Quantum Polynomial Hierarchies: Karp-Lipton, Error Reduction, and Lower Bounds

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Abstract

The Polynomial-Time Hierarchy (PH) is a staple of classical complexity theory, with applications spanning randomized computation to circuit lower bounds to "quantum advantage" analyses for near-term quantum computers. Quantumly, however, despite the fact that at least *four* definitions of quantum PH exist, it has been challenging to prove analogues for these of even basic facts from PH. This work studies three quantum-verifier based generalizations of PH, two of which are from [Gharibian, Santha, Sikora, Sundaram, Yirka, 2022] and use classical strings (QCPH) and quantum mixed states (QPH) as proofs, and one of which is new to this work, utilizing quantum pure states (pureQPH) as proofs. We first resolve several open problems from [GSSSY22], including a collapse theorem and a Karp-Lipton theorem for QCPH. Then, for our new class pureQPH, we show one-sided error reduction pureQPH, as well as the first bounds relating these quantum variants of PH, namely $\mathsf{QCPH} \subseteq \mathsf{pureQPH} \subseteq \mathsf{EXP}^{\mathsf{PP}}$.

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1 Introduction

Introduced by Stockmeyer in 1976 [28], the Polynomial-Time Hierarchy (PH) is one of the foundation stones of classical complexity theory. Intuitively, the levels of PH, denoted Σ_i^p (respectively, Π_i^p) for $i \geq 1$, yield progressively harder, yet natural, "steps up" from NP (respectively, coNP). Specifically, a Σ_i^p verifier is a deterministic poly-time Turing Machine M which, given input $x \in \{0,1\}^n$, takes in i proofs $y_i \in \{0,1\}^{\mathrm{poly}(n)}$, and satisfies:

$$\text{if } x \text{ is a YES input:} \quad \exists y_1 \forall y_2 \exists y_3 \cdots Q_i y_i \text{ s.t. } M(x, y_1, \ldots, y_i) = 1 \tag{1}$$

$$\text{if } x \text{ is a NO input:} \quad \forall y_1 \exists y_2 \forall y_3 \cdots \overline{Q_i} y_i \text{ s.t. } M(x, y_1, \ldots, y_i) = 0. \tag{2}$$

Above, Q_i is $\forall$ ($\exists$) if i is even (odd). PH has played a prominent (and often surprising!) role in capturing the complexity of various computing setups, including the power of BPP [26, 23], low-depth classical circuits [11], counting classes [29], and even near-term quantum computers [9, 1, 8].

In contrast, the role of *quantum* analogues of PH in quantum complexity theory remains embarrassingly unknown. So, where is the bottleneck? *Defining* "quantum PH" is not the problem – indeed, Yamakami [31], Lockhart and González-Guillén [18], and Gharibian, Santha, Sikora, Sundaram and Yirka [13] all gave different definitions of quantum PH. Instead, the difficulty lies in proving even basic properties of quantum PH, which often runs up against difficult phenomena lurking about open problems such as $\exists \cdot \mathsf{BPP} \stackrel{?}{=} \mathsf{MA}$ and $\mathsf{QMA} \stackrel{?}{=} \mathsf{QMA}(2)$.

In this work, we resolve open questions regarding some fundamental properties of quantum PH. We focus on three definitions of quantum PH, chosen because they naturally generalize[1] QCMA and QMA. The first two definitions are from [13] (formal definitions in Section 2), and the third is new to this work. The definitions all use a poly-time uniformly generated quantum verifier V, and are given as follows (for brevity, here we only state the YES case definitions):

- QCPH: $\exists y_1 \forall y_2 \exists y_3 \cdots Q_i y_i$ s.t. $V(x, y_1, \ldots, y_i)$ outputs 1 with probability $\geq 2/3$.
- QPH: $\exists \rho_1 \forall \rho_2 \exists \rho_3 \cdots Q_i \rho_i$ s.t. $V(x, \rho_1, \ldots, \rho_i)$ outputs 1 with probability $\geq 2/3$.
- pureQPH: $\exists |\psi_1\rangle \forall |\psi_2\rangle \exists |\psi_3\rangle \cdots Q_i |\psi_i\rangle$ s.t. $V(x, |\psi_1\rangle, \ldots, |\psi_i\rangle)$ outputs 1 with probability $\geq 2/3$.

In words, QCPH, QPH, and pureQPH utilize poly-size quantum verifiers taking in classical, mixed quantum, and pure quantum proofs, respectively. It is immediate from the definitions that $\mathsf{QCMA} \subseteq \mathsf{QCPH}$, $\mathsf{QMA} \subseteq \mathsf{QPH}$, and $\mathsf{QMA} \subseteq \mathsf{pureQPH}$. Beyond this, not much is clear. For example, a standard use of PH is via its collapse theorem – if for any i, $\Sigma_i^p = \Pi_i^p$, then $\mathsf{PH} = \Sigma_i^p$. Do any of QCPH, QPH, or pureQPH satisfy such a collapse theorem? Does error reduction hold for QPH or pureQPH? What is the relationship between QCPH, QPH, and pureQPH? Note that standard convexity arguments (as used for e.g. QMA) cannot be used to argue $\mathsf{QPH} = \mathsf{pureQPH}$, due to the presence of alternating quantifiers (which make the verification non-convex in the proofs). Can one recover celebrated results for these hierarchies analogous to the Karp-Lipton [20] Theorem for PH?

Our results. *1. Collapse Theorem for* QCPH. We first resolve an open question of [13] by giving a collapse theorem for QCPH.

[1] QCMA and QMA are quantum generalizations of Merlin-Arthur (MA), with a classical proof and quantum verifier and a quantum proof and quantum verifier, respectively.

▶ **Theorem 1.** *If for any $k \geq 1$,* $\mathsf{QC}\Sigma_k = \mathsf{QC}\Pi_k$*, then* $\mathsf{QCPH} = \mathsf{QC}\Sigma_k$*.*

This is in contrast to QPH, for which a collapse theorem is believed difficult to show, as it would imply a subsequent collapse[2] $\mathsf{QMA}(2) \subseteq \mathsf{PSPACE}$.

2. Quantum-Classical Karp-Lipton Theorem for QCPH*.* The celebrated Karp-Lipton theorem [20] states that if SAT can be solved by polynomial-size circuits, then PH collapses to Σ_2^p. We next leverage Theorem 1 and other techniques to obtain a Karp-Lipton Theorem for QCPH:

▶ **Theorem 2** (Karp-Lipton for QCPH)**.** *If* $\mathsf{QCMA} \subseteq \mathsf{BQP}_{/\mathrm{mpoly}}$*, then* $\mathsf{QCPH} = \mathsf{QC}\Sigma_2 = \mathsf{QC}\Pi_2$*.*

Here, $\mathsf{BQP}_{/\mathrm{mpoly}}$ is BQP with poly-size classical advice (Definition 11). In words, Theorem 2 says QCMA cannot be solved by (even non-uniformly generated) poly-size quantum circuits, unless QCPH collapses to its second level. This resolves a second open question of [13].

3. Error reduction for pureQPH*.* While error reduction for QCPH follows from parallel repetition (due to its classical proofs), achieving it for pureQPH is non-trivial for the same reason it is non-trivial for QMA(2) – the tensor product structure between proofs is not necessarily preserved when postselecting on measurements across proof copies in the NO case. Here, we show *one-sided* error reduction for pureQPH (e.g. *exponentially* small soundness):

▶ **Theorem 3.** *For all $i > 0$ and $c - s \geq 1/p(n)$ for some polynomial p,*

1. *For even $i > 0$:*
   a. $\mathsf{pureQ}\Sigma_i(c,s) \subseteq \mathsf{pureQ}\Sigma_i^{\mathsf{SEP}}(1/np(n)^2, 1/e^n)$
   b. $\mathsf{pureQ}\Pi_i(c,s) \subseteq \mathsf{pureQ}\Pi_i^{\mathsf{SEP}}(1-1/e^n, 1-1/np(n)^2)$
2. *For odd $i > 0$:*
   a. $\mathsf{pureQ}\Sigma_i(c,s) \subseteq \mathsf{pureQ}\Sigma_i^{\mathsf{SEP}}(1-1/e^n, 1-1/np(n)^2)$
   b. $\mathsf{pureQ}\Pi_i(c,s) \subseteq \mathsf{pureQ}\Pi_i^{\mathsf{SEP}}(1/np(n)^2, 1/e^n)$

Above, $\mathsf{pureQ}\Sigma_i^{\mathsf{SEP}}$ and $\mathsf{pureQ}\Pi_i^{\mathsf{SEP}}$ have the promise that in the YES case, the verifier's acceptance measurement is separable (see Section 4.2). We remark the proof of this uses a new *asymmetric* version of the Harrow-Montanaro [16] Product Test, which may be of independent interest (see Lemma 13). The reason we are unable to recover exponentially small error simultaneously for both completeness and soundness is because our approach requires the final quantifier to be $\exists$.

