Shortest Path Separators in Unit Disk Graphs

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Abstract

We introduce a new balanced separator theorem for unit-disk graphs involving two shortest paths combined with the 1-hop neighbours of those paths and two other vertices. This answers an open problem of Yan, Xiang and Dragan [CGTA '12] and improves their result that requires removing the 3-hop neighbourhood of two shortest paths. Our proof uses very different ideas, including Delaunay triangulations and a generalization of the celebrated balanced separator theorem of Lipton and Tarjan [J. Appl. Math. '79] to systems of non-intersecting paths.

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1 Introduction

A *geometric intersection graph* is an undirected graph where each vertex corresponds to a geometric object, and edges indicate which pairs of objects intersect each other. One common type of geometric intersection graph is the unit disk graph, which are geometric intersection graphs of disks with diameter 1. These graphs arise in modeling wireless communication. Such graphs also appear in applications such as VLSI design. These graphs have been extensively studied in the computational geometry community.

There have been many papers studying different optimization problems in unit disk graphs [13, 44, 29, 8, 10], as they are one of the simplest types of geometric intersection graph. As such, results for unit disk graphs are often used as a starting point to generalize to more general geometric intersection graphs with more complicated objects [28, 18, 7, 27, 6]. Over the years, researchers have built up a toolbox of techniques that can be applied to unit disk graphs. Some of the techniques have been inspired by the more geometric aspects of these graphs, such as geometric data structures [7, 6], planar spanners [8], and well separated pair decompositions [23]. Other techniques have been inspired by planar graph tools, such as planar separators [40, 14], shortest path seperators [44], and more recently VC-dimension in planar graphs [36, 34, 10]. Thus, there is much value in deepening our understanding of unit disk graphs, and in particular further building up its toolbox, as we do in this paper.

Separator theorems. In graphs, a *separator* is a small set of vertices whose removal splits the graph into smaller components. Separators are very useful for designing divide-and-conquer algorithms. Planar graphs are well-known for admitting good separators. The first separator theorem for planar graphs was due to Lipton and Tarjan [37], who proved that every planar graph on n vertices admits a separator of size $O(\sqrt{n})$ that can be computed in $O(n)$ time. Since then, many variants of separator theorems have been proven for planar graphs [37, 39, 21, 25, 42, 33, 11]. Some of these results can be naturally extended to graphs with bounded genus [16, 24] or to minor-free graphs [2, 30, 43].

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For unit disk graphs, many different separators exist, such as line separators [4] and clique separators [14]. When the unit disks have low *ply*[1], good separators are also known to exist [40, 41]. Separators are also known for more general objects, such as fat objects [5] or low-density objects [26]. Intersection graphs induced by arbitrary curves in the plane, also known as string graphs, have been studied in a series of works [19, 20, 38, 35]. Remarkably, Lee [35] proved that every string graph having m edges admits a balanced separator with $O(\sqrt{m})$ vertices.

Shortest path separators. The usefulness of a separator does not necessarily only depend on the number of vertices. An excellent example of good separators with large size are *shortest path separators*, i.e. separators consisting of a constant number of shortest paths. Thorup showed that every planar graph admits a separator consisting of at most two shortest paths [42]. Similar results have been proven for minor-free graphs [1]. Shortest path separators have been used extensively in distance-related problems in planar graphs, such as distance oracles [42, 32], planar emulators [11], and even network design problems [31, 22].

A natural question to ask is whether shortest path separator theorems can be adapted to unit disk graphs. Naively, such separators cannot exist, as the clique on n vertices is realizable as a unit disk graph for which no such separator can exist. However, we can strengthen the separator by also removing vertices in the k-neighborhood of the shortest path, *i.e.* vertices that are at a distance of at most k from the shortest path. Yan, Xiang and Dragan [44] proved that every unit disk graph admits a shortest path 3-neighborhood separator, that is, by removing two shortest paths and all vertices in the 3-hop neighborhood of any vertex on the shortest paths, the remaining graph is disconnected with every component having size at most $2/3$ of the vertices of the original graph. They left open the question of whether there exists a shortest path 1-neighborhood separator.

Our results. We answer the open question of Yan, Xiang and Dragan [44] in the affirmative. We show that every unit disk graph has a 1-neighborhood separator. In particular, it suffices to only remove the 1-neighborhood of two shortest paths plus the 1-neighborhood of two other vertices. While the proof of Yan, Xiang, and Dragan manipulates crossings in the intersection graph, our proof uses very different ideas involving paths in Delaunay triangulations and a generalization of the shortest path separators of Lipton and Tarjan to sets of *weakly non-crossing paths* in a triangulated planar graph that may be of independent interest. This result is optimal since as mentioned earlier, a shortest path 0-neighborhood separator does not exist for a clique, which is realizable as a unit disk graph.

▶ **Theorem 1.** *Every unit disk graph admits a shortest path 1-neighborhood separator.*

A first attempt. To illustrate the difficulty and to fully appreciate our contribution, let us consider one approach to construct shortest path 1-neighborhood separators for a unit disk graph $G = (V, E)$. To do so, we will make two overly wishful assumptions (that would be great if they always held).

