

Sparse Outerstring Graphs Have Logarithmic Treewidth

Shinwoo An ✉

Pohang University of Science and Technology, South Korea

Eunjin Oh ✉

Pohang University of Science and Technology, South Korea

Jie Xue ✉

New York University Shanghai, China

Abstract

An outerstring graph is the intersection graph of curves lying inside a disk with one endpoint on the boundary of the disk. We show that an outerstring graph with n vertices has treewidth $O(\alpha \log n)$, where α denotes the arboricity of the graph, with an almost matching lower bound of $\Omega(\alpha \log(n/\alpha))$. As a corollary, we show that a t -biclique-free outerstring graph has treewidth $O(t(\log t) \log n)$. This leads to polynomial-time algorithms for most of the central NP-complete problems such as INDEPENDENT SET, VERTEX COVER, DOMINATING SET, FEEDBACK VERTEX SET, COLORING for sparse outerstring graphs. Also, we can obtain subexponential-time (exact, parameterized, and approximation) algorithms for various NP-complete problems such as VERTEX COVER, FEEDBACK VERTEX SET and CYCLE PACKING for (not necessarily sparse) outerstring graphs.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases Outerstring graphs, geometric intersection graphs, treewidth

Digital Object Identifier 10.4230/LIPIcs.ESA.2024.10

Related Version *Full Version*: <https://arxiv.org/abs/2406.17424>

Funding The work by An and Oh was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. RS-2024-00358505).

1 Introduction

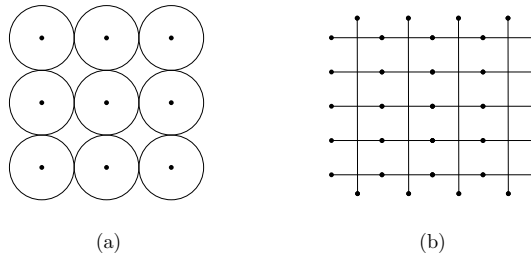
The *intersection graph* of a family \mathcal{F} of geometric objects is the graph $G = (V, E)$ such that every vertex of G corresponds to an object of \mathcal{F} , and two vertices of G are connected by an edge if and only if their corresponding objects intersect. In this case, \mathcal{F} is called a *geometric representation* of G . Notice that a geometric representation of an intersection graph is not necessarily unique. There are several popular classes of intersection graphs such as string graphs, unit disk graphs, and disk graphs. In the case that \mathcal{F} is a family of curves in the plane, its intersection graph is called a *string graph*. Similarly, the intersection graph of a family of unit disks (or disk graphs) is called a *unit disk graph* (or a *disk graph*). Notice that a unit disk graph is a disk graph, and a disk graph is a string graph¹.

Geometric intersection graphs have been studied extensively as early as in the 1960s, motivated by the connection with integrated RC circuits [19, 44]. Most NP-complete problems on general graphs remain NP-complete in geometric intersection graphs (or even in unit disk graphs). However, recently, it is known that lots of NP-complete problems can be solved in subexponential time on geometric intersection graphs [3, 4, 5, 9, 16, 17, 30, 36, 40]. For instance, VERTEX COVER, 3-COLORING and FEEDBACK VERTEX SET can be solved

¹ We can represent each disk as a densely spiral-shaped string that covers the interior of the disk.



10:2 Sparse Outerstring Graphs Have Logarithmic Treewidth



■ **Figure 1** (a) A $\sqrt{n} \times \sqrt{n}$ grid is a unit disk graph of treewidth $\Theta(\sqrt{n})$. (b) A sparse axis-parallel segment graph of treewidth $\Theta(\sqrt{n})$. It does not contain $K_{2,2}$ as a subgraph. The horizontal segments form \sqrt{n} rows, and each row consists of $\Theta(\sqrt{n})$ horizontal segments.

in $2^{O(n^{2/3} \text{polylog } n)}$ time for string graphs with n vertices [9]. In the case of disk graphs and unit disk graphs, one can obtain even stronger results. There are subexponential-time *parameterized* algorithms for numerous NP-complete problems in this case. For instance, one can solve VERTEX COVER and FEEDBACK VERTEX SET in $2^{O(\sqrt{k})} n^{O(1)}$ time for unit disk graphs [4, 16], and in $2^{O(k^c)} n^{O(1)}$ time for disk graphs [3, 36] for some constant $c < 1$, where k denotes the output size.

All of the algorithms mentioned above use the fact that string graphs have treewidth sublinear in the number of vertices if they do not have a large clique (or biclique).² For a string graph G with n vertices that does not contain a biclique of size t has a balanced separator of size $O(\sqrt{t(\log t)n})$ [35], and thus its treewidth is $O(\sqrt{t(\log t)n})$. For most central NP-complete problems, there are $2^{O(tw)} n^{O(1)}$ -time algorithms for any graphs of treewidth tw . For subexponential-time algorithms for string graphs given in [9], the authors produce several instances of *sparse* string graphs using the branching technique. Then they apply $2^{O(tw)} n^{O(1)}$ -time algorithms to the instances. The subexponential-time parameterized algorithms for unit disk graphs and disk graphs mentioned above use similar approaches. Using branching, they produce several instances of *sparse* disk graphs. In this case, using the sparsity and the geometric properties of disk graphs, they show that the treewidth of the disk graph depends only on the parameter and the measure for the sparsity of the graph.

Motivated by these algorithmic applications, an optimal bound on the treewidth of a sparse string graph has been studied extensively for the last two decades [22, 35, 37, 38]. A t -biclique-subgraph-free ($K_{t,t}$ -free) string graph with n vertices has treewidth $O_t(\sqrt{n})$, while a general (not necessarily sparse) string graph with m edges has treewidth $O(\sqrt{m})$. These bounds are tight for string graphs [35]. One might hope to obtain a better bound for a special subclass of string graphs, which might lead to faster algorithms for central NP-hard problems. However, this bound is tight even for sparse unit disk graphs and the intersection graph of axis-parallel segments. In particular, a $\sqrt{n} \times \sqrt{n}$ grid is a unit disk graph of treewidth $\Theta(\sqrt{n})$. Also, there is an axis-parallel segment graph of treewidth $\Theta(\sqrt{n})$ which does not contain a $K_{2,2}$ as a subgraph. See Figure 1.

In this paper, we focus on outerstring graphs. An *outerstring graph* is the intersection graph of curves lying inside a disk with one endpoint on the boundary of the disk. Outerstring graphs have been studied for about 30 years since they were introduced by Kratochvíl [33]. There are numerous works on the combinatorial properties of outerstring graphs [7, 12, 23, 24, 27, 42], and efficient algorithms for outerstring graphs [6, 10, 29]. Basically, RECOGNITION is

² The definition of the treewidth is given in Section 2.

NP-hard for outerstring graphs [33, 39], and all combinatorial properties and algorithms of [6] are applicable to outerstring graphs without explicitly provided geometric representation. On the other hand, the algorithms presented in [10, 29] run in time polynomial in the complexity of a geometric representation. Despite these efforts, we are not able to find any (exact, parameterized, or approximation) polynomial-time algorithms for central NP-hard problems specialized to outerstring graphs.

As the concept of the treewidth is a key to obtaining efficient algorithms for geometric intersection graphs, a natural direction for this problem is to analyze a tight bound on the treewidth of an outerstring graph. Fox and Pach [23] showed that an outerstring graph with m edges has treewidth $O(\min\{\Delta, \sqrt{m}\})$, where Δ denotes the maximum degree of the graph.³ Moreover, they showed that this bound is tight for outerstring graphs as a split graph containing a clique of size $\Theta(m)$ has treewidth $\Theta(m)$, and the Cayley graph with vertex set \mathbb{Z}_n such that any two vertices of cyclic distance at most $\Delta/2$ are adjacent has treewidth $\Theta(\Delta)$. A split graph and such a Cayley graph are all outerstring graphs. Although these examples show that the bound of $O(\min\{\Delta, \sqrt{m}\})$ is tight, these are dense graphs that contain a clique of size $\Theta(\sqrt{m})$. Indeed, what we need for algorithmic applications is a bound on the treewidth of sparse outerstring graphs, for instance, t -biclique-free outerstring graphs. There is still room for the improvement of the bound in the sparse regime.

Our results. In this paper, we show that an outerstring graph G has treewidth $O(\alpha \log n)$, where α is the arboricity of G . The arboricity of G is defined as the maximum average degree of the subgraphs of G . Using the previous structural results about outerstring graphs [24, 35], we show that an outerstring graph which does not contain $K_{t,t}$ as a subgraph has arboricity $O(t \log t)$. Thus our main result implies that a t -biclique-free outerstring graph G has treewidth $O(t(\log t) \log n)$. We emphasize that all of our algorithmic applications mentioned below work even without a geometric representation of an outerstring graph.

First, we can obtain polynomial-time algorithms for any problem that admits a single-exponential-time algorithm parameterized by treewidth, including INDEPENDENT SET, HAMILTONIAN CYCLE, DOMINATING SET, and FEEDBACK VERTEX SET, for t -biclique-free outerstring graphs for a fixed constant t . All algorithms but the algorithm for INDEPENDENT SET are the first polynomial-time algorithms for these problems in t -biclique-free outerstring graphs. Moreover, the algorithm for INDEPENDENT SET is the first polynomial-time *robust* algorithm that does not require the geometric representation of the graph⁴.

