

Parameterized Vertex Integrity Revisited

Tesshu Hanaka  



Department of Informatics, Kyushu University, Fukuoka, Japan

Michael Lampis  

Université Paris-Dauphine, PSL University, CNRS UMR7243, LAMSADE, Paris, France

Manolis Vasilakis  

Université Paris-Dauphine, PSL University, CNRS UMR7243, LAMSADE, Paris, France

Kanae Yoshiwatari  

Department of Mathematical Informatics, Nagoya University, Aichi, Japan

Abstract

Vertex integrity is a graph parameter that measures the connectivity of a graph. Informally, its meaning is that a graph has small vertex integrity if it has a small separator whose removal disconnects the graph into connected components which are themselves also small. Graphs with low vertex integrity are very structured; this renders many hard problems tractable and has recently attracted interest in this notion from the parameterized complexity community. In this paper we revisit the NP-complete problem of computing the vertex integrity of a given graph from the point of view of structural parameterizations. We present a number of new results, which also answer some recently posed open questions from the literature. Specifically, we show that unweighted vertex integrity is $W[1]$ -hard parameterized by treedepth; we show that the problem remains $W[1]$ -hard if we parameterize by feedback edge set size (via a reduction from a BIN PACKING variant which may be of independent interest); and complementing this we show that the problem is FPT by max-leaf number. Furthermore, for weighted vertex integrity, we show that the problem admits a single-exponential FPT algorithm parameterized by vertex cover or by modular width, the latter result improving upon a previous algorithm which required weights to be polynomially bounded.

2012 ACM Subject Classification Theory of computation \rightarrow Parameterized complexity and exact algorithms

Keywords and phrases Parameterized Complexity, Treedepth, Vertex Integrity

Digital Object Identifier 10.4230/LIPIcs.MFCS.2024.58

Related Version *Full Version*: <https://arxiv.org/abs/2402.09971>

Funding This work is partially supported by ANR projects ANR-21-CE48-0022 (S-EX-AP-PE-AL) and ANR-18-CE40-0025-01 (ASSK), and JSPS KAKENHI Grant Numbers JP21K17707, JP21H05852, JP22H00513, and JP23H04388.

1 Introduction

The *vertex integrity* of a graph is a vulnerability measure indicating how easy it is to break down the graph into small pieces. More precisely, the vertex integrity $vi(G)$ of a graph G is defined as $vi(G) = \min_{S \subseteq V(G)} \{|S| + \max_{D \in cc(G-S)} |D|\}$, that is, to calculate the vertex integrity of a graph we must find a separator that minimizes the size of the separator itself plus the size of the largest remaining connected component. Intuitively, a graph has low vertex integrity not only when it contains a small separator, but more strongly when it contains a small separator such that its removal leaves a collection of small connected components.

Vertex integrity was first introduced more than thirty years ago by Barefoot et al. [1], but has recently received particular attention from the parameterized complexity community since it can be considered as a very natural structural parameter: when a graph has vertex



© Tesshu Hanaka, Michael Lampis, Manolis Vasilakis, and Kanae Yoshiwatari; licensed under Creative Commons License CC-BY 4.0

49th International Symposium on Mathematical Foundations of Computer Science (MFCS 2024).

Editors: Rastislav Kráľovič and Antonín Kučera; Article No. 58; pp. 58:1–58:14

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

integrity k , large classes of NP-hard problems admit FPT¹ algorithms with running times of the form $f(k)n^{\mathcal{O}(1)}$ [27]. Note that vertex integrity has a clear relationship with other, well-known structural parameters (see also Figure 1): it is more restrictive than treedepth, pathwidth, and treewidth (all these parameters are upper-bounded by vertex integrity) but more general than vertex cover (a graph of vertex cover k has vertex integrity at most $k + 1$). “Price of generality” questions, where one seeks to discover for a given problem the most general parameter for which an FPT algorithm is possible, are a central topic in structural parameterized complexity, and vertex integrity therefore plays a role as a natural stepping stone in the hierarchy of standard parameters [3, 15, 16, 20, 22, 27].

The investigation of the parameterized complexity aspects of vertex integrity is, therefore, an active field of research, but it is important to remember that a prerequisite for any such parameter to be useful is that it should be tractable to calculate the parameter itself (before we try to use it to solve other problems). Since, unsurprisingly, computing the vertex integrity exactly is NP-complete [6], in this paper we focus on this problem from the point of view of parameterized complexity. We consider both the unweighted, as well as a natural weighted variant of the problem. Formally, we want to solve the following:

UNWEIGHTED (WEIGHTED) VERTEX INTEGRITY

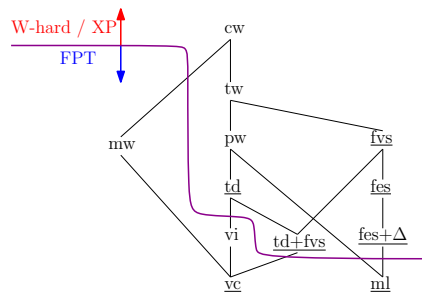
Instance: A graph G (with binary vertex weights $w : V(G) \rightarrow \mathbb{Z}^+$), an integer k .
Goal: Determine whether $\text{vi}(G) \leq k$ ($\text{wvi}(G) \leq k$).

The point of view we adopt is that of structural parameterized complexity, where vertex integrity is the target problem we are trying to solve, and not necessarily the parameter. Instead, we parameterize by standard structural width measures, such as variations of treewidth. The questions we would like to address are of several forms:

1. For which structural parameters is it FPT to compute the vertex integrity?
2. For which such parameters is it possible to obtain an FPT algorithm with single-exponential complexity?
3. For which parameters can the weighted version of the problem be handled as well as the unweighted version?

To put these questions in context, we recall some facts from the state of the art. When the parameter k is the vertex integrity itself, Fellows and Stueckle show an $\mathcal{O}(k^{3k}n)$ -time algorithm for UNWEIGHTED VERTEX INTEGRITY [13], and later Drange et al. proposed an $\mathcal{O}(k^{k+1}n)$ -time algorithm even for WEIGHTED VERTEX INTEGRITY [9], so this problem is FPT. More recently, Gima et al. [21] took up the study of vertex integrity in the same structurally parameterized spirit as the one we adopt here and presented numerous results which already give some answers to the questions we posed above. In particular, for the first question they showed that UNWEIGHTED VERTEX INTEGRITY is W[1]-hard by pathwidth (and hence by treewidth); for the second question they showed that the problem admits a single-exponential algorithm for parameter modular-width; and for the third question they showed that the problem is (weakly) NP-hard on sub-divided stars, which rules out FPT algorithms for most structural parameters.