4. Upper and lower bounds on pureQPH*.* Having introduced pureQPH in this work, we next give bounds on its power.

▶ **Theorem 4.** $\mathsf{QCPH} \subseteq \mathsf{pureQPH} \subseteq \mathsf{EXP}^{\mathsf{PP}}$.

While the upper bound above is not difficult to show (Theorem 17; this may be viewed as an "exponential analogue" of Toda's theorem), the lower bound is surprisingly subtle. The naive strategy of replacing each proof y_i of QCPH with pure state proof $|\psi_i\rangle$, which is then measured in the standard basis, does not work, as the measurement gives rise to mixed states. Mixed states, in turn, are difficult to handle in QPH, as the latter is not a convex optimization due to alternating quantifiers. We remark that while $\mathsf{QPH} \subseteq \mathsf{pureQPH}$ follows easily via purification of proofs, we do *not* know how to show the analogous lower bound $\mathsf{QCPH} \subseteq \mathsf{QPH}$ (our approach uses our asymmetric product test, which requires pure states).

[2] QMA(2) is QMA with two proofs in tensor product. Since its introduction in 2001 by Kobayashi, Matsumoto, and Yamakami [21, 22], its complexity remains stubbornly open. The current best bounds are $\mathsf{QMA} \subseteq \mathsf{QMA}(2) \subseteq \mathsf{Q}\Sigma_3 \subseteq \mathsf{NEXP}$, where the second and third containments are from[13].

Related Work. Yamakami [31] gave the first definition of a quantum PH, which takes quantum inputs (in contrast, we use classical inputs). The same paper [31] also discusses a variant of pureQPH (which they call QPH). However, their QPH is very powerful and its first level already captures QMA(2) (which is contained in the third level of pureQPH), and error reduction is trivial for this complexity class. Gharibian and Kempe [12] defined and obtained hardness of approximation results for $\mathsf{QC\Sigma}_2$, obtaining the first hardness of approximation results for a quantum complexity class. (See [7] for a recent extension to QCMA-hardness of approximation results.) Lockhart and González-Guillén [18] defined a class QCPH' similar to QCPH, except using existential and universal *operators*. Thus, in [18] $\mathsf{QC\Sigma}_1' = \exists \cdot \mathsf{BQP}$, which is not known to equal QCMA (for the same reason $\exists \cdot \mathsf{BPP} \stackrel{?}{=} \mathsf{MA}$ remains open). In exchange for not capturing QCMA, however, the benefit of QCPH' is that its properties are easier to prove than QCPH. Gharibian, Santha, Sikora, Sundaram and Yirka [13] defined QCPH and QPH, and showed weaker variants of the Karp-Lipton theorem (Precise-QCMA $\subseteq \mathsf{BQP}_{/\mathrm{mpoly}}$ implies $\mathsf{QC\Sigma}_2 = \mathsf{QC\Pi}_2$) and Toda's theorem [29] ($\mathsf{QCPH} \subseteq \mathsf{P}^{\mathsf{PP}^{\mathsf{PP}}}$). They also showed $\mathsf{QMA}(2) \subseteq \mathsf{Q\Sigma}_3 \subseteq \mathsf{NEXP}$, giving the first class sitting between QMA(2) and NEXP, and observed that $\mathsf{Q\Sigma}_2 = \mathsf{Q\Pi}_2 = \mathrm{QRG}(1) \subseteq \mathsf{PSPACE}$ (due to work of Jain and Watrous [19]). Finally, Aaronson, Ingram and Kretschmer [4] showed that relative to a random oracle, PP is not in the "QMA hierarchy", i.e. in $\mathsf{QMA}^{\mathsf{QMA}^{\cdots}}$. The relationship between this QMA hierarchy and any of QCPH, QPH, or pureQPH remains open.

The Karp-Lipton theorem has been studied in the setting of quantum advice. Prior works by Aaronson and Drucker [3], and Aaronson, Cojocaru, Gheorghiu, and Kashefi [2] studied the consequences of solving NP-complete problems using polynomial-sized quantum circuits with polynomial quantum advice. Nishimura and Yamakami [25] define the *language* class BQP with classical advice and compare it to BQP with quantum advice. In this paper, however, we study promise problems in BQP with classical advice (called $\mathsf{BQP}_{/\mathrm{mpoly}}$).

Finally, the Product Test was first introduced by Mintert, Kuś and Buchleitner [24], and rigorously analyzed and strikingly leveraged by Harrow and Montanaro[16] to show $\mathsf{QMA}(k) = \mathsf{QMA}(2)$ for polynomial k, as well as error reduction for QMA(2). (See also Soleimanifar and Wright [27].)

Concurrent Work. We mention two concurrent works on quantum variants of the polynomial hierarchy. First, our collapse theorem for QCPH (Theorem 1) was proven concurrently by Falor, Ge, and Natarajan [10]. Second, we upper bound QCPH by showing that for all k, $\mathsf{QC\Pi}_k \subseteq \mathsf{pureQ\Sigma}_k \subseteq \mathsf{pureQ\Sigma}_i \subseteq \mathsf{NEXP}^{\mathsf{NP}^{i-1}}$, implying $\mathsf{QCPH} \subseteq \mathsf{pureQPH} \subseteq \mathsf{EXP}^{\mathsf{PP}}$ (Theorem 4 and Theorem 17). Grewal and Yirka [15] show the stronger bound $\mathsf{QCPH} \subseteq \mathsf{QPH}$, at the expense of the minor caveat that their proof does not obtain level-wise containment for all k, but rather requires constant factor blowup in level. Beyond this, our papers diverge. We show a Quantum-Classical Karp-Lipton Theorem for QCPH and error reduction for pureQPH. Grewal and Yirka [15] define a new quantum polynomial hierarchy called the *entangled quantum polynomial hierarchy* (QEPH), which allows entanglement across alternatively quantified quantum proofs. They show that QEPH collapses to its second level (even with polynomially many proofs), and is equal to QRG(1), the class of one round quantum-refereed games. They also define a generalization of QCPH, denoted DistributionQCPH, in which proofs are not strings but *distributions* over strings. They show QCPH=DistributionQCPH.

Techniques. We sketch our approach for each result mentioned above.

1. Collapse Theorem for QCPH. Collapse theorems for PH are shown via inductive argument – by fixing an arbitrary proof for the first quantifier of Σ_i^p, one obtains an instance of Π_{i-1}^p. Reference [13] noted this approach does not obviously work for QCPH, as fixing the first

proof of $\mathsf{QC}\Sigma_i$ does *not* necessarily yield a valid $\mathsf{QC}\Pi_{i-1}$ instance (i.e. the latter might not satisfy the desired promise gap). We bypass this obstacle by observing that even if most choices for existentially quantified proofs are problematic, there always exists *at least one* "good" choice, for which the recursion works. Formally, we are implicitly using a *promise* version of NP which is robustly[3] defined relative to any promise oracle.

2. Quantum-Classical Karp-Lipton Theorem. The classical Karp-Lipton theorem crucially uses the search-to-decision reduction for SAT. Given a (non-uniform) circuit family that can decide a SAT instance, we can use the circuit family to find a witness for a SAT instance. However, this search-to-decision reduction does not work in the quantum-classical setting since we are working with promise problems instead of languages. As a result, we cannot replicate the classical proof in the quantum-classical setting. Instead, we first convert the QCMA problem (obtained by fixing the universally quantified proof) to a UniqueQCMA (UQCMA) problem. For this, we use the quantum-classical analogue of Valiant-Vazirani's isolation lemma [30] given by Aharonov, Ben-Or, Brandão, and Sattath [6]. We then use a single-query (quantum) search-to-decision reduction for UQCMA that was presented in a recent work by Irani, Natarajan, Nirkhe, Rao and Yuen [17].

