A Faster Algorithm for the 4-Coloring Problem

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Abstract

We explore the 4-coloring problem, a fundamental combinatorial NP-hard problem. Given a graph G, the 4-coloring problem asks whether there exists a function f from the vertex set of G to $\{1, 2, 3, 4\}$ such that $f(u) \neq f(v)$ for each edge uv of G. Such function f is referred to as a 4-coloring of G. The fastest known algorithm for the 4-coloring problem, introduced by Fomin, Gaspers, and Saurabh (COCOON 2007), exhibits a time complexity of $O(1.7272^n)$ and exponential space.

In this paper, we propose an enhanced algorithm for the 4-coloring problem with a time complexity of $O(1.7159^n)$ and polynomial space. Our algorithm is deterministic and built upon a novel method. Specifically, inspired by previous algorithmic approaches for the 4-coloring problem, such as the aforementioned $O(1.7272^n)$ time algorithm, we consider the instance (G, I, S), where G is a graph and I, S are subsets of its vertex set representing vertices colored with 1 and vertices unable to be colored with 1, respectively. For a given instance (G, I, S), we aim to determine the existence of a 4-coloring f of G such that $f(v) = 1$ for $v \in I$ and $f(v) \neq 1$ for $v \in S$.

Our key innovation lies in recognizing that, leveraging certain combinatorial properties, the instance (G, I, S) can be efficiently solved when $G - I - S$ is a union of K_3's and K_4's (where K_3 and K_4 denote complete graphs with 3 and 4 vertices, respectively). The ability to efficiently solve instances (G, I, S), where $G - I - S$ is comprised solely of K_3's and K_4's, enables us to devise a branching algorithm capable of efficiently addressing instances (G, I, S), where $G - I - S$ is not a union of K_3's and K_4's (the other case). Based on this innovative method, we derive our final enhanced algorithm.

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1 Introduction

Consider a graph $G = (V, E)$. A function $f : V \to \{1, 2, \ldots, k\}$ is a k-coloring of G if $f(u) \neq f(v)$ for $uv \in E$. The smallest k, for which such a k-coloring of G exists, is termed the chromatic number of G. The chromatic number problem, identified by Karp [25] as

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one of the 21 seminal NP-complete problems, aims to determine chromatic number of G. The chromatic number problem is notoriously challenging. The best-known polynomial time algorithm achieves only an approximation ratio of $O(n \log^{-3} n (\log \log n)^2)$ [23], where n denotes the number of vertices in the graph. Furthermore, it is NP-hard to approximate within $n^{1-\varepsilon}$ [38] for any positive constant ε.

The k-coloring problem serves as the decision version of the chromatic number problem. Specifically, given a positive integer k and a graph G, the k-coloring problem inquires whether there is a k-coloring of G. When $k \geq 3$, the k-coloring problem is NP-hard[27, 35]. Designing algorithms for the k-coloring problem, including heuristic algorithms [6, 11, 17, 24, 28, 9, 29, 32] and exact algorithms [7, 36, 26, 14, 5, 2, 34, 12, 8, 15, 4], has garnered significant attention. Among exact algorithms addressing graphs with n vertices, the fastest known algorithm [3] for the k-coloring problem operates with a time and space complexity of $O^*(2^n)$ (O^* suppresses a polynomial factor about n). Subsequently, their work [4] further reduces the space complexity to $O(1.2916^n)$. The fastest algorithm with polynomial space complexity achieves a time complexity of $O(2.2356^n)$ [3, 20]. Recently, Björklund, Curticapean, Husfeldt, Kaski, and Pratt [1], in their unofficially published work (reprinted on arXiv), give a $O(1.99982^n)$ time complexity algorithm for the k-coloring problem, assuming the validity of the asymptotic rank conjecture. Specifically, in Theorem 3 of [1], they show that if the asymptotic rank conjecture is true over a field of characteristic zero, then the chromatic number of an n-vertex graph can be computed deterministically in $O(1.99982^n)$ time. As of now, due to the undetermined status of the asymptotic rank conjecture, no known algorithm for the k-coloring problem has a time complexity of $O(a^n)$, where $a < 2$. However, intriguingly, for small values of k, there exist algorithms for the k-coloring problem with a time complexity of $O(a^n)$, where $a < 2$.

Specifically, as far as is currently known, algorithms with time complexity $O(a^n)$, where $a < 2$, for the k-coloring problem are only feasible when $k \leq 6$. The pursuit of fast algorithms for these k-coloring problems has been a longstanding research focus. For the 3-coloring problem, significant progress has been made over time. The time complexity has improved successively to $O(1.4423^n)$ (in 1976) [26, 31], $O(1.415^n)$ (in 1994) [33], and $O(1.3289^n)$ (in 2001) [12], which remains the fastest known algorithm for the 3-coloring problem. Similarly, advancements have been made for the 4-coloring problem. The time complexity has evolved successively to $O^*(2^n)$ (in 1976) [26], $O(1.8072^n)$ (in 2001) [12], $O(1.7504^n)$ (in 2004) [8], and $O(1.7272^n)$ (in 2007) [15], establishing the fastest known algorithms for the 4-coloring problem. Meijer [30], in their unofficially published work (reprinted on arXiv), recently introduces faster algorithms for the 3-coloring and 4-coloring problem, achieving time complexities of $O(1.3217^n)$ and $O(1.7247^n)$, respectively. And recently, Clinch, Gaspers, Saffidine, and Zhang[10], in their unofficially published work (reprinted on arXiv), introduces a new method for analyzing the running time of branching algorithms. This method improves the time complexity of the 4-coloring problem algorithm described in [15] to $O(1.7215^n)$. For the 5-coloring and 6-coloring problems, Zamir [37] recently proposed groundbreaking algorithms, breaking the 2^n Barrier. Zamir's algorithms for the 5-coloring and 6-coloring problems exhibit time complexities of $O((2-\varepsilon)^n)$, where ε is a constant greater than 0, establishing them as the fastest known algorithms for these problems. This paper aims to present a faster algorithm for the 4-coloring problem.

Detailed description of the fastest known algorithm for the 4-Coloring Problem. In the framework of [15] (the framework of the fastest known algorithm for the 4-coloring problem), the approach involves utilizing the algorithm for the 3-coloring problem to address a specific case. Thus, we initially introduce the framework of [12] (the framework of the fastest known algorithm for the 3-coloring problem). Within the framework of [12], the algorithm commences with addressing $(3, 2)$-CSP, as defined in Definition 1. Notably, the k-coloring problem can be naturally transformed into $(k, 2)$-CSP. To enhance efficiency, a subset D of the vertex set of G is selected to enumerate all potential 3-colorings of $G[D]$ (the subgraph of G induced by D). Each feasible 3-coloring of $G[D]$ leads to an instance of $(3, 2)$-CSP, which is subsequently solved by the algorithm tailored for $(3, 2)$-CSP.

▶ **Definition 1** (CSP). *For a positive integer n, $[n]$ represents the set $\{1, 2, \ldots, n\}$, and specially $[0] = \emptyset$. An instance of constraint satisfaction problem (CSP) is presented by (X, R), where $X = \{(x_1, D_1), (x_2, D_2), \ldots, (x_r, D_r)\}$ represents the set of ordered pairs of variable and corresponding finite discrete-valued domain, and $R = \{RC_1, RC_2, \ldots, RC_p\}$ represents the set of constraints. Each $RC_i = \{(x_{j_1}, c_{i,j_1}), (x_{j_2}, c_{i,j_2}), \ldots, (x_{j_{q_i}}, c_{i,j_{q_i}})\}$ in R, satisfies that $c_{i,j_z} \in D_{j_z}$ for $z \in [q_i]$ and $j_1, j_2, \ldots, j_{q_i} \in [r]$. An assignment of (X, R) is a function f on $\{x_1, x_2, \ldots, x_r\}$ such that $f(x_i) \in D_i$ for $i \in [r]$. For an assignment f of (X, R) and a constraint $RC_i \in R$, RC_i is satisfied by f if there exists a $z \in [q_i]$ such that $f(x_{j_z}) \neq c_{i,j_z}$. A satisfying assignment f of (X, R) is an assignment such that RC_i is satisfied by f for $RC_i \in R$. Given an instance (X, R), CSP asks whether there exists a satisfying assignment of (X, R). (d, s)-CSP is the CSP such that $|D_i| \leq d$ for $i \in [r]$ and $|RC_j| \leq s$ for $j \in [p]$.*

In the framework of [15], the focus is on the instance (G, I, S), where G is a graph, and I and S are disjoint subsets of its vertex set, representing vertices colored with 1 and vertices unable to be colored with 1, respectively. This involves determining the existence of a 4-coloring f of G such that $f(v) = 1$ for $v \in I$ and $f(v) \neq 1$ for $v \in S$. Based on the instance (G, I, S), one of the three following strategies is adopted: (1) choose a maximum degree vertex v in $G - I - S$ and branch on v by considering whether v is colored with 1 or not; (2) enumerating all bound size maximal independent sets of $G - I - S$, transform the instance (G, I, S) to instances of the 3-coloring problem, which are then solved by an algorithm for the 3-coloring problem; (3) use a dynamic program over the path decomposition. Due to the necessity of employing dynamic programming, the required space can be exponential in the worst case.

In the aforementioned unofficially published work of Meijer [30], a framework similar to [12] is employed. Differently, Meijer introduces enhancements in selecting the aforementioned set D, leading to an improved algorithm for the 3-coloring problem. Since the framework of [15] utilizes a 3-coloring problem algorithm to address a specific case, Meijer's improvements in the 3-coloring problem algorithm consequently enhance the algorithm for the 4-coloring problem as well. The utilization of the framework of [15] implies that the required space can still be exponential in the worst case.

Our Result and Contribution. In our work, we also address the aforementioned instance (G, I, S). Our primary contribution lies in effectively solving instances (G, I, S), where $G - I - S$ is a union of K_3's and K_4's. Specifically, when $G - I - S$ exhibits this structure, we leverage combinatorial properties to transform the instance (G, I, S) into a significantly reduced number of instances of $(3, 2)$-CSP. Conversely, when $G - I - S$ does not conform to this structure, we employ a branching algorithm by considering whether a vertex v is colored with 1 or not. The effectiveness of solving instances (G, I, S), where $G - I - S$ is a union of K_3's and K_4's, enables the design of an efficient branching algorithm for cases that $G - I - S$ deviates from this structure. Notably, we introduce a new method that enables us to develop a faster algorithm for the 4-coloring problem. As a result, we present an algorithm with polynomial space usage and a time complexity of $O(1.7159^n)$, where n is the number of vertices of the given graph, as formally demonstrated in Theorem 8. A comparison between our result and previous findings is provided in Table 1.

Table 1 Comparison between previous works and our result.