Let $\mathcal{T}$ be a shortest path tree of G starting at a fixed vertex $s \in V$. We will wishfully assume that the induced drawing of $\mathcal{T}$ in the plane has no crossings (assumption 1). Next, we will assume that we can triangulate $\mathcal{T}$ to get a graph $G_{\mathcal{T}}$ (assumption 2) such that every edge of the triangulated graph $G_{\mathcal{T}}$ is an edge in G. Now, we can use the shortest path separator theorem of Lipton and Tarjan [37], on $G_{\mathcal{T}}$ with spanning tree $\mathcal{T}$ to get a Jordan

[1] the number of disks intersecting any given point

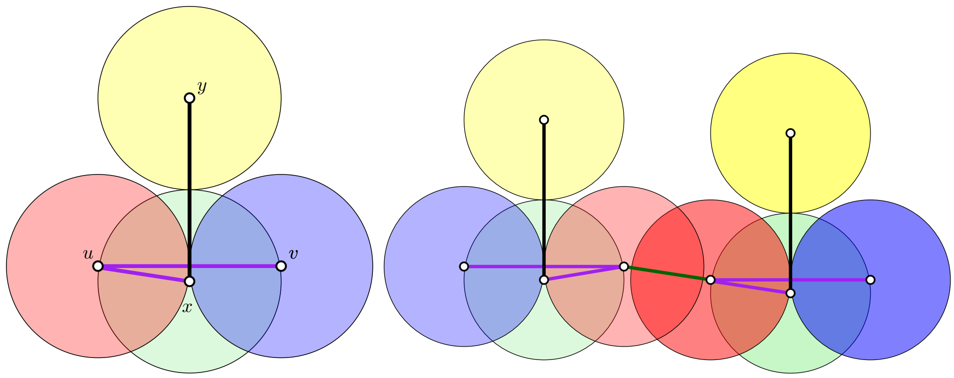

Figure 1 (Left) The points $u, v, x, y \in S$ are drawn with circles of radius 1/2. The unique shortest path tree in G with starting vertex u has a crossing edge. **(Right)** Two reflected copies results in a graph where no plane shortest path tree exists, regardless of starting vertex.

curve C that is a separator for $G_{\mathcal{T}}$. We can prove (and do so in Lemma 3) that all edges $uv \in E$ have the property that for all other edges crossing the line segment between u and v the crossing edge has at least one end point adjacent to either u or v (we call this property *cross-dominating* or *crominating*[2] for short). Thus the cycle C is in fact also a shortest path 1-neighborhood separator of G.

1. Our first assumption was that we could find a shortest path tree $\mathcal{T}$ of G whose natural embedding has no crossings. This is not always the case, there are examples (see Figure 1) of unit disk graphs G where no such shortest path tree exists. Instead, we will construct a non-crossing path system Π consisting of *pseudo-shortest paths*, i.e., for every vertex $u \in V$ we will find a path $\Pi[u]$ to s such that $\Pi[u]$ consists only of vertices on the shortest path between u and s and 1-neighbors of the shortest path. We show an extension of the planar separator algorithm to find a balanced separator in *path systems* of planar graphs.
2. Our second assumption was that we could triangulate the tree $\mathcal{T}$ to get a graph $G_{\mathcal{T}}$ such that every edge of the triangulation is an edge in G. We instead prove that all edges of the Delaunay triangulation have the crominating property, and furthermore, we show that we can construct Π using only edges of the Delaunay triangulation of the centers of the disk.[3]

2 Preliminaries

Crossings.[4] For two line segments uv and pq, we say that uv *crosses* pq if there is a point on both line segments that is not an endpoint of either uv or pq. Given two simple polygonal chains $P = [p_0, p_1, \ldots, p_\ell]$ and $Q = [q_0, q_1, \ldots, q_\ell]$, we say that P and Q have a *forward crossing* if the following hold:

1. $p_i = q_i$ for all $1 \leq i \leq \ell - 1$
2. The cyclic order of p_0, q_0, p_2 around p_1 is the same as the cyclic ordering of $p_\ell, q_\ell, p_{\ell-2}$ around $p_{\ell-1}$.

[2] This is a perfectly cromulent portmanteau to use.
[3] In fact, it is possible to show that there exists a triangulation (specifically the *edge constrained Delaunay triangulation* of Chew [12]) of any plane tree such that all edges are crominating. However, we will not discuss this further since the edge constrained Delaunay triangulation of a spanning subset of edges of a Delaunay triangulation is the original Delaunay triangulation.
[4] Our definitions of *crossing* are special cases of the definition from [17].

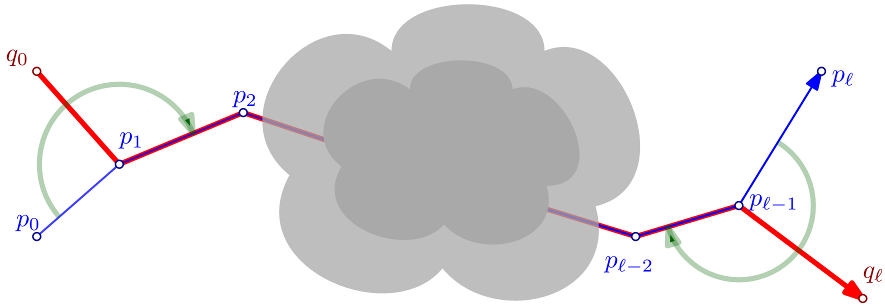

Figure 2 An example of a forward crossing between the red and blue chain. The cloud obscures the shared middle part of the polygonal chains.

