Art Galleries and Mobile Guards: Revisiting O'Rourke's Proof

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Abstract

O'Rourke (1983) proved that every n-vertex polygon, with $n \geqslant 4$, can be guarded by $\lfloor \frac{n}{4} \rfloor$ edges or diagonals – a variant of Chvátal's theorem for sufficiency of $\lfloor \frac{n}{3} \rfloor$ vertices. We present a short proof for a somewhat stronger result that allows us to impose some constraints on the guards. We prove that for every given subset V of vertices, the polygon can be guarded by $\lfloor \frac{n+2|V|}{4} \rfloor$ edges or diagonals that include at least one edge or diagonal incident to every vertex of V. This bound is the best achievable given the constraint for V. Our proof is by induction and suggests a simple linear-time algorithm after triangulating the polygon. The sufficiency of $\lfloor \frac{n}{4} \rfloor$ guards is a special case of the new result where V is the empty set.

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1 Introduction

The problems of guarding polygons with vertices, edges, and diagonals have a rich background. They are usually referred to as the "art gallery problems"; see, for instance, the book by O'Rourke [8] and the survey by Shermer [9]. This type of problems has been initiated as early as 1973 by Victor Klee [4, 8, 9] and still is an active research area from both combinatorial and computational points of view, see e.g. [1, 2, 6, 10].

In 1983, O'Rourke [7] proved that $\lfloor \frac{n}{4} \rfloor$ mobile guards are always sufficient to *guard* – see the interior of – an n-vertex polygon with $n \geqslant 4$. A *mobile guard* is an edge or a diagonal of the polygon.[1] This bound is the best achievable as some polygons need these many mobile guards, see Figure 1(a) for an example, attributed to Toussaint [8, 9]. This is a variant of Chvátal's theorem (1975) on the sufficiency of $\lfloor \frac{n}{3} \rfloor$ vertices to guard the polygon [4]. A short proof of Chvátal's theorem was given by Fisk in 1987 [5] and it has been extended by Michael and Pinciu in 2016 [6].

To the best of our knowledge, O'Rourke's original proof serves as the only proof of the sufficiency of $\lfloor \frac{n}{4} \rfloor$ mobile guards for almost 40 years. In this note we present an alternative proof for a more general result. The new proof is significantly shorter than the original proof, and allows us to impose some constraints on the guards. The trick is to find a suitable extension of the claim.

Let P be a simple polygon and G be a triangulation of P. To guard P it suffices to have one vertex of each triangular face of G on some mobile guard. We use the terms *edge* and *diagonal* to refer to the edges and diagonals of P. A *guard set* is a set of mobile guards that guard P.

[1] O'Rourke uses the terms "mobile guard" and "diagonal guard" interchangeably.

O'Rourke's Proof Technique

This proof is by induction, and it is mainly combinatorial. It involves careful edge contractions in G, which result in a somewhat different triangulation, without affecting the guarding process. To incorporate induction, it finds a diagonal d in G that cuts off 5, 6, 7, or 8 edges of P, and then applies the induction hypothesis on the remaining portion of P. The proof then involves thorough case analysis, crafted nicely, to patch the two parts together.

The existence of d is implied by a useful result of Chvátal [4] that, for any $k \geqslant 2$, G has a diagonal that cuts off at least k and at most $2k-2$ edges P. The original proof of Chvátal shows this for $k=4$, O'Rourke applies it for $k=5$, but it can be generalized to any $k \geqslant 2$.

New Proof Technique

We first find a suitable extension of the result that incorporates a stronger induction hypothesis. This allows us to cut a smaller portion of P that results in fewer cases with shorter proofs. We use a diagonal d that cuts off 3 or 4 edges of P. Such a diagonal exists by Chvátal's result for $k=3$. In Lemma 2 we give an alternative existential proof, that is just as simple, but uses the dual of G.

2 The New Proof

In this section we prove the following theorem. The sufficiency of $\lfloor \frac{n}{4} \rfloor$ guards is a special case of the theorem where V is the empty set.

▶ **Theorem 1.** *Let P be a polygon with $n \geqslant 4$ vertices and V be a subset of the vertices. Then $\lfloor \frac{n+2|V|}{4} \rfloor$ mobile guards are sufficient to guard P such that each vertex in V is incident to a mobile guard.*

The bound in Theorem 1 is the best achievable given the constraint for V; see Figure 1(b). It is well-known that the dual of any triangulation of P is a tree T of degree at most three. Moreover, each edge of T corresponds to a diagonal of P. Each leaf of T corresponds to an *ear* of P – a triangle formed by three consecutive vertices of P.

