Point-To-Set Principle and Constructive Dimension Faithfulness

Satyadev Nandakumar
Department of Computer Science and Engineering, Indian Institute of Technology Kanpur, India

Subin Pulari
Department of Computer Science and Engineering, Indian Institute of Technology Kanpur, India

Akhil S
Department of Computer Science and Engineering, Indian Institute of Technology Kanpur, India

Abstract

Hausdorff Φ-dimension is a notion of Hausdorff dimension developed using a restricted class of coverings of a set. We introduce a constructive analogue of Φ-dimension using the notion of constructive Φ-s-supergales. We prove a Point-to-Set Principle for Φ-dimension, through which we get Point-to-Set Principles for Hausdorff dimension, continued-fraction dimension and dimension of Cantor coverings as special cases. We also provide a Kolmogorov complexity characterization of constructive Φ-dimension.

A class of covering sets Φ is said to be "faithful" to Hausdorff dimension if the Φ-dimension and Hausdorff dimension coincide for every set. Similarly, Φ is said to be "faithful" to constructive dimension if the constructive Φ-dimension and constructive dimension coincide for every set. Using the Point-to-Set Principle for Cantor coverings and a new technique for the construction of sequences satisfying a certain Kolmogorov complexity condition, we show that the notions of "faithfulness" of Cantor coverings at the Hausdorff and constructive levels are equivalent.

We adapt the result by Albeverio, Ivanenko, Lebid, and Torbin [1] to derive the necessary and sufficient conditions for the constructive dimension faithfulness of the coverings generated by the Cantor series expansion, based on the terms of the expansion.

2012 ACM Subject Classification Theory of computation → Computability; Mathematics of computing → Information theory

Keywords and phrases Kolmogorov complexity, Constructive dimension, Faithfulness, Point to set principle, Continued fraction dimension, Cantor series expansion

Digital Object Identifier 10.4230/LIPIcs.MFCS.2024.76

Related Version *Full Version*: https://arxiv.org/abs/2403.08278

Acknowledgements We gratefully acknowledge the anonymous reviewers for their invaluable insights and constructive feedback, which has significantly strengthened this manuscript.

1 Introduction

1.1 Faithfulness in dimension

In the study of randomness and information, an important concept is the preservation of randomness across multiple representations of the same object. Martin-Löf randomness, and computable randomness, for example, are preserved among different base-b representations of the same real (see Downey and Hirschfeldt [5], Nies [27], Staiger [31]) and when we convert from the base-b expansion to the continued fraction expansion ([23, 26, 25]).

A quantification of this notion is whether the *rate* of information is preserved across multiple representations. This rate is studied using a constructive analogue of Hausdorff dimension called Constructive dimension [11, 21]. Hitchcock and Mayordomo [8] show that constructive dimension is preserved across base-b representations. However, in a recent work,

49th International Symposium on Mathematical Foundations of Computer Science (MFCS 2024).
Editors: Rastislav Královič and Antonín Kučera; Article No. 76; pp. 76:1–76:15
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Akhil, Nandakumar and Vishnoi [24] show that the rate of information is not preserved across all representations. In particular, they show that constructive dimension is *not* preserved when we convert from base-b representation to continued fraction representation of the same real.

This raises the following question: *Under which settings is the effective rate of information – i.e., constructive dimension – preserved when we change representations of the same real?* Since constructive dimension is a constructive analogue of Hausdorff dimension, this question is a constructive analogue of the concept of "faithfulness" of Hausdorff dimension.

A family of covering sets Φ is "faithful" to Hausdorff dimension if the dimension of every set $\mathcal{F}$ defined using covers constructed using Φ, called the Hausdorff Φ-dimension, coincides with the Hausdorff dimension of $\mathcal{F}$. Faithfulness is well-studied as determining the Hausdorff dimension of a set is often a difficult problem, and faithful coverings help simplify the calculation. This notion is introduced in a work of Besicovitch [3], which shows that the class of dyadic intervals is faithful for Hausdorff dimension. Rogers and Taylor [29] further develop the idea to show that all covering families generated by comparable net measures are faithful for Hausdorff dimension. This implies that the class of covers generated by the base b expansion of reals for any $b \in \mathbb{N} \setminus \{1\}$ is faithful for Hausdorff dimension. However, not all coverings are faithful for Hausdorff dimension. A natural example is the continued fraction representation, which is not faithful for Hausdorff dimension [28]. Faithfulness of Hausdorff dimension has then been studied in various settings [1, 2, 9, 28].

1.2 Constructive Dimension Faithfulness

In this work, we introduce a constructive analogue of Hausdorff Φ-dimension which we call constructive Φ-dimension. A family of covering sets Φ is "faithful" to constructive dimension if the constructive Φ-dimension of every set $\mathcal{F}$ coincides with the constructive dimension of $\mathcal{F}$. Mayordomo and Hitchcock [8] show that all base-b representations of reals, which are faithful for Hausdorff dimension, are also faithful for constructive dimension. On the other hand, Nandakumar, Akhil, and Vishnoi's work shows that the continued fraction expansion, which is not faithful for Hausdorff dimension is also not faithful for constructive dimension [24]. This raises the natural question: *Are faithfulness with respect to Hausdorff dimension and faithfulness with respect to constructive dimension equivalent notions?* A positive answer to this question implies that Hausdorff dimension faithfulness, a geometric notion, can be studied using the tools from information theory. Conversely, the faithfulness results of Hausdorff dimension can help us understand the settings under which constructive dimension is invariant for *every* individual real.

In this work, we show that for the most inclusive generalization of base-b expansions under which faithfulness has been studied classically, namely, for classes of coverings generated by the Cantor series expansions, the notions of Hausdorff faithfulness and constructive faithfulness are indeed equivalent. The Cantor series expansion, introduced by Georg Cantor [4], uses a sequence of natural numbers $Q = \{n_k\}_{k\in\mathbb{N}}$ as the terms of representation. Whereas base-b representation use exponentials with respect to a fixed b, $\{b^n\}_{n\in\mathbb{N}}$, the Cantor series representation $Q = \{n_k\}_{k\in\mathbb{N}}$ uses factorials $\{n_1 \dots n_k\}_{k\in\mathbb{N}}$ as the basis for representation. This class is of additional interest as there are Cantor expansions that are faithful as well as non faithful for Hausdorff dimension, depending on the Cantor series representation $\{n_k\}_{k\in\mathbb{N}}$ in consideration [1]. To establish our result, we use a Φ-dimensional analogue of the Point-to-Set Principle.

1.3 Point-to-Set Principle and Faithfulness

The Point-to-Set principle introduced by J. Lutz and N. Lutz [13] relates the Hausdorff dimension of a set of n-dimensional reals with the constructive dimensions of points in the set, relative to a minimizing oracle. This theorem has been instrumental in answering open questions in classical fractal geometry using the theory of computing (See [14, 20, 19, 18, 13]). Mayordomo, Lutz, and Lutz [15] extend this work to arbitrary separable metric spaces.