See Figure 2 for an example of a forward crossing. The polygonal chains have a *backwards crossing* if P and the reversal of Q have a forward crossing. Finally, we say that P and Q *cross* if subpaths of P and Q have a forward crossing, a backwards crossing, or two edges p_ip_{i+1} and q_jq_{j+1} that cross. Equivalently P and Q do not cross if and only if there exists a small local perturbation of the vertices of each polygonal chain such that the two chains share no point in common (this is true as we only consider paths without spurs or forks [9, Section 3.1]). If a collection Π of polygonal chains are non-crossing (i.e. no pair of paths $P, Q \in \Pi$ cross), then we can also perturb the vertices of all the chains so that no two chains intersect (even at end points). We note that all chains that we will discuss are fairly nice, and amenable to these two characterizations of crossing. We encourage the interested reader to see the discussion in Chang, Erickson, and Xu [9] for a thoroughly comprehensive discussion of crossing and non-crossing polygonal chains.

Let G be a given graph with a straight-line embedding in $\mathbb{R}^2$. Then each edge of G embeds onto a line segment and each path in G embeds onto a polygonal chain. We say that two edges of G cross if their corresponding line segments cross. Similarly, two paths in G cross if their corresponding polygonal chains cross.

Shortest path separators. Lipton and Tarjan [37] showed that a connected triangulated planar graph on n vertices with arbitrary weights on the vertices has a *balanced shortest path separator* which is a Jordan curve consisting of two shortest paths and one edge such that the interior and exterior of the curve each have at most 2/3 of the total weight of the vertices of the graph.

▶ **Theorem 2** (Balanced shortest path separator of Lipton-Tarjan [37]). *Let $G = (V, E)$ be a maximally triangulated planar graph with n vertices and let T be a spanning shortest path tree of G rooted at s. Suppose that each vertex of the graph has a weight, and let W be the total weight of all the vertices. Then there exists an edge $uv \in E$ such that the Jordan curve C defined by the edge uv along with the path between u and v in T separates the graph into an interior A and exterior B that each have weight at most $2W/3$.*

Path systems. Given a graph $G = (V, E)$ and a fixed vertex $s \in V$, we define a *path system to s* as a function Π defined on some vertex set $V' \subseteq V$ that maps each vertex $u \in V'$ to a directed path $\Pi[u]$ from u to s in G. For this paper, we will always assume that our paths are *simple*, that is no vertex is visited more than once on the path. We will abuse notation and use Π to refer to the collection of paths to s. When G is planar, we say that the path system is *non-crossing* if for every pair of vertices $u, v \in V(G)$, the paths $\Pi[u]$ and $\Pi[v]$ are non-crossing. A path system is a *spanning* path system if every vertex in V is on at least one path of Π.

Pseudo-shortest paths. Given a graph G, a *pseudo-shortest path* P from u to v for two vertices $u, v \in V$ is a path that starts at u, ends at v, contains all vertices of a shortest path P' in G, and that every vertex on P but not on P' is adjacent to some vertex on P'. Note that pseudo-shortest paths are closed under concatenations, i.e. if we have a pseudo-shortest path from u to v, and a pseudo-shortest path from v to w, and a shortest path from u to w goes through v, then the concatenation of the paths is a pseudo-shortest path from u to w.

Unit disk graphs. Given a set S of points in $\mathbb{R}^2$, a unit disk graph $G = G(S) = (V, E)$ is a graph with vertices $V = S$, and edges between every pair of points $u, v \in S$ with distance less than 1, i.e. $|u - v| \leq 1$. Note that by this definition, a unit disk has diameter 1. Observe that unit disk graphs have a natural embedding in the plane, albeit with potentially many crossing edges. The following simple lemma about paths that cross the interior of a disk has been discovered in previous works (e.g. [44]); we include a proof here for the sake of completeness.

▶ **Lemma 3.** *Let $G = (V, E)$ be a unit disk graph, and let $u, v, x \in V$ be three distinct vertices with the edge $uv \in E$ where the straight line edge defining uv intersects the unit disk centered at x. Then x has an edge with at least one of u or v.*

Proof. Consider a point p on uv that lies in the unit disk centered at x. Since the edge uv has length at most 1, p is also in the unit disk of either u or v, so x has an edge with either u or v. ◀

3 Path systems of pseudo-shortest paths in unit disk graphs

In this section, we prove the existence of a path system Π of pseudo-shortest paths consisting of Delaunay triangulation edges for a unit disk graph G, and a fixed starting vertex s. We let W_a denote the set of vertices at distance a from a root vertex s. We will show how to build the paths $\Pi[u]$ for a vertex $u \in W_a$ by first a pseudo-shortest path to the nearest neighbor $w \in W_{a-1}$, then extending it to a pseudo-shortest path to s from w.