3. Error reduction for pureQPH*.* As with QMA(2), the challenge with error reduction via parallel repetition for pureQPH is the following: Given proof $|\psi\rangle_{A_1,B_1} \otimes |\phi\rangle_{A_2,B_2}$, postselecting on a joint measurement outcome on registers $\{A_1, B_1\}$ may entangle registers $\{A_2, B_2\}$. To overcome this, we give an asymmetric version of the Product Test [16], denoted APT. The APT takes in an n-system state $|\psi\rangle$ in register A, and (ideally) m copies of $|\psi\rangle$ in register B. It picks a random subsystem i of A, as well as a random copy j of i in B, and applies the SWAP test (Figure 1) between them. We prove (Lemma 13) that if this test passes with high probability, then $|\psi\rangle_A \approx |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$ for some $\{|\psi_i\rangle\}_{i=1}^n$, i.e. $|\psi\rangle_A$ was of tensor product form.

With the APT in hand, we can show error reduction for (e.g.) $\mathsf{pureQ}\Pi_i$. Here, the aim of an honest ith (existentially quantified) prover is to send many copies of proofs 1 through $i-1$. With probability 1/2, the verifier runs the APT with register A being proofs 1 to $i-1$ and register B being all their copies bundled with the ith proof, and with probability 1/2, the verifier runs parallel repetition on all copies of proofs bundled with the ith proof. This crucially leverages the fact that the ith proof is existential in the YES case, and thus can be assumed to be of this ideal form. In the NO case, however, the ith proof is universally quantified – thus, we cannot assume anything about its structure, which is why we do not get error reduction for the soundness parameter (in this case).

4. Upper and lower bounds on pureQPH*.* We focus on the more difficult direction, $\mathsf{QCPH} \subseteq \mathsf{pureQPH}$, for which we actually show that for all even $k \geq 2$, $\mathsf{QC}\Pi_k \subseteq \mathsf{pureQPH}_k$ (Lemma 16). When simulating QCPH, the challenge is again how to deal with universally quantified proofs, denoted by index set $U \subseteq [k]$. Unlike existentially quantified proofs, which can be assumed to be set honestly to some optimal string y_i, for any index $j \in U$ the proof $|\psi_j\rangle$ can be any pure quantum state. This causes two problems: (1) Measuring $|\psi_j\rangle$ in the standard basis yields a *distribution* D_j over strings, and due to non-convexity of pureQPH, it is not clear if

[3] Formally, let M be a deterministic machine with access to a promise oracle O. We say M is robust [14] if, regardless of how any invalid queries to O (i.e. queries violating the promise gap of O) are answered, M returns the same answer. One can define "PromiseNP" for non-deterministic M with access to O similarly: In the YES case, M has at least one robust accepting branch, and in the NO case, all branches of M are robust and rejecting. For clarity, we do not formally define and use PromiseNP in this work, but the viewpoint sketched here is equivalent to our approach for Theorem 1.

this can help a cheating prover succeed with higher probability that having sent a string. (2) Conditioned on distribution D_j, what should the next, existentially quantified, prover $j+1$ set its optimal proof/string to? We overcome these obstacles as follows. The initial setup is similar to Theorem 3 – prover k (which is existentially quantified in the YES case) send copies of all previous proofs 1 to $k-1$, and with probability 1/2, we run the APT. However, now with probability 1/2, we measure *all* proofs in the standard basis. The key step is to immediately *accept* if for any universally quantified proof index $i \in U$, measuring proof $|\psi_i\rangle$ does *not* match all of its copies bundled in proof k. In contrast, for existentially quantified proofs, we *reject* if a mismatch occurs. Finally, assuming no mismatches occur, we simply run the original QCPH verifier on the corresponding strings obtained via measurement. Showing correctness is subtle, and requires a careful analysis for both YES and NO cases, since recall the location of universally quantified proofs changes between cases.

Discussion and open questions. Many questions remain open for QCPH, QPH, and pureQPH. Perhaps the most frustrating for QCPH is a lack of a genuine Toda's theorem – [13] shows $\mathsf{QCPH} \subseteq \mathsf{P}^{\mathsf{PP}^{\mathsf{PP}}}$, but what one really wants is containment in P^{PP}. Is this possible? And if not, can one show an oracle separation between QCPH and P^{PP}? Moving to QPH, its role in this mess remains rather murky. Is $\mathsf{pureQPH} \subseteq \mathsf{QPH}$ (recall the converse direction follows via purification)? For this, our proof technique for $\mathsf{QCPH} \subseteq \mathsf{pureQPH}$ appears not to apply, as it requires pure states for the SWAP test. QPH *does* have one advantage over pureQPH, however – while the third level of both contains QMA(2), only $\mathsf{Q}\Sigma_3$ is known to be in NEXP [13], providing a class "between" QMA(2) and NEXP. Reference [13]'s proof breaks down for pureQPH, as its semidefinite-programming approach requires *mixed* state proofs.[4] Finally, for pureQPH, can one show two-sided error reduction? Is there a collapse theorem for pureQPH? Can one improve our bound $\mathsf{pureQPH} \subseteq \mathsf{EXP}^{\mathsf{PP}}$? As a first step, is $\mathsf{pureQ}\Sigma_3 \subseteq \mathsf{NEXP}$? If not, this would suggest the combination of "unentanglement" across proofs and alternating quantifiers yields a surprisingly powerful proof system, as it trivially holds that $\mathsf{QMA}(2) \subseteq \mathsf{NEXP}$.

Organization. Section 2 begins with notation and definitions. Section 3.1 and Section 3.2 give our collapse theorem and Karp-Lipton theorem for QCPH, respectively. Section 4 shows error reduction for pureQPH. Section 5 gives upper and lower bounds for pureQPH.

2 Preliminaries

Notation. Let $\mathrm{conv}(S)$ denote the convex hull of set S. Then, the set of separable operators acting on $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_i}$ is

$$\mathrm{conv}\left(|\psi_1\rangle\langle\psi_1| \otimes \cdots \otimes |\psi_i\rangle\langle\psi_i| \mid \forall j \in [i], |\psi_j\rangle \in \mathbb{C}^{d_j} \text{ is a unit vector}\right). \tag{3}$$

Throughout this paper, we study *promise problems*. A promise problem is a pair $A = (A_{\mathrm{yes}}, A_{\mathrm{no}})$ such that $A_{\mathrm{yes}}, A_{\mathrm{no}} \subseteq \{0,1\}^*$ and $A_{\mathrm{yes}} \cap A_{\mathrm{no}} = \emptyset$, but $A_{\mathrm{yes}} \cup A_{\mathrm{no}} = \{0,1\}^*$ does not necessarily hold.

[4] Briefly, in NEXP one can guess the first existentially quantified proof of $\mathsf{Q}\Sigma_3$, leaving a $\mathsf{Q}\Pi_2$ computation. Since we are using mixed states, via duality theory one can rephrase this via an exponential-side SDP, which can be solved in exponential time. Note this "convexification" does not seem to apply for larger values of k.

2.1 Quantum-Classical Polynomial Hierarchy (QCPH)

We first recall the quantum analogue of PH that generalizes QCMA, i.e. has classical proofs [13].

▶ **Definition 5** ($\mathsf{QC\Sigma}_i$). *Let $A = (A_{\text{yes}}, A_{\text{no}})$ be a promise problem. We say that A is in $\mathsf{QC\Sigma}_i(c,s)$ for poly-time computable functions $c, s : \mathbb{N} \mapsto [0,1]$ if there exists a poly-bounded function $p : \mathbb{N} \mapsto \mathbb{N}$ and a poly-time uniform family of quantum circuits $\{V_n\}_{n\in\mathbb{N}}$ such that for every n-bit input x, V_n takes in classical proofs $y_1 \in \{0,1\}^{p(n)}, \ldots, y_i \in \{0,1\}^{p(n)}$ and outputs a single qubit, such that:*

- *Completeness: $x \in A_{\text{yes}} \Rightarrow \exists y_1 \forall y_2 \ldots Q_i y_i$ s.t. $\text{Prob}[V_n \text{ accepts } (y_1, \ldots, y_i)] \geq c$.*
- *Soundness: $x \in A_{\text{no}} \Rightarrow \forall y_1 \exists y_2 \ldots \overline{Q}_i y_i$ s.t. $\text{Prob}[V_n \text{ accepts } (y_1, \ldots, y_i)] \leq s$.*

Here, Q_i equals $\exists$ when m is odd and equals $\forall$ otherwise and $\overline{Q}_i$ is the complementary quantifier to Q_i. Finally, define

$$\mathsf{QC\Sigma}_i := \bigcup_{c-s\in\Omega(1/\text{poly}(n))} \mathsf{QC\Sigma}_i(c,s). \tag{4}$$

Comments: Note that the first level of this hierarchy corresponds to QCMA. The complement of the i^{th} level of the hierarchy, $\mathsf{QC\Sigma}_i$, is the class $\mathsf{QC\Pi}_i$ defined next.