In this work, we first prove the Point-to-Set principle for Φ-dimension (Theorem 27) and show that this generalizes the original Point-to-Set principle. We then develop a combinatorial construction of sequences having Kolmogorov complexities that grow at the same rate as a given sequence relative to any given oracle (Theorem 36). This new combinatorial construction may be of independent interest in the study of randomness. Using these new tools, we show that under the setting of covers generated by Cantor series expansions, the notions of constructive faithfulness and Hausdorff dimension faithfulness are equivalent (Theorem 51). We then adapt the result by Albeverio, Ivanenko, Lebid, and Torbin [1] to derive a loglimit condition for the constructive dimension faithfulness of the coverings generated by the Cantor series expansions (Theorem 53).

Our main results include the following.

1. We introduce the notion of constructive Φ-dimension using that subsumes base-b, continued fraction, and Cantor covering dimension. We also give an equivalent Kolmogorov Complexity characterization of constructive Φ-dimension. We prove a Point-to-Set principle for Φ-dimension. This generalizes the original Point-to-Set Principle and yields new Point-to-Set principles for the dimensions of continued fractions and Cantor series representations.
2. Using the point-to-set principle, we characterize constructive faithfulness for Cantor series expansions using a log limit condition of the terms appearing in the series. This generalizes the invariance result of constructive dimension under base b representations to all Cantor series expansions that obey this log limit condition. Moreover, it implies that for any Cantor series expansion that does not obey the log limit condition, there are sequences whose Cantor series dimension is different from its constructive dimension.

The recent works of J. Lutz, N. Lutz, Stull, Mayordomo and others study the "point-to-set principle" of how constructive Hausdorff dimension of points may be used to compute the classical Hausdorff dimension of arbitrary sets. In addition to the generalization of this point-to-set principle to Φ-systems, our final result may be viewed as a new *point-to-set phenomenon* for the notion of "faithfulness": here, equality of the constructive Cantor series dimensions and constructive dimensions of *every point* yield equality for the classical Cantor series and Hausdorff dimensions of *every set*, and conversely.

2 Preliminaries

2.1 Notation

We use Σ to denote the binary alphabet $\{0,1\}$, Σ^* represents the set of finite binary strings, and Σ^∞ represents the set of infinite binary sequences. We use $|x|$ to denote the length of a finite string $x \in \Sigma^*$. For an infinite sequence $X = X_0X_1X_2\dots$, we use $X \upharpoonright n$ to denote the finite string consisting of the first n symbols of X. When $n \geq m$ we also use the notation $X[m,n]$ to denote the substring $X_mX_{m+1}\dots X_n$ of $X \in \Sigma^\infty$. We call two sets U and V to be incomparable if $U \not\subseteq V$ and $V \not\subseteq U$. For a set $U \subseteq \mathbb{R}$, we denote $|U|$ to denote the diameter of U, that is $|U| = \sup_{x,y\in U} d(x,y)$, where d is the Euclidean metric.

We use $\emptyset$ to denote the empty set, and we assume $|\emptyset| = 0$. For any finite collection of sets $\{U\}$, we use $\#(U)$ to denote the number of elements in U. Given infinite sequences $X_1, \dots X_n$, we define the *interleaved sequence* $X_1 \oplus X_2 \oplus \dots X_n$ to be the interleaved sequence $X = X_1[0]X_2[0]\dots X_n[0]X_1[1]\dots X_n[1]\dots$. For some fixed $n \in \mathbb{N}$, we use $\mathbb{X} \subseteq \mathbb{R}^n$ to denote the metric space under consideration. We call a set of strings $\mathcal{P} \subset \Sigma^*$ to be prefix free if there are no two strings $\sigma, \tau \in \mathcal{P}$ such that σ is a proper prefix of τ. Given $n \in \mathbb{N}$ we use $[n]$ to denote $\{0, 1, \dots n-1\}$. Kolmogorov Complexity represents the amount of information contained in a finite string. For more details on Kolmogorov Complexity, see [5, 10, 27, 30].

▶ **Definition 1.** *The Kolmogorov complexity of $\sigma \in \Sigma^*$ is defined as $K(\sigma) = \min_{\pi \in \Sigma^*} \{|\pi| \mid U(\pi) = \sigma\}$, where U is a fixed universal prefix free Turing machine.*

2.2 Hausdorff Dimension

The following definitions are originally given by Hausdorff [7]. We take the definitions from Falconer [6].

▶ **Definition 2** (Hausdorff [7])**.** *Given a set $\mathcal{F} \subseteq \mathbb{X}$, a collection of sets $\{U_i\}_{i\in\mathbb{N}}$ where for each $i \in \mathbb{N}$, $U_i \subseteq \mathbb{X}$ is called a* δ-cover of $\mathcal{F}$ *if for all $i \in \mathbb{N}$, $|U_i| \leq \delta$ and $\mathcal{F} \subseteq \bigcup_{i\in\mathbb{N}} U_i$.*

▶ **Definition 3** (Hausdorff [7])**.** *Given an $\mathcal{F} \subseteq \mathbb{X}$, for any $s > 0$, define*

$$\mathcal{H}^s_\delta(\mathcal{F}) = \inf\left\{\sum_i |U_i|^s : \{U_i\}_{i\in\mathbb{N}} \text{ is a } \delta\text{-cover of } \mathcal{F}\right\}.$$

As δ decreases, the set of admissible δ covers decreases. Hence $\mathcal{H}^s_\delta(\mathcal{F})$ increases.

▶ **Definition 4** (Hausdorff [7])**.** *For $s \in (0, \infty)$, the* s-dimensional Hausdorff outer measure *of $\mathcal{F}$ is defined as:*

$$\mathcal{H}^s(\mathcal{F}) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(\mathcal{F}).$$

Observe that for any $t > s$, if $\mathcal{H}^s(\mathcal{F}) < \infty$, then $\mathcal{H}^t(\mathcal{F}) = 0$ (see Section 2.2 in [6]). Finally, we have the following definition of Hausdorff dimension.

▶ **Definition 5** (Hausdorff [7])**.** *For any $\mathcal{F} \subset \mathbb{X}$, the* Hausdorff dimension *of $\mathcal{F}$ is defined as:*

$$\dim(\mathcal{F}) = \inf\{s \geq 0 : \mathcal{H}^s(\mathcal{F}) = 0\}.$$

2.3 Constructive dimension

Lutz [11] defines the notion of effective (equivalently, constructive) dimension of an infinite binary sequence using the notion of lower semicomputable s-gales.

▶ **Definition 6** (Lutz [11])**.** *For $s \in [0, \infty)$, a binary s-gale is a function $d : \Sigma^* \to [0, \infty)$ such that $d(\lambda) < \infty$ and for all $w \in \Sigma^*$, $d(w) = 2^s . \sum_{i\in\{0,1\}} d(wi)$.*

The *success set* of d is $S^\infty(d) = \left\{X \in \Sigma^\infty \mid \limsup_{n\to\infty} d(X \upharpoonright n) = \infty\right\}$.