Year	Time complexity	Space complexity	Reference
2001	$O(1.8072^n)$	polynomial	[12]
2004	$O(1.7504^n)$	polynomial	[8]
2007	$O(1.7272^n)$	exponential	[15]
2023	$O(1.7247^n)$	exponential	[30]1)
2024	$O(1.7215^n)$	exponential	[10]1)
2024	$O(1.7159^n)$	polynomial	Ours

1) Both works are unofficially published.

2 Preliminaries

Consider an instance (X, R) of CSP, where $X = \{(x_1, D_1), (x_2, D_2), \ldots, (x_r, D_r)\}$. x_i is termed a $|D_i|$-variable for $i \in [r]$. For a $RC_i = \{(x_{j_1}, c_{i,j_1}), (x_{j_2}, c_{i,j_2}), \ldots, (x_{j_{q_i}}, c_{i,j_{q_i}})\}$ in R and a variable x_z, where $z \in [r]$, we say that RC_i contains x_z if there is a $t \in [q_i]$ such that $x_{j_t} = x_z$. By a known result obtained by Eppstein [12], all 2-variables of a given $(k, 2)$-CSP instance can be removed by replacing them with new constraints. Moreover, as detailed in Lemma 2 below, he showed that the time complexity of his algorithm for $(3, 2)$-CSP mainly depends on the number of 3-variables in the given instance.

▶ **Lemma 2** ([12]). *A $(3, 2)$-CSP instance with n 3-variables, can be solved within $O(1.3645^n)$ time complexity and polynomial space.*

In this paper, we exclusively deal with undirected simple graphs. Consider a graph G. $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively. For $S_1, S_2 \subseteq V(G)$, denote $E_G(S_1, S_2) = \{uv \in E(G) \mid u \in S_1, v \in S_2\}$. The cycle with n vertices, also called the n-cycle, is represented by C_n, and the complete graph with n vertices is denoted as K_n. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of neighbors of v, $N_G[v] = N_G(v) \cup \{v\}$, and $d_G(v)$ represents the degree of v. For a subset $S \subseteq V(G)$, $N_G[S] = \bigcup_{v \in S} N_G[v]$ and $N_G(S) = N_G[S] \setminus S$. When the context is clear, we abbreviate $E_G(S_1, S_2)$, $N_G(v)$, $N_G[v]$, $d_G(v)$, $N_G[S]$, and $N_G(S)$ as $E(S_1, S_2)$, $N(v)$, $N[v]$, $d(v)$, $N[S]$, and $N(S)$, respectively. A vertex $v \in V(G)$ is called a $d(v)$-vertex of G. Let $\delta(G) = \min\{d(v) \mid v \in V(G)\}$ and $\Delta(G) = \max\{d(v) \mid v \in V(G)\}$. The vertex-induced subgraph $G[S]$ of G with respect to a subset $S \subseteq V(G)$ consists of all vertices in S and all edges of G with both endpoints in S. For a $S \subseteq V(G)$, $G - S$ refers to the vertex-induced subgraph $G[V(G) \setminus S]$. The graph H is considered a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We say that G contains a graph H if there exists a subgraph of G that is isomorphic to H. G is defined as connected if there exists a path between any pair of vertices u and v in $V(G)$. A component of G refers to a connected subgraph that is not a subgraph of any larger connected subgraph of G. If a component C of G is isomorphic to K_3 (K_4), it is referred to as a K_3-component (K_4-component). $I \subseteq V(G)$ is an independent set of G if $uv \notin E(G)$ for $u, v \in V(G)$. For an instance $P = (G, I, S)$, denote $\mathcal{G}(P) = G - I - S$. Notably, $v \in I$ is the vertex colored 1, and for a 4-coloring of G, the set of vertices colored 1 is an independent set. Consequently, we can directly put $N_G(I)$ into S. Then we only consider the instance $P = (G, I, S)$ where $S \supseteq N_G(I)$. For a positive integer a and a nonnegative integer b, $\binom{a}{b}$ represents the binomial coefficient equal to the number of ways to pack a items out b items. For a real number x, $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively denote the largest integer less than or equal to x, and the smallest integer greater than or equal to x.

2.1 Maximal independent set and independent dominating set

Consider a graph G. A maximal independent set I of G is an independent set satisfying that I is not a proper subset of any other independent set of G. An independent dominating set ID of G is an independent set satisfying $N[ID] = V(G)$. Actually, I is a maximal independent set of G if and only if I is an independent dominating set of G. We define independent domination number $i(G)$ as the minimum cardinality of any independent dominating set of G, i.e., $i(G) = \min\{|I| \mid I \text{ is an independent dominating set of } G\}$, and $\mathcal{MI}(G)$ as the set of all maximal independent sets of G.

▶ **Lemma 3** ([22]). *$i(C_n) = \lceil \frac{n}{3} \rceil$ for $n \geq 3$.*

▶ **Lemma 4** ([18]). *(1) $|\mathcal{MI}(C_3)| = 3, |\mathcal{MI}(C_4)| = 2, |\mathcal{MI}(C_5)| = 5$.*
(2) $|\mathcal{MI}(C_n)| = |\mathcal{MI}(C_{n-2})| + |\mathcal{MI}(C_{n-3})|$ for $n \geq 6$.

2.2 Measure and conquer method for branching algorithm

When utilizing branching algorithms, accurately assessing the size of the search tree is crucial. To achieve this, the measure and conquer method [16] utilizes a measure μ, where $T(\mu)$ denotes the upper bound on the size of the search tree produced by a branching algorithm on an instance with a measure no greater than μ. Typically, a branching operation involves dividing the instance into t branches, each reducing the measure by at least β_i in the i-th branch. This branching operation can be represented by the following recurrence relation:

$$T(\mu) \leq T(\mu - \beta_1) + T(\mu - \beta_2) + \ldots + T(\mu - \beta_t). \tag{1}$$

The largest root of the function $g(x) = 1 - \sum_{i=1}^{t} x^{-\beta_i}$ is termed the *branching factor* of the recurrence. If the branching factor of any branching operation within the algorithm is less than r, then the algorithm's time complexity can be expressed as $T(\mu) = O(r^\mu)$. To obtain the bound r^μ, a constraint akin to $\sum_{i=1}^{t} r^{-\beta_i} \leq 1$ is formulated for each recurrence relation of branching operations akin to Equation (1). Subsequently, all constraints are utilized in a numerical program to determine a minimum r. For a comprehensive exploration of the measure and conquer method for the branching algorithms please refer to [16].

Consider an instance $P = (G, I, S)$, where $\mathcal{G}(P)$ has n_3 K_3-components and n_4 K_4-components. To define our measure for P, we introduce real number parameters ω_i for $i \geq 0$, ω_{k3}, ω_{k4} and ω_{c3}. Specifically, we directly set $\omega_0 = 0$, and $\omega_i = 1$ for $i \geq 3$. Let $\mathcal{C} = \{C \text{ is a component of } \mathcal{G}(P) \mid C \text{ is neither } K_3 \text{ nor } K_4\}$. In this paper, we define the measure $\mu(P)$ of P as follows:

$$\mu(P) = |S|\omega_{c3} + n_3\omega_{k3} + n_4\omega_{k4} + \sum_{C \in \mathcal{C}} \sum_{v \in V(C)} \omega_{d_C(v)}. \tag{2}$$

To find the minimum r, we need to solve an optimal program. Usually, the optimal programs derived by the measure and conquer method, including the one in this paper, are quasi-convex programs, which can be solved by the method in [13]. In our final conclusion, we present the values of all parameters in Table 2. We note that, in the time complexity analysis in [21, 19], instead of solving the quasi-convex program, Gaspers and Sorkin present a method that only requires solving a convex program, which is easier and supported by more existing solvers. Their method is also suitable for the time complexity analysis of other branching algorithms, including our algorithm. In this paper, we provide the constraints in their original form (the quasi-convex program form) and give the details about obtaining the corresponding convex program in the Appendix (Appendix A). To ensure that the measure doesn't increase in the branching algorithm, we present the following constraints:

$$\begin{cases} 0 \le \omega_{c3} \le \omega_1 \le \omega_2 \le \omega_3, \\ \omega_3 - \omega_2 \le \omega_2 - \omega_1 \le \omega_1, \\ 3\omega_2 \ge \omega_{k3}, \\ 4\omega_3 \ge \omega_{k4}. \end{cases} \tag{3}$$

3 The algorithm for the 4-coloring problem

We present the algorithm for the 4-coloring problem in this section. For a 4-coloring f of a graph G, denote $V_i(f) = \{v \in V(G) \mid f(v) = i\}$ for $i \in [4]$. Consider an instance $P = (G, I, S)$. A 4-coloring of P is a 4-coloring of G such that $f(v) = 1$ for $v \in I$ and $f(v) \ne 1$ for $v \in S$. Recall that $\mathcal{G}(P) = G - I - S$.

▶ **Observation 5.** *Let $P = (G, I, S)$ be an instance and D be a subset of $V(\mathcal{G}(P))$. If there is a 4-coloring of P, then there is a 4-coloring f of P such that $V_1(f) \cap N_G[v] \ne \emptyset$ for $v \in D$.*

Proof. Let g be a 4-coloring of P such that $|\{v \in D \mid V_1(g) \cap N_G[v] = \emptyset\}|$ is minimized. We show that g is the desired 4-coloring. Suppose, for contradiction, that $|\{v \in D \mid V_1(g) \cap N_G[v] = \emptyset\}| > 0$. Consider a $v \in D$ with $V_1(g) \cap N_G[v] = \emptyset$. It is clear that the function f defined by $f(v) = 1$ and $f(x) = g(x)$ otherwise, constitutes a 4-coloring of P. Since $|\{v \in D \mid V_1(f) \cap N_G[v] = \emptyset\}| < |\{v \in D \mid V_1(g) \cap N_G[v] = \emptyset\}|$, we encounter a contradiction. Thus, this observation holds. ◀

Observation 5 suggests that when we want to determine the existence of 4-colorings of an instance $P = (G, I, S)$ with a subset D of $V(\mathcal{G}(P))$, we focus solely on functions f such that $V_1(f) \cap N_G[v] \ne \emptyset$ for $v \in D$. The selection of D varies depending on the case. We consider two scenarios: (1) $V(\mathcal{G}(P)) = \emptyset$, or each component of $\mathcal{G}(P)$ is K_4 or K_3; (2) there exists a component of $\mathcal{G}(P)$ that is neither K_4 nor K_3.