¹ We assume the reader is familiar with the basics of parameterized complexity, as given e.g. in [7]. We give precise definitions of all parameters in the next section.



■ **Figure 1** The parameterized complexity of UNWEIGHTED VERTEX INTEGRITY, with the underlined parameters indicating our results. A connection between two parameters implies that the one above generalizes the one below; that is, the one above is upper-bounded by a function of the one below. Regarding the presented parameters, vc , vi , td , pw , tw , cw , ml , fes , fvs , Δ , and mw stand for vertex cover, vertex integrity, treedepth, pathwidth, treewidth, cliquewidth, max-leaf number, feedback edge set, feedback vertex set, maximum degree, and modular width respectively. All of our FPT algorithms have single-exponential parametric dependence, while the ones for vc and mw extend to the weighted case as well.

Our results. Although the results of [21] are rather comprehensive, they leave open several important questions about the complexity of vertex integrity. In this paper we resolve the questions explicitly left open by [21] and go on to present several other results that further clarify the picture for vertex integrity. In particular, our results are as follows (see also Figure 1):

The first question we tackle is an explicit open problem from [21]: is UNWEIGHTED VERTEX INTEGRITY FPT parameterized by treedepth? This is a very natural question, because treedepth is the most well-known parameter that sits between pathwidth, where the problem is $W[1]$ -hard by [21], and vertex integrity itself, where the problem is FPT. We resolve this question via a reduction from BOUNDED DEGREE VERTEX DELETION, showing that UNWEIGHTED VERTEX INTEGRITY is $W[1]$ -hard for treedepth (Theorem 2).

A second question left open by [21] is the complexity of UNWEIGHTED VERTEX INTEGRITY for parameter feedback vertex set. Taking a closer look at our reduction from BOUNDED DEGREE VERTEX DELETION, which is known to be $W[1]$ -hard for this parameter, we observe that it also settles this question, showing that UNWEIGHTED VERTEX INTEGRITY is also hard. However, in this case we are motivated to dig a little deeper and consider a parameter, feedback edge set, which is a natural restriction of feedback vertex set and typically makes most problems FPT. Our second result is to show that UNWEIGHTED VERTEX INTEGRITY is in fact $W[1]$ -hard even when parameterized by feedback edge set and the maximum degree of the input graph (Theorem 7). We achieve this via a reduction from UNARY BIN PACKING parameterized by the number of bins, which is $W[1]$ -hard [23]. An aspect of our reduction which may be of independent interest is that we use a variant of UNARY BIN PACKING where we are given a choice of only two possible bins per item (we observe that the reduction of [23] applies to this variant).

We complement these mostly negative results with a fixed-parameter tractability result for a more restrictive parameter: we show that UNWEIGHTED VERTEX INTEGRITY is FPT by max-leaf number (Theorem 10) indeed by a single-exponential FPT algorithm. Note that when a graph has bounded max-leaf number, then it has bounded degree and bounded feedback edge set number, therefore this parameterization is a special case of the one considered in Theorem 7. Hence, this positive result closely complements the problem's hardness in the more general case.

Moving on, we consider the parameterization by modular width, and take a second look at the $2^{\mathcal{O}(\text{mw})}n^{\mathcal{O}(1)}$ algorithm provided by [21], which is able to handle the weighted case of the problem, but only for polynomially-bounded weights. Resolving another open problem posed by [21], we show how to extend their algorithm to handle the general case of weights encoded in binary (Theorem 13).

Finally, we ask the question of whether a single-exponential FPT algorithm is possible for parameters other than max-leaf and modular width. We answer this affirmatively for vertex cover, even in the weighted case (Theorem 16), obtaining a faster and simpler algorithm for the unweighted case (Theorem 14).

Related work. The concept of vertex integrity is natural enough that it has appeared in many slight variations under different names in the literature. We mention in particular, the fracture number [10], which is the minimum k such that it is possible to delete k vertices from a graph so that all remaining components have size at most k , and the starwidth [29], which is the minimum width of a tree decomposition that is a star. Both of these are easily seen to be at most a constant factor away from vertex integrity. Similarly, the safe set number [2, 14] seeks a separator such that every component of the separator is only connected to smaller components. These concepts are so natural that sometimes they are used as parameters without an explicit name, for example [4] uses the parameter “size of a deletion set to a collection of components of bounded size”. As observed by [21], despite these similarities, sometimes computing these parameters can have different complexity, especially when weights are allowed. Another closely related computational problem, that we also use, is the COMPONENT ORDER CONNECTIVITY [9] problem, where we are given explicit distinct bounds on the size of the separator sought and the allowed size of the remaining components.

2 Preliminaries

Throughout the paper we use standard graph notation [8] and assume familiarity with the basic notions of parameterized complexity [7]. All graphs considered are undirected without loops. Given a graph G , Δ denotes its maximum degree; if we are additionally given $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S , while $G - S$ denotes $G[V(G) \setminus S]$. Furthermore, given a weight function $w : V(G) \rightarrow \mathbb{Z}^+$, $w(S)$ denotes the sum of the weights of the vertices of S , that is $w(S) = \sum_{s \in S} w(s)$. For $x, y \in \mathbb{Z}$, let $[x, y] = \{z \in \mathbb{Z} : x \leq z \leq y\}$, while $[x] = [1, x]$. For a set of integers $S \subseteq \mathbb{Z}^+$, let $\Sigma(S)$ denote the sum of its elements, i.e. $\Sigma(S) = \sum_{s \in S} s$, while $\binom{S}{c}$ denotes the set of subsets of S of size c , i.e. $\binom{S}{c} = \{S' \subseteq S : |S'| = c\}$. Proofs of statements marked with (\star) are presented in the full version of the paper.