For $\mathcal{F} \subseteq [0, 1]$, $\mathcal{G}(\mathcal{F})$ denotes the set of all $s \in [0, \infty)$ such that there exists a lower semicomputable (Definition 24) binary s-gale d with $\mathcal{F} \subseteq S^\infty(d)$.

▶ **Definition 7** (Lutz [11]). *The* constructive dimension *or* effective Hausdorff dimension *of* $\mathcal{F} \subseteq [0,1]$ *is*

$$\mathrm{cdim}(\mathcal{F}) = \inf \mathcal{G}(\mathcal{F}).$$

The constructive dimension of a sequence $X \in \Sigma^\infty$ *is* $\mathrm{cdim}(X) = \mathrm{cdim}(\{X\})$.

Mayordomo [21] extends the result by Lutz [12] to give the following Kolmogorov complexity characterization of constructive dimension of infinite binary sequences.

▶ **Theorem 8** (Lutz [12], Mayordomo [21]). *For any* $X \in \Sigma^\infty$,

$$\mathrm{cdim}(X) = \liminf_{n\to\infty} \frac{K(X \upharpoonright n)}{n}.$$

Mayordomo [22] later gave the following Kolmogorov complexity characterization of constructive dimension of points in $\mathbb{R}^n$.

▶ **Definition 9** (Mayordomo [22]). *For any* $x \in \mathbb{R}^n$,

$$\mathrm{cdim}(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r}.$$

where $K_r(x) = \min_{q\in\mathbb{Q}^n} \{K(q) : |x - q| < 2^{-r}\}$.

Constructive dimension also works in the Euclidean space. For a real $x = (x_1, ...x_n) \in \mathbb{R}^n$, let binary expansions of the fractional parts of each of the coordinates of x be $S_1 \in \Sigma^\infty, \dots, S_n \in \Sigma^\infty$ respectively. Then $\mathrm{cdim}(x) = n \cdot \mathrm{cdim}(X)$ where X is the interleaved sequence $X = S_1 \oplus S_2 \cdots \oplus S_n$ [16].

We now state some useful properties of constructive dimension. Lutz [11] shows that the constructive dimension of a set is always greater than or equal to its Hausdorff dimension.

▶ **Lemma 10** (Lutz [11]). *For any* $\mathcal{F} \subseteq \mathbb{X}$, $\dim(\mathcal{F}) \leq \mathrm{cdim}(\mathcal{F})$.

Further, Lutz[11] also shows that the constructive dimension of a set is the supremum of the constructive dimensions of points in the set.

▶ **Lemma 11** (Lutz [11]). *For any* $\mathcal{F} \subseteq \mathbb{X}$, $\mathrm{cdim}(\mathcal{F}) = \sup_{x\in\mathcal{F}} \mathrm{cdim}(x)$.

3 Hausdorff Φ-dimension and Effective Φ-dimension

Hausdorff dimension is defined using the notion of s-dimensional outer measures, where a cover is taken as the of union of a collection of covering sets $\{U_i\}_{i\in\mathbb{N}}$. Here a covering set U_i can be any arbitrary subset of the space (see Section 2.2). We define the general notion of Hausdorff Φ-dimension by restricting the class of admissible covers to Φ-covers, which are the union of sets from a family of covering sets Φ.

3.1 Family of covering sets

In this work, we consider a *family of covering sets* which satisfy the properties given below.

▶ **Definition 12** (Family of covering sets Φ). *We consider the space* $\mathbb{X} \subseteq \mathbb{R}^\eta$ *where* $\eta \in \mathbb{N}$. *A countable family of sets* $\Phi = \bigcup_{n\in\mathbb{Z}} \{U_i^n\}_{i\in\mathbb{N}}$, *where for each* $i \in \mathbb{N}, n \in \mathbb{Z}$, $U_i^n \subseteq \mathbb{X}$, *is called a family of covering sets if it satisfies the following properties:*

- *Increasing Monotonicity: For every* $n \in \mathbb{Z}$, $U \in \{U_i^n\}_{i\in\mathbb{N}}$ *and* $m \leq n$, *there is a unique* $V \in \{U_i^m\}_{i\in\mathbb{N}}$ *such that* $U \subseteq V$.
- *Fineness : Given any* $\epsilon > 0$, *and* $x \in \mathbb{X}$, *there exists a* $U \in \Phi$ *such that* $|U| < \epsilon$ *and* $x \in U$.

Note that the number of sets $\{U_i^n\}$ in some level $n \in \mathbb{Z}$ can also be finite and bounded by m. The definition still holds because in this case we take $U_j^n = \emptyset$ for $j > m$. Note that from Increasing monotonicity property, it follows that all elements $\{U_i^n\}$ in a particular level $n \in \mathbb{Z}$ are incomparable.

We now define the notion of a Φ-cover of a set.

► **Definition 13** (Φ-cover). *Let* $\Phi = \bigcup_{n\in\mathbb{Z}}\{U_i^n\}_{i\in\mathbb{N}}$ *be a family of covering sets. A* Φ *-cover of a set* $\mathcal{F} \subseteq \mathbb{X}$ *is collection of sets* $\{V_j\}_{j\in\mathbb{N}} \subseteq \Phi$ *such that* $\{V_j\}_{j\in\mathbb{N}}$ *covers* $\mathcal{F}$, *that is* $\mathcal{F} \subseteq \bigcup\limits_{j\in\mathbb{N}} V_j$

Note: Mayordomo [22] gives a definition of *Nice covers* of a metric space. They then give the definition of constructive dimension on a metric space with a nice cover. We note here that the notion of Family of covering sets is incomparable with the notion of Nice covers. Our definition does not require the c-cover property and the Decreasing monotonicity property of nice covers. Therefore, our notion includes the setting of continued fraction dimension, which is not captured by Nice covers. Also, the Fineness property required in our definition is not there in the definition of nice covers. The notion of Increasing monotonicity is present in both settings.

3.2 Hausdorff Φ-dimension

Recall from Definition 13 that a Φ-cover of $\mathcal{F}$ is a collection of sets from Φ that covers $\mathcal{F}$. We call this as a δ-cover if the diameter of elements in the cover are less than δ.

► **Definition 14.** *Let* Φ *be a family of covering sets defined over* $\mathbb{X}$. *Given a set* $\mathcal{F} \subseteq \mathbb{X}$, *a* Φ*-cover* $\{U_i\}_{i\in\mathbb{N}}$ *of* $\mathcal{F}$ *is called a* δ-cover of $\mathcal{F}$ using Φ *if for all* $i \in \mathbb{N}$, $|U_i| \leq \delta$.

