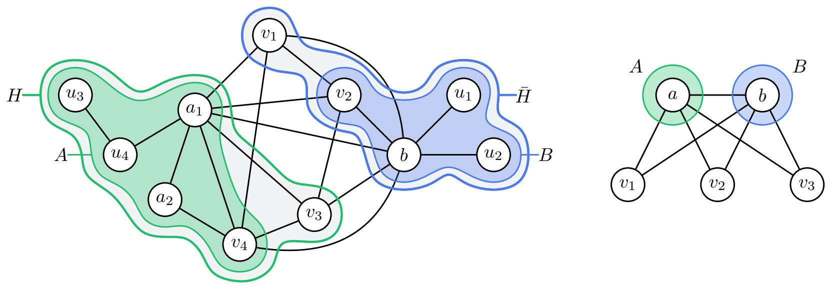

Figure 5 Two cases where A and B are linked, convex and saturated. On the left (follow-up of Figure 3), A and B can be separated (two half-spaces are drawn). On the right, any bipartition of the vertices will contain one of the forbidden pair v_1v_2, v_1v_3 or v_2v_3. Thus, A and B are not separable.

3.3 Testing bipartiteness

Let A, B be two linked, disjoint and saturated subsets of $V(G)$. By definition of saturation, A and B are convex. We characterize the separability of A and B using an equivalence relation $\equiv_{AB}$ on $\overline{A \cup B}$ and a graph G_{AB} defined from $\equiv_{AB}$. More precisely, we prove in Theorem 31 that A and B are separable if and only if G_{AB} is bipartite and no two $\equiv_{AB}$-equivalent vertices form a forbidden pair of $\mathrm{MFS}(A, B)$.

As a preliminary step though, we give properties of G and $N(A \cup B)$ in terms of A and B. We start with a statement that holds for every convex set.

▶ **Proposition 17.** *Let $C \subseteq V(G)$ be a convex set, and let u, v be two distinct vertices of $V(G) \setminus C$. Then:*

(1) *if u, v are not adjacent, then $h(uv) \cap C \neq \emptyset$ if and only if there exists $u', v' \in N(C)$ such that $u'v' \notin E(G)$ and $u', v' \in h(uv)$;*

(2) *if u, v are adjacent, then $F(C, u) \setminus F(C, v) \neq \emptyset$ entails $u \in h(C \cup v)$.*

Leveraging from the fact that A, B are linked and saturated, we use Proposition 17 to show that every vertex in $N(A \cup B)$ is adjacent to both A and B.

▶ **Lemma 18.** *For every $v \in N(A \cup B)$, $F(A, B) \cup F(B, A) \subseteq N(v)$. Therefore, the following properties hold for A (and symmetrically for B):*

(1) *$N(A) = F(B, A) \cup N(A \cup B)$;*

(2) *$F(A, \overline{A}) = F(A, N(A \cup B))$ is a clique.*

Proof. Assume for contradiction there exists $v \in N(A \cup B)$ such that $F(A,B) \cup F(B,A) \nsubseteq N(v)$. We have two cases:

(1) $F(A,B) \nsubseteq N(v)$ and $F(B,A) \nsubseteq N(v)$. Suppose w.l.o.g. that $v \in N(A)$. There exists $b \in F(B,A)$ such that $b \notin N(v)$. Then, we deduce by Proposition 17 that $h(bv) \cap A \neq \emptyset$ and $v \in A/B$.

(2) $F(A,B) \subseteq N(v)$ and $F(B,A) \nsubseteq N(v)$ (w.l.o.g.). Since $F(A,B) \subseteq N(v)$, $v \in N(A)$ holds. Thus, $v \in A/B$ again follows from Proposition 17.

In both cases, we obtain $v \in A/B$ with $v \notin A$. This contradicts A being saturated. We derive $F(A,B) \cup F(B,A) \subseteq N(v)$. Therefore, every $v \in N(A) \setminus B$ also lies in $N(B) \setminus A$ so that $N(A) \cap N(B) = N(A \cup B)$ holds along with $N(A) = F(B,A) \cup N(A \cup B)$ and $F(A, \overline{A}) = F(A, N(A \cup B))$. To see that $F(A, \overline{A})$ is a clique, observe that $B \cup N(A \cup B)$ is connected since B is convex. We deduce that $B \cup N(A \cup B)$ is included in a connected component of $G - A$. Since $F(A, \overline{A}) = F(A, N(A \cup B))$ and A is convex as it saturated, we obtain from Theorem 8 that $F(A, \overline{A})$ is a clique. ◀

Proposition 17 and Lemma 18 have two consequences. First, we can characterize $\mathrm{MFS}(A,B)$ as the set of pairs uv the closure of which contains non-adjacent vertices of $N(A \cup B)$, or in other words, a forbidden pair within $N(A \cup B)$.