Our main result can be used for obtaining various algorithms for (possibly dense) outerstring graphs. We design subexponential-time FPT algorithms for VERTEX COVER and FEEDBACK VERTEX SET on general outerstring graphs work in $2^{O(\sqrt{k} \log^2 k)} n^{O(1)}$ time, where k is the solution size. It is known that VERTEX COVER can be solved in time polynomial in the complexity of a geometric representation of an outerstring graph [6, 10, 29]. But the complexity of a geometric representation of an outerstring graph can be exponential in the number of vertices [7]. For string graphs, VERTEX COVER can be solved in $2^{O(k^{2/3})} n^{O(1)}$ time [9]. On the other hand, FEEDBACK VERTEX SET can be solved in $2^{O(n^{2/3})}$ time for string graphs, and no subexponential-time algorithm parameterized by the solution size was known for this problem on string graphs and outerstring graphs prior to our work.

³ Precisely, they showed that an outerstring has a balanced separator of size $O(\min\{\Delta, \sqrt{m}\})$. By [18], this implies that the treewidth of an outerstring graph is $O(\min\{\Delta, \sqrt{m}\})$.

⁴ Our algorithms for the other problems are also robust. There is an FPT algorithm on INDEPENDENT SET parameterized by the complexity of the geometric representation of an outerstring graph [29].

In addition, we can obtain (non-parameterized) $2^{O(\sqrt{n} \log^2 n)}$ -time algorithms for MAXIMUM INDUCED MATCHING and LIST 3-COLORING on (general) outerstring graphs. These improve the algorithms for these problems running in $2^{O(n^{2/3})}$ time [9]. (But the algorithms in [9] also work for string graphs.) Finally, we can design a 4-approximation algorithm for CYCLE PACKING works in $n^{O(\log \log n)}$ time. Prior to this work, neither approximation nor parameterized algorithms for CYCLE PACKING on outerstring graphs were known. For general graphs, CYCLE PACKING admits an $O(\sqrt{\log n})$ -approximation algorithm [34], and it is quasi-NP-hard to approximate within a factor of $O(\log^{1/2-\varepsilon} n)$ for any $\varepsilon > 0$ [25].

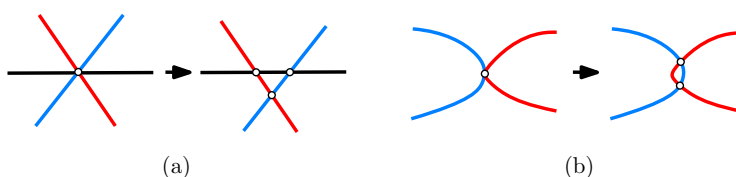
We believe that our result can serve as a starting point for designing efficient algorithms for outerstring graphs. Apart from these applications, we also believe that the main result itself is interesting. Only a few natural classes of graphs such as sparse O_k -free graphs, even-hole-free graphs of bounded degree, even-hole-free graphs of bounded clique number, and (theta, triangle)-free graphs are known to have logarithmic treewidth [1, 2, 8, 13].

Related work. Outerstring graphs were introduced by [33]. The class of outerstring graphs is a broad subclass of string graphs, which includes split graphs (graphs whose vertex set can be decomposed into a clique and an independent set), incomparability graphs (graphs representing the incomparability of elements in a partially ordered set), circle graphs (intersection graphs of chords of a circle), and ray graphs (intersection graph of rays starting from the x -axis). Moreover, several real-world problems such as PCP routing and railway dispatching can be stated in terms of outerstring graphs [20, 31].

Lots of NP-hard problems remain NP-hard on outerstring graphs. For instance, MINIMUM CLIQUE COVER, COLORING, and DOMINATING SET, and HAMILTONIAN CYCLE are NP-hard even on circle graphs [15, 26, 28]. Also, MAXIMUM CLIQUE is NP-hard even on ray graphs [11]. RECOGNITION is also NP-hard for outerstring graphs [33, 39]. On the other hand, INDEPENDENT SET can be solved in time polynomial in the complexity of a geometric representation of an outerstring graph if its geometric representation is given [29]. However, there is an outerstring graph that does not admit geometric representation of polynomial complexity [7]. It is still unknown if INDEPENDENT SET can be solved in time polynomial in the number of vertices in the case that geometric representation is not given.

2 Preliminaries

A *curve* is the *image* of a continuous function from a unit interval to \mathbb{R}^2 . Under this definition, note that the union of pairwise intersecting curves is also a curve. A simple closed curve C partitions the plane into two disjoint regions. We call the bounded region the *interior* of C and denote it by $\text{int}(C)$. We say a collection Γ of curves are *grounded* on a simple closed curve C if every curve is contained in the closure of $\text{int}(C)$, and one of its endpoints is contained in C . In this case, we say C is a *ground* of the collection of curves. Also, for a curve of Γ , one of its endpoints lying on C is called a *ground point* of the curve. For an index set $I = \{1, 2, 3, \dots\}$ and a collection $\Gamma = \{\gamma_i\}_{i \in I}$ of curves grounded on C , we say a graph G as an *outerstring graph* if it is identical to the intersection graph of Γ . In this case, we say Γ is a *geometric representation* of G . Throughout this paper, we assume the general position assumption that no three curves of Γ intersect at a single point, and two curves of Γ cross at their intersection points. These assumptions can be achieved by slightly adjusting the curves without changing the intersection graph. See Figure 2. In the following, to distinguish the curves of Γ from other curves not necessarily in Γ , we call the curves of Γ the *strings* of Γ .



■ **Figure 2** We can always assume the general position assumption.

An induced subgraph of an outerstring graph is an outerstring graph as an outerstring graph is a geometric intersection graph. Similarly, an *induced minor* of an outerstring graph is an outerstring graph although a minor of an outerstring graph is not necessarily an outerstring graph. An induced minor of a graph G is a graph that can be obtained from G by deleting vertices and contracting edges. Whenever we deal with an induced subgraph of G , we assume that its underlying geometric representation is a subset of Γ . Similarly, the underlying geometric representation of an induced minor of G is a set of curves obtained from Γ by removing curves and taking the union of curves. In the case that several curves are merged into a single curve, we choose the ground point of any of them as the ground point of the new curve.

Let G be the intersection graph of a collection Γ of grounded curves. For a subset Γ' of Γ , the intersection graph of Γ' is an induced subgraph of G by curves in Γ' . We use $G[\Gamma']$ to denote the intersection graph of Γ' . Recall that a vertex of G corresponds to a curve of Γ . For any subset U of the vertex set of G , the *geometric representation* of U is defined as the union of the curves corresponding to the vertices of U .

Treewidth and brambles. A key notion we use in this paper is the *treewidth* of a graph, which is defined as follows. A *tree decomposition* of an undirected graph $G = (V, E)$ is defined as a pair (T, β) , where T is a tree and β is a mapping from nodes of T to subsets of V (we call $\beta(t)$ of $t \in T$ bag) with the following conditions.

- For any vertex $u \in V$, there is at least one bag which contains u .
- For any edge $(u, v) \in E$, there is at least one bag which contains both u and v .
- For any vertex $u \in V$, the nodes of T containing u in their bags are connected in T .

The *width* of (T, β) is defined as the size of its largest bag minus one, and the *treewidth* of G is the minimum width of a tree decomposition of G . Notice that the treewidth of $K_{t,t}$ is $\Theta(t)$, and thus any graph containing $K_{t,t}$ as a subgraph has treewidth $\Omega(t)$.

Although our main focus is to analyze a tight bound of the treewidth of an outerstring graph, we do not use this definition directly. Instead, we use an alternative characterization of treewidth using the notion of *brambles*. A *bramble* \mathcal{X} of a graph G is a family of connected subgraphs of G that all touch each other. Here, we say a subgraph X of G *touches* a subgraph X' of G if there is a common vertex in $V(X) \cap V(X')$, or there is an edge with one endpoint in $V(X)$ and one endpoint in $V(X')$. A subset Y of $V(G)$ is a *hitting set* of the bramble if $V(X) \cap V(Y)$ contains a common vertex for each subgraph X of \mathcal{X} . Then the *order* of bramble is defined as the smallest size of a hitting set of the bramble. Let \mathcal{X} be a bramble of G of maximum order. It is known that the order of \mathcal{X} is the treewidth of G plus one [14].

Sparse outerstring graphs. Our main result focuses on *sparse* outerstring graphs. There are several different definitions of the sparsity of a graph. As the measure for the sparsity of a graph, we mainly use the *arboricity*, which is defined as the minimum number of forests into which its edges can be partitioned. Equivalently, it is half of the maximum average degree of the subgraphs of the graph.