► **Definition 15.** *Given an* $\mathcal{F} \subseteq \mathbb{X}$, *for any* $s > 0$, *we define*

$$\mathcal{H}_\delta^s(\mathcal{F}, \Phi) = \inf\left\{\sum_i |U_i|^s : \{U_i\}_{i\in\mathbb{N}} \text{ is a } \delta\text{-cover of } \mathcal{F} \text{ using } \Phi\right\}.$$

From the fineness property given in Definition 12, it follows that for any $\mathcal{F} \subseteq \mathbb{X}$, and $\delta > 0$, δ-covers of $\mathcal{F}$ using Φ always exist.

As δ decreases, the set of admissible δ-covers using Φ decreases. Hence $\mathcal{H}_\delta^s(\mathcal{F}, \Phi)$ increases.

► **Definition 16.** *For* $s \in (0, \infty)$, *define the* s-dimensional Φ outer measure *of* $\mathcal{F}$ *as:*

$$\mathcal{H}^s(\mathcal{F}, \Phi) = \lim_{\delta\to 0} \mathcal{H}_\delta^s(\mathcal{F}, \Phi).$$

Observe that as with the case of classical Hausdorff dimension, for any $t > s$, if $\mathcal{H}^s(\mathcal{F}, \Phi) < \infty$, then $\mathcal{H}^t(\mathcal{F}, \Phi) = 0$ (see Section 2.2 in [6]).

Finally, we have the following definition of Hausdorff Φ-dimension.

► **Definition 17.** *For any* $\mathcal{F} \subset \mathbb{X}$, *the* Hausdorff Φ-dimension *of* $\mathcal{F}$ *is defined as:*

$$\dim_\Phi(\mathcal{F}) = \inf\{s \geq 0 : \mathcal{H}^s(\mathcal{F}, \Phi) = 0\}.$$

3.3 Effective Φ-dimension

We first formulate the notion of a Φ-s-supergale. A Φ-s-supergale can be seen as a gambling strategy where the bets are placed on the covering sets from Φ. The definitions in this subsection are adaptations from Mayordomo [22].

▶ **Definition 18** (Mayordomo [22]). *Let $\Phi = \bigcup_{n\in\mathbb{Z}}\{U_i^n\}_{i\in\mathbb{N}}$ be a family of covering sets from Definition 12. For $s \in [0,\infty)$, a Φ-s-supergale is a function $d : \Phi \to [0,\infty)$ such that:*

- $\sum_{U\in\{U_i^0\}_{i\in\mathbb{N}}} d(U)|U|^s < \infty$ *and*
- *For all $n \in \mathbb{N}$ and all $U \in \{U_i^n\}_{i\in\mathbb{N}}$, the following condition holds:*

$$d(U).|U|^s \geq \sum_{V\in\{U_i^{n+1}\}_{i\in\mathbb{N}}, V\subseteq U} d(V)|V|^s \,.$$

The following is the generalization of Kraft inequality for s-supergales from Mayordomo [22].

▶ **Lemma 19** (Generalisation of Kraft inequality [22]). *Let d be a Φ-s-supergale. Then for every $\mathcal{E} \subseteq \Phi$ such that the sets in $\mathcal{E}$ are incomparable, we have that*

$$\sum_{V\in\mathcal{E}} d(V)|V|^s \leq \sum_{U\in\{U_i^0\}_{i\in\mathbb{N}}} d(U)|U|^s.$$

▶ **Definition 20** (Mayordomo [22]). *Given $x \in \mathbb{X}$, a Φ-representation of x is a sequence $(U_n)_{n\in\mathbb{Z}}$ such that for each $n \in \mathbb{Z}$, $U_n \in \{U_i^n\}_{i\in\mathbb{N}}$ and $x \in \cap_n U_n$.*

Note that the same x can have multiple Φ-representations. Given $x \in \mathbb{X}$, let $\mathcal{R}(x)$ be the set of Φ-representations of x.

▶ **Definition 21** (Mayordomo [22]). *A Φ-s-supergale d succeeds on $x \in \mathbb{X}$ if there is a $(U_n)_{n\in\mathbb{Z}} \in \mathcal{R}(x)$ such that $\limsup_{n\to\infty} d(U_n) = \infty$.*

Equivalently, a Φ-s-supergale d succeeds on a point $x \in \mathbb{X}$ iff for every $k \in \mathbb{N}$, there exists a $U \in \Phi$ such that $x \in U$ and $d(U) > 2^k$.

▶ **Definition 22.** *The* success set *of d is $S^\infty(d) = \{x \in \mathbb{X} \mid d \text{ succeeds on } x\}$.*

To define constructive Φ-dimension, we require some additional computability restrictions over Φ. It is an adaptation of the definition from [22].

▶ **Definition 23** (Family of computable covering sets Φ). *We consider the space $\mathbb{X} \subseteq \mathbb{R}^\eta$ where $\eta \in \mathbb{N}$ and a family of covering sets $\Phi = \bigcup_{n\in\mathbb{Z}}\{U_i^n\}_{i\in\mathbb{N}}$ from Definition 12. We call Φ to be a family of computable covering sets if it satisfies the following additional properties:*

- *Computable diameter: For every $n \in \mathbb{Z}$ and $i \in \mathbb{N}$, $|U_i^n|$ is computable.*
- *Computable subsets: For every $n \in \mathbb{Z}$, and $i \in \mathbb{N}$, the set $\{j \in \mathbb{N} : U_j^{n+1} \subseteq U_i^n\}$ is uniformly computable.*

In definition 23, when we say $|U_i^n|$ is computable, we mean that there is a turing machine that on input n, i, r outputs a $q \in Q$ such that $||U_i^n| - q| < 2^r$. The set $\{j \in \mathbb{N} : U_j^{n+1} \subseteq U_i^n\}$ is uniformly computable if there is a turing machine which on input i, j, n decides if $U_j^{n+1} \subseteq U_i^n$.

We use constructive Φ-s-gales to define the notion of constructive Φ-dimension. For a Φ-s-gale d to be constructive, we require the gale function d to be lower semicomputable. Note that a lower semicomputable supergale actually takes as input (i, n) where $i \in \mathbb{N}, n \in \mathbb{Z}$ to place bets on U_i^n. We omit this technicality in this paper and keep the domain of the gale as Φ for the sake of simplicity.

▶ **Definition 24.** *A function* $d : \Phi \longrightarrow [0, \infty)$ *is called* lower semicomputable *if there exists a total computable function* $\hat{d} : \Phi \times \mathbb{N} \longrightarrow \mathbb{Q} \cap [0, \infty)$ *such that the following two conditions hold.*

- ***Monotonicity*** *: For all* $U \in \Phi$ *and for all* $n \in \mathbb{N}$*, we have* $\hat{d}(U, n) \leq \hat{d}(U, n+1) \leq d(U)$*.*
- ***Convergence*** *: For all* $U \in \Phi$*,* $\lim_{n\to\infty} \hat{d}(U, n) = d(U)$*.*

For $\mathcal{F} \subseteq \mathbb{X}$, let $\mathcal{G}_\Phi(\mathcal{F})$ denote the set of all $s \in [0, \infty)$ such that there exists a lower semicomputable Φ-s-supergale d with $\mathcal{F} \subseteq S^\infty(d)$.