2.1 Vertex Integrity

For a vertex-weighted graph G with $w : V(G) \rightarrow \mathbb{Z}^+$, we define its *weighted vertex integrity*, denoted by $\text{wvi}(G)$, as

$$\text{wvi}(G) = \min_{S \subseteq V(G)} \left\{ w(S) + \max_{D \in \text{cc}(G-S)} w(D) \right\},$$

where $\text{cc}(G - S)$ is the set of connected components of $G - S$. A set S such that $w(S) + \max_{D \in \text{cc}(G-S)} w(D) \leq k$ is called a $\text{wvi}(k)$ -set. The *vertex integrity* of an unweighted graph G , denoted by $\text{vi}(G)$, is defined in an analogous way, by setting $w(v) = 1$ for all $v \in V(G)$. In that case, $S \subseteq V(G)$ is a $\text{vi}(k)$ -set if $|S| + \max_{D \in \text{cc}(G-S)} |D| \leq k$.

A vertex $v \in S$ is called *redundant* if at most one connected component of $G - S$ contains neighbors of v . A set $S \subseteq V(G)$ is *irredundant* if S contains no redundant vertex. Notice that it suffices to only search for irredundant $\text{wvi}(k)$ -sets when solving VERTEX INTEGRITY, since if $v \in S$ is redundant and S is a $\text{wvi}(k)$ -set, that is the case for set $S \setminus \{v\}$ as well.

► **Proposition 1** ([9, 21]). *A graph with a $\text{wvi}(k)$ -set has an irredundant $\text{wvi}(k)$ -set.*

2.2 Graph Parameters

We use several standard graph parameters, so we recall here their definitions and known relations between them. A graph G has *feedback vertex* (respectively *edge*) *set* k if there exists a set of k vertices (respectively edges) such that removing them from G destroys all cycles. We use $\text{fvs}(G)$ and $\text{fes}(G)$ to denote these parameters. Note that even though computing $\text{fvs}(G)$ is NP-complete [24], in all connected graphs with m edges and n vertices $\text{fes}(G) = m - n + 1$. The *vertex cover* of a graph G , denoted by $\text{vc}(G)$, is the size of the smallest set whose removal destroys all edges. The *treedepth* of a graph G can be defined recursively as follows: $\text{td}(K_1) = 1$; if G is disconnected $\text{td}(G)$ is equal to the maximum of the treedepth of its connected components; otherwise $\text{td}(G) = \min_{v \in V(G)} \text{td}(G - v) + 1$. The *max-leaf number* of a graph G , denoted by $\text{ml}(G)$, is the maximum number of leaves of any spanning tree of G .

A module of a graph $G = (V, E)$ is a set of vertices $M \subseteq V$ such that for all $x \in V \setminus M$ we have that x is either adjacent to all vertices of M or to none. The *modular width* of a graph $G = (V, E)$ ([17, 18]) is the smallest integer k such that, either $|V| \leq k$, or V can be partitioned into at most $k' \leq k$ sets $V_1, \dots, V_{k'}$, with the following two properties: (i) for all $i \in [k']$, V_i is a module of G , (ii) for all $i \in [k']$, $G[V_i]$ has modular width at most k .

Let us also briefly explain the relations depicted in Figure 1. Clearly, for all G , we have $\text{fvs}(G) \leq \text{fes}(G)$, because we can remove from the graph one endpoint of each edge of the feedback edge set. It is known that if a graph has $\text{ml}(G) = k$, then G contains at most $\mathcal{O}(k)$ vertices of degree 3 or more (Lemma 8), and clearly such a graph has maximum degree at most k . Since vertices of degree at most 2 are irrelevant for fes , we conclude that the parameterization by ml is more restrictive than that for $\text{fes} + \Delta$. It is also not hard to see that for all G , $\text{td}(G) \leq \text{vi}(G) \leq \text{vc}(G) + 1$. Note also that even though vc can be seen as a parameter more restrictive than mw , when a graph has vertex cover k , the best we can say is that its modular width is at most $2^k + k$ [26]. As a result, the algorithm of Theorem 13 does not imply a single-exponential FPT algorithm for parameter vc (but does suffice to show that the problem is FPT). We also note that the reductions of Theorem 2 (for td) and Theorem 7 (for $\text{fes} + \Delta$) are complementary and cannot be subsumed by a single reduction. The reason for this is that if in a graph we bound simultaneously the treedepth and the maximum degree, then we actually bound the size of the graph (rendering all problems FPT).

3 Treedepth

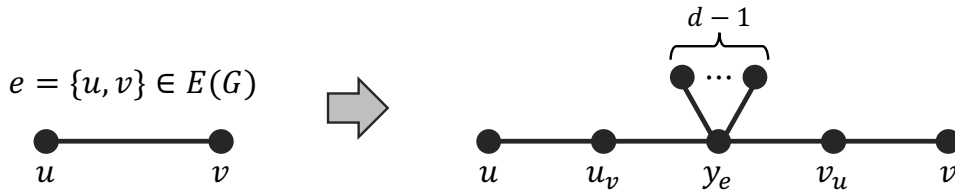
Our main result in this section is the following theorem, resolving a question of [21]. We obtain it via a parameter-preserving reduction from BOUNDED DEGREE VERTEX DELETION, which is known to be $W[1]$ -hard parameterized by treedepth plus feedback vertex set [19].

► **Theorem 2.** *UNWEIGHTED VERTEX INTEGRITY is $W[1]$ -hard parameterized by $\text{td} + \text{fvs}$. Moreover, it cannot be solved in time $f(\text{td})n^{o(\text{td})}$ under the ETH.*

Proof. First we define the closely related COMPONENT ORDER CONNECTIVITY problem: given a graph G as well as integers ℓ and p , we want to determine whether there exists $S \subseteq V(G)$ such that $|S| \leq p$ and all components of $G - S$ have size at most ℓ . We will proceed in two steps: we first reduce BOUNDED DEGREE VERTEX DELETION to COMPONENT ORDER CONNECTIVITY, and then employ the reduction of [21] that reduces the latter to UNWEIGHTED VERTEX INTEGRITY. Notice that [21, Lemma 4.4] creates an equivalent instance of UNWEIGHTED VERTEX INTEGRITY by solely adding disjoint stars and leaves in the vertices of the initial graph, therefore it suffices to prove the statement for COMPONENT ORDER CONNECTIVITY instead.