3.1 $V(\mathcal{G}(P)) = \emptyset$, or each component of $\mathcal{G}(P)$ is K_4 or K_3

Consider an instance $P = (G, I, S)$ such that either $V(\mathcal{G}(P)) = \emptyset$ or $\mathcal{G}(P)$ is a union of n_3 K_3's and n_4 K_4's. Regard the case that $V(\mathcal{G}(P)) = \emptyset$ as the case that $n_3 = n_4 = 0$. Let the n_3 K_3's be denoted as $K_3^1, K_3^2, \ldots, K_3^{n_3}$, and the n_4 K_4's as $K_4^1, K_4^2, \ldots, K_4^{n_4}$. Assume $V(K_3^i) = \{v_{i,1}, v_{i,2}, v_{i,3}\}$ for $i \in [n_3]$ and $V(K_4^j) = \{u_{j,1}, u_{j,2}, u_{j,3}, u_{j,4}\}$ $j \in [n_4]$. Denote $D_u = \{1, 2, 3, 4\}$ for $u \in V(\mathcal{G}(P))$, and $D_v = \{2, 3, 4\}$ for $v \in S$. We transform P into a $(4, 2)$-CSP instance $U = (X, R)$, where $X = \{(v, D_v) \mid v \in V(\mathcal{G}(P)) \cup S\}$ and $R = \{\{(u, i), (v, i)\} \mid i \in D_u \cap D_v, uv \in E(G), u, v \in S \cup V(\mathcal{G}(P))\}$. Notably, f is a 4-coloring of P if and only if f is a satisfying assignment of U.

If $V(\mathcal{G}(P)) = \emptyset$, then U is a $(3, 2)$-CSP instance with $|S|$ 3-variables. By Lemma 2, U can be solved within $O(1.3645^{|S|})$ time complexity. Thus P can be solved within $O(1.3645^{|S|})$ time complexity. To bound our algorithm's time complexity by $O(r^{\mu(P)})$, it is necessary to ensure that $r^{\mu(P)} \ge 1.3645^{|S|}$. Given that $\mu(P) = |S|\omega_{c3}$ by Equation (2), we have the following constraint:

$$\omega_{c3} \ln r \ge \ln 1.3645. \tag{4}$$

If $V(\mathcal{G}(P)) \ne \emptyset$, we aim to reduce the number of $(3, 2)$-CSP instances derived from U by utilizing combinatorial properties. For a satisfying assignment f of U and a set A of some variables, let $f(A) = \{f(x) \mid x \in A\}$. For a satisfying assignment f of U, denote

$L_i(f) = \{j \in [n_3] \mid i \notin f(V(K_3^j))\}$ for $i \in \{2,3,4\}$, i.e., $L_i(f)$ is the set of integers $j \in [n_3]$ such that no vertex in the component K_3^j is assigned the color i under f. Considering $\bigcup_{j\in[n_3]} V(K_3^j)$ as analogous to D in Observation 5, we only consider satisfying assignments f of U such that $1 \in f(V(K_3^j))$ for $j \in [n_3]$. So, for the satisfying assignment f of U to consider, we have $|L_2(f)| + |L_3(f)| + |L_4(f)| = n_3$. Since colors 2, 3, 4 are symmetrical in U, we only search for satisfying assignments f of U such that $1 \in f(V(K_3^j))$ for $j \in [n_3]$ and $|L_2(f)| \le \frac{n_3}{3}$. Enumerating all possible $L_2(f)$, there are $\sum_{i=0}^{\lfloor \frac{n_3}{3} \rfloor} \binom{n_3}{i}$ cases that need to be searched. Let $L_2(f) = \{i_1, i_2, \ldots, i_t\}$ and consider to search for such satisfying assignment f.

Step 1. Consider a K_3^j, where $j \in L_2(f)$. For $v \in V(K_3^j)$, we can restrict D_v to $D'_v = \{1,3,4\}$, resulting in $f(V(K_3^j)) = \{1,3,4\}$. For $u \in N_G(V(K_3^j)) \cap S$, u can't be assigned color 1, meaning that we are free to use color 1 for any of the three vertices in $V(K_3^j)$. Thus we only need to determine which vertices in K_3^j are assigned colors 3 and 4. Thus, we can use two variables to represent the status of K_3^j. We construct two variables $x_{j,3}$, $x_{j,4}$, and two corresponding discrete-valued domains $D_{x_{j,3}} = \{v_{j,1}, v_{j,2}, v_{j,3}\}$ and $D_{x_{j,4}} = \{v_{j,1}, v_{j,2}, v_{j,3}\}$. Specifically, $x_{j,c}$ is assigned a value $v_{j,z}$, where $c \in \{3,4\}$ and $z \in [3]$, meaning that we assign color c to the vertex $v_{j,z}$. Meanwhile the $v_{j,z}$, where $z \in [3]$, and which is unassigned to $x_{j,3}$ and $x_{j,4}$, is colored 1. Thus, we construct constraints $RC_{j,z} = \{(x_{j,3}, v_{j,z}), (x_{j,4}, v_{j,z})\}$ for $z \in [3]$, and $RC_{j,z,c,v} = \{(x_{j,c}, v_{j,z}), (v, c)\}$ for $v \in N_G(v_{j,z}) \cap S$, $z \in [3]$, $c \in \{3,4\}$. Then, we remove all variables in $\bigcup_{j\in L_2(f)} V(K_3^j)$, all constraints containing at least one variable in $\bigcup_{j\in L_2(f)} V(K_3^j)$, and add all variables $x_{j,3}$, $x_{j,4}$ for $j \in L_2(f)$, all constraints $RC_{j,z}, RC_{j,z,c,v}$ for $v \in N_G(v_{j,z}) \cap S$, $j \in L_2(f)$, $z \in [3]$, $c \in \{3,4\}$.

Step 2. Consider a K_3^j, where $j \notin L_2(f)$. It clear that $\{1,2\} \subseteq f(V(K_3^j))$, resulting in three cases: (1) $\{f(v_{j,1}), f(v_{j,2})\} = \{1,2\}$, $f(v_{j,3}) \in \{3,4\}$; (2) $\{f(v_{j,1}), f(v_{j,3})\} = \{1,2\}$, $f(v_{j,2}) \in \{3,4\}$; (3) $\{f(v_{j,2}), f(v_{j,3})\} = \{1,2\}$, $f(v_{j,1}) \in \{3,4\}$. Correspondingly, we restrict $D_{v_{j,1}}, D_{v_{j,2}}, D_{v_{j,3}}$ into: (1) $\{1,2\}$,$\{1,2\}$,$\{3,4\}$; (2) $\{1,2\}$,$\{3,4\}$,$\{1,2\}$; (3) $\{3,4\}$,$\{1,2\}$,$\{1,2\}$. We enumerate all the possibilities for each K_3^j with $j \notin L_2(f)$.

Step 3. Consider a K_4^j, where $j \in [n_4]$. It is clear that $f(V(K_4^j)) = \{1,2,3,4\}$, resulting in that there are exactly two vertices in K_4^j assigned color in $\{1,2\}$ and exactly two vertices in K_4^j assigned color in $\{3,4\}$. Correspondingly, we choose two vertices w_1, w_2 of K_4^j, and assume $V(K_4^j) \setminus \{w_1, w_2\} = \{w_3, w_4\}$. We restrict D_{w_1}, D_{w_2} both into $\{1,2\}$, and D_{w_3}, D_{w_4} both into $\{3,4\}$. There are $\binom{4}{2} = 6$ cases for choosing such w_1, w_2. We enumerate all the possibilities for each K_4^j with $j \in [n_4]$.

After enumerating all possible $L_2(f)$ and completing Steps 1-3, we transform U into $\sum_{i=0}^{\lfloor \frac{n_3}{3} \rfloor} \binom{n_3}{i} 3^{n_3-i} 6^{n_4}$ $(3,2)$-CSP instances. In Step 1, each K_3^j with $j \in L_2(f)$ is transformed into two 3-variables, and in Steps 2 and 3, each 4-variable is restricted to a 2-variable. Consequently, the obtained $(3,2)$-instance has $|S| + 2|L_2(f)|$ 3-variables. Specifically, for $i \in \{0, 1, \ldots, \lfloor \frac{n_3}{3} \rfloor\}$, there are exactly $\binom{n_3}{i} 3^{n_3-i} 6^{n_4}$ $(3,2)$-CSP instances with $|S| + 2i$ 3-variables. Thus, by Lemma 2, the time complexity of solving P is $O(\sum_{i=0}^{\lfloor \frac{n_3}{3} \rfloor} \binom{n_3}{i} 3^{n_3-i} 6^{n_4} 1.3645^{|S|+2i})$. Notably, $\mu(P) = n_3\omega_{k3} + n_4\omega_{k4} + |S|\omega_{c3}$ by Equation (2). Thus, we need to satisfy $6^{n_4} 1.3645^{|S|} \sum_{i=0}^{\lfloor \frac{n_3}{3} \rfloor} \binom{n_3}{i} 3^{n_3-i}\, 1.3645^{2i} = O(r^{n_3\omega_{k3}+n_4\omega_{k4}+|S|\omega_{c3}})$. Considering Stirling's formula, $n! = \Theta(\sqrt{2n\pi}\left(\frac{n}{e}\right)^n)$, we have $\binom{n_3}{i} = O((\frac{1}{\beta^\beta(1-\beta)^{1-\beta}})^{n_3})$, where $\beta = \frac{i}{n_3}$ and $i \in \{0, 1, \ldots, n_3\}$, resulting in $\binom{n_3}{i} 3^{n_3-i} 1.3645^{2i} = (\frac{1}{\beta^\beta(1-\beta)^{1-\beta}} 3^{1-\beta} 1.3645^{2\beta})^{n_3}$. Upon inspection, when $0 \le \beta \le \frac{1}{3}$, we have $\frac{1}{\beta^\beta(1-\beta)^{1-\beta}} 3^{1-\beta} 1.3645^{2\beta} < 4.837$. And especially, the expression $\frac{1}{\beta^\beta(1-\beta)^{1-\beta}} 3^{1-\beta} 1.3645^{2\beta}$ attains a maximum when $\beta = \frac{1}{3}$. Therefore,

$6^{n_4}1.3645^{|S|}\sum_{i=0}^{\lfloor\frac{n_3}{3}\rfloor}\binom{n_3}{i}3^{n_3-i}1.3645^{2i} = O(4.837^{n_3}6^{n_4}1.3645^{|S|})$. Then, we need to satisfy $4.837^{n_3}6^{n_4}1.3645^{|S|} = O(r^{n_3\omega_{k3}+n_4\omega_{k4}+|S|\omega_{c3}})$. By setting $r^{\omega_{k3}} \geq 4.837, r^{\omega_{k4}} \geq 6$, and $r^{\omega_{c3}} \geq 1.3645$, we obtain the desired result, $4.837^{n_3}6^{n_4}1.3645^{|S|} = O(r^{n_3\omega_{k3}+n_4\omega_{k4}+|S|\omega_{c3}})$. Notably, $r^{\omega_{c3}} \geq 1.3645$ is already obtained from Equation (4). Thus we present the following constraints:

$$\begin{cases}\omega_{k3}\ln r \geq \ln 4.837,\\ \omega_{k4}\ln r \geq \ln 6.\end{cases} \tag{5}$$

3.2 There exists a component of $\mathcal{G}(P)$ that is neither K_4 nor K_3

Recall that $P = (G, I, S)$ and $S \supseteq N_G(I)$. Consider a component C of $\mathcal{G}(P)$ that is neither a K_4 nor a K_3. For a vertex $v \in V(\mathcal{G}(P))$, "take v" means including v in I (coloring v with 1). Conversely, "discard v" signifies placing v into S (disabling v from being colored with 1). Notably, if $\delta(C) = 0$, meaning $|V(C)| = 1$, we can directly apply Observation 5 and utilize the following reduction rule.