▶ **Definition 6** ($\mathsf{QC\Pi}_i$). *Let $A = (A_{\text{yes}}, A_{\text{no}})$ be a promise problem. We say that $A \in \mathsf{QC\Pi}_i(c,s)$ for poly-time computable functions $c, s : \mathbb{N} \mapsto [0,1]$ if there exists a polynomially bounded function $p : \mathbb{N} \mapsto \mathbb{N}$ and a poly-time uniform family of quantum circuits $\{V_n\}_{n\in\mathbb{N}}$ such that for every n-bit input x, V_n takes in classical proofs $y_1 \in \{0,1\}^{p(n)}, \ldots, y_i \in \{0,1\}^{p(n)}$ and outputs a single qubit, such that:*

- *Completeness: $x \in A_{\text{yes}} \Rightarrow \forall y_1 \exists y_2 \ldots Q_i y_i$ s.t. $\text{Prob}[V_n \text{ accepts } (y_1, \ldots, y_i)] \geq c$.*
- *Soundness: $x \in A_{\text{no}} \Rightarrow \exists y_1 \forall y_2 \ldots \overline{Q}_i y_i$ s.t. s.t. $\text{Prob}[V_n \text{ accepts } (y_1, \ldots, y_i)] \leq s$.*

Here, Q_i equals $\forall$ when m is odd and equals $\exists$ otherwise, and $\overline{Q}_i$ is the complementary quantifier to Q_i. Finally, define

$$\mathsf{QC\Pi}_i := \bigcup_{c-s\in\Omega(1/\text{poly}(n))} \mathsf{QC\Pi}_i(c,s). \tag{5}$$

Now the corresponding quantum-classical polynomial hierarchy is defined as follows.

▶ **Definition 7** (Quantum-Classical Polynomial Hierarchy (QCPH)).

$$\mathsf{QCPH} = \bigcup_{m\in\mathbb{N}} \mathsf{QC\Sigma}_i = \bigcup_{m\in\mathbb{N}} \mathsf{QC\Pi}_i. \tag{6}$$

2.2 (Pure-State) Quantum Polynomial Hierarchy (pureQPH)

Next, we introduce Quantum PH with pure-state proofs (for clarity, the definition below is new to this work). Prior work [13] defined QPH using mixed-state quantum proofs. Unlike QMA or QMA(2) (where one may argue due to convexity that pure states suffice) it is not clear how to use convexity arguments in the presence of alternating quantifiers.

▶ **Definition 8** ($\mathsf{pureQ\Sigma}_i$). *A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is in $\mathsf{pureQ\Sigma}_i(c,s)$ for poly-time computable functions $c, s : \mathbb{N} \mapsto [0,1]$ if there exists a polynomially bounded function $p : \mathbb{N} \mapsto \mathbb{N}$ and a poly-time uniform family of quantum circuits $\{V_n\}_{n\in\mathbb{N}}$ such that for every n-bit input x, V_n takes $p(n)$-qubit states $|\psi_1\rangle, \ldots, |\psi_i\rangle$ as quantum proofs and outputs a single qubit, then:*

- *Completeness: If* $x \in A_{\text{yes}}$*, then* $\exists|\psi_1\rangle\forall|\psi_2\rangle \dots Q_i|\psi_i\rangle$*:* V_n *accepts* $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_i\rangle$ *with probability* $\geq c$*.*
- *Soundness: If* $x \in A_{\text{no}}$*, then* $\forall|\psi_1\rangle\exists|\psi_2\rangle \dots \overline{Q}_i|\psi_i\rangle$*:* V_n *accepts* $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_i\rangle$ *with probability* $\leq s$*.*

Here, Q_i *equals* $\forall$ *when* m *is even and equals* $\exists$ *otherwise, and* $\overline{Q}_i$ *is the complementary quantifier to* Q_i*. Define*

$$\mathsf{pureQ\Sigma}_i = \bigcup_{c-s\in\Omega(1/\operatorname{poly}(n))} \mathsf{pureQ\Sigma}_i(c,s). \tag{7}$$

▶ **Definition 9** ($\mathsf{pureQ\Pi}_i$)**.** *A promise problem* $A = (A_{\text{yes}}, A_{\text{no}})$ *is in* $\mathsf{pureQ\Pi}_i(c,s)$ *for poly-time computable functions* $c, s : \mathbb{N} \mapsto [0,1]$ *if there exists a polynomially bounded function* $p : \mathbb{N} \mapsto \mathbb{N}$ *and a poly-time uniform family of quantum circuits* $\{V_n\}_{n\in\mathbb{N}}$ *such that for every* n*-bit input* x*,* V_n *takes* $p(n)$*-qubit states* $|\psi_1\rangle, \dots, |\psi_i\rangle$ *as quantum proofs and outputs a single qubit, then:*

- *Completeness: If* $x \in A_{\text{yes}}$*, then* $\forall|\psi_1\rangle\exists|\psi_2\rangle \dots Q_i|\psi_i\rangle$*:* V_n *accepts* $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_i\rangle$ *with probability* $\geq c$*.*
- *Soundness: If* $x \in A_{\text{no}}$*, then* $\exists|\psi_1\rangle\forall|\psi_2\rangle \dots \overline{Q}_i|\psi_i\rangle$*:* V_n *accepts* $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_i\rangle$ *with probability* $\leq s$*.*

Here, Q_i *equals* $\exists$ *when* m *is even and equals* $\forall$ *otherwise, and* $\overline{Q}_i$ *is the complementary quantifier to* Q_i*. Define*

$$\mathsf{pureQ\Pi}_i = \bigcup_{c-s\in\Omega(1/\operatorname{poly}(n))} \mathsf{pureQ\Pi}_i(c,s). \tag{8}$$

The Quantum Polynomial Hierarchy with pure-state proofs can now be defined as follows.

▶ **Definition 10** (Pure Quantum Poly-Hierarchy (pureQPH))**.**

$$\mathsf{pureQPH} = \bigcup_{m\in\mathbb{N}} \mathsf{pureQ\Sigma}_i = \bigcup_{m\in\mathbb{N}} \mathsf{pureQ\Pi}_i.$$

2.3 Other complexity classes

▶ **Definition 11** ($\mathsf{BQP}_{/\text{mpoly}}$)**.** *A promise problem* $\Pi = (A_{\text{yes}}, A_{\text{no}})$ *is in* $\mathsf{BQP}_{/\text{mpoly}}$ *if there exists a poly-sized family of quantum circuits* $\{C_n\}_{n\in\mathbb{N}}$ *and a collection of binary advice strings* $\{a_n\}_{n\in\mathbb{N}}$ *with* $|a_n| = \operatorname{poly}(n)$*, such that for all* $n \in \mathbb{N}$ *and all strings* $x \in \{0,1\}^n$*,*

$$\Pr[C_n(|x\rangle, |a_n\rangle) = 1] \geq 2/3 \text{ if } x \in A_{\text{yes}} \quad \text{and} \quad \Pr[C_n(|x\rangle, |a_n\rangle) = 1] \leq 1/3 \text{ if } x \in A_{\text{no}}.$$

3 Collapse theorems and Quantum Karp-Lipton

3.1 Collapse Theorem for QCPH

For the quantum-classical hierarchy, QCPH, we now show a quantum analogue of the standard collapse theorem for classical PH, i.e. $\Sigma_2^p = \Pi_2^p$ implies $\mathsf{PH} = \Sigma_2^p$, resolving an open question of [13].