▶ **Lemma 19.** *Let G be a connected graph and let A, B be two linked, disjoint and saturated subsets of $V(G)$. The following equality holds:*

$$\mathrm{MFS}(A,B) = \{uv \subseteq \overline{A \cup B} : h(uv) \cap N(A \cup B) \textit{ is not a clique}\}$$

In particular, $X \subseteq \overline{A \cup B}$ is forbidden if and only if it includes a forbidden pair of $\mathrm{MFS}(A,B)$.

As another consequence, we can describe $N(A \cup B)$ and its interactions with A and B depending on whether it is a clique or not.

▶ **Lemma 20.** *Let G be a connected graph and let A, B be two linked, disjoint and saturated subsets of $V(G)$. Then either $N(A \cup B)$ is a clique or for every $u, v \in N(A \cup B)$, $F(A,v) = F(A,u)$ and $F(B,v) = F(B,u)$.*

Proof. Suppose that $N(A \cup B)$ is not a clique and let u, v be two non-adjacent vertices of $N(A \cup B)$. We first prove that $F(A,v) = F(A,u)$ and $F(B,v) = F(B,u)$. Assume for contradiction that $F(A,v) \neq F(A,u)$. We have $F(A,v) \setminus F(A,u) \neq \emptyset$ or $F(A,u) \setminus F(A,v) \neq \emptyset$. By Proposition 17, we deduce $v \in h(A \cup u)$ or $u \in h(A \cup v)$. Since u, v are not adjacent, $uv \in \mathrm{MFS}(A,B)$ by Lemma 19 and we obtain $h(A \cup u) \cap B \neq \emptyset$ or $h(A \cup v) \cap B \neq \emptyset$. Thus, either $u \in A/B$ or $v \in A/B$. This contradicts A being saturated. We obtain $F(A,v) = F(A,u)$, and $F(B,v) = F(B,u)$ using the same argument on B.

Now, let $w \in N(A \cup B)$ such that $w \neq u, v$. If w is not adjacent to u or v, then $F(A,w) = F(A,u) = F(A,v)$ and $F(B,w) = F(B,u) = F(B,v)$ readily holds by previous argument. Therefore, suppose that w is adjacent to both u and v. We prove that: (1) $F(A,w) \setminus F(A,u) = \emptyset$ and (2) $F(A,u) \setminus F(A,w) = \emptyset$.

(1) Assume for contradiction that $F(A,w) \setminus F(A,u) \neq \emptyset$. Then, $w \in h(A \cup u)$ by Proposition 17. But since, $F(A,u) = F(A,v)$, we deduce $F(A,w) \setminus F(A,v) \neq \emptyset$ and hence $w \in h(A \cup v)$. Because $uv \in \mathrm{MFS}(A,B)$ and $w \in h(A \cup u) \cap h(A \cup v)$, $w \notin A$ is a contradiction with A being saturated. We deduce that $F(A,w) \setminus F(A,u) = \emptyset$ must hold.

(2) Again, suppose for contradiction that $F(A,u) \setminus F(A,w) \neq \emptyset$. By Proposition 5, we obtain $u \in h(A \cup w)$ and since $F(A,u) = F(A,v)$, $v \in h(A \cup w)$ also holds. Since $uv \in \mathrm{MFS}(A,B)$, we obtain $w \in B/A$, a contradiction with B being saturated.