We give a parameterized reduction from BOUNDED DEGREE VERTEX DELETION, which is $W[1]$ -hard by treedepth plus feedback vertex set number [19] and cannot be solved in time $f(\text{td})n^{o(\text{td})}$ under the ETH [28]. In BOUNDED DEGREE VERTEX DELETION we are given a graph $G = (V, E)$ and two integers k and d , and we are asked to determine whether there exists $S \subseteq V$ of size $|S| \leq k$ such that the maximum degree of $G - S$ is at most d . In the following, let $n = |V(G)|$ and $m = |E(G)|$.

Given an instance (G, k, d) of BOUNDED DEGREE VERTEX DELETION, we construct an equivalent instance (G', ℓ, p) of COMPONENT ORDER CONNECTIVITY. We construct G' from G as follows: We subdivide every edge $e = \{u, v\} \in E(G)$ three times, thus replacing it with a path on vertices $u, u_v, y_e, v_u,$ and v , where $T_e = \{u_v, y_e, v_u\}$. Next, we attach $d - 1$ leaves to y_e (see Figure 2). This concludes the construction of G' . Notice that the subdivision of the edges three times and the attachment of pendant vertices does not change the feedback vertex set number, while the treedepth is only increased by an additive constant. Thus, it holds that $\text{fvs}(G') = \text{fvs}(G)$ and $\text{td}(G') = \text{td}(G) + \mathcal{O}(1)$.



■ **Figure 2** Edge gadget for edge $e = \{u, v\} \in E(G)$.

In the following, we show that (G, k, d) is a yes-instance of BOUNDED DEGREE VERTEX DELETION if and only if (G', ℓ, p) is a yes-instance of COMPONENT ORDER CONNECTIVITY, where $\ell = d + 1$ and $p = k + m$.

For the forward direction, let S be a set of vertices of size at most k such that the maximum degree of $G - S$ is at most d . We will construct a set $S' \subseteq V(G')$ such that $|S'| \leq p$ and every connected component of $G' - S'$ has size at most ℓ . Initially set $S' = S$. Then, add one vertex to S' per edge $e = \{u, v\} \in E(G)$ as follows. If $u, v \in S$ or $u, v \notin S$, we add y_e to S' . Otherwise, if $u \in S$ and $v \notin S$, we add v_u to S' ; symmetrically, if $u \notin S$ and $v \in S$, we add u_v instead. Notice that $|S'| = |S| + m \leq k + m = p$, therefore it suffices to show that the size of each connected component of $G' - S'$ is at most $\ell = d + 1$.

Consider a connected component D of $G' - S'$. Assume that D does not contain any vertices of $V \setminus S$. If D is a leaf it holds that $|D| \leq d + 1$. Alternatively, D is a subgraph of the graph induced by u_v (or v_u), y_e , and its attached leaves, for some $e = \{u, v\} \in E(G)$, in which case $|D| \leq d + 1$. Now assume that D contains $u \in V \setminus S$. Notice that u is the only vertex of $V \setminus S$ present in D , since $S' \cap T_e \neq \emptyset$ for all $e \in E(G)$. Moreover, let $N(u) \setminus S = \{u_i : i \in [q]\}$ denote its neighbors in $G - S$, where $q \leq d$ since the maximum degree of $G - S$ is at most d . In that case, it follows that D consists of u , as well as the vertices u_{u_i} for all $i \in [q]$. Consequently, $|D| = q + 1 \leq d + 1$.

For the converse direction, assume there exists $S' \subseteq V(G')$ such that $|S'| \leq p = k + m$ and $|D| \leq \ell = d + 1$, for all connected components $D \in \text{cc}(G' - S')$. Assume that S' does not contain any leaves; if it does, substitute them with their single neighbor. Moreover, $S' \cap T_e \neq \emptyset$ for all $e \in E(G)$, since otherwise $G' - S'$ has a component of size at least $d + 2 > \ell$, which is a contradiction. Assume without loss of generality that $|S' \cap T_e| = 1$, for all $e = \{u, v\} \in E(G)$; if that is not the case, there is always a vertex of $\{u_v, v_u\}$, say u_v , such that $u_v \in S'$ and $S' \cap \{y_e, v_u\} \neq \emptyset$, in which case one may consider the deletion set $(S' \cup \{u\}) \setminus \{u_v\}$ instead (the argument is symmetric in case $v_u \in S'$).

Let $S = S' \cap V$, where $|S| \leq k$. We will prove that $G - S$ has maximum degree at most d . Let D_u denote the connected component of $G' - S'$ that contains $u \in V \setminus S'$; in fact this is the only vertex of $V \setminus S'$ present in D_u , since $S' \cap T_e \neq \emptyset$ for all $e \in E(G)$. Notice that for all $e = \{u, v\} \in E(G)$ where $u, v \notin S'$, it holds that $y_e \in S'$: if that were not the case, then either D_u or D_v contains at least $d + 2 > \ell$ vertices, due to $\{u, u_v, y_e\}$ or $\{v, v_u, y_e\}$ and the leaves of y_e respectively. For $u \in V \setminus S$, let $N(u) \setminus S = \{u_i : i \in [q]\}$, for some integer q , denote its neighbors in $G - S$, where $e_i = \{u, u_i\} \in E(G)$ for $i \in [q]$. It suffices to show that $q \leq d$. Assume that this is not the case, i.e. $q > d$. Then, since $S' \cap T_{e_i} = \{y_{e_i}\}$ for $i \in [q]$, it follows that D_u contains vertices u and u_{u_i} , therefore $|D_u| \geq q + 1 > d + 1 = \ell$, which is a contradiction. Consequently, $|N(u) \setminus S| \leq d$ for all $u \in V \setminus S$, i.e. $G - S$ has maximum degree d . ◀

4 Feedback Edge Set plus Maximum Degree

In this section we prove that VERTEX INTEGRITY is W[1]-hard parameterized by $\text{fes} + \Delta$. Since our reduction is significantly more involved than the one of Theorem 2, we proceed in several steps. We start from an instance of UNARY BIN PACKING where the parameter is the number of bins and consider a variant where we are also supplied in the input, for each item, a choice of two possible bins to place it. We first observe that the reduction of [23] shows that this variant is also W[1]-hard. We then reduce this to a *semi-weighted* version of VERTEX INTEGRITY, where placing a vertex in the separator always costs 1, but vertices have weights which they contribute to their components if they are not part of the separator, and where we are prescribed the size of the separator to use (this is called the COMPONENT ORDER CONNECTIVITY problem). Subsequently, we show how to remove the weights and the prescription on the separator size to obtain hardness for VERTEX INTEGRITY.