▶ **Lemma 12.** *If for any* $k \geq 1$*,* $\mathsf{QC\Sigma}_k = \mathsf{QC\Pi}_k$*, then for all* $i \geq k$*,* $\mathsf{QC\Sigma}_i = \mathsf{QC\Pi}_i = \mathsf{QC\Sigma}_k$*.*

Proof. We proceed by induction. For $j \geq k$, define $P(j) := \mathsf{QC\Sigma}_j = \mathsf{QC\Pi}_j = \mathsf{QC\Sigma}_k$. The base case $P(k)$ holds by the assumption of the lemma. For the inductive case, assume $P(j)$ holds for all $k \leq j \leq i-1$. We show $P(j)$ holds for $j = i$. Consider arbitrary promise problem $L = (L_{\text{yes}}, L_{\text{no}}, L_{\text{inv}}) \in \mathsf{QC\Sigma}_i$ and let $\{V_n\}$ be the verifier circuits for the promise problem. Define new promise problem $L' = (L'_{\text{yes}}, L'_{\text{no}}, L'_{\text{inv}})$:

$$L'_{\text{yes}} = \left\{ (x, y_1) \mid \forall y_2 \exists y_3 \ldots Q_i y_i \ \Pr[V(x, y_1, y_2, \ldots, y_i) = 1] \geq \frac{2}{3} \right\} \tag{9}$$

$$L'_{\text{no}} = \left\{ (x, y_1) \mid \exists y_2 \forall y_3 \ldots \overline{Q}_i y_i \ \Pr[V(x, y_1, y_2, \ldots, y_i) = 1] \leq \frac{1}{3} \right\} \tag{10}$$

$$L'_{\text{inv}} = \{0,1\}^* \setminus (L'_{\text{yes}} \cup L'_{\text{no}}). \tag{11}$$

Clearly, $L'_{\text{yes}} \cap L'_{\text{no}} = \emptyset$, and so $(L'_{\text{yes}}, L'_{\text{no}}, L'_{\text{inv}}) \in \mathsf{QC\Pi}_{i-1}$. By the induction hypothesis, there exists promise problem $L'' = (L''_{\text{yes}}, L''_{\text{no}}, L''_{\text{inv}}) \in \mathsf{QC\Sigma}_{i-1}$ such that $L'_{\text{no}} \subseteq L''_{\text{no}}$ and $L'_{\text{yes}} \subseteq L''_{\text{yes}}$. Letting $\{V''_n\}$ denote the verification circuits for L'', we have

$$\begin{aligned} (x, y_1) \in L'_{\text{yes}} \ &\Rightarrow \ (x, y_1) \in L''_{\text{yes}} \ \Rightarrow \ \exists y_2 \forall y_3 \ldots Q_i y_i \colon \Pr[V''(x, y_1, \ldots, y_i) = 1] \geq \tfrac{2}{3}, \\ (x, y_1) \in L'_{\text{no}} \ &\Rightarrow \ (x, y_1) \in L''_{\text{no}} \ \Rightarrow \ \forall y_2 \exists y_3 \ldots \overline{Q}_i y_i \colon \Pr[V''(x, y_1, \ldots, y_i) = 1] \leq \tfrac{1}{3}. \end{aligned} \tag{12}$$

Now considering again $L = (L_{yes}, L_{\text{no}}, L_{\text{inv}})$, we have

$$\begin{aligned} x \in L_{\text{yes}} \ &\Rightarrow \ \exists y_1 \colon (x, y_1) \in L'_{\text{yes}} \ \Rightarrow \ \exists y_1 \exists y_2 \forall y_3 \ldots Q_{i-1} y_i \Pr[V'(x, y_1, \ldots, y_i) = 1] \geq \tfrac{2}{3} \\ x \in L_{\text{no}} \ &\Rightarrow \ \forall y_1 \colon (x, y_1) \in L'_{\text{no}} \ \Rightarrow \ \forall y_1 \forall y_2 \exists y_3 \ldots \overline{Q}_{i-1} y_i \Pr[V'(x, y_1, \ldots, y_i) = 1] \leq \tfrac{1}{3}. \end{aligned} \tag{13}$$

We conclude $L \in \mathsf{QC\Sigma}_{i-1} = \mathsf{QC\Sigma}_i$. So $\mathsf{QC\Sigma}_j = \mathsf{QC\Sigma}_k$. Similarly, $\mathsf{QC\Pi}_i \subseteq \mathsf{QC\Pi}_{i-1} = \mathsf{QC\Sigma}_k$. Thus, $P(i)$ holds, as claimed. ◀

Theorem 1 follows from Lemma 12.

3.2 Quantum-Classical Karp-Lipton Theorem

We will show that if there exists a polynomial size circuit family $\{C_n\}_{n \in \mathbb{N}}$ that can decide a QCMA complete problem, then $\mathsf{QC\Pi}_2 \subseteq \mathsf{QC\Sigma}_2$. Using Theorem 1, it follows that if $\mathsf{QCMA} \subseteq \mathsf{BQP}_{/\text{mpoly}}$, then QCPH collapses to the second level.

▶ **Theorem 2** (Karp-Lipton for QCPH). *If* $\mathsf{QCMA} \subseteq \mathsf{BQP}_{/\text{mpoly}}$, *then* $\mathsf{QCPH} = \mathsf{QC\Sigma}_2 = \mathsf{QC\Pi}_2$.

The formal proof is presented in the full version of the paper, together with some immediate applications of the quantum-classical Karp-Lipton theorem. Here, we present an overview of the proof.

Proof-sketch. Let $L = (L_{\text{yes}}, L_{\text{no}}, L_{\text{inv}}) \in \mathsf{QC\Pi}_2$, and let V be the corresponding (quantum) verifier. Since the proofs are classical, we assume V has small error. We show that $L \in \mathsf{QC\Sigma}_2$ by using the following (quantum) verifier V': it takes as input an instance x, a quantum circuit C, a string y_1. The verifier V' runs the circuit C on input (x, y_1) and receives a string y_2. It then accepts x if V accepts (x, y_1, y_2). If $x \in L_{\text{no}}$, then there exists a y_1 such that for all y_2, $V(x, y_1, y_2)$ accepts with negligible probability. Therefore, for any circuit C, there exists a y_1 such that $V'(x, C, y_1) = V(x, y_1, C(x, y_1))$ is 1 with very low probability.

For any $x \in L_{\text{yes}}$, we have that for all y_1, the pair (x, y_1) is a YES-instance of a QCMA problem. Using the [6] isolation procedure, we obtain an instance $\phi_{(x,y_1)}$ such that with non-negligible probability $\phi_{(x,y_1)}$ has a unique witness. Next, using the assumption that $\mathsf{QCMA} \subseteq \mathsf{BQP}_{/\text{mpoly}}$, we get that there exists a circuit $\tilde{C}$ that can decide $\phi_{(x,y_1)}$. Finally, using the UQCMA search-to-decision procedure of [17], we can use C to find the unique

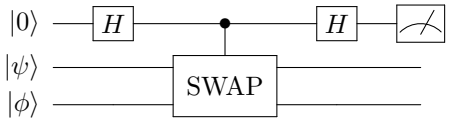

Figure 1 The SWAP test, whose output is the measurement result on the first wire.

witness for $\phi_{(x,y_1)}$. Let C be the circuit that, on input, (x, y_1), first performs the witness isolation from [6], followed by the UQCMA search-to-decision reduction from [17]. Putting these together, we get that there exists a circuit C that, for any y_1, finds a y_2 with non-negligible probability such that $V(x, y_1, y_2) = 1$. Therefore, there exists C such that for any y_1, $V'(x, C, y_1) = 1$ with non-negligible probability. ◀

4 Error reduction for pureQPH

We next study (weak) error reduction for pureQPH (pure proofs). For this, we first require an asymmetric generalization of the Product Test [24, 16], given in Section 4.1. We then give one-sided error reduction results in Section 4.2.

4.1 Asymmetric product test

We first give a generalization of the Product Test [24, 16], which we denote the Asymmetric Product Test (APT), stated as follows:

1. The input is $|\psi\rangle \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$ in register A, and $|\phi\rangle \in \mathbb{C}^{d^m}$ in register B, where for brevity $d := d_1 \cdots d_n$. We think of B as encoding m copies of A.
2. Choose $(i, j) \in [n] \times [m]$ uniformly at random.
3. Run the SWAP Test (Figure 1) between the ith register of A, and in B, the ith register of the jth copy of A.
4. Accept if the SWAP Test outputs 0, reject otherwise.