3.1 Nearest neighbors, Voronoi cells, and Delaunay paths

Consider an edge $uw \in E$. Either u and w are in adjacent Voronoi cells, or the line segment uw crosses a sequence of Voronoi cells of the points $u = v_1, v_2, \ldots, v_\ell = w$ for some ℓ. As the Voronoi diagram is dual to the Delaunay triangulation, between each pair of Voronoi cells there is a Delaunay edge. This induces a path in the Delaunay triangulation $\mathrm{DT}(G)$ between u and w, which we call the *Delaunay path between u and w*. Such paths were first considered by Dickerson and Drysdale [15], and also later by Cabello and Jejčič [3].

▶ **Lemma 4** (Dickerson-Drysdale [15]; Cabello-Jejčič [3]). *Let $uw \in E$. Let $P = [u = v_1, v_2, \ldots, v_\ell = w]$ be the Delaunay path between u and w. Then the following holds:*

1. *$|v_i - v_j| \leq 1$ for all $1 \leq i < j \leq \ell$, i.e. all pairs of vertices of P are connected to each other in the unit disk graph.*
2. *All vertices of P lie inside the disk with uw as diameter.*
3. *For all $1 \leq i < j < k \leq \ell$, we have $|v_i - v_j| < |v_i - v_k|$.*

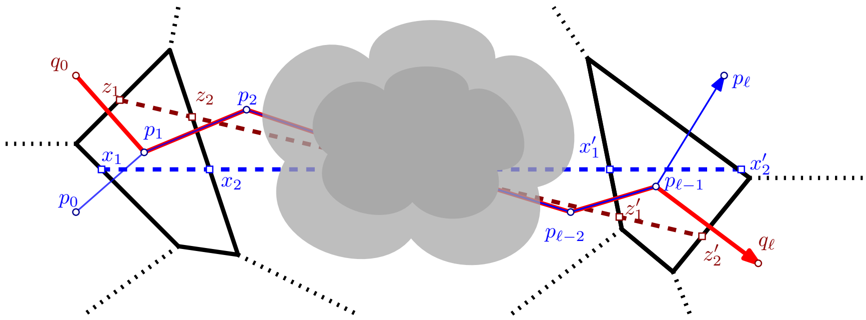

Figure 3 The points x_1, z_1, z_2, x_2 are clockwise around the Voronoi cell, leading to z_1 and z_2 having positive y-coordinates. The cloud covers the intermediate vertices on the Delaunay paths.

In particular, if $u \in W_a$ and w is the nearest[5] neighbor of u in W_{a-1}, then we denote by $\Delta[u]$ the Delaunay path between u and w, and call it the *Delaunay path for* u. In this case, it follows immediately from Lemma 4 that for all $i < \ell$, we have $v_i \in W_a$. Furthermore, $\Delta[u]$ is a pseudo-shortest path from u to w.

3.2 Non-crossing property of Delaunay paths

We now prove that the Delaunay paths do not cross.

▶ **Lemma 5.** *Let $u_1, u_2 \in W_a$ and w_1, w_2 be the corresponding nearest neighbors in W_{a-1}. Then the Delaunay paths for u_1 and u_2 do not cross.*

Proof. See Figure 3 throughout this proof. Consider the line segments u_1w_1 and u_2w_2. Note that these two line segments don't cross as if they did, then either w_1 would be closer to u_2 or w_1 would be closer to u_2. By rotating the plane and a suitable shift, we may assume that u_1w_1 is horizontal line on the x axis. Now let's consider the Delaunay paths $P = \Delta[u_1]$ from u_1 to w_1 and $Q = \Delta[u_2]$ from u_2 to w_2. We assume for sake of contradiction that P and Q cross. Since P and Q consists of edges in a Delaunay triangulation, there can be no simple crossings, thus there is some subpath $P' = [p_0, p_1, \ldots, p_\ell]$ of P and (possibly reversed) subpath $Q' = [q_0, q_1, \ldots q_\ell]$ of Q that form a forward crossing.

Consider the Voronoi cell of $p = p_1 = q_1$. Let x_1 and x_2 be the intersection of the line segment u_1w_1 with the Voronoi cell p, and z_1 and z_2 be the intersection of the line segment u_2w_2 with the Voronoi cell of p. Without loss of generality, we may assume the cyclic ordering p_0, q_0, p_2 around p_1 is clockwise (or we can reflect everything about the x axis). This implies that the Voronoi edges between p_1 and p_0, p_1 and q_0, p_1 and p_2 are also ordered clockwise about the Voronoi cell for p_1. As P' and Q' are parts of Delaunay paths, we know that x_1 lies on the Voronoi edge between p_1 and p_0, z_1 lies on the Voronoi edge between p_1 and q_0, and both z_2 and x_2 lie on the Voronoi edge between x_2 and p_2. Furthermore, since x_1x_2 and z_1z_2 don't intersect (as they are subsets of u_1w_1 and u_2w_2), we can conclude that x_1, z_1, z_2, x_2 is the cyclic ordering in the clockwise direction on the Voronoi cell of p_1. Since x_1 and x_2 are on the x axis, we can thus conclude that z_1 and z_2 have positive y-value.

Now let's consider the Voronoi cell of $p' = p_{\ell-1} = q_{\ell-1}$, and define x'_1, x'_2, z'_1, z'_2 analogously for this cell as we did before for the Voronoi cell of p. Since P' and Q' form a forward crossing, this implies $p_\ell, q_\ell, p_{\ell-2}$ are oriented clockwise as well. By the same argument as before about

[5] In this section, *nearest* refers to distances in the Euclidean metric, not in the graph metric.