Reduction Rule. If there is a vertex $v \in V(\mathcal{G}(P))$ with degree 0, we directly take v.

Therefore, we focus solely on the case that $\delta(C) \geq 1$. To formulate the branching algorithm, we delineate it into five cases: (1) $\delta(C) = 1$; (2) $\delta(C) = \Delta(C) = 2$; (3) $\delta(C) = 2$ and $3 \leq \Delta(C) \leq 4$; (4) $\delta(C) \geq 2$ and $\Delta(C) \geq 5$; (5) $3 \leq \delta(C) \leq \Delta(C) \leq 4$.

Case 1. $\delta(C) = 1$. Let $v \in V(C)$ be a vertex with degree 1, $N_C(v) = \{u\}$, and $D = \{v\}$. By Observation 5, we only consider taking v or u, resulting in two subinstances: $P_1 = (G, I \cup \{v\}, S \cup N_C(v))$ and $P_2 = (G, I \cup \{u\}, S \cup N_C(u))$. Notably $\omega_i \leq \omega_{i+1}$ for $i \geq 0$ by Equation (3). Since $d_C(v) = 1$ and $d_C(u) \geq 1$, we have $\mu(P) - \mu(P_i) \geq 2\omega_1 - \omega_{c3}$ for $i \in [2]$. Therefore, we obtain the following constraint:

$$2r^{-2\omega_1+\omega_{c3}} \leq 1. \tag{6}$$

Case 2. $\delta(C) = \Delta(C) = 2$. In this scenario, C is an n-cycle, denoted as C_n. As C_n cannot be a K_3, we have $n \geq 4$. Let $D = V(C_n)$ and by Observation 5, we only need to search for a 4-coloring f of P such that $V_1(f) \cap N_G[v] \neq \emptyset$ for $v \in D$, meaning $V_1(f)$ represents a maximal independent set of C_n. Let $\mathcal{MI}(C_n) = \{I_1, I_2, \ldots, I_t\}$. As a result, there are $t = |\mathcal{MI}(C_n)|$ instances: $P_i = (G, I \cup I_i, S \cup N_C(I_i))$ for $i \in [t]$. Since I_i is a maximal independent set, we have $V(C_n) = N_C[I_i]$. Consequently, $\mu(P) - \mu(P_i) = n\omega_2 - (n - |I_i|)\omega_{c3}$. By Lemma 3, $\mu(P) - \mu(P_i) \geq n(\omega_2 - \omega_{c3}) + i(C_n)\omega_{c3} = n(\omega_2 - \omega_{c3}) + \lceil\frac{n}{3}\rceil\omega_{c3}$ for $i \in [t]$. Therefore, we have the constraint $|\mathcal{MI}(C_n)|r^{-n(\omega_2-\omega_{c3})-\lceil\frac{n}{3}\rceil\omega_{c3}} \leq 1$. By Lemma 4, $|\mathcal{MI}(C_n)| = |\mathcal{MI}(C_{n-2})| + |\mathcal{MI}(C_{n-3})| = |\mathcal{MI}(C_{n-4})| + |\mathcal{MI}(C_{n-5})| + |\mathcal{MI}(C_{n-3})|$ for $n \geq 9$. Hence,

$$|\mathcal{MI}(C_n)|r^{-n(\omega_2-\omega_{c3})-\lceil\frac{n}{3}\rceil\omega_{c3}} \leq \sum_{j=3}^{5}|\mathcal{MI}(C_{n-j})|r^{-(n-j)(\omega_2-\omega_{c3})-\lceil\frac{n-j}{3}\rceil\omega_{c3}} \cdot r^{-j(\omega_2-\omega_{c3})-\omega_{c3}}$$

for $n \geq 9$. Thus, it suffices to satisfy the following constraints:

$$\begin{cases}|\mathcal{MI}(C_j)|r^{-j(\omega_2-\omega_{c3})-\lceil\frac{j}{3}\rceil\omega_{c3}} \leq 1, \quad \text{for } j \in \{4,5,\ldots,8\}\\ \displaystyle\sum_{j=3}^{5} r^{-j(\omega_2-\omega_{c3})-\omega_{c3}} \leq 1.\end{cases} \tag{7}$$

Case 3. $\delta(C) = 2$ and $3 \leq \Delta(C) \leq 4$. Let $V' = \{v \in V(C) \mid d_C(v) = 2$ and there is a vertex $x \in N_C(v)$ with $d_C(x) \geq 3\}$. Since $\delta(C) = 2$ and $3 \leq \Delta(C) \leq 4$, we have $V' \neq \emptyset$.

Case 3.1. There exists a $v \in V'$, w.l.o.g., assuming $N_C(v) = \{v_1, v_2\}$ and $d_C(v_1) \geq 3$, such that at least one of the following conditions holds: (1) $v_1v_2 \notin E(G)$; (2) $d_C(v_1) = 4$; (3) $d_C(v_2) = 4$; (4) $d_C(v_i) \geq 3$ for $i \in [2]$. By Observation 5, we consider searching for a 4-coloring f of P such that $V_1(f) \cap N_G[v] \neq \emptyset$. Regardless of which of (1)-(4) holds, we explore three cases: (i) take v; (ii) discard v and take v_1; (iii) discard v, v_1 and take v_2. This leads to three subinstances: $P_1 = (G, I \cup \{v\}, S \cup N_C(v))$, $P_2 = (G, I \cup \{v_1\}, S \cup N_C(v_1))$ and $P_3 = (G, I \cup \{v_2\}, S \cup N_C(v_2) \cup \{v_1\})$. Recall that $0 \leq \omega_{c3} \leq \omega_1 \leq \omega_2 \leq 1$ by Equation (3). Notably, $\mu(P) - \mu(P_1) \geq \omega_2 + \sum_{u \in N_C(v)}(\omega_{d_C(u)} - \omega_{c3})$, $\mu(P) - \mu(P_2) \geq 1 + \sum_{u \in N_C(v_1)}(\omega_{d_C(u)} - \omega_{c3})$ and $\mu(P) - \mu(P_3) \geq \omega_{d_C(v_2)} + \sum_{u \in N_C(v_2) \cup \{v_1\}}(\omega_{d_C(u)} - \omega_{c3})$. For each condition (1)-(4), we respectively have the following constraints:

$$\begin{cases} r^{-(1+2\omega_2-2\omega_{c3})} + r^{-(1+3\omega_2-3\omega_{c3})} + r^{-(1+3\omega_2-3\omega_{c3})} \leq 1, \\ r^{-(1+2\omega_2-2\omega_{c3})} + r^{-(1+4\omega_2-4\omega_{c3})} + r^{-(1+2\omega_2-2\omega_{c3})} \leq 1, \\ r^{-(2+\omega_2-2\omega_{c3})} + r^{-(1+3\omega_2-3\omega_{c3})} + r^{-(2+3\omega_2-4\omega_{c3})} \leq 1, \\ r^{-(2+\omega_2-2\omega_{c3})} + r^{-(1+3\omega_2-3\omega_{c3})} + r^{-(2+2\omega_2-3\omega_{c3})} \leq 1. \end{cases} \tag{8}$$

Case 3.2. The complementary case (Case 3.1 doesn't hold). Let $v \in V'$ and $N_C(v) = \{v_1, v_2\}$. Since Case 3.1 doesn't apply, all of the following conditions hold: (1) $v_1v_2 \in E(G)$; (2) $d_C(v_1) \leq 3$; (3) $d_C(v_2) \leq 3$. (4) $d_C(v_i) = 2$ for some $i \in [2]$, w.l.o.g., assume $d_C(v_1) = 2$. By the definition of V', there exists $i \in [2]$ such that $d_C(v_i) \geq 3$, resulting in $d_C(v_2) = 3$. Now, let $N_C(v_2) = \{v_1, v, w\}$. If $d_C(w) = 2$, consider $N_C(w) = \{v_2, v_3\}$. Since $d_C(v_2) = 3$ and $d_C(w) = 2$, we infer that $w \in V'$. It is clear that $v_3v_2 \notin E(G)$. Considering w as analogous to the vertex v specified in Case 3.1, we encounter a contradiction to the assumption that (1) of Case 3.1 doesn't hold. Therefore, we conclude that $d_C(w) \geq 3$.

Case 3.2.1. $d_C(w) = 4$. We consider either taking w or discarding it, resulting in two subinstances: $P_1 = (G, I \cup \{w\}, S \cup N_C(w))$ and $P_2 = (G, I, S \cup \{w\})$. Notably, $d_{\mathcal{G}(P_1)}(v) = d_{\mathcal{G}(P_1)}(v_1) = 1$, and $\mathcal{G}(P_2)[\{v, v_1, v_2\}]$ is a K_3-component of $\mathcal{G}(P_2)$. Consequently, $\mu(P) - \mu(P_1) \geq 2 + 3\omega_2 + 2\omega_2 - 4\omega_{c3} - 2\omega_1$ and $\mu(P) - \mu(P_2) \geq 2 + 2\omega_2 - \omega_{c3} - \omega_{k3}$. Thus, we have the following constraint:

$$r^{-2-5\omega_2+4\omega_{c3}+2\omega_1} + r^{-2-2\omega_2+\omega_{c3}+\omega_{k3}} \leq 1. \tag{9}$$

Case 3.2.2. $d_C(w) = 3$ and there is a vertex $w_1 \in N_C(w)$ with $d_C(w_1) = 2$. Let $N_C(w) = \{v_2, w_1, w_2\}$. Since $d_C(w) = 3$, we have $w_1 \in V'$. Since Case 3.1 doesn't hold, we have $wx \in E(G)$ and $d_C(x) = 2$, where $x \in N_C(w_1) \setminus \{w\}$, implying that $x = w_2$ and $d_C(w_2) = 2$. By Observation 5, we only consider to search for a 4-coloring f of P such that $V_1(f) \cap N_G[v_1] \neq \emptyset$. Thus we consider three cases: (1) take v_2; (2) discard v_2 and take v; (3) discard v_2, v and take v_1. Correspondingly, there are three subinstances: $P_1 = (G, I \cup \{v_2\}, S \cup N_C(v_2))$, $P_2 = (G, I \cup \{v\}, S \cup N_C(v))$ and $P_3 = (G, I \cup \{v_1\}, S \cup N_C(v_1))$. Notably, $d_{\mathcal{G}(P_1)}(w_1) = d_{\mathcal{G}(P_1)}(w_2) = 1$, $\mathcal{G}(P_2)[\{w, w_1, w_2\}]$ and $\mathcal{G}(P_3)[\{w, w_1, w_2\}]$ are respectively both K_3-component of $\mathcal{G}(P_2)$ and $\mathcal{G}(P_3)$. Consequently, $\mu(P) - \mu(P_1) \geq 2 + 4\omega_2 - 3\omega_{c3} - 2\omega_1$ and $\mu(P) - \mu(P_i) \geq 2 + 4\omega_2 - 2\omega_{c3} - \omega_{k3}$ for $i \in \{2, 3\}$. So we have the following constraint:

$$r^{-2-4\omega_2+3\omega_{c3}+2\omega_1} + 2r^{-2-4\omega_2+2\omega_{c3}+\omega_{k3}} \leq 1. \tag{10}$$