In fact, above, one can also assume that $|\phi\rangle$ in the APT is potentially entangled across two registers B and C, as we do for the main lemma of this section, given below.

▶ **Lemma 13.** *[Asymmetric Product Test (APT)] Define $d = d_1 \cdots d_n$. Consider $|\phi\rangle_{BC} \in \mathbb{C}^{d^m} \otimes \mathbb{C}^{d'}$ for some $d' > 0$. Suppose*

$$\max_{|\psi\rangle := |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle \in \mathbb{C}^d} \langle\phi|_{BC}[(|\psi\rangle\langle\psi|^{\otimes m})_B \otimes I_C]|\phi\rangle_{BC} = 1 - \varepsilon \tag{14}$$

for $\varepsilon \geq 0$. Then, given the state $|\eta\rangle_{ABC} := |\psi\rangle_A \otimes |\phi\rangle_{BC}$, the APT accepts with probability at most $1 - \varepsilon/2mn$.

Proof. Let $\{|\alpha_{ik}\rangle\}_{k\in[d_i]}$ be an orthonormal basis of $\mathbb{C}^{d_i}$ with $|\alpha_{i1}\rangle := |\psi_i\rangle$, and define index set

$$X = \{(x_{ij})_{i\in[n],j\in[m]} \mid \forall ij : x_{ij} \in [d_i]\}. \tag{15}$$

Then, rewrite $|\phi\rangle_{BC}$ in the $\{|\alpha_{ik}\rangle\}_k$ bases to obtain:

$$|\phi\rangle = \sum_{x\in X} a_x \left(\bigotimes_{ij} |\alpha_{i,x_{ij}}\rangle\right)_B |\gamma_x\rangle_C \tag{16}$$

for some states $|\gamma_x\rangle_C$. Without loss of generality, consider the swap test between the first register of A, $\mathbb{C}^{d_1}$ (which contains $|\psi_1\rangle$), and the first copy of $\mathbb{C}^{d_1}$ in B. Just prior to the final measurement in the test, we have the state $|\eta'\rangle_{SABC}$ given by (where S encodes the control qubit for SWAP in the SWAP test)

$$\sum_{x\in X} a_x \Big(\frac{1}{2}|0\rangle_S\big(|\psi_1\rangle|\alpha_{1,x_{11}}\rangle + |\alpha_{1,x_{11}}\rangle|\psi_1\rangle\big) + \frac{1}{2}|1\rangle_S\big(|\psi_1\rangle|\alpha_{1,x_{11}}\rangle - |\alpha_{1,x_{11}}\rangle|\psi_1\rangle\big)\Big) \otimes \bigotimes_{i\neq 1 \text{ or } j\neq 1} |\alpha_{i,x_{ij}}\rangle|\gamma_x\rangle =: \sum_{x\in X} a_x|\eta'_x\rangle_{SABC}. \tag{17}$$

We now show that the cross terms of $\langle\eta'|\eta'\rangle$ vanish, as $\langle\eta'_x|\eta'_y\rangle = 0$ with $x \neq y$, where $x, y \in X$: (1) If $x_{ij} \neq y_{ij}$ for $(i,j) \neq (1,1)$, this follows immediately from orthonormality of the basis sets $\{|\alpha_{ik}\rangle\}_{k\in[d_i]}$. (2) The only remaining case is $x_{11} \neq y_{11}$. Without loss of generality, $y_{11} \neq 1$. Then $|\alpha_{1,y_{11}}\rangle$ is orthogonal to $|\psi_1\rangle$, again by choice of our basis set. Hence, $\langle\psi_1, \alpha_{1,x_{11}}|\alpha_{1,y_{11}}, \psi_1\rangle = 0$. Since trivially $\langle\psi_1, \alpha_{1,x_{11}}|\psi_1, \alpha_{1,y_{11}}\rangle = 0$, we again have $\langle\eta'_x|\eta'_y\rangle = 0$.

As for the non-cross-terms, we first have again from our basis choice that for any $x \in X$,

$$\langle\eta'_x|(|0\rangle\langle 0|_S)|\eta'_x\rangle = \begin{cases} 1, & \text{if } x_{11} = 1 \\ \frac{1}{2}, & \text{if } x_{11} \neq 1 \end{cases}. \tag{18}$$

Recall now that the APT selects $i \in [n]$ and $j \in [m]$ uniformly at random and does a SWAP test between $|\psi_i\rangle$ (the ith register of A) and the jth copy of the i register of B. Thus, conditioned on the APT randomly choosing $(i,j) = (1,1)$, we may bound its acceptance probability as

$$\Pr[\mathrm{APT}(|\eta\rangle) = 1 \mid i = 1, j = 1] = \langle\eta'|(|0\rangle\langle 0|_S)|\eta'\rangle = \sum_{\substack{x\in X \\ \text{s.t. } x_{11}=1}} |a_x|^2 + \frac{1}{2}\sum_{\substack{x\in X \\ \text{s.t. } x_{11}\neq 1}} |a_x|^2. \tag{19}$$

By symmetry, an identical argument holds for any pair (i,j), and so

$$\Pr[\mathrm{APT}(|\eta\rangle) = 1] = \frac{1}{mn}\sum_{ij}\left(\sum_{\substack{x\in X \\ \text{s.t. } x_{ij}=1}} |a_x|^2 + \frac{1}{2}\sum_{\substack{x\in X \\ \text{s.t. } x_{ij}\neq 1}} |a_x|^2\right) \tag{20}$$

$$\leq |a_{1^{mn}}|^2 + \frac{2mn-1}{2mn}\sum_{x\in X\setminus\{1^{mn}\}} |a_x|^2 = 1 - \frac{\varepsilon}{2mn}. \tag{21}$$

◀

4.2 One-sided error reduction

With Lemma 13 (APT) in hand, we now show one-sided error reduction for $\mathsf{pureQ\Pi}_i$, which suffices to obtain statements for all desired classes subsequently in Theorem 3. For this, we define classes $\mathsf{pureQ\Sigma}_i^{\mathsf{SEP}}$ and $\mathsf{pureQ\Pi}_i^{\mathsf{SEP}}$ as identical to $\mathsf{pureQ\Sigma}$ and $\mathsf{pureQ\Pi}$, respectively, except the measurement POVM of the verifier in the YES case must additionally be a separable operator relative to the cuts between each of the i proofs $|\psi_1\rangle$ to $|\psi_i\rangle$. (This is analogous to $\mathsf{QMA}(2)$ versus $\mathsf{QMA}^{\mathsf{SEP}}(2)$ [16].)

▶ **Lemma 14.** *[One-sided* $\mathsf{pureQ\Pi}_i$ *amplification] If i is even, then*

$$\mathsf{pureQ\Pi}_i(c,s) \subseteq \mathsf{pureQ\Pi}_i^{\mathsf{SEP}}\left(1 - \frac{1}{e^n}, 1 - \frac{1}{np(n)^2}\right), \tag{22}$$

for all functions c and s such that $c - s \geq 1/p(n)$ for some polynomial p.

Proof. Let $\mathrm{L} = (L_{yes}, L_{\mathrm{no}}, L_{\mathrm{inv}}) \in \mathsf{pureQ\Pi}_i(c,s)$, with verifier V taking in i proofs denoted $|\psi_1\rangle, \ldots |\psi_i\rangle$. Since i is even, the last proof, $|\psi_i\rangle$, is existentially quantified. We define a new verifier V' to decide L in $\mathsf{pureQ\Pi}_i(1-1/\exp, 1-1/\operatorname{poly})$ as follows. V' receives the following proofs from an honest prover:

$$\left(|\psi_1'\rangle \otimes \cdots \otimes |\psi_{i-1}'\rangle\right)_A = (|\psi_1\rangle \otimes \cdots \otimes |\psi_{i-1}\rangle)_A \tag{23}$$

$$|\psi_i'\rangle_{BC} = \left(\bigotimes_{j=1}^{i-1} |\psi_j\rangle\right)_B^{\otimes m} \otimes |\psi_i\rangle_C^{\otimes m} \tag{24}$$

for $m \in \Theta(n(c-s)^{-2})$, and where A, B and C are used to align with the notation of Lemma 13 (which we use shortly). In words, the last prover sends m copies of the first $i-1$ proofs in register B, and m copies of the last proof $|\psi_i\rangle$ in register C. Then, V' acts as follows:

1. With probability 1/2, apply the APT (Lemma 13) between registers A and B. Accept iff the test accepts.
2. With probability 1/2, apply verifier V m times, taking one proof $|\psi_i\rangle$ from each respective subregister of B. Accept iff at least $(c+s)/2$ measurements accept.