We conclude that $F(A,w) = F(A,u)$ holds, and similarly $F(B,w) = F(B,u)$. This concludes the proof. ◀

The two situations obtained from Lemma 20 are illustrated in Figure 6. In the case where

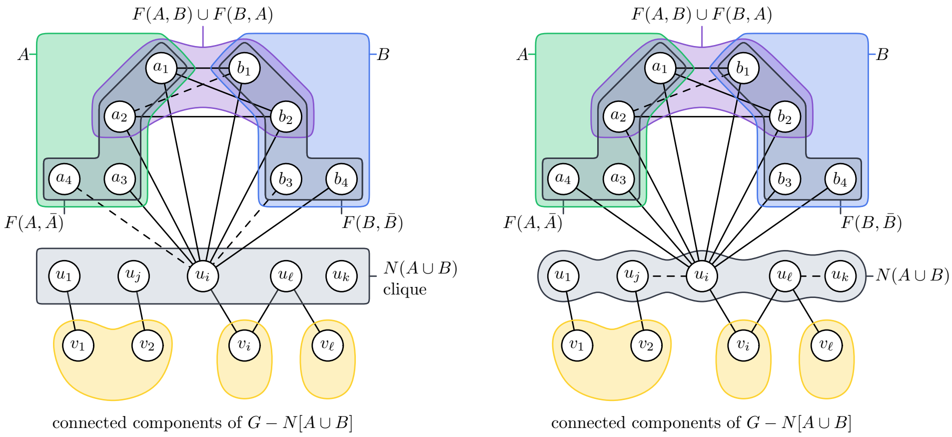

Figure 6 The two possible situations of Lemma 20. On the left, $N(A\cup B)$ is a clique. Each vertex of $N(A\cup B)$, is connected to each vertex of $F(A,B)\cup F(B,A)$ (circled in purple), modelled by u_i. However, it needs not be adjacent to all the vertices of the cliques $F(A,\bar{A})$ and $F(B,\bar{B})$ (the dotted line u_ia_4 indicates a non-edge). On the right, $N(A\cup B)$ is not a clique (for instance, u_i, u_j are not adjacent). Each vertex of $N(A\cup B)$ is complete to $F(A,\bar{A})\cup F(B,\bar{B})$, including $F(A,B)\cup F(B,A)$.

$N(A\cup B)$ is not a clique, Lemma 20 together with Lemma 18 yields the subsequent corollary that will be useful later on.

▶ **Corollary 21.** *If $N(A\cup B)$ is not a clique, then for every clique $K \subseteq N(A\cup B)$, $F(A,\overline{A})\cup K$ (resp. $F(B,\overline{B})\cup K$) is a clique.*

Thanks to Lemmas 19 and 20, we are in position to relate the separability of A, B with (co)bipartiteness. We first address the case where all the vertices left to assign lie in $N(A\cup B)$, i.e., when $\overline{A\cup B} = N(A\cup B)$. Although restricted, this case gives some insights for the general one.

If $N(A\cup B)$ is a clique, then $\mathrm{MFS}(A,B)=\emptyset$ by Lemma 19. Hence every bipartition X, Y of $N(A\cup B)$ readily satisfies $h(A\cup X)\cap B=\emptyset$ and $h(B\cup Y)\cap A=\emptyset$. Therefore, X and Y need only satisfy $h(A\cup X)\cap Y=\emptyset$ and $h(B\cup X)\cap Y=\emptyset$. The trivial bipartition $X=\emptyset$ and $Y=N(A\cup B)$ vacuously obeys this requirement.

On the other hand, when $N(A\cup B)$ is not a clique, the subsequent lemma implies that for any bipartition X, Y of $N(A\cup B)$ into cliques, $A\cup X$ and $B\cup Y$ are convex.

▶ **Lemma 22.** *Let G be a connected graph and let A, B be two linked, disjoint and saturated subsets of $V(G)$ such that $N(A\cup B)$ is not a clique. Then for every clique $K\subseteq N(A\cup B)$, both $A\cup K$ and $B\cup K$ are convex.*

Proof. To check that $A\cup K$ is convex, we verify that $J[u,v]\subseteq A\cup K$ for every $u,v\in A\cup K$. If $u,v\in A$ or $u,v\in K$, then the result holds since A is convex and K is a clique. Consider instead $u\in A, v\in K$. Assume for contradiction $J[u,v]\nsubseteq K\cup A$. There exists a chordless uv-path $u=v_1,\ldots,v_k=v$ such that $v_i\notin K\cup A$ for some $1<i<k$. Consider the least such i. By assumption $v_i\in N(A)$ and $v_{i-1}\in F(A,v_i)$. Morever, since A, B are saturated, $v_i\in N(A\cup B)$ must hold. As $N(A\cup B)$ is not a clique, we obtain by Lemma 20 that $F(A,v_i)=F(A,v)$, meaning that v_{i-1} is adjacent to v. This contradicts v_i being on a chordless uv-path. We deduce that $J[u,v]\subseteq A\cup K$ and $A\cup K$ is convex. ◀

We finally arrive at the following intermediate claim.