Case 3.2.3. $d_C(w) = 3$ and $d_C(x) \geq 3$ for $x \in N_C(w)$. Let $N_C(w) = \{v_2, w_1, w_2\}$. By Observation 5, we only consider to search for a 4-coloring f of P such that $V_1(f) \cap N_G[w] \neq \emptyset$. Thus we consider four cases: (1) take w; (2) discard w and take v_2; (3) discard w, v_2 and take w_1; (4) discard w, v_2, w_1 and take w_2. Correspondingly, there are four subinstances: $P_1 = (G, I \cup \{w\}, S \cup N_C(w))$, $P_2 = (G, I \cup \{v_2\}, S \cup N_C(v_2))$, $P_3 = (G, I \cup \{w_1\}, S \cup N_C(w_1) \cup \{v_2\})$ and $P_4 = (G, I \cup \{w_2\}, S \cup N_C(w_2) \cup \{v_2, w_1\})$. Notably, $d_{\mathcal{G}(P_1)}(v) = d_{\mathcal{G}(P_1)}(v_1) = d_{\mathcal{G}(P_3)}(v) = d_{\mathcal{G}(P_3)}(v_1) = d_{\mathcal{G}(P_4)}(v) = d_{\mathcal{G}(P_4)}(v_1) = 1$. Consequently, $\mu(P) - \mu(P_1) \geq 4 + 2\omega_2 - 3\omega_{c3} - 2\omega_1$, $\mu(P) - \mu(P_2) \geq 2 + 2\omega_2 - 3\omega_{c3}$, and $\mu(P) - \mu(P_i) \geq 3 + 4\omega_2 - 4\omega_{c3} - 2\omega_1$ for $i \in \{3, 4\}$. Thus we have the following constraint:

$$r^{-4-2\omega_2+3\omega_{c3}+2\omega_1} + r^{-2-2\omega_2+3\omega_{c3}} + 2r^{-3-4\omega_2+4\omega_{c3}+2\omega_1} \leq 1. \tag{11}$$

Case 4. $\delta(C) \geq 2$ and $\Delta(C) \geq 5$. Let $v \in V(C)$ be a vertex with maximum degree. We consider taking v or discarding v. Thus, there are two subinstances $P_1 = (G, I \cup \{v\}, S \cup N_C(v))$ and $P_2 = (G, I, S \cup \{v\})$. Since $\delta(C) \geq 2$ and $d_C(v) \geq 5$, we have $\mu(P) - \mu(P_1) \geq 1 + 5\omega_2 - 5\omega_{c3}$ and $\mu(P) - \mu(P_2) \geq 1 - \omega_{c3}$. Therefore, we obtain the following constraint:

$$r^{-1-5\omega_2+5\omega_{c3}} + r^{-1+\omega_{c3}} \leq 1. \tag{12}$$

Case 5. $3 \leq \delta(C) \leq \Delta(C) \leq 4$. In this case, we initially address Cases 5.1-5.4, handling situations that C includes specific structures. Then, assuming that Cases 5.1-5.4 don't hold, we proceed to discuss two subcases: $\delta(C) = 3$ in Case 5.5 and $\delta(C) = \Delta(C) = 4$ in Case 5.6.

Case 5.1. There is a vertex v in C with $N_C(v) = \{v_1, v_2, v_3\}$, such that $d_C(v_1) = 4$, and v_i, where $i \in \{2, 3\}$, satisfies that $d_C(v_i) = 4$ or $v_i v_1 \notin E(G)$. By Observation 5, we only consider searching for a 4-coloring f of P such that $V_1(f) \cap N_G[v] \neq \emptyset$. Thus, we consider four cases: (1) take v; (2) discard v and take v_1; (3) discard v, v_1 and take v_2; (4) discard v, v_1, v_2 and take v_3. Correspondingly, there are four subinstances: $P_1 = (G, I \cup \{v\}, S \cup N_C(v))$, $P_2 = (G, I \cup \{v_1\}, S \cup N_C(v_1))$, $P_3 = (G, I \cup \{v_2\}, S \cup N_C(v_2) \cup \{v_1\})$ and $P_4 = (G, I \cup \{v_3\}, S \cup N_C(v_3) \cup \{v_1, v_2\})$. Notably, $d_C(v) = 3$, $d_C(v_1) = 4$, and $d_C(v_i) = 4$ or $v_i v_1 \notin E(G)$ for $i \in \{2, 3\}$. Consequently, we have $\mu(P) - \mu(P_1) \geq 4 - 3\omega_{c3}$ and $\mu(P) - \mu(P_i) \geq 5 - 4\omega_{c3}$ for $i \in \{2, 3, 4\}$. Thus we have the following constraint:

$$r^{-4+3\omega_{c3}} + 3r^{-5+4\omega_{c3}} \leq 1. \tag{13}$$

Case 5.2. There exist vertices $v_1, v_2, v_3, v_4 \in V(C)$ such that $C[\{v_1, v_2, v_3, v_4\}]$ is a K_4. Notably, since C is not a K_4, we have $|N_C(\{v_1, v_2, v_3, v_4\})| \geq 1$. Let $V' = \{v_i \mid d_C(v_i) = 4, i \in [4]\}$.

Case 5.2.1. $|V'| \geq 3$. As $C[\{v_1, v_2, v_3, v_4\}]$ forms a K_4, in any 4-coloring f of P, one of v_1, v_2, v_3, v_4 must be colored 1. Since there are at least three 4-vertices among v_1, v_2, v_3, v_4, we have the following constraint:

$$3r^{-5+4\omega_{c3}} + r^{-4+3\omega_{c3}} \leq 1. \tag{14}$$

Case 5.2.2. $|V'| \leq 2$ and $|N_C(\{v_1, v_2, v_3, v_4\})| = 1$. Assume $N_C(\{v_1, v_2, v_3, v_4\}) = \{w\}$. We consider either taking w or discarding it. Thus, there are two subinstances: $P_1 = (G, I \cup \{w\}, S \cup N_C(w))$ and $P_2 = (G, I, S \cup \{w\})$. Notably $\mathcal{G}(P_2)[\{v_1, v_2, v_3, v_4\}]$ is a K_4-component of $\mathcal{G}(P_2)$. Consequently $\mu(P) - \mu(P_1) \geq 4 - 3\omega_{c3}$ and $\mu(P) - \mu(P_2) \geq 5 - \omega_{c3} - \omega_{k4}$. This yields the following constraint:

$$r^{-4+3\omega_{c3}} + r^{-5+\omega_{c3}+\omega_{k4}} \leq 1. \tag{15}$$

Case 5.2.3. $|V'| \le 2$ and $|N_C(\{v_1, v_2, v_3, v_4\})| \ge 2$. In fact, $|N_C(\{v_1, v_2, v_3, v_4\})| = |V'| = 2$. Let $N_C(\{v_1, v_2, v_3, v_4\}) = \{w_1, w_2\}$, and w.l.o.g., assume $w_i v_i \in E(G)$ for $i \in [2]$. Notably, $d_C(v_i) = 4$ for $i \in [2]$. If $d_C(w_i) = 3$ for some $i \in [2]$, with considering w_i and v_i as analogous to the vertices v and v_1 specified in Case 5.1, we get a contradiction to that Case 5.1 doesn't hold. Thus we have $d_C(w_i) = 4$ for $i \in [2]$. Consider the following three cases: (1) take w_1; (2) discard w_1 and take w_2; (3) discard w_1, w_2. Correspondingly, there are three subinstances: $P_1 = (G, I \cup \{w_1\}, S \cup N_C(w_1)), P_2 = (G, I \cup \{w_2\}, S \cup N_C(w_2) \cup \{w_1\})$ and $P_3 = (G, I, S \cup \{w_1, w_2\})$. Notably $\mathcal{G}(P_3)[\{v_1, v_2, v_3, v_4\}]$ is a K_4-component of $\mathcal{G}(P_3)$. Consequently $\mu(P) - \mu(P_i) \ge 5 - 4\omega_{c3}$ for $i \in [2]$ and $\mu(P) - \mu(P_3) \ge 6 - 2\omega_{c3} - \omega_{k4}$. This leads to the following constraint:

$$2r^{-5+4\omega_{c3}} + r^{-6+2\omega_{c3}+\omega_{k4}} \le 1. \tag{16}$$

Case 5.3. There is a vertex v of C with $N_C(v) = \{v_1, v_2, v_3\}$, such that $d_C(v_1) = d_C(v_2) = 4$. Since Case 5.1 doesn't hold, we have $d_C(v_3) = 3$. If $v_3 v_i \notin E(G)$, for some $i \in [2]$, then considering v and v_i as analogous to the vertices v and v_1 specified in Case 5.1, we get a contradiction to that Case 5.1 doesn't hold. Consequently $v_3v_1, v_3v_2 \in E(G)$. Since C doesn't contain a K_4, we have $v_1v_2 \notin E(G)$. Let $N_C(v_1) = \{v, v_3, w_1, w_2\}$, and notably $w_i \notin N_C[v]$ for $i \in [2]$. Let $D = \{v, v_1\}$ and by Observation 5, we only consider to search for a 4-coloring f of P such that $V_1(f) \cap N_G[x] \neq \emptyset$ for $x \in D$. Thus when we discard v, v_1, v_3, we need to take v_2 and one of w_1, w_2. Specifically we consider the following five cases: (1) take v; (2) discard v and take v_3; (3) discard v, v_3 and take v_1; (4) discard v, v_3, v_1 and take v_2, w_1; (5) discard v, v_3, v_1, w_1 and take v_2, w_2. Correspondingly, we have the following five subinstances: $P_1 = (G, I \cup \{v\}, S \cup N_C(v))$, $P_2 = (G, I \cup \{v_3\}, S \cup N_C(v_3))$, $P_3 = (G, I \cup \{v_1\}, S \cup N_C(v_1))$, $P_4 = (G, I \cup \{v_2, w_1\}, S \cup N_C(\{v_2, w_1\}))$, $P_5 = (G, I \cup \{v_2, w_2\}, S \cup N_C(\{v_2, w_2\}) \cup \{w_1\})$. Notably, if $w_1 \in N_G(v_2)$ or $w_2 \in N_G(v_2)$, then case (4) or (5) need not be considered. Thus when $w_i \in N_G(v_2)$ for some $i \in [2]$, the worst-case scenario imposes the constraint: $2r^{-4+3\omega_{c3}} + r^{-5+4\omega_{c3}} + r^{-6+4\omega_{c3}} \le 1$. Now consider $w_1, w_2 \notin N_C(v_2)$. First we have $\mu(P) - \mu(P_i) \ge 4 - 3\omega_{c3}$ for $i \in [2]$, $\mu(P) - \mu(P_3) \ge 5 - 4\omega_{c3}$. We present the following Claim 6 to estimate $\mu(P) - \mu(P_i)$ for $i \in \{4, 5\}$. Since $v_1v_2 \notin E(G)$ and $|N_C(w_i) \backslash (N_C(v_2) \cup \{v_1\})| \ge 1$ for $i \in [2]$ by Claim 6, we have $|N_C(\{v_2, w_i\})| \ge 6$ for $i \in [2]$. Thus $\mu(P) - \mu(P_i) \ge 8 - 6\omega_{c3}$ for $i \in \{4, 5\}$. Then we have the following constraints:

$$\begin{cases} 2r^{-4+3\omega_{c3}} + r^{-5+4\omega_{c3}} + r^{-6+4\omega_{c3}} \le 1, \\ 2r^{-4+3\omega_{c3}} + r^{-5+4\omega_{c3}} + 2r^{-8+6\omega_{c3}} \le 1. \end{cases} \tag{17}$$

$\triangleright$ Claim 6. Given vertices v, v_1, v_2, v_3, and w_1, w_2 defined in Case 5.3, since Cases 5.1-5.2 don't hold and $w_1, w_2 \notin N_C(v_2)$, it follows that $|N_C(w_i) \setminus (N_C(v_2) \cup \{v_1\})| \ge 1$ for $i \in [2]$.

Proof. The proof of $|N_C(w_2) \setminus (N_C(v_2) \cup \{v_1\})| \ge 1$ is the same as the proof of $|N_C(w_1) \setminus (N_C(v_2) \cup \{v_1\})| \ge 1$. Thus, our aim is to demonstrate that $|N_C(w_1) \setminus (N_C(v_2) \cup \{v_1\})| \ge 1$.

- If $d_C(w_1) = 4$, then $|N_C(w_1) \setminus (N_C(v_2) \cup \{v_1\})| \ge 1$.
- If $d_C(w_1) = 3$ and $|N_C(w_1) \setminus (N_C(v_2) \cup \{v_1\})| = 0$, then $(N_C(w_1) \setminus \{v_1\}) \subseteq N_C(v_2)$.

Notably $N_C(v_1) = \{v, v_3, w_1, w_2\}$. Consequently, if there exists a vertex $x \in N_C(w_1) \cap N_C(v_1)$, then $x = w_2$ and $x \in N_C(v_2)$, leading to a contradiction to $w_2 \notin N_C(v_2)$. Thus $xv_1 \notin E(G)$ for $x \in N_C(w_1) \setminus \{v_1\}$. Considering w_1 and v_1 as analogous to the vertices v and v_1 specified in Case 5.1, we get a contradiction to the assumption that Case 5.1 doesn't hold. This yields $|N_C(w_1) \setminus (N_C(v_2) \cup v_1)| \ge 1$. $\triangleleft$

Case 5.4. There is a 3-cycle of C containing a 3-vertex of C. Assume that C_3 is a 3-cycle of C such that $|D|$ is maximum, where $D = \{w \in V(C_3) \mid d_C(w) = 3\}$. Let $V(C_3) = \{v_1, v_2, v_3\}$. By Observation 5, we only consider to search for a 4-coloring f of P such that $V_1(f) \cap N_G[w] \neq \emptyset$ for $w \in D$. Let $T = N_C(D) \setminus V(C_3)$. Then when we discard v_1, v_2, v_3, we directly take vertices in T. Specifically, we consider the following four cases: (1) take v_1; (2) discard v_1 and take v_2; (3) discard v_1, v_2 and take v_3; (4) discard v_1, v_2, v_3 and take T. Correspondingly, we have the following four subinstances: $P_1 = (G, I \cup \{v_1\}, S \cup N_C(v_1))$, $P_2 = (G, I \cup \{v_2\}, S \cup N_C(v_2))$, $P_3 = (G, I \cup \{v_3\}, S \cup N_C(v_3))$, and $P_4 = (G, I \cup T, S \cup N_C(T) \cup V(C_3))$. If T is not an independent set of G, we only need to consider (1)-(3). Thus, when T is not an independent set of G, the worst-case scenario imposes the constraint: $3r^{-4+3\omega_{c3}} \leq 1$. Now Consider that T is an independent set of G. We present Claim 7 to estimate $\mu(P) - \mu(P_i)$ for $i \in [4]$. Since $|D| \geq 2, |T| \geq 2$ by Claim 7 and $3 \leq \delta(C) \leq \Delta(C) \leq 4$, we have either $|D| = 2$, $|N_C(T) \cup V(C_3)| \geq 4$, or $|D| = 3$, $|N_C(T) \cup V(C_3)| \geq 5$. When $|D| = 2$, w.l.o.g., assume $d_C(v_3) = 4$. Since $|D| = 2$ or $|D| = 3$, one of the following holds: (i) $\mu(P) - \mu(P_i) \geq 4 - 3\omega_{c3}$ for $i \in [2]$, $\mu(P) - \mu(P_3) \geq 5 - 4\omega_{c3}$ and $\mu(P) - \mu(P_4) \geq 6 - 4\omega_{c3}$; (ii) $\mu(P) - \mu(P_i) \geq 4 - 3\omega_{c3}$ for $i \in [3]$ and $\mu(P) - \mu(P_4) \geq 7 - 5\omega_{c3}$. Then we have the following constraints:

$$\begin{cases} 3r^{-4+3\omega_{c3}} \leq 1, \\ 2r^{-4+3\omega_{c3}} + r^{-5+4\omega_{c3}} + r^{-6+4\omega_{c3}} \leq 1, \\ 3r^{-4+3\omega_{c3}} + r^{-7+5\omega_{c3}} \leq 1. \end{cases} \tag{18}$$

▷ Claim 7. Given C_3 and T as defined in Case 5.4, since Cases 5.2-5.3 don't hold, we have $|D| \geq 2$ and $|T| \geq 2$.

Proof. Notably $|D| \geq 1$. If $|D| = 1$, w.l.o.g., assuming $d_C(v_1) = 3$, then $v_2, v_3 \in N_C(v_1)$, with $d_C(v_2) = d_C(v_3) = 4$, contradicts with that Case 5.3 doesn't hold. Thus $|D| \geq 2$.

- If $|D| = 2$, suppose to the contrary that $|T| \leq 1$. W.l.o.g., assume $d_C(v_1) = d_C(v_2) = 3$, $d_C(v_3) = 4$ and $N_C(\{v_1, v_2\}) = \{v_3, w\}$. If $d_C(w) = 3$, then $C[\{v_1, v_2, w\}]$ forms a 3-cycle of C with $d_C(v_1) = d_C(v_2) = d_C(w) = 3$, contradicting the maximality of $|D|$. Thus, $d_C(w) = 4$. Since $d_C(v_1) = 3, d_C(w) = 4$, and $|D| = 2$, $N_C(v_1)$ contains two 4-vertices of C, leading to a contradiction to that Case 5.3 doesn't hold.
- If $|D| = 3$, suppose to the contrary that $|T| \leq 1$. Then $|N_C(\{v_1, v_2, v_3\})| = 1$, implying $C[N_C[\{v_1, v_2, v_3\}]]$ is a K_4, contradicting that Case 5.2 doesn't hold. ◁

Case 5.5. $\delta(C) = 3$. If $\Delta(C) = 4$, then there exists $uw \in E(G)$ such that $d_C(u) = 3$ and $d_C(w) = 4$. Assume $N_C(u) = \{w, u_1, u_2\}$. Since Case 5.4 doesn't hold, $N_C(u)$ is an independent set of C. Thus $d_C(w) = 4$ and $u_1w, u_2w \notin E(G)$ contradicts the assumption that Case 5.1 doesn't hold. So $\delta(C) = \Delta(C) = 3$. Let v be a 3-vertex of C with $N_C(v) = \{v_1, v_2, v_3\}$. Since Case 5.4 doesn't hold, $N_C(v)$ is an independent set of C. By Observation 5, we only need to search for a 4-coloring f of P such that $V_1(f) \cap N_G[v] \neq \emptyset$. Thus, we consider the following four cases: (1) take v; (2) discard v and take v_1; (3) discard v, v_1 and take v_2; (4) discard v, v_1, v_2 and take v_3. Correspondingly, there are four subinstances: $P_1 = (G, I \cup \{v\}, S \cup N_C(v))$, $P_2 = (G, I \cup \{v_1\}, S \cup N_C(v_1))$, $P_3 = (G, I \cup \{v_2\}, S \cup N_C(v_2) \cup \{v_1\})$, and $P_4 = (G, I \cup \{v_3\}, S \cup N_C(v_3) \cup \{v_1, v_2\})$. Since $N_C(v)$ is an independent set in C, we have $|N_C(v_2) \cup \{v_1\}| = 4$ and $|N_C(v_3) \cup \{v_1, v_2\}| = 5$. Recall the definition of $E_G(S_1, S_2)$ in the Preliminary. Since Case 5.4 doesn't hold, we have $|E_C(N_C[v], V(\mathcal{G}(P_i)))| \geq 6$ for $i \in [2]$ and $|E_C(N_C[v_2] \cup \{v_1\}, V(\mathcal{G}(P_3)))| \geq 3$. Notably, a vertex $w \in V(\mathcal{G}(P_i))$ contributes $1 - \omega_{d_{\mathcal{G}(P_i)}(w)}$ weight to $\mu(P) - \mu(P_i)$ for $i \in [4]$. Since $1 - \omega_2 \leq \omega_2 - \omega_1 \leq \omega_1 - \omega_0$ by

Equation (3), we have $1 - \omega_{d_{\mathcal{G}(P_i)}(w)} \geq (d_{\mathcal{G}(P)}(w) - d_{\mathcal{G}(P_i)}(w))(1 - \omega_2)$ for $w \in V(\mathcal{G}(P_i))$ and $i \in [4]$. Since $|E_C(N_C[v], V(\mathcal{G}(P_i)))| \geq 6$ for $i \in [2]$ and $|E_C(N_C[v_2] \cup \{v_1\}, V(\mathcal{G}(P_3)))| \geq 3$, we have $\mu(P) - \mu(P_j) \geq 4 - 3\omega_{c3} + 6(1 - \omega_2)$ for $j \in [2]$, $\mu(P) - \mu(P_3) \geq 5 - 4\omega_{c3} + 3(1 - \omega_2)$, and $\mu(P) - \mu(P_4) \geq 6 - 5\omega_{c3}$. Thus, we have the following constraint:

$$2r^{-4+3\omega_{c3}-6(1-\omega_2)} + r^{-5+4\omega_{c3}-3(1-\omega_2)} + r^{-6+5\omega_{c3}} \leq 1. \tag{19}$$

Case 5.6. $\delta(C) = \Delta(C) = 4$. Let v be a vertex in C such that $|E(C[N_C(v)])|$ is minimized, and denote the induced subgraph by $G' = C[N_C(v)]$. Let $N_C(v) = \{v_1, v_2, v_3, v_4\}$. If $\Delta(G') = 3$, w.l.o.g., assume $d_{G'}(v_1) = 3$, i.e., $N_C(v_1) = \{v, v_2, v_3, v_4\}$. Since there is no K_4 in C, we have $v_2v_3, v_2v_4, v_3v_4 \notin E(G)$. Notably $|E(G')| = 3$. Let $N_C(v_2) = \{v, v_1, w_1, w_2\}$. It is clear that $w_1, w_2 \notin N_C(\{v, v_1\})$. Thus $|E(C[N_C(v_2)])| \leq 2 < |E(G')|$, which contradicts the minimality of $|E(G')|$. Hence, $\Delta(G') \leq 2$.