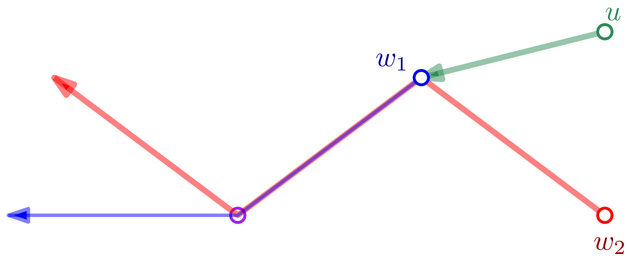

Figure 4 The paths $\Pi[w_1]$ (in blue) and $\Pi[w_2]$ (in red) for $w_1, w_2 \in W_{a-1}$ don't cross. However, the Delaunay path $\Delta[u]$ consisting of $u \in W_a$ to w_1 (in green) concatenated with $\Pi[w_1]$ forms a path that crosses $\Pi[w_2]$.

cyclic orderings, this implies that z_1' and z_2' must have a negative y value. However, z_2 had a positive y-value and z_1' had a negative y-value. This implies that the ray $\vec{r}(z_1, z_2)$ from z_1 to z_2 and the ray $\vec{r}(z_2', z_1')$ from z_2' to z_1' intersect the line between x_1 and x_2'. By symmetry, we can also show that the ray $\vec{r}(x_1, x_2)$ and the ray $\vec{r}(x_2', x_1')$ intersects the line between z_1 and z_2'. Thus we conclude that the line segment between z_1 and z_2' intersect the line segment between x_1 and x_2', a contradiction. ◀

We can also show that Delaunay paths between different levels do not cross either.

▶ **Lemma 6.** *Let $u_1 \in W_a$ and $u_2 \in W_b$ for some postive integers a and b with $a < b$. Then the Delaunay paths $\Delta[u_1]$ and $\Delta[u_2]$ do not cross.*

Proof. Let w_1 be the nearest neighbor of u_1 in W_{a-1}, and w_2 be the nearest neighbor of u_2 in W_{b-1}. We consider the Delaunay paths $\Delta[u_1] = [p_1 = u_1, p_2, \dots, p_k = w_1]$ and $\Delta[u_2] = [q_1 = u_2, q_2, \dots q_\ell = w_2]$. First observe that since $\Delta[u_1]$ and $\Delta[u_2]$ are Delaunay paths, they contain subsets of edges in the Delaunay triangulation. They can only have forward or backwards crossings. However, by Lemma 4, q_i for $i < \ell$ is on W_b and q_ℓ is in W_{b-1}, while all of the vertices of $\Delta[u_1]$ are on level a or level $a-1$. Since $a < b$, $\Delta[u_1]$ and $\Delta[u_2]$ can share at most one vertex in common, namely when $b-1 = a$ and q_ℓ is the common vertex. Since q_ℓ is an endpoint of $\Delta[u_1]$, a forward (or backward) crossing between $\Delta[u_1]$ and $\Delta[u_2]$ is not possible. ◀

3.3 Constructing pseudo-shortest paths from Delaunay paths

We have shown via Lemma 5 and Lemma 6 that no two Delaunay paths cross. This naturally suggests an intuitive way to inductively build $\Pi[u]$ based on distance from s. Suppose we have constructed a non-crossing path system $\Pi_{\leq a-1}$ of pseudo-shortest paths of vertices at distance at most $a-1$ from s by finding paths $\Pi[v]$ for all $v \in \bigcup_{i=0}^{a-1} W_i$, and we wanted to construct $\Pi[u]$ for $u \in W_a$. The natural method is to define $\Pi[u]$ to be the concatenation of the Delaunay path $\Delta[u]$ that goes from u to a vertex $w \in W_{a-1}$ with the path $\Pi[w]$. Clearly this is a pseudo-shortest path, as it is the concatenation of a pseudo-shortest path from u to w and from w to s.

Unfortunately, carelessly extending paths in this manner may in fact create crossings! Consider two vertices $w_1, w_2 \in W_{a-1}$ that have non-crossing paths $\Pi[w_1]$ and $\Pi[w_2]$ where $w_1 \in \Pi[w_2]$. If we attach the Delaunay path $\Delta[u]$ for a vertex $u \in W_a$ that has nearest neighbor w_1, then this might induce a crossing as pictured in Figure 4. Thankfully, with a little more care, we show that there is a way to extend the Delaunay path $\Delta[u]$ for all $u \in W_a$ into full paths that don't cross each other or $\Pi_{\leq a-1}$ with the following lemma.