4.1 Preliminary Tools

Unary Bin Packing. Given a set $S = \{s_1, \dots, s_n\}$ of integers in unary (i.e. $s_i = \mathcal{O}(n^c)$ for some constant c), as well as $k \in \mathbb{Z}^+$, UNARY BIN PACKING asks whether we can partition S into k subsets S_1, \dots, S_k , such that $\Sigma(S_i) = \Sigma(S)/k$, for all $i \in [k]$. This problem is well known to be W[1]-hard parameterized by the number of bins k [23]. We formally define a restricted version where every item is allowed to choose between *exactly two* bins, and by delving deeper into the proof of [23] we observe that an analogous hardness result follows.

RESTRICTED UNARY BIN PACKING

- Instance:** A set $S = \{s_1, \dots, s_n\}$ of integers in unary, $k \in \mathbb{Z}^+$, as well as a function $f : S \rightarrow \binom{[k]}{2}$.
- Goal:** Determine whether we can partition S into k subsets S_1, \dots, S_k , such that for all $i \in [k]$ it holds that (i) $\Sigma(S_i) = \Sigma(S)/k$, and (ii) $\forall s \in S_i, i \in f(s)$.

► **Theorem 3** (\star). RESTRICTED UNARY BIN PACKING is $W[1]$ -hard parameterized by the number of bins.

Semi-weighted problems. In this section we study semi-weighted versions of COMPONENT ORDER CONNECTIVITY and VERTEX INTEGRITY, which we first formally define. Then, we prove that the first can be reduced to the latter, while retaining the size of the minimum feedback edge set and the maximum degree.

SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY

Instance: A vertex-weighted graph $G = (V, E, w)$, as well as integers $\ell, p \in \mathbb{Z}^+$.
Goal: Determine whether there exists $S \subseteq V$ of size $|S| \leq p$, such that $w(D) \leq \ell$ for all $D \in \text{cc}(G - S)$.

SEMI-WEIGHTED VERTEX INTEGRITY

Instance: A vertex-weighted graph $G = (V, E, w)$, as well as an integer $\ell \in \mathbb{Z}^+$.
Goal: Determine whether there exists $S \subseteq V$ such that $|S| + w(D) \leq \ell$ for all $D \in \text{cc}(G - S)$.

► **Theorem 4** (\star). SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY parameterized by $\text{fes} + \Delta$ is *fpt-reducible* to SEMI-WEIGHTED VERTEX INTEGRITY parameterized by $\text{fes} + \Delta$.

4.2 Hardness Result

Using the results of Section 4.1, we proceed to proving the main theorem of this section. To this end, we present a reduction from RESTRICTED UNARY BIN PACKING to SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY such that for the produced graph G it holds that $\text{fes}(G) + \Delta(G) \leq f(k)$, for some function f and k denoting the number of bins of the RESTRICTED UNARY BIN PACKING instance.

We first provide a sketch of our reduction. For every bin of the RESTRICTED UNARY BIN PACKING instance, we introduce a clique of $\mathcal{O}(k)$ heavy vertices, and then connect any pair of such cliques via two paths. The weights are set in such a way that an optimal solution will only delete vertices from said paths. In order to construct a path for a pair of bins, we compute the set of all subset sums of the items that can be placed in these two bins, and introduce a vertex of medium weight per such subset sum. Moreover, every such vertex corresponding to subset sum s is preceded by exactly s vertices of weight 1. An optimal solution will cut the path in such a way that the number of vertices of weight 1 will be partitioned between the two bins, encoding the subset sum of the elements that are placed on each bin. The second path that we introduce has balancing purposes, allowing us to exactly count the number of vertices of medium weight that every connected component will end up with.

► **Theorem 5** (\star). SEMI-WEIGHTED COMPONENT ORDER CONNECTIVITY is $W[1]$ -hard parameterized by $\text{fes} + \Delta$.

By Theorems 4 and 5, the hardness of SEMI-WEIGHTED VERTEX INTEGRITY follows.

► **Theorem 6.** SEMI-WEIGHTED VERTEX INTEGRITY is $W[1]$ -hard parameterized by $\text{fes} + \Delta$.

Moreover, we can easily reduce an instance (G, w, k) of SEMI-WEIGHTED VERTEX INTEGRITY to an instance (G', k) of UNWEIGHTED VERTEX INTEGRITY by attaching a path on $w(v) - 1$ vertices to each vertex v (we assume that $w(v) \leq k$, otherwise v belongs to the deletion set). Thus, $\text{fes}(G') = \text{fes}(G)$ and $\Delta(G') = \Delta(G) + 1$, and due to Theorem 6 the main result of this section follows.

► **Theorem 7.** UNWEIGHTED VERTEX INTEGRITY is $W[1]$ -hard parameterized by $\text{fes} + \Delta$.

5 Max-Leaf Number

In this section, we consider UNWEIGHTED VERTEX INTEGRITY parameterized by the max-leaf number. For a connected graph G we denote by $\text{ml}(G)$ the maximum number of leaves of any spanning tree of G . This is a well-studied but very restricted parameter [11, 12, 26]. In particular, it is known that if a graph G has $\text{ml}(G) \leq k$, then in fact G is a subdivision of a graph on $\mathcal{O}(k)$ vertices [25]. We are motivated to study this parameter because in a sense it lies close to the intractability boundary established in Section 4. Observe that if a graph is a sub-division of a graph on k vertices, then it has maximum degree at most k and feedback edge set at most k^2 ; however, graphs of small feedback edge set and small degree do not necessarily have small max-leaf number (consider a long path where we attach a leaf to each vertex). Interestingly, the graphs we construct in Section 4.2 *do* have small max-leaf number, if we consider semi-weighted instances. However, adding the necessary simple gadgets in order to simulate weights increases the max-leaf number of the graphs of our reduction. It is thus a natural question whether this is necessary. In this section, we show that indeed this is inevitable, as VERTEX INTEGRITY is FPT parameterized by ml .