▶ **Definition 25.** *The* constructive Φ-dimension *of* $\mathcal{F} \subseteq \mathbb{X}$ *is*

$$\mathrm{cdim}_\Phi(\mathcal{F}) = \inf \mathcal{G}_\Phi(\mathcal{F}).$$

The constructive Φ dimension *of a point* $x \in \mathbb{X}$ *is defined by* $\mathrm{cdim}_\Phi(\{x\})$*, the constructive* Φ*-dimension of the singleton set containing* x*.*

This definition can easily be relativized with respect to an oracle $A \subseteq \mathbb{N}$ by giving the s-supergale an additional oracle access to set $A \subseteq \mathbb{N}$. We denote this using $\mathrm{cdim}^A_\Phi(\mathcal{F})$.

We now show that the constructive Φ-dimension of a set is the supremum of constructive Φ-dimensions of points in the set. The proof is a straightforward adaptation of proof of Theorem 11 by Lutz [11].

▶ **Theorem 26.** *For any family of computable covering sets* Φ *defined over the space* $\mathbb{X}$*, for any* $\mathcal{F} \subseteq \mathbb{X}$*, we have*

$$\mathrm{cdim}_\Phi(\mathcal{F}) = \sup_{x\in\mathcal{F}} \mathrm{cdim}_\Phi(x).$$

4 Point-to-set principle for Φ-dimension

Let $\Phi = \bigcup_{n\in\mathbb{Z}} \{U^n_i\}_{i\in\mathbb{N}}$ be a family of computable covering sets from Definition 23. In this work, we introduce the Point-to-Set principle for Φ-dimension. We show that the Hausdorff Φ-dimension of any set $\mathcal{F} \subseteq \mathbb{X}$ is equal to the relative constructive Φ-dimensions of elements in the set, relative to a minimizing oracle A.

▶ **Theorem 27.** *For a family of computable covering sets* Φ *over the space* $\mathbb{X}$*, for all* $\mathcal{F} \subseteq \mathbb{X}$*,*

$$\dim_\Phi(\mathcal{F}) = \min_{A\subseteq\mathbb{N}} \sup_{x\in\mathcal{F}} \mathrm{cdim}^A_\Phi(x).$$

4.1 Point to Set Principle for constructive dimension

▶ **Definition 28** (Dyadic Family of covers)**.** *Consider the space* $\mathbb{X} = \mathbb{R}^n$*. The dyadic family of covers is the set of coverings* $\Phi_B = \bigcup_{r\in\mathbb{N}} \{[\frac{m_1}{2^r}, \frac{m_1+1}{2^r}] \times \cdots \times [\frac{m_n}{2^r}, \frac{m_n+1}{2^r}]\}_{m_1, m_2 \ldots, m_n \in [2^r]}$*.*

It is straightforward to verify that Φ_B is a family of computable covering sets from Definition 23. Besicovitch [3] gave the following Φ-dimension characterization of Hausdorff dimension.

▶ **Lemma 29** (Besicovitch [3])**.** *For all* $\mathcal{F} \subseteq \mathbb{R}^n$*, we have* $\dim(\mathcal{F}) = \dim_{\Phi_B}(\mathcal{F})$*.*

Similarly, we have the following Φ-dimension characterization of Constructive dimension.

▶ **Lemma 30** (Lutz and Mayordomo [16])**.** *For all* $\mathcal{F} \subseteq \mathbb{R}^n$*, we have* $\mathrm{cdim}(\mathcal{F}) = \mathrm{cdim}_{\Phi_B}(\mathcal{F})$*.*

From Theorem 27 for Φ_B and using Lemma 29 and 30, we have the following point-to-set principle from [13] relating Hausdorff and Constructive dimensions.

▶ **Corollary 31** (J.Lutz and N.Lutz [13]). *For all $\mathcal{F} \subseteq \mathbb{X}$,*

$$\dim(\mathcal{F}) = \min_{A \subseteq \mathbb{N}} \sup_{x \in \mathcal{F}} \operatorname{cdim}^A(x).$$

4.2 Point to Set Principle for Continued Fraction dimension

The sequence $Y = [a_1, a_2, \dots]$ where each $a_i \in \mathbb{N}$ is the continued fraction expansion of the number $y = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$. Given $u = [a_1, a_2 \dots a_n] \in \mathbb{N}^*$, the cylinder set of u, C_u is defined as $C_u = [[a_1, a_2, \dots a_n], [a_1, a_2, \dots a_n + 1]]$ when n is even and $C_u = [[a_1, a_2, \dots a_n + 1], [a_1, a_2, \dots a_n]]$ when n is odd.

The notion of constructive continued fraction dimension was introduced by Nandakumar and Vishnoi [26] using continued fraction s-gales. Akhil, Nandakumar and Vishnoi [24] showed that this notion is different from that of constructive dimension.

Consider Φ_{CF} to be the set of covers generated by the continued fraction cylinders, that is $\Phi_{CF} = \bigcup_{n \in \mathbb{Z}} \{C_{[a_1, a_2, \dots a_n]}\}_{a_1 \dots a_n \in \mathbb{N}}$. It is routine to verify that this is a family of computable covering sets from Definition 23. From Theorem 27, we therefore have the following point-to-set principle for Continued fraction dimension.

▶ **Corollary 32.** *For all $\mathcal{F} \subseteq \mathbb{X}$, $\dim_{CF}(\mathcal{F}) = \min_{A \subseteq \mathbb{N}} \sup_{x \in \mathcal{F}} \operatorname{cdim}^A_{CF}(x)$.*

4.3 Effective Φ-dimension using Kolmogorov Complexity

We give an equivalent formulation of constructive Φ-dimension of a point using Kolmogorov complexity. For this, we require some additional properties for the space Φ.

▶ **Definition 33** (Family of finitely intersecting computable covering sets Φ). *We consider the space $\mathbb{X} \subseteq \mathbb{R}^\eta$ where $\eta \in \mathbb{N}$ and a family of computable covering sets $\Phi = \bigcup_{n \in \mathbb{Z}} \{U_i^n\}_{i \in \mathbb{N}}$ from Definition 23. We say that Φ is a family of finitely intersecting computable covering sets if it satisfies the following additional properties:*

- *Density of Rational points: For each $U \in \Phi$, there exists a $q \in \mathbb{Q}^n$ such that $q \in U$.*
- *Finite intersection: There exists a constant $c \in \mathbb{N}$ such that for any collection $\{U_i\} \subseteq \Phi$ satisfying*
  (1) *$U_i \not\subseteq U_j$ for all $i \neq j$, and*
  (2) *$\bigcap_i U_i \neq \emptyset$,*
  we have $\#(\{U_i\}) \leq c$.
- *Membership test: There is a computable function that given $i \in \mathbb{N}$ and $n \in \mathbb{Z}$ and a $q \in \mathbb{Q}^n$ checks if $q \in U_i^n$.*

The Finite intersection property states that the cardinality of any collection of incomparable sets from Φ having non empty intersection is bounded by a constant.