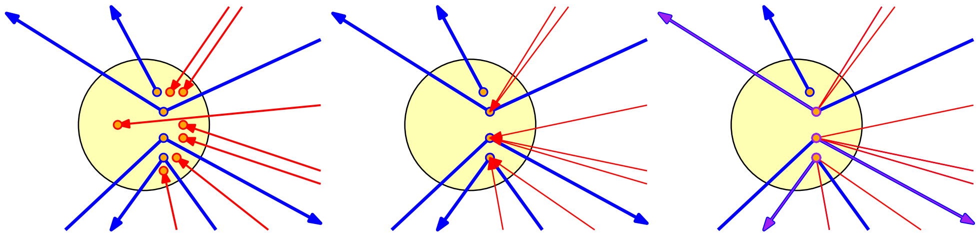

Figure 5 (Left) Step 1: The yellow disk represent the node w; blue paths represent paths in Π that goes through w (Π^w in the case of Lemma 8); red paths represent a non-crossing collection of paths that end at w (Δ^w in the case of Lemma 8). **(Middle)** Step 2: Snapping every red path to the vertex corresponding to the first blue path clockwise around the boundary. **(Right)** Step 3: Extending every red path by the continuation of the blue path marked in purple.

▶ **Lemma 7** (Path Extension Lemma). *Let $G = (V, E)$ be a planar graph, and Π be a non-crossing path system to $s \in V$ that contains a path $\Pi[w]$ from w to s for some $v \in V$. Suppose we had a non-crossing collection of paths $\mathcal{P}$ that end at w where no path in $\mathcal{P}$ crosses any of the paths of Π. Then we can extend each path of $\mathcal{P}$ to end at s without creating any crossing.*

Proof. The extension proceeds as follows and is illustrated in Figure 5:

1. Consider a fixed perturbation to the paths of Π and $\mathcal{P}$ such that none of the paths share any point in common, and all end points of w lie in a small ball of radius ε for an arbitrarily small $\varepsilon > 0$. This is possible because all paths are non-intersecting. See [9, Section 4] for discussions of algorithms for constructing the perturbation.
2. For every path $P \in \mathcal{P}$, snap the end point of the path to the end point of the path of Π that is clockwise around the ε-radius ball. We omit the formal description of this step, as it is more instructive to observe the illustration in Figure 5.
3. Extend all paths $P \in \mathcal{P}$ by the forward continuation of the path of Π we have snapped to.

Observe that by construction, our extended (perturbed) paths are non-crossing. ◀

This gives us a way to construct our path system $\Pi_{\leq a-1}$ to $\Pi_{\leq a}$ with Delaunay paths.

▶ **Lemma 8.** *Given a set of non-crossing pseudo-shortest paths $\Pi_{\leq a-1}$, we can extend each $\Delta[u]$ for all $u \in W_a$ to a pseudo-shortest path $\Pi[u]$ in a way such that no two paths intersect each other or paths of $\Pi_{\leq a-1}$.*

Proof. Fix a vertex $w \in W_{a-1}$ and consider the pseudo-shortest paths

$$\Pi^w = \{\Pi[v] \in \Pi_{\leq a-1} \mid \Pi[v] \text{ passes through } w\}.$$

Also consider the following Delaunay paths.

$$\Delta^w = \{\Delta[u] \mid u \in W_a \text{ and the nearest neighbor of } u \text{ in } W_{a-1} \text{ is } w\}.$$

Note that no path of Π^w crosses any path of Δ^w, and both are non-crossing path systems.

We repeatedly apply Lemma 7 to each $w \in W_{a-1}$. Doing so, we construct a path $\Pi[u]$ for each $u \in W_a$ that is crossing free with all paths of $\Pi_{\leq a-1}$. ◀

By induction on distance from s the lemma below follows.

▶ **Lemma 9** (Spanning non-crossing path systems of psuedo-shortest paths). *Let G be a unit disk graph on the point set S and let s be a fixed source vertex. There exists a spanning non-crossing path system Π of pseudo-shortest paths rooted at s using only edges in the Delaunay triangulation of S.*

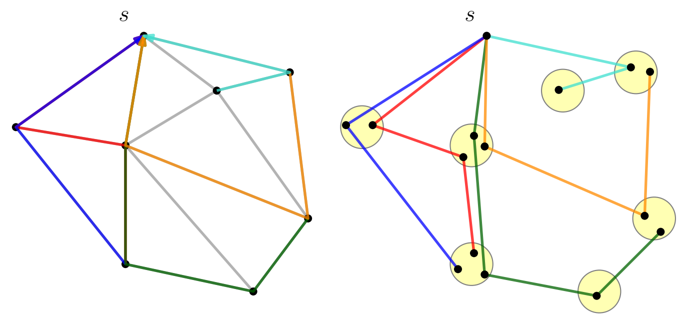

Figure 6 **(Left)** A path system Π to a vertex s in a Delaunay triangulation. **(Middle)** A perturbation of all points on all paths at vertices other than s by at most ε such that no two paths intersect except at s. **(Right)** A triangulation of the perturbation of the paths using only previously existing edges.

4 Shortest path separators in non-crossing spanning path systems

In this section we show how to find the shortest path separators on embedded triangulated planar graphs $G = (V, E)$ when given a spanning non-crossing path systems Π to $s \in V$. We aim to apply the lemma of Lipton and Tarjan albeit onto a triangulated graph G' containing a slightly perturbed version of Π. Formally we define a *perturbation of a path system* Π *rooted at* s as the collection of all paths with vertices except s perturbed within a small ball of radius $\varepsilon > 0$ such that none of the paths are crossing. Observe that a single vertex $u \in V$ can be perturbed to many copies $u_1, ..., u_k$ corresponding to the same vertex $u \in V$. See Figure 6 for an example of this.