Correctness. *YES case.* Since the i^{th} prover is existentially quantified, it sends the state in Equation (24). Thus, the APT accepts with certainty. Similarly, for parallel repetition of V, each repetition is independent, hence the overall verifier accepts with probability at least $1 - \exp(-(c-s)^2 m/2)$, as desired.

NO case. Now the i^{th} prover is universally quantified, hence it can send us a state entangled across BC.

Suppose the APT accepts with probability $1-\varepsilon$. By Lemma 13,

$$|\psi_i'\rangle_{BC} = \alpha \left(\bigotimes_{j=1}^{i-1} |\psi_j\rangle\right)_B^{\otimes m} |\phi\rangle_C + \beta|\gamma\rangle_{BC} =: \alpha|\eta\rangle_B|\phi\rangle_C + \beta|\gamma\rangle_{BC} \tag{25}$$

for $|\alpha|^2 \geq 1 - 2m(i-1)\varepsilon$, and arbitrary states $|\phi\rangle, |\gamma\rangle$ satisfying that $|\eta\rangle_A|\phi\rangle_B$ is orthogonal to $|\gamma\rangle_C$. We have

$$\Pr[\text{parallel repetition of } V \text{ accepts } |\eta\rangle_A|\phi\rangle_B] \leq e^{\frac{-(c-s)^2 m}{2}}. \tag{26}$$

We conclude the acceptance probability of V' is at most

$$\frac{1}{2}\Big((1-\varepsilon) + (1-2m(i-1)\varepsilon)e^{-(c-s)^2 m/2} + 2m(i-1)\varepsilon \tag{27}$$

$$+ 2\sqrt{2m(i-1)\varepsilon + (2m(i-1))^2\varepsilon}\Big) \leq 1 - \frac{\varepsilon}{2}, \tag{28}$$

where the maximum is attained when $\varepsilon = \Theta(1/m(i-1))$.

Finally, that the measurement operator for the YES case is separable follows via the argument of Harrow and Montanaro for $\mathsf{QMA}(2)$ amplification [16], since our use of the APT is agnostic to whether proofs are universally or existentially quantified. ◀

Lemma 14 now easily generalizes to cover all classes regarding $\mathsf{pureQPH}$ we are concerned with:

▶ **Theorem 3.** *For all $i > 0$ and $c - s \geq 1/p(n)$ for some polynomial p,*

1. *For even $i > 0$:*
   a. $\mathsf{pureQ\Sigma}_i(c,s) \subseteq \mathsf{pureQ\Sigma}_i^{\mathsf{SEP}}(1/np(n)^2, 1/e^n)$
   b. $\mathsf{pureQ\Pi}_i(c,s) \subseteq \mathsf{pureQ\Pi}_i^{\mathsf{SEP}}(1-1/e^n, 1-1/np(n)^2)$

2. *For odd $i > 0$:*
   a. $\mathsf{pureQ\Sigma}_i(c,s) \subseteq \mathsf{pureQ\Sigma}_i^{\mathsf{SEP}}(1-1/e^n, 1-1/np(n)^2)$
   b. $\mathsf{pureQ\Pi}_i(c,s) \subseteq \mathsf{pureQ\Pi}_i^{\mathsf{SEP}}(1/np(n)^2, 1/e^n)$

Proof. Statement $1b$ is from Lemma 14, and $1a$ follows from $1b$ since we can get a $\mathsf{pureQ\Sigma}_i$ verifier by flipping the answer of a $\mathsf{pureQ\Pi}_i$ verifier corresponding to the complement of our promise problem. The remaining cases are analogous: $2a$ follows from $1b$, and $2b$ from $1a$. ◀

5 Upper and lower bounds on pureQPH

5.1 Lower bound: QCPH versus QPH

We first give a lower bound on $\mathsf{pureQPH}$, by showing that alternatingly-quantified classical proofs can be replaced by pure-state quantum proofs.

▶ **Theorem 15.** $\mathsf{QCPH} \subseteq \mathsf{pureQPH}$.

This follows immediately from the following lemma.

▶ **Lemma 16.** *For all even $k \geq 2$, $\mathsf{QC\Pi}_k \subseteq \mathsf{pureQ\Pi}_k$.*

Proof. That k is even implies the kth proof is existentially quantified in the YES case, a fact we will leverage. To begin, Let V be a verifier for $\mathsf{QC\Pi}_k$, so that in the YES case $\forall_1 x_1 \ldots \exists_k x_k : \Pr[V(x_1,\ldots,x_k)=1] \geq c$ and in the NO case $\exists x_1 \ldots \forall x_k : \Pr[V(x_1,\ldots,x_k)=1] \leq s$, where we may assume without loss of generality that c and s are exponentially close to 1 and 0, respectively. We construct a $\mathsf{pureQPH}$ verifier V' as follows:

- V' receives k proofs, $|\psi_1\rangle \otimes \cdots \otimes |\psi_k\rangle$.
- The last proof $|\psi_k\rangle$ consists of two registers denoted A and B. We think of A as containing m copies of proofs $1,\ldots,k-1$ (as in the APT, Lemma 13), and B as containing the kth proof for V, x_k.
- V' acts as follows:
  1. With probability $1/2$, run the APT (Lemma 13), and accept if and only if the APT accepts.
  2. With probability $1/2$, measure proofs $|\psi_1\rangle \otimes \cdots \otimes |\psi_{k-1}\rangle$ in the standard basis to obtain strings $x_1,\ldots,x_{k-1}$, respectively, similarly measure all copies of these proofs in A, and finally measure B in the standard basis to obtain $x_{k,B}$. Let $U = \{1,3\ldots,k-1\}$ denote the indices of universally quantified proofs (in the YES case). Then:
     a. If there exists an $i \in [k-1]$ such that the strings obtained by measuring all copies of $|\psi_i\rangle$ did *not* equal x_i, let i denote the minimal such index. Accept if $i \in U$, and reject otherwise.
     b. Otherwise, simulate $V(x_1,\ldots,x_{k-1},x_{k,B})$.

Correctness strategy. Since we are trying to simulate $\mathsf{QC\Pi}_k$, ideally we want all proofs to be strings. This can be assumed without loss of generality for existentially quantified proofs, but not for universally quantified proofs, which can be set to any pure state by definition of $\mathsf{pureQ\Pi}_k$. So, for any $i \in U$, write $|\psi_i\rangle = \sum_x \alpha_{i,x}|x\rangle$, where we view $|\alpha_{i,x}|^2$ as a distribution over strings x for proof i. Denote for any $i \in U$ by x_i^* the amplitude of highest weight, i.e. $x_i^* = \arg\max_x |\alpha_{i,x}|$ (ties broken arbitrarily). The key idea is that the existentially quantified proof at index $i+1$ will now send string y_{i+1}^*, where y_{i+1}^* is the same string that a prover for the *original* $\mathsf{QC\Pi}_k$ verifier V would have sent in response to x_i^* on proof i. (For clarity, if $i+1=k$, then y_{i+1}^* is sent in register B.)

YES case. Since the kth proof is existentially quantified, for Step 1 (APT), we may assume all copies in $|\psi_{k,A}\rangle$ (the state in register A) are correctly set, so the APT accepts with probability 1, leaving all proofs invariant. As for Step 2, for each $i \in U$, let X_i be the random variable resulting from measuring $|\psi_i\rangle$ in the standard basis, and X_{ij} the random variables for the jth copy of $|\psi_i\rangle$ in register A. The verification can now fail in one of two ways:

1. There exists an $i \in U$ such that we did not measure the "right" result, i.e. $X_i \neq x_i^*$, but that all measured copies of $|\psi_i\rangle$ returned the same string, i.e. $\forall j\ X_{ij} = X_i$. In this case, our strategy for setting the existentially quantified proof $i+1$ is not necessarily the correct response to X_i. Thus, when Step 2(b) is run, we have no guarantee for the acceptance probability of V.
2. Measuring all $i \in U$ yields the desired outcomes x_i^* (as well as $\forall j\ X_{ij} = X_i$), but V nevertheless rejects due to imperfect completeness, i.e. $c < 1$.