▶ **Lemma 23.** *Let G be a connected graph and let A, B be two disjoint, linked and saturated subsets of $V(G)$. If $\overline{A \cup B} = N(A \cup B)$, then A and B are separable if and only if $N(A \cup B)$ is cobipartite.*

Proof. We start with the only if part. Let $H = A \cup X$, $\overline{H} = B \cup Y$ be half-spaces separating A and B. By assumption, X contains no forbidden pair of $\text{MFS}(A, B)$. Since $X \subseteq N(A \cup B)$, we deduce from Lemma 19 that X is a clique. In the same way, we deduce that Y is a clique. As X, Y is a bipartition of $\overline{A \cup B} = N(A \cup B)$, we deduce that $N(A \cup B)$ is cobipartite.

We proceed to the if part. If $N(A \cup B)$ is cobipartite, we have two cases: either $N(A \cup B)$ is a clique or it is not. If $N(A \cup B)$ is a clique, then (resp. $A \cup N(A \cup B)$ and B) are half-spaces separating A and B. If $N(A \cup B)$ is not a clique, the fact that $A \cup X$ and $B \cup Y$ are half-spaces for all bipartitions X, Y of $N(A \cup B)$ into cliques follows from Lemma 22. ◀

Let us consider now that there are vertices outside of $N(A \cup B)$, i.e., $N(A \cup B) \subset \overline{A \cup B}$. First, if $N(A \cup B)$ is a clique, $\text{MFS}(A, B) = \emptyset$ still holds by Lemma 19. In this case, the same reasoning as before applies, and $A, B \cup \overline{A \cup B}$ is a half-space separation of A, B. Suppose on the other hand that $N(A \cup B)$ is not a clique. If it is not cobipartite, then any bipartition of $N(A \cup B)$ will contain a pair of non-adjacent vertices, and hence a forbidden pair, again due to Lemma 19. In other words, if $N(A \cup B)$ is not cobipartite, A and B are not separable. However, there are also cases where $N(A \cup B)$ is cobipartite, yet A and B are not separable. This is the case for the graphs of Figure 7, that we will use to illustrate the steps of the upcoming discussion. This happens because when picking an element v in a

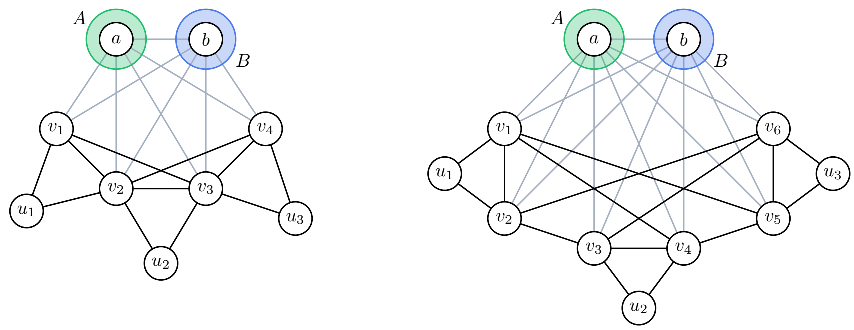

Figure 7 Two examples where A and B are linked and saturated, yet not separable despite $N(A \cup B)$ being cobipartite. For readability, the edges incident to a and b are clearer. Remark that since $N(A \cup B)$ is not a clique, both a and b are complete to $N(A \cup B)$ in virtue of Lemma 20.

connected component S of $G - N[A \cup B]$, $h(A \cup v)$ and $h(B \cup v)$ will share elements from $N(S)$, regardless of the structure of $N(A \cup B)$ (clique or not).

▶ **Lemma 24.** *Let G be a connected graph and let A, B be two linked, disjoint and saturated subsets of $V(G)$. Let S be a connected component of $G - N[A \cup B]$. Then, for every $v \in S$, $N(S) \subseteq h(A \cup v) \cap h(B \cup v) \cap N(A \cup B)$.*

Using Lemma 24, we define an equivalence relation on $\overline{A \cup B}$ that will help us characterize the separability of A and B. Every half-space separation H, $\overline{H}$ of A and B, if any, can be written as $H = A \cup X$ and $\overline{H} = B \cup Y$ where X, Y is a bipartition of $\overline{A \cup B}$. Since $H \cap \bar{H} = \emptyset$, we have $H \cap Y = h(A \cup X) \cap Y = \emptyset$ and similarly $\overline{H} \cap X = h(B \cup Y) \cap X = \emptyset$. As a direct application of Lemma 24, we deduce:

(1) For each connected component S of $G - N[A \cup B]$, either $N[S] \subseteq X$ or $N[S] \subseteq Y$;

(2) If $S_1, \dots, S_k$ is a sequence of (not necessarily distinct) connected components of $G - N[A \cup B]$ such that $N(S_i) \cap N(S_{i+1}) \neq \emptyset$ for each $1 \leq i < k$, then $\bigcup_{i=1}^{k} N[S_i]$ must be included in one of X or Y. We call such a sequence an intersecting sequence of connected components.

Given an intersecting sequence $S_1, \dots, S_k$ of connected components of $G - N[A \cup B]$, we say for brevity that u, v belongs to the sequence $S_1, \dots, S_k$ if there exists $1 \leq i, j \leq k$ such that $u \in N[S_i]$ and $v \in N[S_j]$. Let us define the equivalence relation $\equiv_{AB}$ on $\overline{A \cup B}$ such that, for all $u, v \in \overline{A \cup B}$:

$$u \equiv_{AB} v \iff u = v \text{ or } u, v \text{ belong to an intersecting sequence of connected components of } G - N[A \cup B]$$

▶ **Proposition 25.** *The relation $\equiv_{AB}$ is an equivalence relation.*

▶ Remark 26. The definition of $\equiv_{AB}$ encompasses the vertices that do not belong to the closed neighborhood of any connected component of $G - N[A \cup B]$, i.e., those vertices v in $N(A \cup B)$ such that $N[v] \subseteq N[A \cup B]$. By definition of $\equiv_{AB}$, they are equivalent to themselves only.

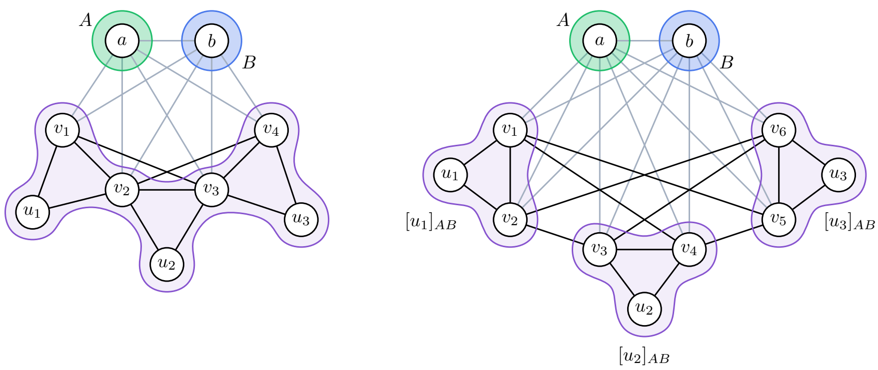

Figure 8 The equivalence relation $\equiv_{AB}$ applied to the graphs of Figure 7. The classes are circled (purple). On the left, there is a unique equivalence class. Remark that, as a consequence, v_1v_4 is a forbidden pair all the while $v_1 \equiv_{AB} v_4$. On the right, there are three classes, $[u_1]_{AB}$, $[u_2]_{AB}$, and $[u_3]_{AB}$.

For $u \in \overline{A \cup B}$, let $[u]_{AB}$ be the equivalence class of u: $[u]_{AB} = \{v \in \overline{A \cup B} : u \equiv_{AB} v\}$. In Figure 8, we give the equivalence classes induced by $\equiv_{AB}$ on the graphs of Figure 7. The next lemma is a direct yet important consequence of the above discussion and Lemma 24.

▶ **Lemma 27.** *Let G be a connected graph and let A, B be two disjoint, linked and saturated subsets of $V(G)$. For every bipartition X, Y of $\overline{A \cup B}$, we have $h(A \cup X) \cap Y = \emptyset$ and $h(B \cup Y) \cap X = \emptyset$ only if for each $v \in \overline{A \cup B}$, either $[v]_{AB} \subseteq X$ or $[v]_{AB} \subseteq Y$.*

We consider $\equiv_{AB}$ together with $\mathrm{MFS}(A, B)$. Remind that $\mathrm{MFS}(A, B)$ consists in pairs of vertices only, thanks to Lemma 19. Hence, a forbidden pair $uv \in \mathrm{MFS}(A, B)$ falls into exactly one of the following cases regarding equivalence classes:

(1) Either $u \equiv_{AB} v$ so that the equivalence class $[u]_{AB}$ prevents separation of A and B on its own (see Proposition 28 below). This case happens for instance in the graph on the left of Figure 8: $v_1 \equiv_{AB} v_4$ yet $v_1v_4 \in \mathrm{MFS}(A, B)$.