The following lemma shows that this graph can be constructed while ensuring the additional edges we add are between vertices corresponding to edges in the original graph G.

▶ **Lemma 10.** *Given a triangulated embedded planar graph $G = (V, E)$, and a non-crossing spanning path system Π to a vertex $s \in V$, there exists a planar graph $G' = (V', E')$ that is the triangulation of the perturbation of Π such that every edge $e \in E'$ belongs to one of the following categories:*

- *Edges E_{PATH} between vertices $u_i, v_j \in V'$ that correspond to $u, v \in V$ where $uv \in \Pi$.*
- *Edges E_{VERTEX} between $v_i, v_j \in V'$ that correspond to the same vertex $v \in V$.*
- *Edges E_{TRI} between vertices $u_i, v_j \in V'$ that correspond to $u, v \in V$ where $uv \in E$.*

Proof. We describe a construction of G'. Let V' be the vertices of the perturbation of Π, consisting of all vertices $u \in V$ with $u \neq s$ that are perturbed to $Q_u = \{u_1, u_2, ..., u_{\ell(u)}\}$ where $\ell(u)$ is the number of paths using vertex u in Π. We say an edge $u_i, v_j \in V'$ is *faithful* if $u_i \in Q_u$ and $v_j \in Q_v$ and $uv \in G$. Let E_{PATH} denote the edges that correspond to the perturbed paths of Π. Observe that the edges of E_{PATH} are faithful.

For all perturbations of the same vertex $u \in V$ to $Q_u = \{u_1, u_2, ..., u_{\ell(u)}\} \subset V'$, add a maximal set of planar edges that do not cross the edges of E_{PATH} (i.e. a restricted triangulation in the local neighborhood of the perturbation), and let E^u_{VERTEX} denote these added edges. Define $E_{\text{VERTEX}} = \bigcup_{u \in V} E^u_{\text{VERTEX}}$.

Now let's consider an edge $uv \in E$ with no path in Π using uv. We claim that there exists a perturbed vertex $u_i \in Q_u$ and $v_j \in Q_v$ such that the edge $u_i v_j$ does not intersect edges of $E_{\text{VERTEX}} \cup F$ where F is any non-crossing collection of faithful edges. Thus it is possible to add a collection of non-crossing faithful edges E^1_{TRI} that do not intersect $E_{\text{VERTEX}} \cup E_{\text{PATH}}$ so that every edge $uv \in E$ has a faithful edge.

Now consider the faces in the graph $(V', E_{\text{PATH}} \cup E_{\text{VERTEX}} \cup E^1_{\text{TRI}})$, they are either: (1) a face that consists of three faithful edges corresponding to a triangle uvw in G joined by chains of vertices of V' corresponding to vertices u, v, or w (2) a face with two faithful edges corresponding to the same edge $uv \in E$. Either case is easy to triangulate with an additional set of non-crossing faithful edges E^2_{TRI} since any non-crossing triangulation can only consist of faithful edges. Let $E_{\text{TRI}} = E^1_{\text{TRI}} \cup E^2_{\text{TRI}}$. ◀

With this lemma we present our result for any non-crossing spanning path system of a triangulated planar graph.

▶ **Lemma 11** (Balanced separators for non-crossing path systems). *Given a triangulated embedded planar graph $G = (V, E)$, and a non-crossing spanning path system Π to a vertex $s \in V$, there exists a Jordan curve C with the following properties:*

1. *There are at most $2n/3$ vertices in $V_{\text{INSIDE}}(C)$ and $V_{\text{OUTSIDE}}(C)$.*
2. *C is defined by two paths P_u from u to s and P_v from v to s and either one edge $uv \in E$ or $u = v$.*
3. *P_u is the suffix of a path $\Pi[u']$ and P_v is a suffix of a path $\Pi[v']$ for some $u', v' \in V$.*

Proof. Begin by constructing the triangulated perturbed graph $G' = (V', E')$ from G and Π with Lemma 10. Let T be the tree defined by the perturbed paths of Π in G' rooted at s.

Observe that T is spanning and the fundamental cycles of T contain s. For vertices $u_1, ..., u_{\ell(u)} \in V'$ that correspond to the vertex $u \in V$, we arbitrarily choose one vertex (say u_1) and give it weight 1, and give all other vertices weight 0. We do this for all vertices of V. Now we can apply the weighted separator theorem of Theorem 2 to G' with this weight function, and spanning tree defined by the perturbed paths of Π to get a balanced separator C satisfying condition 1. Observe that C corresponds to a fundamental cycle of the perturbed paths, which is an edge $u_iv_j \in E'$, and two paths that end at s and correspond to paths of Π satisfying condition 3. Finally observe that the u_i and v_j correspond either to two vertices $u, v \in V$ with $uv \in E$ or to the same vertex $u \in V$. In either case condition 2 is satisfied. ◀

5 Constructing the shortest path 1-neighborhood separator

5.1 Delaunay edges are crominating

Recall that a pair of vertices $u, v \in V$ is crominating if for all edges of G intersecting the line segment between u and v has one end point adjacent to u or v. We will first show that Delaunay edges are crominating.