Given family of finitely intersecting computable covering sets Φ over a space $\mathbb{X}$, and an $r \in \mathbb{N}$, we define the notion of Kolmogorov Complexity of a point X at precision r with respect to Φ. We denote this using $K_r(X, \Phi)$.

▶ **Definition 34.** *Given an $r \in \mathbb{N}$, and $x \in \mathbb{X}$, define*

$$K_r(x, \Phi) = \min_{U \in \Phi}\{K(U) \mid x \in U \text{ and } |U| < 2^{-r}\}.$$

where for $U \in \Phi$, $K(U)$ is defined as $K(U) = \min\{K(q) \mid q \in U \cap \mathbb{Q}^n\}$.

For any family of finitely intersecting computable covering sets Φ, we have the following Kolmogorov Complexity characterization of constructive Φ-dimension.

▶ **Theorem 35.** *Given a family of finitely intersecting computable covering sets Φ, over a space $\mathbb{X}$. For any $x \in \mathbb{X}$,*

$$\mathrm{cdim}_\Phi(x) = \liminf_{r \to \infty} \frac{K_r(x, \Phi)}{r}$$

5 Kolmogorov Complexity Construction

In this section we give a technical construction which is crucial in proving the results in section 6. Theorem 36 says that given an infinite sequence X and an oracle A, for any oracle B, there exists a sequence Y whose relativized Kolmogorov complexity (of prefixes) with respect to B is similar to the relativized Kolmogorov complexity (of prefixes) of X with respect to A.

▶ **Theorem 36.** *For all $X \in \Sigma^\infty$ and $A \in \Sigma^\infty$, given a $B \in \Sigma^\infty$, there exists a $Y \in \Sigma^\infty$ such that for all $n \in \mathbb{N}$, $|K^A(X \upharpoonright n) - K^B(Y \upharpoonright n)| = o(n)$ and $\mathrm{cdim}^B(Y) = \mathrm{cdim}(Y)$.*

6 Equivalence of Faithfulness of Cantor Coverings at Constructive and Hausdorff Levels

In this section, we show that when the class of covers Φ is generated by computable Cantor series expansions, the faithfulness at the Hausdorff and constructive levels are equivalent notions.

6.1 Faithfulness of Family of Coverings

We will first see the definition of Hausdorff dimension faithfulness. We then introduce the corresponding notion at the effective level, which we call constructive dimension faithfulness.

A family of covering sets Φ is said to be *faithful* with respect to Hausdorff dimension if the Φ dimension of every set in the space is the same as its Hausdorff dimension.

▶ **Definition 37.** *A family of covering sets Φ over the space $\mathbb{X}$ is said to be* faithful with respect to Hausdorff dimension *if for all $\mathcal{F} \subseteq \mathbb{X}$, $\dim_\Phi(\mathcal{F}) = \dim(\mathcal{F})$.*

We extend the definition to the constructive level as well. A family of computable covering sets Φ is defined to be *faithful* with respect to constructive dimension if the constructive Φ dimension of every set is the same as its constructive dimension.

▶ **Definition 38.** *A family of computable covering sets Φ is said to be* faithful with respect to constructive dimension *if for all $\mathcal{F} \subseteq \mathbb{X}$, $\mathrm{cdim}_\Phi(\mathcal{F}) = \mathrm{cdim}(\mathcal{F})$.*

The following lemma follows from Theorem 26 and Lemma 11. It states that constructive dimension faithfulness can be equivalently stated in terms of preservation of constructive dimensions of points in the set.

▶ **Lemma 39.** *A constructive family of covers Φ is faithful with respect to Constructive dimension if and only if for all $x \in \mathbb{X}$, $\mathrm{cdim}_\Phi(x) = \mathrm{cdim}(x)$.*

The following lemma states that the Φ-dimension of a set is always greater than or equal to its Hausdorff dimension. Similarly, the constructive $\Phi-$dimension of a set is always greater than or equal to its constructive dimension.

▶ **Lemma 40.** *For any family of covering sets Φ over $\mathbb{X}$, for all $\mathcal{F} \subseteq \mathbb{X}$, $\dim_\Phi(\mathcal{F}) \geq \dim(\mathcal{F})$.*

▶ **Lemma 41.** *For any family of finitely intersecting computable covering sets Φ over $\mathbb{X}$, for all $\mathcal{F} \subseteq \mathbb{X}$, $\mathrm{cdim}_\Phi(\mathcal{F}) \geq \mathrm{cdim}(\mathcal{F})$.*

Therefore we have that Φ is not faithful for Hausdorff dimension if and only if there exists an $\mathcal{F} \subset \mathbb{X}$ such that $\dim_\Phi(\mathcal{F}) > \dim(\mathcal{F})$. Similarly, Φ is not faithful for Constructive dimension if and only if there exists an $\mathcal{F} \subset \mathbb{X}$ such that $\mathrm{cdim}_\Phi(\mathcal{F}) > \mathrm{cdim}(\mathcal{F})$.

6.2 Cantor coverings over unit interval

We consider the faithfulness of family of coverings generated by the computable Cantor series expansion [4]. We call such class of coverings as *Cantor coverings.*

Given a sequence $Q = \{n_k\}_{k\in\mathbb{N}}$ with $n_k \in \mathbb{N} \setminus \{1\}$, the expression

$$x = \sum_{k=1}^{\infty} \frac{\alpha_k}{n_1.n_2 \dots n_k}$$

where $\alpha_k \in [n_k]$ is called the cantor series expansion of the real number $x \in [0, 1]$.

▶ **Definition 42** (Cantor Coverings Φ_Q). *The class of Cantor coverings Φ_Q over the space $\mathbb{X} = [0, 1]$ generated by the Cantor series expansion $Q = \{n_k\}_{k\in\mathbb{N}}$ is the set of intervals*

$$\bigcup_{k\in\mathbb{Z}} \{[\frac{m}{n_0.n_1.n_2 \dots n_k}, \frac{m+1}{n_0.n_1.n_2 \dots n_k}]\}_{m\in[n_0.n_1.n_2 \dots n_k]}$$

with n_0 taken as 1.

▶ **Definition 43** (Computable Cantor Coverings). *The cantor series expansion $Q = \{n_k\}_{k\in\mathbb{N}}$ is said to be computable if there exists a machine that generates n_k given k. We call the class of Cantor coverings Φ_Q generated by a computable Cantor series expansion Q as a class of Computable Cantor Coverings over $\mathbb{X} = [0, 1]$.*

It is routine to verify that for any computable Cantor series expansion $Q = \{n_k\}_{k\in\mathbb{N}}$, the Cantor covering Φ_Q is a family of finitely intersecting computable covering sets from Definition 33. Therefore, from Theorem 27, we have the following Point-to-Set principle for Cantor covering dimension.