10:6 Sparse Outerstring Graphs Have Logarithmic Treewidth

Other measures about the sparsity of graphs are *biclique-freeness* and *degeneracy*. For an integer t , we say a graph is *t-biclique-free* if it does not contain a (not necessarily induced) subgraph isomorphic to $K_{t,t}$. We say t as the *size* of the biclique $K_{t,t}$. We say a graph G is *t-degenerate* if every subgraph of G has a vertex of degree at most t . In the case of outerstring graphs, all concepts mentioned above are equivalent up to log factors as we will see in Lemma 1. Moreover, the bounds stated in the following lemma are all tight up to constant factors. For the first two statements, as mentioned in [22], the construction of [21, 41] shows that there are t -biclique-free incomparability graphs with n vertices and $\Theta((t \log t)n)$ edges. Moreover, every vertex in this graph has degree $\Omega(t \log t)$. It is known that an incomparability graph is an outerstring graph [23], and thus the first two bounds in the following lemma are tight. We give a tight lower bound for the last two cases in Section 4.

► **Lemma 1.** *For an outerstring graph G , the following statements hold.*

- *If G is t -biclique-free, then G is $O(t \log t)$ -degenerate.*
- *If G is t -biclique-free, G has arboricity $O(t \log t)$.*
- *If G has arboricity t , then G is $2t$ -biclique-free.*
- *If G is t -degenerate, then G is $2t$ -biclique-free.*

Proof. For the first two statements, assume that G is t -biclique-free. Consider an induced subgraph H of G . Note that H is also a t -biclique-free outerstring graph. By [24] and [35], the number of edges of H is $O(t(\log t)|V(H)|)$. Therefore, the average degree of H is $O(t \log t)$, and thus H has a vertex of degree $O(t \log t)$. Hence, G is $O(t \log t)$ -degenerate and it has arboricity $O(t \log t)$.

Now we consider the contrapositives of the last two statements. If G contains a subgraph H isomorphic to $K_{2t,2t}$, the average degree of H is large than t . Therefore, G has arboricity greater than t , and thus the third statement holds. Moreover, it does not contain a vertex of degree at most t , and thus G is not t -degenerate. Therefore, the lemma holds. ◀

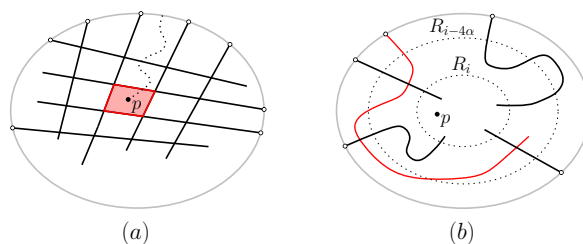
3 Upper Bound on the Treewidth of an Outerstring Graph

In this section, we show that an outerstring graph G has treewidth $O(\alpha \log n)$, where α denotes the arboricity of G . Let Γ be a geometric representation of G , which is a collection of curves grounded on a ground C . A key of our proof lies in defining a new notion, called the *crossing-level*, which is a variant of the level in an arrangement. The *arrangement* of Γ is the subdivision of $\text{int}(C)$ formed by the curves of Γ into vertices, edges, and faces. Note that the degree of each vertex of the arrangement is at most four by the general position assumption. For an illustration of the crossing-level, see Figure 3(a).

► **Definition 2.** *For a point p in $\text{int}(C)$, the crossing-level of p in Γ is the smallest number of different strings that one must cross to reach the ground from p .⁵ The maximum crossing-level of Γ is the maximum of the crossing-levels of all points of $\text{int}(C)$.*

Our proof consists of three steps. We first show that the maximum crossing-level of Γ is at most $3\alpha \log n$. In the second step, we show that an outerstring graph G contains a clique minor of size $\Omega(tw)$, where tw denotes the treewidth of G . Then in the third step, using the result of the second step, we show that the maximum crossing-level of Γ is at least $\Omega(tw)$. By combining the two claims, we conclude that the treewidth of G is $O(\alpha \log n)$.

⁵ In the case that p lies on a curve of Γ , the starting point p is not considered as a crossing point.



■ **Figure 3** (a) The red region depicts U_2 . Jordan curve R_2 consists of the parts of four strings. (b) All black strings are contained in both Γ_i and $\Gamma_{i-4\alpha}$, and the red string is contained in Γ_i only.

3.1 Step 1. Upper Bound on the Maximum Crossing-Level

In this subsection, we show that the maximum crossing-level of Γ is $O(\alpha \log n)$. We use the following observation which immediately comes from the definition of the crossing-level.

► **Observation 3.** *Any two points in the same face of the arrangement of Γ have the same crossing-level. Also, any two points of the same edge of the arrangement of Γ have the same crossing-level unless one of the points is an endpoint of the edge.*

Let p be a point of $\text{int}(C)$ that achieves the maximum crossing-level of Γ , and let r be the crossing-level of p . For each index $0 \leq i \leq r$, let U_i be the set of points in $\text{int}(C)$ of crossing-level at least i . Clearly, U_0 coincides with $\text{int}(C)$. Notice that U_i contains p for all indices $0 \leq i \leq r$. Let R_i be the boundary of the connected component of U_i containing p . Due to Observation 3, R_i is a Jordan curve consisting of parts of strings of Γ for $i > 1$, and R_0 coincides with the ground. See also Figure 3(a). Then let Γ_i be the set of strings of Γ intersecting R_i . Since R_0, R_1, \dots, R_r are concentric, $\Gamma_r \subseteq \Gamma_{r-1} \subseteq \dots \subseteq \Gamma_0 = \Gamma$. In the following lemma, we show that the size of Γ_i is (almost) geometrically decreasing. Since $|\Gamma_0| = n$, this gives an upper bound of r in terms of input size n and arboricity α .

► **Lemma 4.** *For each index $i \geq 4\alpha$, where α is the arboricity of G , we have $2|\Gamma_i| \leq |\Gamma_{i-4\alpha}|$.*

Proof. We consider the region bounded by R_i and $R_{i-4\alpha}$. See Figure 3(b). Note that a string of Γ_i intersects both R_i and $R_{i-4\alpha}$ by definition. Each string γ of Γ_i is intersected by at least 4α different strings of Γ in the region bounded by R_i and $R_{i-4\alpha}$. Otherwise, the first intersection point of γ with R_i from the ground point of γ would have the crossing-level less than i , which contradicts the fact that this intersection point lies on R_i . Also, observe that a string of Γ intersecting γ in the region bounded by R_i and $R_{i-4\alpha}$ is contained in $\Gamma_{i-4\alpha}$. Let \mathcal{P} be the set of all pairs (γ, γ') such that a string γ of Γ_i is intersected by a string γ' of $\Gamma_{i-4\alpha}$. Due to the previous observation, we have $|\mathcal{P}| \geq 4\alpha \cdot |\Gamma_i|$.

Now we show that $|\mathcal{P}| \leq 2\alpha \cdot |\Gamma_{i-4\alpha}|$. To see this, observe that a pair (γ, γ') of \mathcal{P} corresponds to an edge of $G[\Gamma_{i-4\alpha}]$. This is because $\gamma \in \Gamma_i \subseteq \Gamma_{i-4\alpha}$, and $\gamma' \in \Gamma_{i-4\alpha}$. Moreover, by construction, an edge of $G[\Gamma_{i-4\alpha}]$ corresponds to at most two different pairs of \mathcal{P} . Here, notice that two strings of Γ may intersect more than once, but in the construction of \mathcal{P} , for all pairs of \mathcal{P} whose first elements are the same, their second elements must be distinct. Therefore, an edge of $G[\Gamma_{i-4\alpha}]$ corresponds to at most two pairs of \mathcal{P} . Since the arboricity of G is α , any subgraph H of G has at most $\alpha \cdot |V(H)|$ edges. Then the number of edges of $G[\Gamma_{i-4\alpha}]$ is at most $\alpha \cdot |\Gamma_{i-4\alpha}|$. Therefore, we have $|\mathcal{P}| \leq 2\alpha \cdot |\Gamma_{i-4\alpha}|$, and thus the lemma holds. ◀

► **Lemma 5.** *The maximum crossing-level of Γ is at most $O(\alpha \log n)$.*

Proof. For the maximum crossing-level r of Γ , we have $\Gamma_r \neq \emptyset$. By Lemma 4, $2|\Gamma_i| \leq |\Gamma_{i-4\alpha}|$ for all indices $i \geq 4\alpha$. Therefore, $2^{\lfloor r/(4\alpha) \rfloor} \cdot |\Gamma_r| \leq |\Gamma_0| = n$. Thus, $r \leq 4\alpha(\log n + 1)$. ◀

3.2 Step 2. Existence of a Clique Minor of Size tw

In this subsection, we show that an outerstring graph has a clique minor of size $\Omega(tw)$. A *clique minor* of G is a clique formed from G by deleting edges, vertices, and contracting edges. Note that a general “string” graph may not contain a clique minor of size $\Omega(tw)$. For instance, an n by n rectangular grid graph is a string graph since it is a planar graph and all planar graphs are string graphs [43]. The treewidth of this graph is \sqrt{n} but it does not have a size-five clique minor. For any two disjoint sets A and B of vertices of a general graph H with $|A| = |B|$, an (A, B) -*linkage* of H is defined as a set of $|A|$ vertex-disjoint paths connecting every vertex of A and every vertex of B . See Figure 4(a). We say a set Q of vertices of H is *well-linked* if for any two disjoint subsets A and B of Q with $|A| = |B|$, there is an (A, B) -linkage of H . The following lemma is frequently used in literature without formal proof, but to make our paper self-contained, we add a short proof.