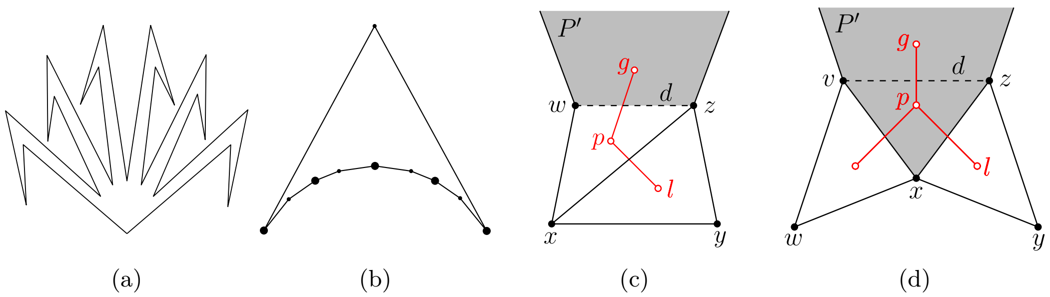

Figure 1 (a) A polygon that needs $\lfloor \frac{n}{4} \rfloor$ guards. (b) A polygon that needs $\lfloor \frac{n+2|V|}{4} \rfloor$ guards; bold vertices are in V. (c)-(d) Illustration of the proofs of Lemma 2 and Theorem 1; The tree T is in red, and the polygon P' is shaded.

▶ **Lemma 2.** *Every triangulation of a polygon with $n \geqslant 6$ vertices has a diagonal that cuts off 3 edges of the polygon, or 4 edges of the polygon that are incident to two adjacent ears.*

Proof. Consider the dual tree T, and root it at a vertex of degree 1. Let l be a deepest leaf, p be its parent, and g be its grandparent. The node p has at most two children, which are leaves due to our choice of l. The diagonal of P that corresponds to the edge (g, p) cuts off 3 edges if p has one child or 4 edges if p has two children; see Figure 1(c) and 1(d). In the latter case the two children correspond to two adjacent ears. ◀

Proof of Theorem 1. Let G be a triangulation of P. We will guard the triangular faces of G by choosing the edges of G (which are edges or diagonals of P) as our mobile guards. We prove by induction on n that P can be guarded by $\lfloor \frac{n+2|V|}{4} \rfloor$ edges of G such that each vertex in V is incident to an edge in the guard set. The base cases are proved at the end. Now we proceed with the induction step. Take a diagonal d in G (as in Lemma 2) that cuts off 3 or 4 edges of P.

Case 1. The diagonal d cuts 3 edges say wx, xy, and yz. Then $d = wz$. Since G is a triangulation, $wy \in G$ or $xz \in G$. Without loss of generality assume that $xz \in G$. Let P' be the polygon with $n - 2$ vertices that is obtained from P by cutting triangles xyz and wxz, as in Figure 1(c). Let G' be the graph obtained by removing vertices x, y and their incident edges from G. We set $V' = V \setminus \{x, y\}$ and consider the following cases.

Case 1(a). ($x \notin V$ and $y \notin V$) If $z \notin V'$ then we add z to V'. Hence $|V'| \leqslant |V| + 1$. By the induction hypothesis we guard P', with respect to G' and V', by at most $\lfloor \frac{(n-2)+2(|V|+1)}{4} \rfloor = \lfloor \frac{n+2|V|}{4} \rfloor$ edges of G'. These mobile guards also guard triangles xyz and wxz, and hence P, because z is incident to a mobile guard.

Case 1(b). ($x \in V$ and/or $y \in V$) Then $|V'| \leqslant |V| - 1$. By the induction hypothesis we guard P', with respect to G' and V', by at most $\lfloor \frac{(n-2)+2(|V|-1)}{4} \rfloor = \lfloor \frac{n+2|V|}{4} \rfloor - 1$ edges of G'. By adding the edge xy to these mobile guards we obtain a desired guard set for P.