▶ **Corollary 44.** *For all $\mathcal{F} \subseteq \mathbb{X}$ and for all computable Cantor coverings Φ_Q,*

$$\dim_{\Phi_Q}(\mathcal{F}) = \min_{A\subseteq\mathbb{N}} \sup_{x\in\mathcal{F}} \mathrm{cdim}^A_{\Phi_Q}(x).$$

6.3 Kolmogorov Complexity Characterization of Cantor Series Dimension

We first show a Kolmogorov complexity characterization of constructive Φ-dimension for computable Cantor coverings.

▶ **Theorem 45.** *For any $x \in \mathbb{X}$, and any computable Cantor coverings Φ_Q generated by $Q = \{n_k\}_{k \in \mathbb{N}}$,*

$$\mathrm{cdim}_{\Phi_Q}(x) = \liminf_{k \to \infty} \frac{K(X \upharpoonright m_k)}{m_k}$$

where X is a binary expansion of x and $m_k = \lfloor \log_2(n_1.n_2 \dots n_k) \rfloor$.

Theorem 8 ensures that when the Kolmogorov complexities of any two $X, Y \in \Sigma^\infty$ align over all finite prefixes, their constructive dimensions become equal. From Theorem 45, we get that when this happens, the constructive Φ-dimensions also become equal.

▶ **Lemma 46.** *For any $x, y \in \mathbb{X}$, $A, B \subseteq \mathbb{N}$ and any class of computable Cantor coverings Φ, if for all n, $|K^A(X \upharpoonright n) - K^B(Y \upharpoonright n)| = o(n)$, then $\mathrm{cdim}^A(x) = \mathrm{cdim}^B(y)$ and $\mathrm{cdim}^A_\Phi(x) = \mathrm{cdim}^B_\Phi(y)$. Here X and Y are the binary expansions of x and y respectively.*

6.4 Faithfulness of Cantor Coverings

Using the Point-to-Set Principle and properties of Kolmogorov complexity, we show that the notions of faithfulness for Cantor coverings at Hausdorff and Constructive levels are equivalent.

We first show that if a class of computable Cantor coverings Φ is faithful with respect to constructive dimension, then Φ is also faithful with respect to Hausdorff dimension.

▶ **Lemma 47.** *For any class of computable Cantor coverings Φ, if for all $\mathcal{F} \subseteq \mathbb{X}$, $\mathrm{cdim}(\mathcal{F}) = \mathrm{cdim}_\Phi(\mathcal{F})$, then for all $\mathcal{F} \subseteq \mathbb{X}$, $\dim(\mathcal{F}) = \dim_\Phi(\mathcal{F})$.*

To prove the converse, we require the construction of set $\mathcal{I}_s$ that contains all points in $\mathbb{X}$ having constructive dimension equal to s.

▶ **Definition 48.** *Given $s \in [0, \infty)$, define $\mathcal{I}_s = \{x \in \mathbb{X} \mid \mathrm{cdim}(x) = s\}$.*

Lutz and Weihrauch [17] showed that the Hausdorff dimension of $\mathcal{I}_s$ is equal to s. We provide a simple alternate proof of this using the point-to-set principle.

▶ **Lemma 49** (Lutz and Weihrauch [17])**.** $\dim(\mathcal{I}_s) = s$.

We now show that if a class of computable Cantor coverings Φ is faithful with respect to Hausdorff dimension, then Φ is also faithful with respect to constructive dimension.

▶ **Lemma 50.** *For any class of computable Cantor coverings Φ, if for all $\mathcal{F} \subseteq \mathbb{X}$, $\dim(\mathcal{F}) = \dim_\Phi(\mathcal{F})$, then for all $\mathcal{F} \subseteq \mathbb{X}$, $\mathrm{cdim}(\mathcal{F}) = \mathrm{cdim}_\Phi(\mathcal{F})$.*

Therefore, we have the following theorem which states that for the classes of Cantor coverings Φ, faithfulness with respect to Hausdorff and Constructive dimensions are equivalent notions.

▶ **Theorem 51.** *For any class of computable Cantor coverings Φ,*

$$\forall \mathcal{F} \subseteq \mathbb{X}\,;\ \dim(\mathcal{F}) = \dim_\Phi(\mathcal{F}) \iff \forall \mathcal{F} \subseteq \mathbb{X}\,;\ \mathrm{cdim}(\mathcal{F}) = \mathrm{cdim}_\Phi(\mathcal{F}).$$

6.5 Log limit condition for faithfulness

Albeverio, Ivanenko, Lebid and Torbin [1] showed that the Cantor coverings Φ_Q generated by the Cantor series expansions of $Q = \{n_k\}_{k\in\mathbb{N}}$ can be faithful as well as non faithful with respect to Hausdorff dimension depending on Q. Interestingly, they showed that the Hausdorff dimension faithfulness of Cantor coverings can be determined using the terms n_k in Q.

▶ **Theorem 52** (Albeverio, Ivanenko, Lebid and Torbin [1]). *A family of Cantor coverings Φ_Q generated by $Q = \{n_k\}_{k\in\mathbb{N}}$ is faithful with respect to Hausdorff dimension if and only if* $\lim\limits_{k\to\infty} \frac{\log n_k}{\log n_1 \cdot n_2 \ldots n_{k-1}} = 0$.

Using the above result and the result that faithfulness at the constructive level is equivalent to faithfulness with respect to Hausdorff dimension (Theorem 51), we have that the condition stated above provides the necessary and sufficient conditions for Cantor series coverings to be faithful for constructive dimension.

▶ **Theorem 53.** *A family of Cantor coverings Φ_Q generated by $Q = \{n_k\}_{k\in\mathbb{N}}$ is faithful with respect to constructive dimension if and only if*

$$\lim_{k\to\infty} \frac{\log n_k}{\log n_1.n_2 \ldots n_{k-1}} = 0. \tag{1}$$

The Cantor series expansion is a generalization of the base-b representation, which is the special case when $n_k = b$ for all $k \in \mathbb{N}$. That is $Q_b = \{b\}_{n\in\mathbb{N}}$. Since the condition in Theorem 53 is satisfied by Q_b for any $b \in \mathbb{N}$, we have the following result by Hitchcock and Mayordomo about the base invariance of constructive dimension.

▶ **Corollary 54** (Hitchcock and Mayordomo [8]). *For any $x \in [0, 1]$ and $k, l \in \mathbb{N} \setminus \{1\}$, $\mathrm{cdim}_{(k)}(x) = \mathrm{cdim}_{(l)}(x)$. where $\mathrm{cdim}_{(k)}(x)$ represents the constructive dimension of x with respect to its base-k representation.*

Note that condition (1) classifies the Cantor series expansions on the basis of constructive dimension faithfulness. As an example, when $n_k = 2^k$, condition (1) holds, and therefore $Q = \{2^k\}_{k\in\mathbb{N}}$ is faithful for constructive dimension. However, when $n_k = 2^{2^k}$, condition (1) does not hold, and therefore $Q = \{2^{2^k}\}_{k\in\mathbb{N}}$ is not faithful for constructive dimension.