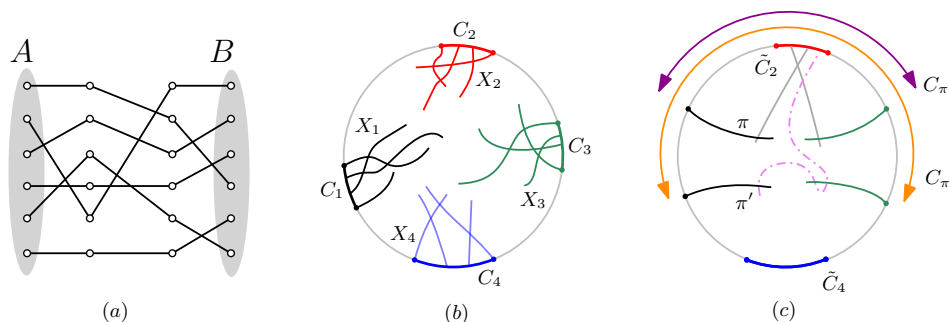
► **Lemma 6** (Folklore). *Any graph H of treewidth tw has a well-linked set of size $\Theta(tw)$.*

Proof. Let \mathcal{X} be a bramble of H of maximum order. Recall that the order of \mathcal{X} is $tw + 1$ by the characterization of the treewidth mentioned in Section 2. Let Q be the smallest hitting set of \mathcal{X} with $|Q| = tw + 1$. We show that Q is well-linked. If this is not the case, there is a witness (A, B) for two disjoint sets A and B of Q with $|A| = |B|$. That is, the maximum number of vertex-disjoint paths between A and B is less than $|A|$. Among all such witnesses (A, B) , we choose the one that minimizes $|A|$. Then no two vertices, one from A and one from B , are adjacent. Then by Menger’s theorem, there is a vertex set X of size less than $|A|$ such that every path between A and B intersects X .

We show that $X \cup (Q \setminus A)$ or $X \cup (Q \setminus B)$ is a hitting set of \mathcal{X} , which contradicts that H is the smallest hitting set of \mathcal{X} . To show this, observe that for an element X of \mathcal{X} , either X intersects X , or it is fully contained in a connected component of $G - X$. The elements of \mathcal{X} of the first type are hit by X , and thus it is hit by both $X \cup (H \setminus A)$ and $X \cup (H \setminus B)$. The elements of \mathcal{X} of the second type must be contained in the same connected component of $G - X$ as they touch each other. Then either $(H \setminus A)$ or $(H \setminus B)$ hits all such elements. Therefore, either $X \cup (H \setminus A)$ or $X \cup (H \setminus B)$ is a hitting set of \mathcal{X} , and thus the lemma holds. ◀

Let X be a well-linked set of G of size $\Theta(tw)$. Without loss of generality, we may assume that the size of X is a power of four. Assume further that they are sorted along C with respect to the ground points of their corresponding curves of Γ . Then we partition X into four equal-sized subsets X_1, X_2, X_3, X_4 so that each subset consists of consecutive vertices of X with respect to their corresponding ground points. See Figure 4(b). By the well-linkedness of X , there are an (X_1, X_3) -linkage \mathcal{P}_h and an (X_2, X_4) -linkage \mathcal{P}_v of G . Observe that if all vertices in the paths of $\mathcal{P}_h \cup \mathcal{P}_v$ are distinct, the geometric representation of every path of \mathcal{P}_h crosses the geometric representation of every path of \mathcal{P}_v . Then we simply pair each path of \mathcal{P}_h with a path of \mathcal{P}_v and then contract all the edges in the two paths to form a single vertex. In this way, we have $\Omega(tw)$ contracted vertices, which form a clique. However, a vertex in a path of \mathcal{P}_h might appear in a path of \mathcal{P}_v . To handle this case, we choose new sets $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ and \tilde{X}_4 using X_1, X_2, X_3, X_4 as follows.

Construction of new sets. Recall that the sizes of X_i ’s are the same. Let k be the size of these sets. For $i \in \{1, 2, 3, 4\}$, let C_i be the minimal subarc of C containing the ground points of all curves corresponding to X_i but not containing the ground point of any curve corresponding to a vertex of $X \setminus X_i$. See Figure 4(b). Notice that C_1, C_2, C_3 and C_4 are



■ **Figure 4** (a) Illustration of an (A, B) -linkage. (b) Partition of X into X_1, X_2, X_3 and X_4 . (c) The case that $C_\pi \subseteq C_{\pi'}$. An internal vertex of π' (corresponding to the pink dotted curve) intersects the geometric representation of π .

pairwise disjoint. We say two vertex sets A and B of G are *separated* by C_1 and C_3 if the ground points of the curves corresponding to the vertices of A are contained in one component of $C \setminus \{C_1, C_3\}$, and the ground points of the curves corresponding to the vertices of B are contained in the other component of $C \setminus \{C_1, C_3\}$. An (A, B) -linkage of G is called a *shortest* (A, B) -linkage if its total length is minimum over all (A, B) linkages of G . Here, the length of a path is defined as the number of vertices in the path.

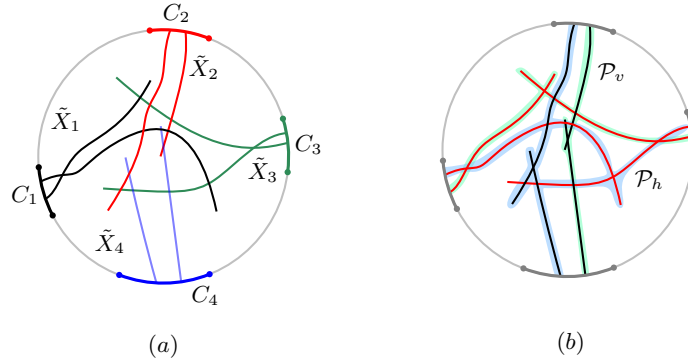
Among all pairs (X'_2, X'_4) of disjoint vertex sets of $V(G) \setminus (X_1 \cup X_3)$ separated by C_1 and C_3 with $|X'_2| = |X'_4| = k$, we choose the one that minimizes the total length of the shortest (X'_2, X'_4) -linkage. Notice that such a pair always exists since (X_2, X_4) satisfies the conditions for being a candidate. Let \tilde{X}_2 and \tilde{X}_4 be the resulting sets. Then let \tilde{C}_2 (and \tilde{C}_4) be the minimal subarc of C containing the ground points of all curves corresponding to \tilde{X}_2 (and \tilde{X}_4) and not containing any other curve of \tilde{X}_4 (and \tilde{X}_2). Then among all pairs (X'_1, X'_3) of disjoint vertex sets of $V(G) \setminus (\tilde{X}_2 \cup \tilde{X}_4)$ separated by \tilde{C}_2 and \tilde{C}_4 with $|X'_1| = |X'_3| = k$, we choose the one that minimizes the total length of the shortest (X'_1, X'_3) -linkage. Notice that such a pair always exists since (X_1, X_3) satisfies the conditions for being a candidate. Let \tilde{X}_1 and \tilde{X}_3 be the resulting sets. By construction, all resulting sets are pairwise disjoint.

Then a shortest $(\tilde{X}_1, \tilde{X}_3)$ -linkage satisfies the following property. By changing the roles of $\tilde{C}_2 \cup \tilde{C}_4$ and $C_1 \cup C_3$, we can show that the following also holds for a shortest $(\tilde{X}_2, \tilde{X}_4)$ -linkage.

► **Lemma 7.** *Let \mathcal{P} be a shortest $(\tilde{X}_1, \tilde{X}_3)$ -linkage. Then the set of paths of \mathcal{P} of length at least three can be decomposed into two subsets such that for any two paths of \mathcal{P} in the same subset, their geometric representations intersect.*

Proof. Let π be a path of \mathcal{P} of length at least three. For any internal vertex of π , its corresponding curve of Γ has the ground point on $\tilde{C}_2 \cup \tilde{C}_4$. Otherwise, we partition π into two subpaths π_1 and π_2 whose common endpoint v lies outside of $\tilde{C}_2 \cup \tilde{C}_4$. Assume, without loss of generality, it lies the subarc of $C \setminus (\tilde{C}_2 \cup \tilde{C}_4)$ containing C_1 . In this case, the other endpoint of π_i ($i=1$ or 2) lies in the other subarc of $C \setminus (\tilde{C}_2 \cup \tilde{C}_4)$. Then we simply replace π with π_i . In this way, we can decrease the total length of paths of \mathcal{P} without any conditions for \mathcal{P} . This contradicts the choice of \mathcal{P} . Then we decompose the paths of \mathcal{P} of length at least three into two subsets \mathcal{P}' and \mathcal{P}'' such that \mathcal{P}' consists of all paths of \mathcal{P} of length at least three containing at least one internal vertex whose corresponding curve has its ground point on \tilde{C}_2 , and \mathcal{P}'' consists of all paths of \mathcal{P} of length at least three not contained in \mathcal{P}' .