Case 5.6.1. $\Delta(G') = 2$. W.l.o.g., assume $d_{G'}(v_1) = 2$ and $N_C(v_1) = \{v, v_2, v_3, w\}$, where $w \neq v_4$. Since there is no K_4 in C, we have $v_2v_3 \notin E(G)$. Let $D = \{v, v_1\}$ and by Observation 5, we only need to search for a 4-coloring f of P such that $V_1(f) \cap N_G[x] \neq \emptyset$ for $x \in D$. Thus when we discard v, v_1, v_2, v_3, we directly take w, v_4. Specifically, we have five cases: (1) take v; (2) discard v and take v_1; (3) discard v, v_1 and take v_2; (4) discard v, v_1, v_2 and take v_3; (5) discard v, v_1, v_2, v_3 and take w, v_4. Correspondingly, there five subinstances: $P_1 = (G, I \cup \{v\}, S \cup N_C(v))$, $P_2 = (G, I \cup \{v_1\}, S \cup N_C(v_1))$, $P_3 = (G, I \cup \{v_2\}, S \cup N_C(v_2))$, $P_4 = (G, I \cup \{v_3\}, S \cup N_C(v_3) \cup \{v_2\})$, $P_5 = (G, I \cup \{v_4, w\}, S \cup N_C(\{w, v_4\}) \cup \{v_2, v_3\})$. Notably we have $\mu(P) - \mu(P_i) \geq 5 - 4\omega_{c3}$ for $i \in [3]$ and $\mu(P) - \mu(P_4) \geq 6 - 5\omega_{c3}$. If $wv_4 \in E(G)$, we only need to consider (1)-(4). Considering the worst case, we assume $wv_4 \notin E(G)$. It is clear that $|N_C(\{w, v_4\}) \cup \{v_2, v_3\}| \geq 5$. Consequently, we have $\mu(P) - \mu(P_5) \geq 7 - 5\omega_{c3}$. Thus we have the following constraint:

$$3r^{-5+4\omega_{c3}} + r^{-6+5\omega_{c3}} + r^{-7+5\omega_{c3}} \leq 1. \tag{20}$$

Case 5.6.2. $\Delta(G') \leq 1$. It is clear that there are two vertices $w_1, w_2 \in V(G')$, assuming $V(G') \setminus \{w_1, w_2\} = \{w_3, w_4\}$, such that $E_C(\{w_1, w_2\}, \{w_3, w_4\}) = \emptyset$. W.l.o.g., we assume $E_C(\{v_1, v_2\}, \{v_3, v_4\}) = \emptyset$. By Observation 5, we only need to search for a 4-coloring f of P such that $V_1(f) \cap N_G[v] \neq \emptyset$. We have the following five cases: (1) take v; (2) discard v and take v_1; (3) discard v, v_1 and take v_3; (4) discard v, v_1, v_3 and take v_2; (5) discard v, v_1, v_2, v_3 and take v_4. Correspondingly, there are five subinstances: $P_1 = (G, I \cup \{v\}, S \cup N_C(v))$, $P_2 = (G, I \cup \{v_1\}, S \cup N_C(v_1))$, $P_3 = (G, I \cup \{v_3\}, S \cup N_C(v_3) \cup \{v_1\})$, $P_4 = (G, I \cup \{v_2\}, S \cup N_C(v_2) \cup \{v_1, v_3\})$, $P_5 = (G, I \cup \{v_4\}, S \cup N_C(\{v_4\}) \cup \{v_1, v_2, v_3\})$. Since $E_C(\{v_1, v_2\}, \{v_3, v_4\}) = \emptyset$, we have $\mu(P) - \mu(P_i) \geq 5 - 4\omega_{c3}$ for $i \in [2]$, $\mu(P) - \mu(P_j) \geq 6 - 5\omega_{c3}$ for $j \in \{3, 4\}$, and $\mu(P) - \mu(P_5) \geq 7 - 6\omega_{c3}$. Thus we have the following constraint:

$$2r^{-5+4\omega_{c3}} + 2r^{-6+5\omega_{c3}} + r^{-7+6\omega_{c3}} \leq 1. \tag{21}$$

▶ **Theorem 8.** *For a graph G, we can test whether there exists a 4-coloring of G within $O(1.7159^{|V(G)|})$ time complexity and polynomial space.*

Proof. We develop a numerical program to find the minimum value of r under Equations (3)-(21). Subsequently, we determine the values of r and all parameters ω_{c3}, ω_{k3}, ω_{k4} and ω_i ($i \geq 0$) as presented in Table 2. Moreover, we can check that, with the parameters in Table 2, Equations (3) to (21) hold. Thus, the time complexity of our algorithm for solving the

Table 2 The values of all parameters.

r	ω_{k3}	ω_{k4}	ω_{c3}	ω_0	ω_1	ω_2	ω_i $(i \geq 3)$
1.7159	2.96427205	3.43996139	0.57560002	0	0.92972213	0.99787746	1

instance $P = (G, I, S)$ is $O(1.7159^{\mu(P)})$. Then, the instance $P = (G, \emptyset, \emptyset)$ can be solved in $O(1.7159^{\mu(P)})$ time complexity and polynomial space. Given that $\mu(P) = \sum_{v \in V(G)} \omega_{d_G(v)} \leq |V(G)|$ by Equation (2), this lemma holds. ◀

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A Details about obtaining the convex program

In this section, we illustrate the process of transforming the problem of finding the minimum r into a convex program using the method in [21, 19], motivated by the goal of eliminating nonconvex constraint functions. The typical nonconvex constraint functions among Equations (3)-(21) are formed as $\sum_{j=1}^{t} r^{-\beta_j}$, where β_j is a linear combination of $\omega_{c3}, \omega_{k3}, \omega_{k4}$, and ω_i (where $i \geq 0$). Specifically, Equations (6)-(21) are the constraints formed as $\sum_{j=1}^{t} r^{-\beta_j}$. To handle these constraints, following the method in [21, 19], we introduce a new parameter λ, representing $\log_2 r$. Then, $\sum_{j=1}^{t} r^{-\beta_j} \leq 1$ can be transformed to $\sum_{j=1}^{t} 2^{-\lambda\beta_j} \leq 1$. Notably, all $\lambda\beta_j$, where $j \in [t]$, are linear combinations of $\lambda\omega_{c3}, \lambda\omega_{k3}, \lambda\omega_{k4}$, and $\lambda\omega_i$ (where $i \geq 0$). Then, we introduce new parameters ω'_{c3}, ω'_{k3}, ω'_{k4}, and ω'_i (where $i \geq 0$) representing $\lambda\omega_{c3}$, $\lambda\omega_{k3}$, $\lambda\omega_{k4}$, and $\lambda\omega_i$ (where $i \geq 0$) respectively, and refer to them along with λ as new parameters.

Now, we can transform Equations (6)-(21) into constraints similar to $\sum_{j=1}^{t} 2^{-\beta'_j}$, where β'_j is a linear combination of the new parameters. For instance, Equation (21) is transformed into the constraint $2 \cdot 2^{-5\lambda+4\omega'_{c3}} + 2 \cdot 2^{-6\lambda+5\omega'_{c3}} + 2^{-7\lambda+6\omega'_{c3}} \leq 1$. A key detail is that $2r^{-5+4\omega_{c3}}$ is transformed into $2 \cdot 2^{-5\lambda+4\omega'_{c3}}$ because $-5+4\omega_{c3}$ actually represents $-5\omega_3 + 4\omega_{c3}$, and ω_3 needs to be transformed to $\lambda\omega_3 = \omega'_3 = \lambda$. Similarly, we transform Equations (3)-(5) into the following constraints:

$$\begin{cases} 0 \leq \omega'_{c3} \leq \omega'_1 \leq \omega'_2 \leq \lambda, \\ \omega'_3 - \omega'_2 \leq \omega'_2 - \omega'_1 \leq \omega'_1, \\ 3\omega'_2 \geq \omega'_{k3}, \\ 4\omega'_3 \geq \omega'_{k4}, \\ \omega'_{c3} \geq \log_2 1.3645 \\ \omega'_{k3} \geq \log_2 4.837, \\ \omega'_{k4} \geq \log_2 6. \end{cases} \tag{22}$$

We now have a convex program: find the minimum λ under these new constraints. Importantly, the optimal value r^* of the original quasi-convex program is 2^{λ^*}, where λ^* is the optimal value of the convex program. Readers interested in further details on constructing the convex program for the measure and conquer method are encouraged to read [21, 19].

B Discussions about the bottleneck

In this section, we present the tight constraints among Equations (4)-(21) and discuss the potential for improving our algorithm's efficiency. The tight constraints within this range are Equations (4), (9)-(11), and (19). Specifically, the values of the left-hand side of these tight constraints are $0.3107881, 0.9990582, 0.9991849, 0.9999997$, and 0.9996582, respectively. Interestingly, even though Equation (4) is tight, it is not the unavoidable bottleneck. We searched for a minimum r under Equations (3)-(21), excluding Equations (9)-(11) and (19). The resulting values of r and all parameters are shown in Table 3. It is verified that the values in Table 3 satisfy all Equations (3)-(21), except for Equations (9)-(11) and (19). Since the r value in Table 3 is less than 1.7159, improving the methods for handling Cases 3.2 and 5.5 can lead to a faster algorithm.