7 Conclusion and Open Problems

We develop a constructive analogue of Φ-dimension and prove a Point-to-Set principle for Φ-dimension. Using this, we show that for Cantor series representations, constructive dimension faithfulness and Hausdorff dimension are equivalent notions. We also provide a loglimit condition for faithfulness of Cantor series expansions.

The following are some problems that remain open

1. Are the faithfulness at constructive and Hausdorff levels equivalent for all computable family of covering sets Φ ?
2. What is the packing dimension analogue of faithfulness, is there any relationship between faithfulness of Hausdorff dimension and packing dimension ?
3. Is there any relationship between faithfulness of constructive dimension and constructive strong dimension ?

References

1 S. Albeverio, Ganna Ivanenko, Mykola Lebid, and Grygoriy Torbin. On the Hausdorff dimension faithfulness and the Cantor series expansion. *Methods of Functional Analysis and Topology*, 26(4):298–310, 2020.

2 Sergio Albeverio and Grygoriy Torbin. Fractal properties of singular probability distributions with independent Q^*-digits. *Bull. Sci. Math.*, 129(4):356–367, 2005. doi:10.1016/j.bulsci.2004.12.001.

3 A. S. Besicovitch. On existence of subsets of finite measure of sets of infinite measure. *Indag. Math.*, 14:339–344, 1952. Nederl. Akad. Wetensch. Proc. Ser. A **55**.

4 George Cantor. Uber die einfachen zahlensysteme, zeit. für math. 14 (1869), 121-128. *Coll. Papers, Berlin*, pages 35–53, 1932.

5 Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic randomness and complexity*. Theory and Applications of Computability. Springer, New York, 2010. doi:10.1007/978-0-387-68441-3.

6 Kenneth Falconer. *Fractal geometry*. John Wiley & Sons, Inc., Hoboken, NJ, second edition, 2003. Mathematical foundations and applications. doi:10.1002/0470013850.

7 F. Hausdorff. Dimension und äusseres Mass. *Mathematische Annalen*, 79:157–179, 1919.

8 John M. Hitchcock and Elvira Mayordomo. Base invariance of feasible dimension. *Inform. Process. Lett.*, 113(14-16):546–551, 2013. doi:10.1016/j.ipl.2013.04.004.

9 M. Kh. Ībragīm and G. M. Torbīn. On a probabilistic approach to the DP-transformations and faithfulness of covering systems for calculating the Hausdorff-Besicovitch dimension. *Teor. Ĭmovīr. Mat. Stat.*, 92:28–40, 2015. doi:10.1090/tpms/980.

10 Ming Li and Paul Vitányi. *An introduction to Kolmogorov complexity and its applications*. Texts in Computer Science. Springer, New York, third edition, 2008. doi:10.1007/978-0-387-49820-1.

11 J. H. Lutz. Dimensions of individual strings and sequences. *Information and Computation*, 187(1):49–79, 2003.

12 Jack H. Lutz. Dimension in complexity classes. *SIAM J. Comput.*, 32(5):1236–1259, 2003. doi:10.1137/S0097539701417723.

13 Jack H. Lutz and Neil Lutz. Algorithmic information, plane kakeya sets, and conditional dimension. *ACM Trans. Comput. Theory*, 10(2):7:1–7:22, 2018. doi:10.1145/3201783.

14 Jack H. Lutz and Neil Lutz. Who asked us? how the theory of computing answers questions about analysis. In Ding-Zhu Du and Jie Wang, editors, *Complexity and Approximation - In Memory of Ker-I Ko*, volume 12000 of *Lecture Notes in Computer Science*, pages 48–56. Springer, 2020. doi:10.1007/978-3-030-41672-0_4.

15 Jack H. Lutz, Neil Lutz, and Elvira Mayordomo. Extending the reach of the point-to-set principle. In Petra Berenbrink and Benjamin Monmege, editors, *39th International Symposium on Theoretical Aspects of Computer Science, STACS 2022, March 15-18, 2022, Marseille, France (Virtual Conference)*, volume 219 of *LIPIcs*, pages 48:1–48:14. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.STACS.2022.48.

16 Jack H. Lutz and Elvira Mayordomo. Dimensions of points in self-similar fractals. *SIAM Journal on Computing*, 38:1080–1112, 2008.

17 Jack H. Lutz and Klaus Weihrauch. Connectivity properties of dimension level sets. *MLQ Math. Log. Q.*, 54(5):483–491, 2008. doi:10.1002/malq.200710060.

18 Neil Lutz. Fractal intersections and products via algorithmic dimension. *ACM Trans. Comput. Theory*, 13(3):Art. 14, 15, 2021. doi:10.1145/3460948.

19 Neil Lutz and D. M. Stull. Bounding the dimension of points on a line. *Inform. and Comput.*, 275:104601, 15, 2020. doi:10.1016/j.ic.2020.104601.

20 Neil Lutz and Donald M. Stull. Projection theorems using effective dimension. In *43rd International Symposium on Mathematical Foundations of Computer Science*, volume 117 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 71, 15. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018.

21 Elvira Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Inform. Process. Lett.*, 84(1):1–3, 2002. doi:10.1016/S0020-0190(02)00343-5.

22 Elvira Mayordomo. Effective hausdorff dimension in general metric spaces. *Theory Comput. Syst.*, 62(7):1620–1636, 2018. doi:10.1007/s00224-018-9848-3.

23 Satyadev Nandakumar. An effective ergodic theorem and some applications. In *STOC'08*, pages 39–44. ACM, New York, 2008. doi:10.1145/1374376.1374383.

24 Satyadev Nandakumar, Akhil S, and Prateek Vishnoi. Effective continued fraction dimension versus effective Hausdorff dimension of reals. In *48th International Symposium on Mathematical Foundations of Computer Science*, volume 272 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Paper No. 70, 15. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2023. doi:10.4230/lipics.mfcs.2023.70.

25 Satyadev Nandakumar and Prateek Vishnoi. Randomness and effective dimension of continued fractions. In *45th International Symposium on Mathematical Foundations of Computer Science*, volume 170 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 73, 13. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2020.

26 Satyadev Nandakumar and Prateek Vishnoi. On continued fraction randomness and normality. *Information and Computation*, 285(part B):104876, 2022. doi:10.1016/j.ic.2022.104876.

27 André Nies. *Computability and randomness*, volume 51. OUP Oxford, 2009.

28 Yuval Peres and Gyorgiy Torbin. Continued fractions and dimensional gaps. In preparation.

29 C. A. Rogers. *Hausdorff measures*. Cambridge University Press, London-New York, 1970.

30 Alexander Shen, Vladimir A Uspensky, and Nikolay Vereshchagin. *Kolmogorov complexity and algorithmic randomness*, volume 220. American Mathematical Society, 2022.

31 Ludwig Staiger. The Kolmogorov complexity of real numbers. *Theor. Comput. Sci.*, 284(2):455–466, 2002. doi:10.1016/S0304-3975(01)00102-5.