(2) Or $u \not\equiv_{AB} v$, so that $[u]_{AB}$ and $[v]_{AB}$ cannot be taken together in any separation of A and B. For example in the graph on the right of Figure 8 we have $v_1 \not\equiv_{AB} v_3$ and $v_1v_3 \in \mathrm{MFS}(A, B)$, which makes $[u_1]_{AB}$ and $[u_2]_{AB}$ incompatible for separating A and B. In this example, all equivalence classes are incompatible, so that A and B not separable.

As for the first case, we have the direct property:

▶ **Proposition 28.** *If $uv \in \mathrm{MFS}(A,B)$ and $u \equiv_{AB} v$, then A, B are not separable.*

For the second case, we can build a graph G_{AB} on the equivalence classes of $\equiv_{AB}$ that makes adjacent every two distinct equivalence classes sharing a forbidden pair. More formally:

$$V(G_{AB}) = \{[v]_{AB} : v \in \overline{A \cup B}\}$$
$$E(G_{AB}) = \{[u]_{AB}[v]_{AB} : u \not\equiv_{AB} v \text{ and } uv \in \mathrm{MFS}(A,B)\}.$$

For the graph on the right of Figure 8, the corresponding graph G_{AB} will be a clique. Figure 9 illustrates the graph G_{AB} on an other example.

▶ Remark 29. In the case where $\overline{A \cup B} = N(A \cup B)$, the equivalence classes $[v]_{AB}$ are precisely the singletons $\{v\}$ for all $v \in N(A \cup B)$. Identifying $[v]_{AB}$ with v, G_{AB} turns out to be precisely the complement of $G[N(A \cup B)]$.

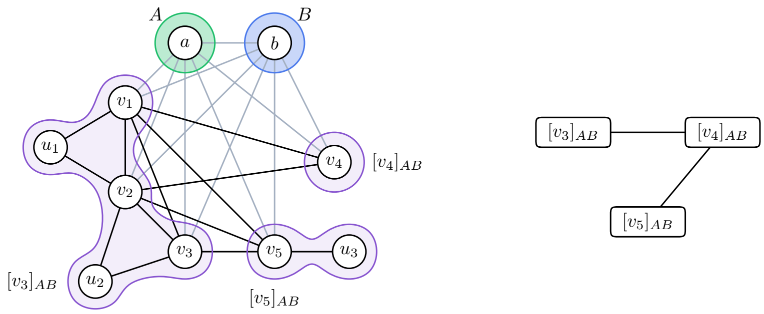

Figure 9 On the left, a graph with linked A and B where the equivalence class are highlighted. Again, the edges incident to a and b are clearer for readability. On the right, the corresponding graph G_{AB}.

Before characterizing the separability of A and B we give a lemma extending Lemma 22.

▶ **Lemma 30.** *Let G be a connected graph and let A, B be two disjoint, linked and saturated subsets of $V(G)$ such that $N(A \cup B)$ is not a clique. Then, for every collection $\mathcal{X}$ of equivalence classes of $\equiv_{AB}$, if $\bigcup \mathcal{X} \cap N(A \cup B)$ is a clique, then both $A \cup \bigcup \mathcal{X}$ and $B \cup \bigcup \mathcal{X}$ are convex.*