▶ **Lemma 12.** *For a unit disk graph $G = (V, E)$, all edges in the Delaunay triangulation $DT(G)$ are crominating.*

Proof. For the sake of contradiction, suppose that there exists Delaunay edge $uv \in DT(G)$ and edge $xy \in E$, such that uv crosses but does not dominate xy. Without loss of generality, we may assume that the unit disk centered at x intersects uv, uv is horizontal, and x lies above uv. Consider if $|u - v| \le 1$ so that $uv \in E$. This would mean that xy would need to intersect either the unit disk at u or v. By Lemma 3, this implies that one of x or y is adjacent to either u or v, a contradiction. Thus we will focus on the case where $|u - v| > 1$.

Let $D^{\uparrow}_{uv}$ (resp. $D^{\downarrow}_{uv}$) be the semidisk above (resp. below) uv with diameter uv. Since $|x - u|, |x - v| > 1$ and the distance between x and line segment uv is at most $1/2$, we have $x \in D^{\uparrow}_{uv}$. Now, let D_x be the disk with center x and radius 1, and let $D^{\downarrow}_x$ be the part of D_x that lies below uv. Clearly $y \in D^{\downarrow}_x$. Since $|x - u|, |x - v| > 1$, it follows that $D^{\downarrow}_x \subset D^{\downarrow}_{uv}$, and thus $y \in D^{\downarrow}_{uv}$. (See Figure 7.)

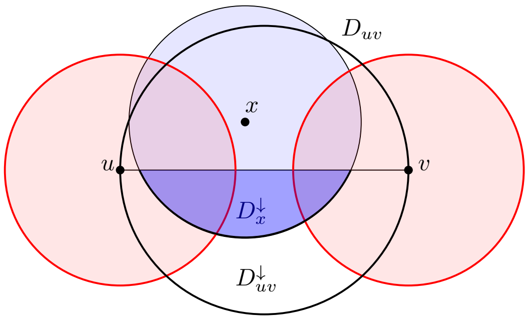

Figure 7 Suppose an edge $xy \in E$ intersects but is not dominated by a Delaunay edge uv. Without loss of generality assume that disk x intersects edge uv and lies above uv. We show that $x \in D_{uv}^{\uparrow}$ and $y \in D_x^{\downarrow} \subset D_{uv}^{\downarrow}$.

Let $z \in S$ be the point above uv such that u, z, v share the same face in $DT(S)$. Let D_{uvz} be the circumcircle through u, v, z. There are two cases to consider.

1. If $z \in D_{uv}^{\uparrow}$, then $D_{uvz} \supset D_{uv}^{\downarrow}$, and thus must contain y. However, this is forbidden by the definition of Delaunay triangulation, a contradiction.
2. If $z \notin D_{uv}^{\uparrow}$, then $D_{uvz} \supset D_{uv}^{\uparrow}$, and thus must contain x, which is a contradiction. ◀

5.2 Putting everything together

We finally have all the pieces we need to prove that unit disk graphs have a shortest path 1-neighborhood separator.

Let $G = (V, E)$ be a unit disk graph on n points. Fix an arbitrary source vertex $s \in V$. By Lemma 9, there exists a spanning non-crossing path system Π of pseudo-shortest paths rooted at s using only the edges in the triangulation of $\mathrm{DT}(G)$[6]. Let C be a Jordan curve according to Lemma 11, and let P_u, P_v be the paths defining C as in Lemma 11. Consider an edge $xy \in E$ that intersects C. There are a few cases:

1. xy intersects the Delaunay edge uv.
2. xy intersects an edge wz of the triangulated outer face.
3. xy intersects an edge wz in either $\Pi[u]$ or $\Pi[v]$.

In the first case, xy is crominated by uv, thus, removing the 1-neighborhood of u and v removes the edge xy. In the second case, note that the edge wz in the triangulated outer face is outside of the Delaunay triangulation and thus outside the convex hull of the point set, so it is impossible for an edge $xy \in E$ to cross. In the third case, assume without loss of generality that $wz \in \Pi[u]$. The edge wz is part of a Delaunay path between two vertices w^* and z^* which are two vertices on the shortest path from u to s. By Lemma 4, this entire edge is covered by the disk centered at w^* and the disk centered at z^*. In particular that means that xy intersects at least one of the disks centered at w^* or z^*, and thus applying Lemma 3 shows that either x or y is a neighbor of w^* or z^*. We can now conclude that removing the 1-neighborhood of u, v, P_u and P_v removes all edges that cross C.

▶ **Theorem 13.** *Every unit disk graph admits a shortest path 1-neighborhood separator.*

[6] Note that $\mathrm{DT}(G)$ may not be triangulated as a planar graph, the outer face is typically not a triangle.

Remarks. We have omitted discussions of runtimes related to the construction of our 1-neighborhood separator, but we note that we can construct the separator in $O(n^2)$ time. Finding the path system takes $O(n^2)$ time. Indeed, the path system has $O(n^2)$ size, although more compact representations of the path system are possible. Furthermore, it can be shown that performing the perturbation can be done in $O(n^2 \log n)$ time by the algorithm of [9], and the separator construction of Lemma 11 can be done in $O(n^2)$. We leave as an open problem as to whether it is possible to construct the separator faster. To contrast, finding shortest path separators in planar graphs can be done in $O(n)$ time using the dual co-tree.

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