We start with a high-level overview of our approach. As mentioned, we will rely on the result of Kleitman and West [25] who showed that if a graph $G = (V, E)$ has $\text{ml}(G) \leq k$, then there exists a set X of size $|X| = \mathcal{O}(k)$ such that all vertices of $V \setminus X$ have degree at most 2. Our main tool is a lemma (Lemma 9) which allows us to “rotate” solutions: whenever we have a cycle in our graph, we can, roughly speaking, exchange every vertex of S in the cycle with the next vertex, until we reach a point where our solution removes strictly more vertices of X . We therefore guess the largest intersection of an optimal separator with X , and can now assume that in every remaining cycle, the separator S is not using any vertices. This allows us to simplify the graph in a way that removes all cycles and reduces the case to a tree, which is polynomial-time solvable.

Let us now give more details. We first recall the result of [25].

► **Lemma 8** (\star). *In any graph G , the set X of vertices of degree at least 3 has size at most $|X| \leq 12\text{ml}(G) + 32$.*

We will solve COMPONENT ORDER CONNECTIVITY: for a given ℓ we want to calculate the minimum number of vertices p such that there exists a separator S of size at most p with all components of $G - S$ having size at most ℓ . To obtain an algorithm for UNWEIGHTED VERTEX INTEGRITY, we will try all possible values of ℓ and select the solution which minimizes $\ell + p$.

Our main lemma is now the following:

► **Lemma 9** (\star). *Let $G = (V, E)$ be a graph and X be a set of vertices such that all vertices of $V \setminus X$ have degree at most 2 in G . For all positive integers ℓ, p , if there exists a separator S of size at most p such that all components of $G - S$ have size at most ℓ , then there exists such a separator S that also satisfies the following property: for every cycle C of G with $C \cap X \neq \emptyset$ we either have $C \cap S = \emptyset$ or $C \cap X \cap S \neq \emptyset$.*

We are now ready to state the main result of this section.

► **Theorem 10** (★). UNWEIGHTED VERTEX INTEGRITY can be solved in time $2^{\mathcal{O}(\text{ml})}n^{\mathcal{O}(1)}$.

6 Modular Width

In this section we revisit an algorithm of [21] establishing that WEIGHTED VERTEX INTEGRITY can be solved in time $2^{\mathcal{O}(\text{mw})}n^{\mathcal{O}(1)}$, on graphs of modular width mw , but only if weights are polynomially bounded in n (or equivalently, if weights are given in unary). It was left as an explicit open problem in [21] whether this algorithm can be extended to the case where weights are given in binary and can therefore be exponential in n . We resolve this problem positively, by showing how the algorithm of [21] can be modified to work also in this case, without a large increase in its complexity.

The high-level idea of the algorithm of [21] is to perform dynamic programming to solve the related WEIGHTED COMPONENT ORDER CONNECTIVITY problem. In this problem we are given a target component weight ℓ and a deletion budget p and are asked if it is possible to delete from the graph a set of vertices with total weight at most p so that the maximum weight of any remaining component is at most ℓ . Using this algorithm as a black box, we can then solve WEIGHTED VERTEX INTEGRITY by iterating over all possible values of ℓ , between 1 and the target vertex integrity. If vertex weights are polynomially bounded, this requires a polynomial number of iterations, giving the algorithm of [21]. However, if weights are given in binary, the target vertex integrity could be exponential in n , so in general, it does not appear possible to guess the weight of the heaviest component in an optimal solution.

Our contribution to the algorithm of [21] is to observe that for graphs of modular width mw the weight of the heaviest component may take at most $2^{\mathcal{O}(\text{mw})}n$ distinct possible values. Hence, for this parameter, guessing the weight of the heaviest component in an optimal solution can be done in FPT time. We can therefore plug in this result to the algorithm of [21] to obtain an algorithm for WEIGHTED VERTEX INTEGRITY with binary weights running in time $2^{\mathcal{O}(\text{mw})}n^{\mathcal{O}(1)}$.

Our observation is based on the following lemma.

► **Lemma 11** (★). Let $G = (V, E)$ be an instance of WEIGHTED VERTEX INTEGRITY. There exists an optimal solution using a separator S such that for all connected components D of $G - S$ and modules M of G we have one of the following: (i) $M \cap D = \emptyset$, (ii) $M \subseteq D$, or (iii) $D \subseteq M$.

We also recall the algorithmic result of [21].

► **Theorem 12** ([21]). There exists an algorithm that takes as input a vertex-weighted graph $G = (V, E)$ and an integer ℓ and computes the minimum integer p such that there exists a separator S of G of weight at most p such that each component of $G - S$ has weight at most ℓ . The algorithm runs in time $2^{\mathcal{O}(\text{mw})}n^{\mathcal{O}(1)}$, where n is the size of the input.

Putting Lemma 11 and Theorem 12 together we obtain the main result of this section.

► **Theorem 13** (★). There exists an algorithm that solves WEIGHTED VERTEX INTEGRITY in time $2^{\mathcal{O}(\text{mw})}n^{\mathcal{O}(1)}$, where mw is the modular width of the input graph, n is the size of the input, and weights are allowed to be written in binary.

7 Vertex Cover Number

In this section, we design single-exponential algorithms for VERTEX INTEGRITY parameterized by vertex cover number. We suppose that a minimum vertex cover C of size vc is given since it can be computed in time $\mathcal{O}(1.2738^{vc} + vcn)$ [5]. We start by presenting an algorithm for UNWEIGHTED VERTEX INTEGRITY, before moving on to the weighted version of the problem.

► **Theorem 14** (★). UNWEIGHTED VERTEX INTEGRITY can be solved in time $5^{vc}n^{\mathcal{O}(1)}$.

We now move on to the weighted case of the problem. It is clear that WEIGHTED VERTEX INTEGRITY is FPT parameterized by vertex cover, due to Theorem 13 and the relation between modular-width and vertex cover. However, this gives a double-exponential dependence on vc , as $mw \leq 2^{vc} + vc$ and there are some graphs for which this is essentially tight. We would like to obtain an algorithm that is as efficient as that of Theorem 14. The algorithm of Theorem 14, however, cannot be applied to the weighted case because the case of the branching where we place a vertex of the independent set in the separator is not guaranteed to make much progress (the vertex could have very small weight compared to our budget).