Proof. Let $\mathcal{X}$ be a collection of equivalences classes such that $\bigcup \mathcal{X} \cap N(A \cup B)$ is a clique and let $C = A \cup \bigcup \mathcal{X}$. We show that C is convex. We put $K = F(A, \bar{A}) \cup (N(A \cup B) \cap C)$. Now, by assumption, $N(A \cup B) \cap C$ is a clique, A and B are linked and saturated and $N(A \cup B)$ is not a clique. Therefore, K is a clique by Corollary 21. In view of Lemma 9, we show that K is a clique separator of G and that $C \setminus K$ is a union of connected components of $G - K$. First, since $F(A, \bar{A}) \subseteq K$ and K is a clique, we have that $G - K$ disconnects $A \setminus K$ from $\overline{A \cup B} \setminus K$. Hence K is a clique separator of G and moreover, $A \setminus K$ is indeed a union of connected components of $G - K$ since $F(A, \bar{A}) \subseteq K$. Now we consider $C \setminus (K \cup A)$. If $C \setminus (K \cup A) = \emptyset$, we deduce $C \subseteq A \cup K$ and the result holds by Lemma 22. Assume that $C \setminus (K \cup A) \neq \emptyset$ and let $S_1, \ldots, S_k$ be the connected components of $G - N[A \cup B]$ such that $C \cap S_i \neq \emptyset$ for each $1 \leq i \leq k$. We have $C \setminus (A \cup K) \subseteq \bigcup_{i=1}^{k} S_i$. We show that $\bigcup_{i=1}^{k} S_i \subseteq C \setminus (A \cup K)$. Let $v \in S_i$ for some $1 \leq i \leq k$. By definition of $\equiv_{AB}$, $S \subseteq [v]_{AB}$ and since $\mathcal{X}$ is a collection of equivalence classes, we obtain $S \subseteq [v]_{AB} \subseteq C \setminus (K \cup A)$. We deduce $C \setminus (A \cup K) \subseteq \bigcup_{i=1}^{k} S_i$ and hence $C \setminus (A \cup K) = \bigcup_{i=1}^{k} S_i$. It remains to show that S_i is a connected component of $G - K$. Since S_i is a connected component of $G - N[A \cup B]$, it is a connected component of

$G - N(S_i)$. Moreover, $N(S_i) \subseteq N(A \cup B)$ by construction. Finally, again by definition of C and $\equiv_{AB}$, $N(S_i) \subseteq C$. Henceforth, $N(S_i) \subseteq N(A \cup B) \cap C$ from which we deduce that S_i is a connected component of $G - K$. Hence, $C = K \cup (A \setminus K) \cup (C \setminus (K \cup A))$ is the union of a clique separator K of G and connected components of $G - K$. Applying Lemma 9, we deduce that C is convex, which concludes the proof. ◀

We can characterize the separability of A, B by generalizing Lemma 23.

▶ **Theorem 31.** *Let G be a connected graph and let A, B be two disjoint, linked and saturated subsets of V. Then A and B are separable if and only if the next conditions hold:*
(1) *for every $v \in \overline{A \cup B}$, $[v]_{AB}$ contains no forbidden pairs;*
(2) *G_{AB} is bipartite.*

Proof. We start with the only if part. Assume A and B are separable and let $H, \bar{H}$ be a half-space separation of A and B with $A \subseteq H$ and $B \subseteq \bar{H}$. Put $X = H \setminus A$ and $Y = \overline{H} \setminus B$. By assumption, $H \cap Y = h(A \cup X) \cap Y = \emptyset$. Hence, by Lemma 27, for each $v \in \overline{A \cup B}$, either $[v]_{AB} \subseteq X$ or $[v]_{AB} \subseteq Y$. Let $\mathcal{X} = \{[v]_{AB} \in V(G_{AB}) : [v]_{AB} \subseteq X\}$ and $\mathcal{Y} = \{[v]_{AB} \in V(G_{AB}) : [v]_{AB} \subseteq Y\}$. Since $H, \overline{H}$ are half-spaces separating A and B, and $X \subseteq H$, $Y \subseteq \overline{H}$, we deduce that neither X nor Y contain a forbidden pair of $\mathrm{MFS}(A, B)$. We derive:
(1) for each $[v]_{AB}$, $[v]_{AB}$ contains no forbidden pair, i.e., item (1) holds;
(2) for each pair of distinct classes $[u]_{AB}, [v]_{AB}$ in X (resp. Y), $[u]_{AB}$ and $[v]_{AB}$ are not adjacent in G_{AB}, i.e., that $\mathcal{X}$ (resp. $\mathcal{Y}$) is an independent set of G_{AB}. Since $\mathcal{X}$, $\mathcal{Y}$ is a partition of G_{AB} into two independent sets, we conclude that G_{AB} is bipartite, and that item (2) of the theorem holds.