Now we show that for any two paths π and π' of \mathcal{P}' , their geometric representations intersect. The other case can be handled symmetrically by changing the roles of \tilde{C}_2 and \tilde{C}_4 . Let C_π be the subarc of C containing \tilde{C}_2 lying between the ground points of the



■ **Figure 5** (a) Illustration of $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ and \tilde{X}_4 . (b) Illustration of length-two paths of \mathcal{P}_v and \mathcal{P}_h . Two geometric representations of two pairs of paths of \mathcal{P}_h and \mathcal{P}_v intersect.

strings corresponding to the endpoints of π . See Figure 4(c). Similarly, let $C_{\pi'}$ be the subarc of C containing \tilde{C}_2 lying between the ground point of the strings corresponding to the endpoints of π . If neither C_{π} nor $C_{\pi'}$ contains the other, π and π' cross because the geometric representation of π (and π') is connected. If this is not the case, without loss of generality, assume that C_{π} is contained in $C_{\pi'}$. Since π' has an internal vertex whose corresponding string has its ground point on \tilde{C}_2 , the geometric representation of π must intersect the string. See the pink dotted curve in Figure 4(c). Therefore, the lemma holds. \blacktriangleleft

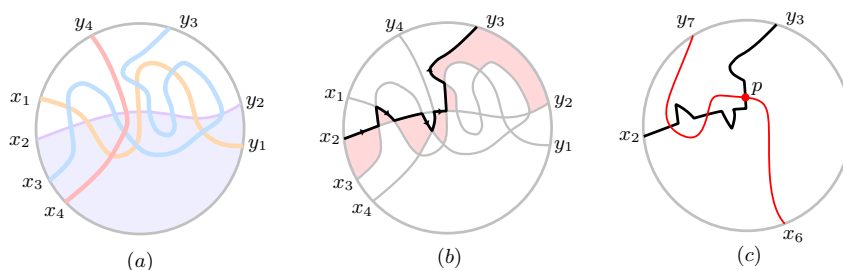
Now we are ready to show that G has a clique minor of size $\Omega(tw)$. Recall that \tilde{X}_i 's have size $k = \Omega(tw)$. Let \mathcal{P}_v be a shortest $(\tilde{X}_1, \tilde{X}_3)$ -linkage, and \mathcal{P}_h be a shortest $(\tilde{X}_2, \tilde{X}_4)$ -linkage. If \mathcal{P}_v (or \mathcal{P}_h) contains $k/2$ paths of length at least three, then we are done. To see this, observe that \mathcal{P}_v (or \mathcal{P}_h) contains at least $k/4$ paths whose geometric representations intersect by Lemma 7. For each such path, we contract all edges in the path into a single vertex. Then at least $k/4$ vertices form a clique, and thus G has a clique minor of size $k/4$.

Thus we assume each of \mathcal{P}_v and \mathcal{P}_h contains at least $k/2$ paths of length *at most* two. Recall that $\tilde{X}_i \cap \tilde{X}_j = \emptyset$ for any two distinct indices $i, j \in \{1, 2, 3, 4\}$. This implies that every such path of \mathcal{P}_v and \mathcal{P}_h has length *exactly* two. Moreover, the paths of $\mathcal{P}_v \cup \mathcal{P}_h$ of length exactly two are pairwise vertex-disjoint since $\tilde{X}_i \cap \tilde{X}_j = \emptyset$. Since the geometric representation of every path of \mathcal{P}_h crosses the geometric representation of every path of \mathcal{P}_v , we pair each path of \mathcal{P}_h with a path of \mathcal{P}_v and then contract all the edges in the two paths to form a single vertex. In this way, we have $k/2$ contracted vertices, which form a clique. See Figure 5. Therefore, in any case, G has a clique minor of size $k/2 = \Omega(tw)$.

3.3 Step 3. Lower Bound on the Maximum Crossing-Level

As the third step, we show that the maximum crossing-level of Γ is $\Omega(tw)$. First, we reduce the general case to the case that Γ consists of *double-grounded circularly ordered curves*, and then we analyze the maximum crossing-level of Γ in this case. A *double-grounded curve* is a curve having both endpoints on C . We say that double-grounded curves $\gamma_1, \gamma_2, \dots, \gamma_k$ are *circularly ordered* if $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{k-1}$ and y_k lie on C in the counterclockwise order, where x_i and y_i denote two endpoints of γ_i for $i = 1, \dots, k$. See Figure 6(a).

Reduction to the double-grounded circularly ordered case. Let k be the largest integer such that G has a clique minor of size $4k$. Notice that $k = \Omega(tw)$ by Section 3.2. Then G has a *model* μ of K_{4k} , that is, there is a function μ that maps the vertices of K_{4k} to



■ **Figure 6** (a) Illustration of double-grounded circularly ordered curves. The colored region is h_2 . (b-c) Illustrates P_1 (black curve) and P_2 (red curve). The crossing-level of p is at least $k/2$.

vertex-disjoint connected subgraphs of G such that, for any two vertices u and v of K_k , $\mu(u)$ and $\mu(v)$ are adjacent in G . In other words, the edges of $\mu(v)$ are contracted into v in the construction of the clique minor of G of size k .

For a vertex v of K_{4k} , recall that the geometric representation of v is defined as the union of the strings of Γ corresponding to the vertices of $\mu(v)$. We choose an arbitrary one as its ground point of v (and its geometric representation). Let $\langle v_1, v_2, \dots, v_{4k} \rangle$ be the sequence of vertices of K_{4k} sorted with respect to their ground points along C in the counterclockwise order. We pair a vertex v_i in the first half of the sequence with v_{i+2k} , and then find a curve in the union of their geometric representations. More specifically, for each index i with $i = 1, \dots, 2k$, consider the union of the geometric representations of v_i and v_{i+2k} . Notice that the union consists of a single connected component since v_i and v_{i+2k} are adjacent in K_{4k} . Let x_i be the ground point of v_i , and let y_i be the ground point of v_{i+2k} . Then the union contains a curve with endpoints x_i and y_i . Let γ_i be a curve between x_i and y_i contained in the union. By construction, $\gamma_1, \gamma_2, \dots, \gamma_{2k}$ are doubly-grounded circularly ordered curves. Due to the following lemma, it is sufficient to show that the maximum crossing-level of k double-grounded circularly ordered curves is $\Omega(k)$.

► **Lemma 8.** *The max crossing-level of Γ is at least the max crossing-level of $\{\gamma_1, \dots, \gamma_{2k}\}$.*

Proof. Let q be a point in $\text{int}(C)$, and let $\ell(q)$ be the crossing-level of q in $\{\gamma_1, \gamma_2, \dots, \gamma_{2k}\}$. We show that the crossing-level of q in Γ is at least $\ell(q)$, which implies the lemma. Consider a curve π connecting q and C . The number of curves of $\{\gamma_1, \gamma_2, \dots, \gamma_{2k}\}$ intersected by π is at least $\ell(q)$. By the construction of $\{\gamma_1, \gamma_2, \dots, \gamma_{2k}\}$, for any point in γ_i and any point in γ_j with $i \neq j$, they come from different strings of Γ . Therefore, the number of strings of Γ intersected by π is at least $\ell(q)$, and thus the lemma holds. ◀

Analysis of the double-grounded circularly ordered case. In the following, we focus on the set $\{\gamma_1, \dots, \gamma_{2k}\}$ of double-grounded circularly ordered curves along C . To make the description easier, we assume that k is even. We give a direction to each path γ_i from x_i to y_i . Then every edge of the arrangement of $\{\gamma_1, \dots, \gamma_{2k}\}$ has its direction. By the general position assumption, every vertex of the arrangement has two incoming arcs and two outgoing arcs. Then γ_i subdivides $\text{int}(C)$ into two regions. The (closed) region lying locally to the right of γ_i is denoted by h_i . See Figure 6(a). Let $H_1 = \{h_1, h_2, \dots, h_k\}$ and $H_2 = \{h_{k+1}, h_{k+2}, \dots, h_{2k}\}$. Let P_i be the set of points contained in exactly $k/2$ regions of H_i and lying on the boundary curves of the regions of H_i for $i = 1, 2$. Then P_i forms a simple curve, but in the following, we prove the weaker property that P_i contains a curve as it is sufficient for our purpose.

► **Lemma 9.** *The set P_i contains a simple curve connecting two points on C for $i = 1, 2$.*

10:12 Sparse Outerstring Graphs Have Logarithmic Treewidth

Proof. We prove this for P_1 only. The other case can be handled similarly. We traverse the arrangement of the curves of $\{\gamma_1, \dots, \gamma_k\}$ along a simple curve as follows. Starting from $x_{k/2}$, we move along $\gamma_{k/2}$ until we reach a vertex of the arrangement. Whenever we meet a vertex of the arrangement, this vertex is an intersection point between the curve we are traversing and another curve in $\{\gamma_1, \dots, \gamma_k\}$. Then we follow the outgoing arc of the vertex lying on the new curve. We repeat this until we reach a point of C . Let ρ be the set of points we have visited in this way. We claim that ρ is a simple curve contained in P_i .