Table 3 The values of all parameters with excluding Equations (9)-(11) and (19).

r	ω_{k3}	ω_{k4}	ω_{c3}	ω_0	ω_1	ω_2	ω_i $(i \geq 3)$
1.7141	2.99988192	3.66711472	0.57672093	0	0.99992243	0.9999775	1

C The intuitions of obtaining Equations (6)-(21)

In this section, we present the intuitions and details behind deriving Equations (6)-(21). These equations are formulated by analyzing the changes in measure between given instances and their subinstances. For example, consider Equation (6). In the Case 1, we obtain two subinstances, P_1 and P_2, by taking either u or v. And we show that $\mu(P) - \mu(P_i) \geq 2\omega_1 - \omega_3$ for $i \in [2]$ in the Case 1. Thus we present Equation (6). Now, let's discuss the intuitions and details behind obtaining the lower bounds of the changes in measure between the given instances and its subinstances.

Consider an instance $P = (G, I, S)$ and denote the maximum degree of $\mathcal{G}(P)$ as Δ. By Equation (2), $\mu(P)$ consists of three parts: (i) the number of K_3-components and K_4-components in $\mathcal{G}(P)$; (ii) the number of vertices in S; (iii) the number of i-vertices, where $i \geq 0$, in non-K_3-components and non-K_4-components of $\mathcal{G}(P)$. When we obtain a subinstance $P' = (G, I', S')$ by taking and discarding some vertices in $V(\mathcal{G}(P))$, the changes in measure $\mu(P) - \mu(P')$ depend on four parts: (i) The numbers of i-vertices in $\mathcal{G}(P)$ that are put into I', denoted by $s_{1,i}$; (ii) The numbers of i-vertices in $\mathcal{G}(P)$ that are put into S', denoted by $s_{2,i}$; (iii) The numbers of non-K_3-components (non-K_4-components) of $\mathcal{G}(P)$ that become K_3-components (K_4-components) in $\mathcal{G}(P')$, denoted by $s_{3,3}$ ($s_{3,4}$); (iv) The numbers of i-vertices with $i \leq 2$ in $\mathcal{G}(P')$ that are j-vertices with $j > i$ in $\mathcal{G}(P)$, denoted by $s_{4,i,j}$. We emphasize that $s_{4,i,j}$ are about the vertices in $\mathcal{G}(P')$, instead of the vertices just in the non-K_3-components and non-K_4-components of $\mathcal{G}(P')$. Specifically, $\mu(P) - \mu(P') = \sum_{i=1}^{\Delta} s_{1,i}\omega_i + \sum_{i=1}^{\Delta} s_{2,i}(\omega_i - \omega_{c3}) + s_{3,3}(3\omega_2 - \omega_{k3}) + s_{3,4}(4 - \omega_{k4}) + \sum_{i=0}^{2}\sum_{j=1}^{\Delta} s_{4,i,j}(\omega_j - \omega_i)$. Let $s_{1,i^+} = \sum_{j=i}^{\Delta} s_{1,j}$ and $s_{2,i^+} = \sum_{j=i}^{\Delta} s_{2,j}$. Consider the lower bounds for s_{1,i^+} and s_{2,i^+}, denoted as $low_{1,i}$ and $low_{2,i}$ respectively. We emphasize that $low_{1,\Delta+1} = low_{2,\Delta+1} = 0$. Since $\omega_i \geq \omega_j$ when $i \geq j$ by Equation (3), $\sum_{i=1}^{\Delta} s_{1,i}\omega_i \geq \sum_{j=0}^{\Delta}(low_{1,j} - low_{1,j+1})\omega_j$ and $\sum_{i=1}^{\Delta} s_{2,i}(\omega_i - \omega_{c3}) \geq \sum_{j=0}^{\Delta}(low_{2,j} - low_{2,j+1})(\omega_j - \omega_{c3})$. This implies that to obtain the lower bounds of $\mu(P) - \mu(P')$, we only need to get the lower bounds of s_{1,i^+} and s_{2,i^+} where $i \geq 0$, and the lower bounds of $s_{3,3}$, $s_{3,4}$, and $s_{4,i,j}$ where $0 \leq i \leq 2$ and $i < j$.

Actually, from the deductions in each case in the main body, we can directly derive the lower bounds of all kinds of "s" (i.e., s_{1,i^+}, s_{2,i^+}, $s_{3,3}$, $s_{3,4}$, and $s_{4,i,j}$). For the reader's convenience in verifying correctness, we provide all the lower bounds for Cases 1-5 below. Specifically, in each case, the content in (i) represents the lower bounds of all kinds of "s" derived by the corresponding subinstance P_i. If we do not provide the lower bounds of a particular type of "s", it means that we adopt the trivial lower bound. Specifically, the trivial lower bound is 0, except for s_{1,i^+} and s_{2,i^+}. It is clear that $s_{1,i^+} \geq s_{1,j^+}$ and $s_{2,i^+} \geq s_{2,j^+}$ when $j \geq i$. If no lower bound of s_{1,j^+} is given for $j > i$, the trivial lower bound of s_{1,i^+} is 0; otherwise, the trivial lower bound of s_{1,i^+} is $\max\{low_{1,j} \mid j > i\}$, where $low_{1,j}$ is the presented lower bound of $s_{1,j}$. The same applies to the trivial lower bound of s_{2,i^+}.

Case 1: (1) $s_{1,1^+} = 1, s_{2,1^+} = 1$; (2) $s_{1,1^+} = 1, s_{2,1^+} \geq 1$.

Case 2: In the main body, we have already provided the exact form of $\mu(P) - \mu(P')$ for each subinstance and determined their lower bounds using the exact form.

Case 3.1: The lower bounds of $\mu(P) - \mu(P')$ are divided into four cases. We give the lower bounds of all kinds "s" in the following (a)-(d).

(a). (1) $s_{1,2^+} = 1, s_{2,2^+} = 2, s_{2,3^+} \geq 1$; (2) $s_{1,3^+} = 1, s_{2,2^+} \geq 3$; (3) $s_{1,2^+} = 1, s_{2,2^+} \geq 3, s_{2,3^+} \geq 1$;

(b). (1) $s_{1,2^+} = 1, s_{2,2^+} = 2, s_{2,4^+} \geq 1$; (2) $s_{1,4^+} = 1, s_{2,2^+} = 4$; (3) $s_{1,2^+} = 1, s_{2,2^+} \geq 2, s_{2,4^+} \geq 1$;

(c). (1) $s_{1,2^+} = 1, s_{2,3^+} = 2$; (2) $s_{1,3^+} = 1, s_{2,2^+} \geq 3$; (3) $s_{1,4^+} = 1, s_{2,2^+} \geq 4, s_{2,3^+} \geq 1$;

(d). (1) $s_{1,2^+} = 1, s_{2,3^+} = 2$; (2) $s_{1,3^+} = 1, s_{2,2^+} \geq 3$; (3) $s_{1,3^+} = 1, s_{2,2^+} \geq 3, s_{2,3^+} \geq 1$.

Case 3.2.1: (1) $s_{1,4^+} = 1, s_{2,2^+} = 4, s_{2,3^+} \geq 1, s_{4,1,2} \geq 2$; (2) $s_{2,4^+} = 1, s_{3,3} \geq 1, s_{4,2,3} \geq 1$.

Case 3.2.2: (1) $s_{1,3^+} = 1, s_{2,2^+} = 3, s_{2,3^+} = 1, s_{4,1,2} \geq 2$; (2)-(3) $s_{1,2^+} = 1, s_{2,2^+} = 2, s_{2,3^+} = 1, s_{3,3} = 1, s_{4,2,3} = 1$.

Case 3.2.3: (1) $s_{1,3^+} = 1, s_{2,3^+} = 3, s_{4,1,2} \geq 2$; (2) $s_{1,3^+} = 1, s_{2,2^+} = 3, s_{2,3^+} = 1$; (3)-(4) $s_{1,3^+} = 1, s_{2,2^+} \geq 4, s_{2,3^+} \geq 2, s_{4,1,2} \geq 2$.

Case 4: (1) $s_{1,5^+} = 1, s_{2,2^+} \geq 5$; (2) $s_{2,5^+} = 1$.

Case 5.1: (1) $s_{1,3^+} = 1, s_{2,3^+} = 3$; (2)-(4) $s_{1,3^+} = 1, s_{2,3^+} \geq 4$.

Case 5.2.1: In this case we take one of $\{v_1, v_2, v_3, v_4\}$, if the vertex we take has degree 4, then we have $s_{1,4^+} = 1, s_{2,3^+} = 4$; and if has degree 3, then we have $s_{1,3^+} = 1, s_{2,3^+} = 3$.

Case 5.2.2: (1) $s_{1,3^+} = 1, s_{2,3^+} \geq 3$; (2) $s_{2,3^+} = 1, s_{3,4} \geq 1$.

Case 5.2.3: (1)-(2) $s_{1,4^+} = 1, s_{2,3^+} \geq 4$; (3) $s_{2,4^+} = 2, s_{3,4} \geq 1$.

Case 5.3: The lower bounds of $\mu(P) - \mu(P')$ are divided into two cases. We give the lower bounds of all kinds "s" in the following (a)-(b).

(a). (1)-(2) $s_{1,3^+} = 1, s_{2,3^+} = 3$; (3) $s_{1,4^+} = 1, s_{2,3^+} = 4$; (4) or (5) $s_{1,3^+} = 2, s_{2,3^+} \geq 4$.

(b). (1)-(2) $s_{1,3^+} = 1, s_{2,3^+} = 3$; (3) $s_{1,4^+} = 1, s_{2,3^+} = 4$; (4)-(5) $s_{1,3^+} = 2, s_{2,3^+} \geq 6$.

Case 5.4: The lower bounds of $\mu(P) - \mu(P')$ are divided into three cases. We give the lower bounds of all kinds "s" in the following (a)-(c).

(a). (1)-(3) $s_{1,3^+} = 1, s_{2,3^+} \geq 3$.

(b). (1)-(2) $s_{1,3^+} = 1, s_{2,3^+} \geq 3$; (3) $s_{1,4^+} = 1, s_{2,3^+} = 4$; (4) $s_{1,3^+} \geq 2, s_{2,3^+} \geq 4$.

(c). (1)-(3) $s_{1,3^+} = 1, s_{2,3^+} = 3$; (4) $s_{1,3^+} \geq 2, s_{2,3^+} \geq 5$.

Case 5.5: We have already provided the comprehensive deduction in the main body.

Case 5.6.1: (1)-(3) $s_{1,4^+} = 1, s_{2,4^+} = 4$; (4) $s_{1,4^+} = 1, s_{2,4^+} = 5$; (5) $s_{1,4^+} = 2, s_{2,4^+} \geq 5$.

Case 5.6.2: (1)-(2) $s_{1,4^+} = 1, s_{2,4^+} = 4$; (3)-(4) $s_{1,4^+} = 1, s_{2,4^+} \geq 5$; (5) $s_{1,4^+} = 1, s_{2,4^+} \geq 6$.