Before we proceed, it is worth thinking a bit about how this can be avoided. One way to obtain a faster FPT algorithm would be, rather than guessing only the intersection of the optimal separator S with the vertex cover C , to also guess how the vertices of $C \setminus S$ are partitioned into connected components in the optimal solution. This would immediately imply the decision for all vertices of the independent set: vertices with neighbors in two components must clearly belong to S , while the others cannot belong to S if S is irredundant. This algorithm would give a complexity of $vc^{\mathcal{O}(vc)}n^{\mathcal{O}(1)}$, however, because the number of partitions of C is slightly super-exponential.

Let us sketch the high level idea of how we handle this. Our first step is, similarly to Theorem 13, to calculate the weight w_{\max} of the most expensive connected component of the optimal solution. For this, there are at most $2^{vc} + n$ possibilities, because this component is either a single vertex, or it has a non-empty intersection with C . However, if we fix its intersection with C , then this fixes its intersection with the independent set: the component must contain (by irredundancy) exactly those vertices of the independent set all of whose neighbors in C are contained in the component. Having fixed a value of w_{\max} we simply seek the best separator so that all components have weight at most w_{\max} . The reason we perform this guessing step is that this version of this problem is easier to decompose: if we have a disconnected graph, we simply calculate the best separator in each part and take the sum (this is not as clear for the initial version of VERTEX INTEGRITY).

Suppose then that we have fixed w_{\max} , how do we find the best partition of C into connected components? We apply a win/win argument: if the optimal partition has a connected component that contains many (say, more than $vc/10$) vertices of C , we simply guess the intersection of the component with C and complete it with vertices from the independent set, as previously, while placing vertices with neighbors inside and outside the component in the separator. If the weight of the component is at most w_{\max} , we recurse in the remaining instance, which has vertex cover at most $9vc/10$. The complexity of this procedure works out as $T(vc) \leq 2^{vc} \cdot T(9vc/10) = 2^{\mathcal{O}(vc)}$.

What if the optimal partition of C only has components with few vertices of C ? In that case we observe that we do not need to compute the full partition (which would take time vc^{vc}), but it suffices to guess a good bipartition of C into two sets, of roughly the same size (say, both sets have size at least $2vc/5$), such that the two sets are a coarsening of the optimal

partition. In other words, we compute two subsets of C , of roughly equal size, such that the intersection of each connected component with C is contained in one of the two sets. This is always possible in this case, because no connected component has a very large intersection with C . Now, all vertices of I which have neighbors on both sides of the bipartition of C must be placed in the separator. But once we do this, we have disconnected the instance into two independent instances, each of vertex cover at most $3vc/5$. The complexity of this procedure again works out as $T(vc) \leq 2^{vc} \cdot 2 \cdot T(3vc/5) = 2^{\mathcal{O}(vc)}$.

Let us now proceed to the technical details. To solve WEIGHTED VERTEX INTEGRITY, we first define the annotated and optimization version of the problem.

ANNOTATED WEIGHTED VERTEX INTEGRITY WITH VERTEX COVER

Instance: A vertex-weighted graph $G = (V, E, w)$, a vertex cover C of G , an integer w_{\max} .
Goal: Find a minimum weight irredundant wvi-set $S \subseteq V \setminus C$ such that $w(D) \leq w_{\max}$ for all $D \in cc(G - S)$. If there is no such S , report NO.

Then we give an algorithm that solves ANNOTATED WEIGHTED VERTEX INTEGRITY WITH VERTEX COVER.

► **Theorem 15** (★). ANNOTATED WEIGHTED VERTEX INTEGRITY WITH VERTEX COVER can be solved in time $2^{\mathcal{O}(|C|)} n^{\mathcal{O}(1)}$.

► **Theorem 16** (★). WEIGHTED VERTEX INTEGRITY can be solved in time $2^{\mathcal{O}(vc)} n^{\mathcal{O}(1)}$.

8 Conclusion

We have presented a number of new results on the parameterized complexity of computing vertex integrity. The main question that remains open is whether the slightly super-exponential $k^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ algorithm, where k is the vertex integrity itself, can be improved to single-exponential. Although we have given such an algorithm for the more restricted parameter vertex cover, we conjecture that for vertex integrity the answer is negative. Complementing this question, it would be interesting to consider approximation algorithms for vertex integrity, whether trying to obtain FPT approximations in cases where the problem is W[1]-hard, or trying to obtain almost-optimal solutions via algorithms that run with a better parameter dependence. Again, a constant-factor or even $(1 + \varepsilon)$ -approximation running in time $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ would be the ideal goal. Do such algorithms exist or can they be ruled out under standard complexity assumptions?

References

- 1 C. A. Barefoot, Roger Entringer, and Henda Swart. Vulnerability in graphs – A comparative survey. In *Proceedings of the first Carbondale combinatorics conference (Carbondale, Ill., 1986)*, volume 1, pages 13–22, 1987.
- 2 Rémy Belmonte, Tesshu Hanaka, Ioannis Katsikarelis, Michael Lampis, Hirotaka Ono, and Yota Otachi. Parameterized complexity of safe set. *J. Graph Algorithms Appl.*, 24(3):215–245, 2020. doi:10.7155/JGAA.00528.
- 3 Matthias Bentert, Klaus Heeger, and Tomohiro Koana. Fully polynomial-time algorithms parameterized by vertex integrity using fast matrix multiplication. In *31st Annual European Symposium on Algorithms, ESA 2023*, volume 274 of *LIPICs*, pages 16:1–16:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2023. doi:10.4230/LIPICs.ESA.2023.16.