We first show that each (directed) arc of ρ is contained in P_i . As $x_{k/2}$ is contained in exactly $k/2$ regions of H_1 , the claim holds for the first arc. Consider two consecutive arcs η and η' of ρ . Assume that the claim holds for η , and we show that the claim also holds for η' . Let h and h' be the two regions of H_1 such that the boundary curve of h contains η , and the boundary curve of h' contains η' . The regions of $H_1 \setminus \{h, h'\}$ containing η also contains η' , and vice versa. On the other hand, there are two possibilities: (1) h contains both η and η' , and h' contains both η and η' , or (2) h contains η only, and h' contains η' only. Therefore, the number of regions of H_1 containing η' (and η) are the same, and thus ρ is contained in P_i . See also Figure 6(b).

Moreover, ρ is a simple curve, that is, it does not contain a cycle. If this is not the case, ρ contains three arcs of the arrangement sharing a common vertex. Two of them are contained in the boundary curve of the same region h of H_1 . Let h' be the region of H_1 whose boundary curve contains the other arc. The regions of $H_1 \setminus \{h'\}$ containing one arc coming from h contain the other arc coming from h , and vice versa. On the other hand, one arc coming from h is not contained in h' , and the other arc coming from h is contained in h' . Therefore, not both arcs coming from h are contained in P_1 , which contradicts that ρ contains both arcs. Therefore, ρ does not contain a cycle, and thus it ends at a point on C . ◀

► **Lemma 10.** *We have $P_1 \cap P_2 \neq \emptyset$, and a point in $P_1 \cap P_2$ has a crossing-level at least $k/2$.*

Proof. Notice that $P_1 \cap C$ consists of two points $x_{k/2}$ and $y_{k/2+1}$. Therefore, P_1 contains a simple curve connecting $x_{k/2}$ and $y_{k/2+1}$. Similarly, P_2 contains a simple curve connecting $x_{3k/2}$ and $y_{3k/2+1}$. Since $x_{k/2}, x_{3k/2}, y_{k/2+1}$ and $y_{3k/2+1}$ lie on C in the counterclockwise order, the two simple curves cross. See Figure 6(c). Notice that an intersection point p of the two simple curves is contained in exactly $k/2$ regions of H_i for $i = 1, 2$.

Now we show that the crossing-level of p is at least $k/2$. Let z be a point in C . It suffices to show that any curve connecting p and z intersects at least $k/2$ curves of $\{\gamma_1, \dots, \gamma_{2k}\}$. For two points x' and y' on C , we let $C[x', y']$ be the circular arc of C from x' to y' in the counterclockwise direction. We show this through case studies of the position of z . Consider the case that $z \in C[x_{2k}, y_1]$. Recall that the common intersection of all regions of $H_1 \cup H_2$ contains $C[x_{2k}, y_1]$. Recall also that the number of regions of H not containing p is exactly k . The region h_i of H not containing p is defined by the double-grounded curve γ_i . Then γ_i separates p and z . In this case, any curve connecting p and z intersects at least k different curves of $\{\gamma_1, \gamma_2, \dots, \gamma_{2k}\}$, and thus we are done.

Now consider the case that $z \in C[x_1, x_k] \cup C[y_1, y_k]$. Then the regions of H_2 containing p do not contain z . Therefore, at least $k/2$ curves of $\{\gamma_{k+1}, \gamma_2, \dots, \gamma_{2k}\}$ separate p and z , and thus any curve connecting p and z intersects at least $k/2$ different curves of $\{\gamma_1, \gamma_2, \dots, \gamma_{2k}\}$.

The other cases can be handled similarly. Specifically, the case that $z \in C[y_{2k}, x_1]$ can be handled symmetrically to the first case by changing the roles of the regions of H not containing p and the regions of H containing p . The case that $z \in C[x_k, x_{2k}] \cup C[y_k, y_{2k}]$ can be handled symmetrically to the second case by changing the roles of H_1 and H_2 . ◀

In this way, we can show that for an outerstring graph G with a geometric representation Γ , the maximum crossing-level in Γ is $\Omega(tw)$, where tw denotes the treewidth of G .

► **Theorem 11.** *The treewidth of an outerstring graph is $O(\alpha \log n)$, where α and n denote the arboricity and the number of vertices of the graph, respectively.*

► **Corollary 12.** *The treewidth of a $K_{t,t}$ -free outerstring graph is $O(t(\log t) \log n)$.*

4 Lower Bound of the Treewidth of an Outerstring Graph

In this section, we show that the bound of Theorem 11 is almost tight by constructing an *outersegment graph* G with treewidth $\Theta(\alpha \log(n/\alpha))$, where α is the arboricity of G . An outersegment graph G is the intersection graph of *segments* grounded on C . We say an outersegment graph is k -directional if it has a geometric representation consisting of grounded segments of k orientations. To make the description easier, we first present a 2-directional outersegment graph with arboricity $O(1)$ of treewidth $\Omega(\log n)$.

► **Lemma 13.** *There is a 2-directional outersegment graph with arboricity $O(1)$ and treewidth $\Omega(\log n)$, where n denotes the number of its vertices.*

Proof. Let $n = 2^m$. We consider the x -axis as the ground, and the region lying above the x -axis as the interior of the ground. A *V-shape curve* is the union of two equal-length outersegments sharing a common ground point, with the other endpoints having the same y -coordinates. Note that the angle of the two outersegments is $\pi/3$. The width of a V-shaped curve is defined as the length of each outersegment. A *folk* is the union of equal-width V-shaped curves which form a simple curve. See Figure 7(a). The *size* of folk is the number of V-shaped curves in the folk, and the *width* of a folk is the size of each V-shaped curve in the folk. For an index i with $1 \leq i \leq m$, let F_i be the folk of width 2^i and size $n/2^i$ whose leftmost point lies on the y -axis. Then we set Γ as the set of all outersegments of the folks F_1, F_2, \dots, F_m . See Figure 7(b). Notice that the ground points of the outersegments of F_i have x -coordinates $2^{i-1} + 2^i \cdot j$ for integers j with $0 \leq j < n/2^i - 1$. Therefore, no two segments from different folks share their ground points.

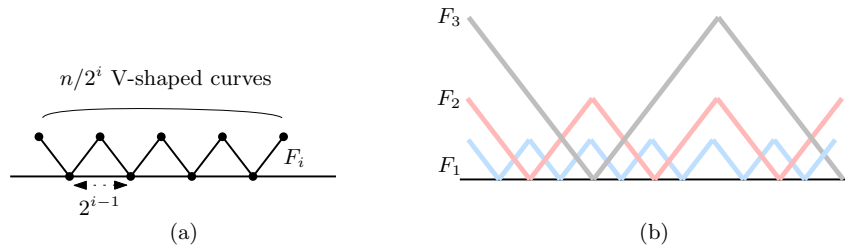
Its intersection graph G is clearly a 2-directional outersegment graph. The number of vertices of G is $\Theta(n)$ since the number of segments of F_i is $2 \cdot (n/2^i)$. The arboricity of G is $O(1)$ since each segment of F_i intersects at most one segment of $F_{i+1} \cup F_{i+2} \cup \dots \cup F_m$ for any index i with $1 \leq i \leq m$. Then the number of edges of any subgraph of G is linear in the number of its vertices, and thus the arboricity of G is $O(1)$. Consider the minor of G obtained by contracting all edges coming from F_i for each index i with $1 \leq i \leq m$. Since the folks are pairwise intersecting, the resulting minor is a clique of size $m = \log n$. Since the treewidth of G is at least the treewidth of any of its minors, the treewidth of G is $\Omega(\log n)$. ◀

Notice that this example is 2-biclique-free. We can generalize this example to an outersegment graph with an arbitrary arboricity α whose treewidth is $\Theta(\alpha \log(n/\alpha))$ by simply copying all segments α times. Notice that we can slightly perturb the copied segments without changing their intersection graphs so that any two segments intersect at most once.

► **Lemma 14.** *For any integer α , there is an outersegment graph with arboricity α and treewidth $\Omega(\alpha \log(n/\alpha))$, where n denotes the number of its vertices.*

Proof. Let Γ be the set of outersegments we constructed in Lemma 13. We copy each outersegment of Γ into α copies. Then these α curves are pairwise intersecting. Then the intersection graph of the resulting outersegments has arboricity α , and it has αn vertices.

10:14 Sparse Outerstring Graphs Have Logarithmic Treewidth



■ **Figure 7** (a) A folk of width 2^{i-1} and size $n/2^i$. (b) Illustration of F_1, F_2 and F_3 .