Case 2. The diagonal d cuts 4 edges say vw, wx, xy, and yz. Then $d = vz$, and, by Lemma 2, vwx and xyz are two adjacent ears in P. Let P' be the polygon with $n - 2$ vertices that is obtained from P by cutting these ears, as in Figure 1(d). Let G' be the graph obtained by removing vertices w, y and their incident edges from G. We set $V' = V \setminus \{w, y\}$ and consider three cases.

Case 2(a). ($w \notin V$ and $y \notin V$) If $x \notin V'$ then we add x to V'. Hence $|V'| \leqslant |V| + 1$. By the hypothesis we guard P', with respect to G' and V', by at most $\lfloor \frac{(n-2)+2(|V|+1)}{4} \rfloor = \lfloor \frac{n+2|V|}{4} \rfloor$ edges of G'. These guards also guard P because x is incident to a guard.

Case 2(b). ($w \in V$ or $y \in V$ but not both) Due to symmetry assume that $w \in V$, and hence $|V'| = |V| - 1$. We guard P' by at most $\lfloor \frac{(n-2)+2(|V|-1)}{4} \rfloor = \lfloor \frac{n+2|V|}{4} \rfloor - 1$ guards, and then add wx to obtain a desired guard set for P.

Case 2(c). ($w \in V$ and $y \in V$) If $x \notin V'$ then we add x to V'. Hence $|V'| \leqslant |V| - 1$. Then we guard P' by a guard set of size at most $\lfloor \frac{(n-2)+2(|V|-1)}{4} \rfloor = \lfloor \frac{n+2|V|}{4} \rfloor - 1$. This guard set has at least one of the two edges of G' incident to x, say the edge xz. We remove xz from this set and add wx and yz. This gives a desired guard set for P (the triangles in G' that were guarded by xz are now guarded by wx or yz).

Base Cases. Each inductive step decreases n down by 2. Thus, for the base case it suffices to consider $n = 4, 5$. First assume that $n = 4$. Then any edge of P (which is also an edge of G) would guard P. If $|V| = 0$ then we guard P by an arbitrary edge. If $|V| = 1$ then we guard P by one edge incident to the only vertex of V. If $|V| = 2, 3, 4$ then we guard P by two opposite edges which are incident to all four vertices; in this case $2 = \lfloor \frac{4+2*2}{4} \rfloor \leqslant \lfloor \frac{4+2|V|}{4} \rfloor$.

Now assume that $n = 5$. Then G has two diagonals of P that are incident to the same vertex x. Hence any edge or diagonal incident to x, or any two arbitrary edges would guard P. If $|V| = 0$ then we guard P by an edge incident to x. If $|V| = 1$ then we guard P by an edge or diagonal that is incident to x and to the vertex of V. If $|V| = 2, 3$ then we guard P by two edges that are incident to all vertices of V, in this case $2 = \lfloor \frac{4+2*2}{4} \rfloor \leqslant \lfloor \frac{4+2|V|}{4} \rfloor$. If $|V| = 4, 5$ then we guard P by three edges incident to all five vertices, in this case $3 = \lfloor \frac{4+2*4}{4} \rfloor \leqslant \lfloor \frac{4+2|V|}{4} \rfloor$. This finishes the proof. ◀

Algorithmic Implication

Our proof of Theorem 1 suggests a linear-time algorithm for finding such a guard set. First we obtain an arbitrary triangulation G of P using Chazelle's linear-time algorithm [3]. Then we maintain two sets D_3 and D_4 of diagonals of G that cut 3 or 4 edges, respectively, of the current polygon. We initialize these sets in linear time by checking, for each vertex v of P, at most 4 vertices that appear before v and at most 4 vertices that appear after v along P. In each induction step, after cutting P by a diagonal d we update D_3 and D_4 in constant time by checking at most 4 vertices that appear before or after each endpoint of d along P'.

Notice that, in view of Lemma 2, one can maintain D_3 and D_4 in linear time also by checking the leaves of the dual tree T iteratively.

3 Concluding Remarks

One can extend Theorem 1 further by imposing more constraints. In particular, if E and F are subsets of the edges of P, then one can show the sufficiency of $\lfloor \frac{n+3|E|+2|V|+|F|}{4} \rfloor$ mobile guards that include every edge of E and at least one endpoint of every edge of F. This can be proven by considering a few more subcases in our proof, however, due to our desire of having a shorter proof we leave this to the interested reader.

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