We move to the if part. Assume both items (1) and item (2) are satisfied. In particular, if $N(A \cup B)$ is a clique, $\mathrm{MFS}(A, B) = \emptyset$ by Lemma 19. Hence, $A \cup N(A \cup B)$ and B (resp. $B \cup N(A \cup B)$ and A) are half-spaces separating A and B. Assume $N(A \cup B)$ is not a clique and let $\mathcal{X}, \mathcal{Y}$ be any bipartition of $V(G_{AB})$ into two independent sets. We show that $\bigcup \mathcal{X}$ contains no forbidden pair. Assume for contradiction there exists a forbidden pair $uv \in \bigcup \mathcal{X}$. We have two cases:
(1) $u \equiv_{AB} v$, but this would contradict item (1) of the statement;
(2) $u \not\equiv_{AB} v$, but this would contradict $\mathcal{X}_A$ being an independent set of G_{AB} by definition of G_{AB}.

By Lemma 19 we deduce that $\bigcup \mathcal{X}$ contains no forbidden pair, and hence that $\bigcup \mathcal{X} \cap N(A \cup B)$ is a clique. Applying Lemma 30, $A \cup \bigcup \mathcal{X}$ is convex. The same reasoning on $B \cup \mathcal{Y}$ yields that $A \cup \bigcup \mathcal{X}$ and $B \cup \mathcal{Y}$ are half-spaces separating A and B. This concludes the proof. ◀

Figure 10 illustrate the conditions of Theorem 31 on the example of Figure 9. We finally argue that the conditions of Theorem 31 can be checked in polynomial time. Since $\mathrm{MFS}(A, B)$ consists in pairs only, it can be computed in polynomial time. Then, we identify the connected components of $G - N[A \cup B]$ in polynomial time by traversing $G - N[A \cup B]$. We then identify the equivalence relation $\equiv_{AB}$ and build G_{AB} accordingly. Testing that no equivalent class contains a forbidden pair can be done in polynomial as well as checking that G_{AB} is bipartite. We deduce:

▶ **Theorem 32.** *Let G be a connected graph and let A, B be two disjoint, linked and saturated subsets of V. Whether A, B can be separated by half-spaces can be checked in polynomial time in the size of G.*

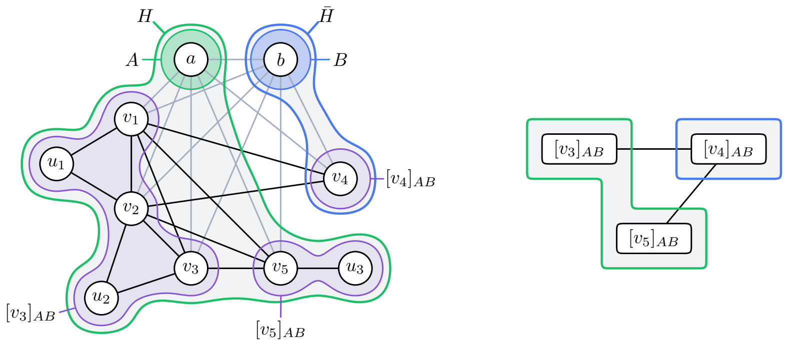

Figure 10 Illustration of Theorem 31 on the graph of Figure 9. A half-space separation of A and B is drawn. Observe that it corresponds to a bipartition of G_{AB} into independent sets.

4 Conclusion

We proved that half-space separability can be tested in polynomial time for monophonic convexity. Using Lemma 30, the algorithm we propose can be adapted to generate a pair of half-spaces separating two sets of vertices, if any. Moreover, we deduce as a corollary that the 2-partition problem can be solved in polynomial time for monophonic convexity, thus answering an open problem in [17].

To decide separability, we used the underlying graph together with the fact that the Carathéodory number is constant for monophonic convexity (Theorem 6, [12]). A natural question is then to investigate to what extent the Carathéodory number can be used to decide separability. However, relying on the problem of 2-coloring 3-uniform hypergraphs [19], we can show that already with Carathéodory number 3, half-space separation in general convexity spaces is out of reach.

▶ **Theorem 33.** *Half-space separation is NP-complete for convexity spaces $(V, \mathcal{C})$ given by V and a hull operator h that computes $h(X)$ in polynomial time in the size of V for all $X \subseteq V$, even if $(V, \mathcal{C})$ has Carathéodory number 3.*

Theorem 33 together with Theorem 1 motivate the next intriguing open problem.

▶ **Open Problem 34.** *Find a natural (graph) convexity with Carathéodory number 2 (e.g. triangle-path convexity [5]) where half-space separation is hard, or show that for all such convexities, half-space separation is tractable.*

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