- 4 Sriram Bhyravarapu, Lawqueen Kanesh, A. Mohanapriya, Nidhi Purohit, N. Sadagopan, and Saket Saurabh. On the parameterized complexity of minus domination. In *SOFSEM 2024: Theory and Practice of Computer Science – 49th International Conference on Current Trends in Theory and Practice of Computer Science, SOFSEM 2024*, volume 14519 of *Lecture Notes in Computer Science*, pages 96–110. Springer, 2024. doi:10.1007/978-3-031-52113-3_7.
- 5 Jianer Chen, Iyad A. Kanj, and Ge Xia. Improved upper bounds for vertex cover. *Theor. Comput. Sci.*, 411(40-42):3736–3756, 2010. doi:10.1016/J.TCS.2010.06.026.
- 6 Lane H. Clark, Roger C. Entringer, and Michael R. Fellows. Computational complexity of integrity. *J. Combin. Math. Combin. Comput.*, 2:179–191, 1987.
- 7 Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015. doi:10.1007/978-3-319-21275-3.
- 8 Reinhard Diestel. *Graph Theory*, volume 173 of *Graduate texts in mathematics*. Springer, 2017. doi:10.1007/978-3-662-53622-3.
- 9 Pål Grønås Drange, Markus S. Dregi, and Pim van ’t Hof. On the computational complexity of vertex integrity and component order connectivity. *Algorithmica*, 76(4):1181–1202, 2016. doi:10.1007/S00453-016-0127-X.
- 10 Pavel Dvorák, Eduard Eiben, Robert Ganian, Dusan Knop, and Sebastian Ordyniak. The complexity landscape of decompositional parameters for ILP: programs with few global variables and constraints. *Artif. Intell.*, 300:103561, 2021. doi:10.1016/J.ARTINT.2021.103561.
- 11 Michael R. Fellows, Bart M. P. Jansen, and Frances A. Rosamond. Towards fully multivariate algorithmics: Parameter ecology and the deconstruction of computational complexity. *Eur. J. Comb.*, 34(3):541–566, 2013. doi:10.1016/J.EJC.2012.04.008.
- 12 Michael R. Fellows, Daniel Lokshtanov, Neeldhara Misra, Matthias Mnich, Frances A. Rosamond, and Saket Saurabh. The complexity ecology of parameters: An illustration using bounded max leaf number. *Theory Comput. Syst.*, 45(4):822–848, 2009. doi:10.1007/S00224-009-9167-9.
- 13 Michael R. Fellows and Sam Stueckle. The immersion order, forbidden subgraphs and the complexity of network integrity. *J. Combin. Math. Combin. Comput.*, 6:23–32, 1989.
- 14 Shinya Fujita and Michitaka Furuya. Safe number and integrity of graphs. *Discret. Appl. Math.*, 247:398–406, 2018. doi:10.1016/J.DAM.2018.03.074.
- 15 Ajinkya Gaikwad and Soumen Maity. Offensive alliances in graphs. *Theor. Comput. Sci.*, 989:114401, 2024. doi:10.1016/J.TCS.2024.114401.
- 16 Ajinkya Gaikwad and Soumen Maity. On structural parameterizations of the harmless set problem. *Algorithmica*, 86(5):1475–1511, 2024. doi:10.1007/S00453-023-01199-9.
- 17 Jakub Gajarský, Michael Lampis, Kazuhisa Makino, Valia Mitsou, and Sebastian Ordyniak. Parameterized algorithms for parity games. In *Mathematical Foundations of Computer Science 2015 – 40th International Symposium, MFCS 2015*, volume 9235 of *Lecture Notes in Computer Science*, pages 336–347. Springer, 2015. doi:10.1007/978-3-662-48054-0_28.
- 18 Jakub Gajarský, Michael Lampis, and Sebastian Ordyniak. Parameterized algorithms for modular-width. In *Parameterized and Exact Computation – 8th International Symposium, IPEC 2013*, volume 8246 of *Lecture Notes in Computer Science*, pages 163–176. Springer, 2013. doi:10.1007/978-3-319-03898-8_15.
- 19 Robert Ganian, Fabian Klute, and Sebastian Ordyniak. On structural parameterizations of the bounded-degree vertex deletion problem. *Algorithmica*, 83(1):297–336, 2021. doi:10.1007/S00453-020-00758-8.
- 20 Tatsuya Gima, Tesshu Hanaka, Masashi Kiyomi, Yasuaki Kobayashi, and Yota Otachi. Exploring the gap between treedepth and vertex cover through vertex integrity. *Theor. Comput. Sci.*, 918:60–76, 2022. doi:10.1016/J.TCS.2022.03.021.
- 21 Tatsuya Gima, Tesshu Hanaka, Yasuaki Kobayashi, Ryota Murai, Hirotaka Ono, and Yota Otachi. Structural parameterizations of vertex integrity. In *WALCOM: Algorithms and Computation – 18th International Conference and Workshops on Algorithms and Computation, WALCOM 2024*, volume 14549 of *Lecture Notes in Computer Science*, pages 406–420. Springer, 2024. doi:10.1007/978-981-97-0566-5_29.

- 22 Tatsuya Gima and Yota Otachi. Extended MSO model checking via small vertex integrity. *Algorithmica*, 86(1):147–170, 2024. doi:10.1007/S00453-023-01161-9.
- 23 Klaus Jansen, Stefan Kratsch, Dániel Marx, and Ildikó Schlotter. Bin packing with fixed number of bins revisited. *J. Comput. Syst. Sci.*, 79(1):39–49, 2013. doi:10.1016/J.JCSS.2012.04.004.
- 24 Richard M. Karp. Reducibility among combinatorial problems. In *Proceedings of a symposium on the Complexity of Computer Computations, held March 20-22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York, USA*, The IBM Research Symposia Series, pages 85–103. Plenum Press, New York, 1972. doi:10.1007/978-1-4684-2001-2_9.
- 25 Daniel J. Kleitman and Douglas B. West. Spanning trees with many leaves. *SIAM J. Discret. Math.*, 4(1):99–106, 1991. doi:10.1137/0404010.
- 26 Michael Lampis. Algorithmic meta-theorems for restrictions of treewidth. *Algorithmica*, 64(1):19–37, 2012. doi:10.1007/S00453-011-9554-X.
- 27 Michael Lampis and Valia Mitsou. Fine-grained meta-theorems for vertex integrity. In *32nd International Symposium on Algorithms and Computation, ISAAC 2021*, volume 212 of *LIPICs*, pages 34:1–34:15. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPICs.ISAAC.2021.34.
- 28 Michael Lampis and Manolis Vasilakis. Structural parameterizations for two bounded degree problems revisited. In *31st Annual European Symposium on Algorithms, ESA 2023*, volume 274 of *LIPICs*, pages 77:1–77:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2023. doi:10.4230/LIPICs.ESA.2023.77.
- 29 Martijn van Ee. Some notes on bounded starwidth graphs. *Inf. Process. Lett.*, 125:9–14, 2017. doi:10.1016/J.IPL.2017.04.011.