Moreover, the resulting set Γ can be decomposed into $\alpha \log n$ pairwise intersecting folks, and they form a clique minor of size $\alpha \log n$. Therefore, the treewidth of the resulting graph is $\alpha \log(N/\alpha)$, where N denotes the number of vertices of the resulting graph. ◀

5 Algorithmic Applications

In this section, we present algorithmic applications of Corollary 12. Immediate consequences of Corollary 12 are as follows. We compute a tree decomposition of width $2 \cdot tw$ in $2^{O(tw)}n$ time [32]. Then lots of central NP-hard problems can be solved in $2^{O(tw)}n^{O(1)}$ time [14] including the problems mentioned in the following corollary (except for COLORING). The definitions of the problems we will mention in this section is as follows.

- INDEPENDENT SET asks for a maximum-sized set of pairwise non-adjacent vertices.
- HAMILTONIAN CYCLE asks for a cycle that visits all the vertices of the graph.
- VERTEX COVER asks for a minimum-sized set of vertices that cover all the edges.
- DOMINATING SET asks for a minimum-sized set D of vertices such that every other vertex is adjacent to D .
- FEEDBACK VERTEX SET asks for a minimum-sized vertex set whose removal makes G acyclic.
- COLORING asks for a coloring of the vertices of G with the smallest number of colors.
- MAXIMUM INDUCED MATCHING asks for a maximum-sized edge set S such that no two such edges are joined by an edge of G .
- LIST 3-COLORING asks for a function $\bar{f} : V(G) \rightarrow \{1, 2, 3\}$ such that $\bar{f}(v) \in f(v)$ for all vertices v of $V(G)$, and $\bar{f}(u) \neq \bar{f}(v)$ for all edges (u, v) of G . An input of this problem consists of a graph G and a function $f : V(G) \rightarrow 2^{\{1,2,3\}}$.
- CYCLE PACKING asks for a maximum-sized set of vertex-disjoint cycles of G .

► **Corollary 15.** *INDEPENDENT SET, HAMILTONIAN CYCLE, DOMINATING SET, FEEDBACK VERTEX SET and COLORING can be solved in polynomial time on t -biclique-free outerstring graphs for a fixed constant t .*

Proof. All problems mentioned above, except for COLORING, admit polynomial-time algorithms for graphs with logarithmic treewidth. More specifically, they are based on dynamic programming over a tree decomposition of width $tw = O(\log n)$, and they run in $2^{O(tw)}n^{O(1)} = n^{O(1)}$ time in our case. For details, see [14, Chapter 7.3 and Chapter 11.2].

In the case of COLORING, recall that a t -biclique-free outerstring graph is $O(t \log t)$ -degenerate by Lemma 1. Thus it is $O(t \log t)$ -colorable. To see this, consider the following constructive argument. In each iteration, we find a minimum degree vertex v and compute a minimum coloring of $G - \{v\}$ recursively. Then since the degree of v is $O(t \log t)$, we can

color v using a color not used by any neighbor of v once we have $\Omega(t \log t)$ colors. This implies that G is $O(t \log t)$ -colorable. For a fixed constant q , we can compute a coloring of a graph with treewidth tw with q colors in $2^{O(tw)} n^{O(1)}$ time [14, Chapter 7.3]. Therefore, we can solve COLORING in polynomial time in the case that t is a constant. ◀

Subexponential-time FPT algorithms for (not necessarily sparse) outerstring graphs.

We can obtain subexponential-time FPT algorithms for VERTEX COVER and FEEDBACK VERTEX SET on general outerstring graphs. To do this, we first compute the polynomial kernel with respect to the solution size k . Then we compute biclique of size $t = \sqrt{k}$ in a brute-force manner, and apply branching with respect to the partial solution in the biclique.

► **Corollary 16.** *VERTEX COVER and FEEDBACK VERTEX SET parameterized by the solution size k can be solved in $2^{O(\sqrt{k} \log^2 k)} n^{O(1)}$ time for (not necessarily sparse) outerstring graphs.*

Proof of Vertex Cover. Let G be an outerstring graph with n vertices. VERTEX COVER admits a kernel of size k [14], that is, there is an instance (H, k') consisting of an induced minor H of G with $O(k)$ vertices and an integer $k' \leq k$ such that H has a vertex cover of size k' if and only if G has a vertex cover of size k . Thus it is sufficient to solve VERTEX COVER for (H, k') . Notice that H is also an outerstring graph.

Let $t = \sqrt{k}$. Given an instance (H, k') , we first compute a biclique of size t in H in a brute-force fashion if it exists. This takes $k^t = 2^{O(\sqrt{k} \log k)}$ time. Let A, B be the sets of $V(H)$ which form a biclique of size t . Notice that a vertex cover of H must contain either A or B . We branch on whether A is contained in an optimal vertex cover, or B is contained in an optimal vertex cover. For this, we produce two instances $(H - A, k' - t)$ and $(H - B, k' - t)$. If H does not contain a biclique of size t , we apply a polynomial-time algorithm for computing a minimum vertex cover for a t -biclique-free outerstring graph.

The number of instances we produce is $2^{O(k/t)}$, which is $2^{O(\sqrt{k})}$. For each instance, we compute a biclique of size t in $2^{O(\sqrt{k} \log k)}$ time. If it does not exist, we compute a minimum vertex cover in $O(2^{t(\log t) \log k}) = 2^{O(\sqrt{k} \log^2 k)}$ time. ◀

Subexponential-time algorithms for (not necessarily sparse) outerstring graphs.

Apart from Corollary 12, the main tools we use for VERTEX COVER and FEEDBACK VERTEX SET are branching and kernelization. For the following two problems, we are not aware of any polynomial kernel although we can apply branching. In this case, we can obtain subexponential-time (non-parameterized) algorithms for those problems.

► **Corollary 17.** *MAXIMUM INDUCED MATCHING and LIST 3-COLORING on outerstring graphs with n vertices can be solved in $2^{O(\sqrt{n} \log^2 n)}$ time.*

Proof of Maximum Induced Matching. Let G be an outerstring graph with n vertices. Let $t = \sqrt{n}$. We first compute a biclique of size t in H in a brute-force fashion if it exists. This takes $n^t = 2^{O(\sqrt{n} \log n)}$ time. Let A, B be the sets of $V(G)$ which form a biclique of size t . Notice that for any induced matching M of G , at most one edge of M is incident to $A \cup B$. We produce t^2 instances for branching on which vertices of $A \cup B$ are involved in a maximum induced matching. For this, we produce two instances $G \setminus A$ and $G \setminus B$, and t^2 instances $(G \setminus (A \cup B)) \cup \{u, v\}$ for all pairs (u, v) with $u \in A$ and $v \in B$. If G does not contain a biclique of size t , we apply a $2^{O(tw)} n^{O(1)}$ -time algorithm for computing a maximum-cardinality induced matching for a t -biclique-free outerstring graph.

The number of instances we produce is $(t^2)^{O(n/t)}$, which is $2^{O(\sqrt{n} \log n)}$. For each instance, we compute a biclique of size t in $2^{O(\sqrt{n} \log n)}$ time. If it does not exist, we compute a maximum induced matching in $O(2^{t(\log t) \log n}) = 2^{O(\sqrt{n} \log^2 n)}$ time. ◀

Approximation algorithms for (not necessarily sparse) outerstring graphs. We present the first constant-factor approximation algorithm for the CYCLE PACKING running in quasi-polynomial time. Our algorithm repeatedly computes a cycle of length at most four and removes it from G until no such cycle exists. Then the remaining graph is 2-biclique-free, and thus we can find a maximum number of cycles in the remaining graph in $2^{O(tw \log tw)}$ time. Then we show that the total number of cycles we have found so far is at least $\text{OPT}/4$.

► **Corollary 18.** *CYCLE PACKING on outerstring graphs with n vertices can be solved approximately with an approximation factor of 4 in $n^{O(\log \log n)}$ time.*

Proof. Let G be an outerstring graph with n vertices. As the first step, we repeatedly compute a cycle of length at most four. We add it to the solution and remove it from G . Then finally we have a graph G_{final} whose shortest cycle has a length larger than four. Notice that this graph does not contain $K_{2,2}$ as a subgraph, and thus its treewidth is $O(\log n)$. It is known that CYCLE PACKING can be solved in $2^{O(tw \log tw)} n^{O(1)}$ time. As the second step, we find a maximum set of vertex-disjoint cycles in G_{final} in $n^{O(\log \log n)}$ time, and return all cycles we computed in the first and second steps as output.