Combining these, via the union bound we thus have for Step 2 that

$$\Pr[\text{reject}] \leq \Pr[\ \exists i \in U : X_i \neq x_i^* \text{ AND } \forall ij : X_i = X_{ij}\ ] + (1-c) \tag{29}$$

$$\leq \min\{p, 1-p\}^m + 1 - c \tag{30}$$

$$\leq 2^{-m} + 1 - c, \tag{31}$$

where $p := \max_i |\alpha_{i,x_i}|^2$, since $\Pr[X_i \neq x_i^*] = \Pr[X_{ij} \neq x_i^*] \leq \max\{p, 1-p\}$ because $|\alpha_{i,y}|^2 \leq 1-p$ for $y \neq x_i^*$. Since we assumed the APT accepts with perfect probability, we conclude V' accepts with probability $\geq \frac{1}{2} + \frac{1}{2}(c - 2^{-m}) =: c'$. (As an aside, recall c is exponentially close to 1.)

NO case. The analysis is more subtle in this case, as the set of indices $U = \{1, 3, \ldots, k-1\}$ now refers to *existentially* quantified proofs. Thus, in Step 2(a) when V' accepts iff $i \in U$, this now means it accepts on existentially quantified proofs. This is because V' does not know whether it is in a YES or NO case. For the same reason, the actions and role of the final proof $|\psi_k\rangle$ on registers A and B remain the same, even though it is now universally quantified. Finally, the strategy of any existential prover $i \in U$ is the same as the YES case: Prover i sends the optimal response y_i^* to universally quantified proof x_{i-1}^*. (If $i = 1$, then there is no universal proof to condition on for $|\psi_1\rangle$.)

To proceed, assume for now that the APT would have succeeded in Step 1 with certainty. In Step 2, define again for all $i \in U$, X_i the random variable resulting from measuring $|\psi_i\rangle$ in the standard basis, and X_{ij} the random variables for the jth copy of $|\psi_i\rangle$ in register A. The verifier can now fail in one of two ways, the first of which differs significantly from the YES case:

1. There exists $i \in U$ such that X_i mismatched one of its copies in A, i.e. $\exists j$ such that $X_i \neq X_{ij}$. *A priori*, this seems like a problem – since $|\psi_k\rangle$ is universally quantified, most choices of $|\psi_k\rangle$ will cause a mismatch with X_i with high probability, causing V' to accept with high probability. The crucial insight is that, in order for $|\psi_1\rangle \otimes \cdots \otimes |\psi_k\rangle$ to pass the APT, it must essentially set each copy of $|\psi_i\rangle$ to string y_i^*. Thus, measuring the A register is highly unlikely to produce mismatches on existentially quantified proofs!
2. Measuring all $i \in U$ will yield the desired outcomes y_i^*, since U is existentially quantified. If in addition $\forall j\ X_{ij} = X_i$, running V may nevertheless accept due to imperfect soundness, i.e. $s > 0$.

Then, by a similar argument as for the YES case that in Step 2, in which we first assume the APT passes with certainty, $\Pr[\text{accept}] \leq 2^{-m} + s$. Now, let us assume the APT accepts with probability $\geq 1 - \varepsilon/2mn$. By Lemma 13,

$$\langle\psi_k|(|\psi_1, \ldots, \psi_{k-1}\rangle\langle\psi_1, \ldots, \psi_{k-1}|_A \otimes I_B)|\psi_k\rangle \geq 1 - \varepsilon. \tag{32}$$

Let $\{|\beta_i\rangle\}$ be an orthonormal basis of register A with $|\beta_1\rangle = |\psi_1, \ldots, \psi_{n-1}\rangle$. Then we can write $|\psi_n\rangle = \sum_i \alpha_i |\beta_i\rangle_A |\gamma_i\rangle_B$ with $|\alpha_1|^2 \geq 1 - \varepsilon$. Hence, V' accepts with probability at most

$$\frac{1}{2}\left(1 - \frac{\varepsilon}{2mn}\right) + \frac{1}{2}\left(\varepsilon + (1-\varepsilon)s\right) \leq 1 - \frac{1}{4mn} + s =: s', \tag{33}$$

where the first inequality follows because, without loss of generality, we may assume $\varepsilon \leq 1/2$, as otherwise the prover cannot hope to succeed make V' accept with probability greater than $3/4$ (whereas $c' \approx 1$). Finally, we can choose c, s, m such that $c' - s' \geq 1/\operatorname{poly}$. ◀

5.2 Upper bound

To complement Section 5.1, we next give a simple but non-trivial upper bound on $\mathsf{pureQPH}$, which may be viewed as an "exponential analogue" of Toda's theorem. For this, let NP^k denote a tower of NP oracles of height k. (For example, $\mathsf{NP}^1 = \mathsf{NP}$ and $\mathsf{NP}^2 = \mathsf{NP}^{\mathsf{NP}}$.) Define NEXP^k analogously, by a tower of NEXP oracles. In [13], it was observed that $\mathsf{Q}\Sigma_i \subseteq \mathsf{NEXP}^i$. We show a sharper bound here.

▶ **Theorem 17.** $\mathsf{pureQPH} \subseteq \mathsf{EXP}^{\mathsf{PP}}$.

▶ **Observation 18.** $\mathsf{pureQ}\Sigma_i \subseteq \mathsf{NEXP}^{\mathsf{NP}^{i-1}}$.

Proof. Replace all proofs by by their exponential-size classical description (up to additive additive inverse exponential additive error in the entries), and simulate the verifier's action on the proofs via exponential-time matrix multiplication. The standard proof technique for showing $\Sigma_i^p \subseteq \mathsf{NP}^i$ now applies, except we only require NEXP at the base level of the oracle tower, i.e. $\mathsf{NEXP}^{\mathsf{NP}^{i-1}}$, since an exponential time base can "inflate" or pad the instance size for its oracle exponentially. ◀

For comparison, the observed bound in [13] of $\mathsf{Q}\Sigma_i \subseteq \mathsf{NEXP}^i$ is overkill, since it allows the first NEXP oracle can use *double* exponential time to process its exponential size input.

▶ **Observation 19.** $\mathsf{NEXP} \subseteq \mathsf{EXP}^{\mathsf{NP}}$.

Proof. Since using an exponential time machine we can "inflate" the instance size to exponential, an NP machine can thereafter simulate the NEXP computation on the inflated instance size. The EXP machine just returns the answer of the NP oracle. ◀

▶ **Observation 20.** $\mathsf{NEXP}^O \subseteq \mathsf{EXP}^{\mathsf{NP}^O}$ *for an oracle to any language* O.

Proof. It is easy to see that the argument in Observation 19 relativizes, since the NP oracle to the EXP machine can make the NEXP queries directly to the oracle O. ◀

▶ **Observation 21.** $\mathsf{EXP}^{\mathsf{P}^{\mathsf{PP}}} \subseteq \mathsf{EXP}^{\mathsf{PP}}$.

Proof. The EXP machine can make at most exponentially many queries to the its oracle, each of which can be of size at most exponential in the size of the input. Therefore an EXP machine can simulate the action of a P^{PP} machine (even on an exponential sized query) by making queries to a PP oracle while simulating the action of a P machine (which will only take time polynomial in size of the query). ◀

▶ **Theorem 22.** *For all* $i \geq 1$, $\mathsf{Q}\Sigma_i \subseteq \mathsf{EXP}^{\mathsf{PP}}$.

Proof. By Observation 18, Observation 20 and Toda's Theorem [29],

$$\mathsf{Q}\Sigma_i \subseteq \mathsf{NEXP}^{\mathsf{NP}^{i-1}} \subseteq \mathsf{EXP}^{\mathsf{NP}^i} \subseteq \mathsf{EXP}^{\mathsf{P}^{\mathsf{PP}}}. \tag{34}$$

The claim now follows from Observation 21. ◀

Theorem 17 now follows immediately from Theorem 22.

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