We can obtain at least $\text{OPT}/4$ vertex-disjoint cycles in this way, where OPT is the maximum number of vertex-disjoint cycles. Let \mathcal{C}_{OPT} be a set of vertex-disjoint cycles with $|\mathcal{C}_{\text{OPT}}| = \text{OPT}$. A cycle of \mathcal{C}_{OPT} not contained in G_{final} intersects at least one cycle we computed in the first step. Since the length of a cycle we computed in the first step is at most four, each such cycle intersects at most four cycles of \mathcal{C}_{OPT} . On the other hand, the number of cycles of \mathcal{C}_{OPT} contained in G_{final} is at most the number of cycles we computed in the second step since we compute a maximum number of vertex-disjoint cycles of G_{final} in the second step. Therefore, the number of vertex-disjoint cycles we have is at least $\text{OPT}/4$. ◀

References

- 1 Pierre Aboulker, Isolde Adler, Eun Jung Kim, Ni Luh Dewi Sintiar, and Nicolas Trotignon. On the tree-width of even-hole-free graphs. *European Journal of Combinatorics*, 98:103394, 2021.
- 2 Tara Abrishami, Maria Chudnovsky, and Kristina Vušković. Induced subgraphs and tree decompositions I. Even-hole-free graphs of bounded degree. *Journal of Combinatorial Theory, Series B*, 157:144–175, 2022.
- 3 Shinwoo An, Kyungjin Cho, and Eunjin Oh. Faster algorithms for cycle hitting problems on disk graphs. In *Proceedings of the 18th Algorithms and Data Structures Symposium (WADS 2023)*, pages 29–42, 2023.
- 4 Shinwoo An and Eunjin Oh. Feedback vertex set on geometric intersection graphs. In *Proceedings of the 32nd International Symposium on Algorithms and Computation (ISAAC 2021)*, 2021.
- 5 Shinwoo An and Eunjin Oh. Eth-Tight Algorithm for Cycle Packing on Unit Disk Graphs. In *40th International Symposium on Computational Geometry (SoCG 2024)*, 2024.
- 6 Sujoy Bhore, Satyabrata Jana, Supantha Pandit, and Sasanka Roy. The balanced connected subgraph problem for geometric intersection graphs. *Theoretical Computer Science*, 929:69–80, 2022.
- 7 Therese Biedl, Ahmad Biniiaz, and Martin Derka. On the size of outer-string representations. In *Proceedings of the 16th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2018)*, pages 10:1–10:14, 2018.
- 8 Marthe Bonamy, Édouard Bonnet, Hugues Déprés, Louis Esperet, Colin Geniet, Claire Hilaire, Stéphan Thomassé, and Alexandra Wesolek. Sparse graphs with bounded induced cycle packing number have logarithmic treewidth. In *Proceedings of the 2023 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2023)*, pages 3006–3028. SIAM, 2023.

- 9 Édouard Bonnet and Paweł Rzażewski. Optimality program in segment and string graphs. *Algorithmica*, 81:3047–3073, 2019.
- 10 Prosenjit Bose, Paz Carmi, J Mark Keil, Anil Maheshwari, Saeed Mehrabi, Debajyoti Mondal, and Michiel Smid. Computing maximum independent set on outerstring graphs and their relatives. *Computational Geometry*, 103:101852, 2022.
- 11 Sergio Cabello, Jean Cardinal, and Stefan Langerman. The clique problem in ray intersection graphs. *Discrete & Computational Geometry*, 50:771–783, 2013.
- 12 Jean Cardinal, Stefan Felsner, Tillmann Miltzow, Casey Tompkins, and Birgit Vogtenhuber. Intersection graphs of rays and grounded segments. *Journal of Graph Algorithms and Applications*, 22(2):273–295, 2018.
- 13 Maria Chudnovsky, Tara Abrishami, Sepehr Hajebi, and Sophie Spirkl. Induced subgraphs and tree decompositions III. Three-path-configurations and logarithmic treewidth. *Advances in Combinatorics*, 2022:6, 2022.
- 14 Marek Cygan, Fedor V Fomin, Łukasz Kowalik, Daniel Lokshantov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*, volume 5. Springer, 2015.
- 15 Peter Damaschke. The Hamiltonian circuit problem for circle graphs is NP-complete. *Information Processing Letters*, 32(1):1–2, 1989.
- 16 Mark De Berg, Hans L Bodlaender, Sándor Kisfaludi-Bak, Dániel Marx, and Tom C Van Der Zanden. A framework for exponential-time-hypothesis-tight algorithms and lower bounds in geometric intersection graphs. *SIAM Journal on Computing*, 49(6):1291–1331, 2020.
- 17 Mark de Berg, Sándor Kisfaludi-Bak, Morteza Monemizadeh, and Leonidas Theodorou. Clique-based separators for geometric intersection graphs. *Algorithmica*, 85(6):1652–1678, 2023.
- 18 Zdeněk Dvořák and Sergey Norin. Treewidth of graphs with balanced separations. *Journal of Combinatorial Theory, Series B*, 137:137–144, 2019.
- 19 Gideon Ehrlich, Shimon Even, and Robert Endre Tarjan. Intersection graphs of curves in the plane. *Journal of Combinatorial Theory, Series B*, 21(1):8–20, 1976.
- 20 Holger Flier, Matúš Mihalák, Anita Schöbel, Peter Widmayer, and Anna Zych. Vertex disjoint paths for dispatching in railways. In *Proceedings of the 10th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS’10)*, 2010.
- 21 Jacob Fox. A bipartite analogue of Dilworth’s theorem. *Order*, 23(2-3):197–209, 2006.
- 22 Jacob Fox and János Pach. A separator theorem for string graphs and its applications. *Combinatorics, Probability and Computing*, 19(3):371–390, 2010.
- 23 Jacob Fox and János Pach. Coloring K_k -free intersection graphs of geometric objects in the plane. *European Journal of Combinatorics*, 33:853–866, 2012.
- 24 Jacob Fox and János Pach. String graphs and incomparability graphs. *Advances in Mathematics*, 230(3):1381–1401, 2012.
- 25 Zachary Friggstad and Mohammad R Salavatipour. Approximability of packing disjoint cycles. *Algorithmica*, 60(2):395–400, 2011.
- 26 Michael R Garey, David S Johnson, Gary L Miller, and Christos H Papadimitriou. The complexity of coloring circular arcs and chords. *SIAM Journal on Algebraic Discrete Methods*, 1(2):216–227, 1980.
- 27 Tomáš Gavenčiak, Przemysław Gordinowicz, Vít Jelínek, Pavel Klavík, and Jan Kratochvíl. Cops and robbers on intersection graphs. *European Journal of Combinatorics*, 72:45–69, 2018.
- 28 J Mark Keil. The complexity of domination problems in circle graphs. *Discrete Applied Mathematics*, 42(1):51–63, 1993.
- 29 J Mark Keil, Joseph SB Mitchell, Dinabandhu Pradhan, and Martin Vatshelle. An algorithm for the maximum weight independent set problem on outerstring graphs. *Computational Geometry*, 60:19–25, 2017.

- 30 Sándor Kisfaludi-Bak, Karolina Okrasa, and Paweł Rzażewski. Computing list homomorphisms in geometric intersection graphs. In *Proceedings of the International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2022)*, pages 313–327. Springer, 2022.
- 31 Hui Kong, Qiang Ma, Tan Yan, and Martin DF Wong. An optimal algorithm for finding disjoint rectangles and its application to PCB routing. In *Proceedings of the 47th Design Automation Conference (DAC 2010)*, pages 212–217, 2010.
- 32 Tuukka Korhonen. A single-exponential time 2-approximation algorithm for treewidth. *SIAM Journal on Computing*, pages FOCS21–174, 2023.
- 33 Jan Kratochvíl. String graphs I. The number of critical nonstring graphs is infinite. *Journal of Combinatorial Theory, Series B*, 52(1):53–66, 1991.
- 34 Michael Krivelevich, Zeev Nutov, Mohammad R Salavatipour, Jacques Verstraete, and Raphael Yuster. Approximation algorithms and hardness results for cycle packing problems. *ACM Transactions on Algorithms (TALG)*, 3(4):48–es, 2007.
- 35 James R Lee. Separators in region intersection graphs. In *Proceedings of the 8th Innovations in Theoretical Computer Science Conference (ITCS 2017)*, 2017.
- 36 Daniel Lokshtanov, Fahad Panolan, Saket Saurabh, Jie Xue, and Meirav Zehavi. Subexponential parameterized algorithms on disk graphs. In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2022)*, pages 2005–2031. SIAM, 2022.
- 37 Jiří Matoušek. Near-optimal separators in string graphs. *Combinatorics, Probability and Computing*, 23(1):135–139, 2014.
- 38 Jiří Matoušek. String graphs and separators. In *Geometry, Structure and Randomness in Combinatorics*, pages 61–97. Springer, 2014.
- 39 Matthias Middendorf and Frank Pfeiffer. Weakly transitive orientations, Hasse diagrams and string graphs. *Discrete Mathematics*, 111(1-3):393–400, 1993.
- 40 Karolina Okrasa and Paweł Rzażewski. Subexponential algorithms for variants of the homomorphism problem in string graphs. *Journal of Computer and System Sciences*, 109:126–144, 2020.
- 41 János Pach and Géza Tóth. Comment on fox news. *Geombinatorics*, 15:150–154, 2006.
- 42 Alexandre Rok and Bartosz Walczak. Outerstring graphs are χ -bounded. *SIAM Journal on Discrete Mathematics*, 33(4):2181–2199, 2019.
- 43 Marcus Schaefer and Daniel Štefankovič. Decidability of string graphs. *Journal of Computer and System Sciences*, 2(68):319–334, 2004.
- 44 Frank W. Sinden. Topology of thin film RC circuits. *Bell System Technical Journal*, 45:1639–1662, 1966.