Finding a Maximum Restricted t-Matching via Boolean Edge-CSP

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Abstract

The problem of finding a maximum 2-matching without short cycles has received significant attention due to its relevance to the Hamilton cycle problem. This problem is generalized to finding a maximum t-matching which excludes specified complete t-partite subgraphs, where t is a fixed positive integer. The polynomial solvability of this generalized problem remains an open question. In this paper, we present polynomial-time algorithms for the following two cases of this problem: in the first case the forbidden complete t-partite subgraphs are edge-disjoint; and in the second case the maximum degree of the input graph is at most $2t - 1$. Our result for the first case extends the previous work of Nam (1994) showing the polynomial solvability of the problem of finding a maximum 2-matching without cycles of length four, where the cycles of length four are vertex-disjoint. The second result expands upon the works of Bérczi and Végh (2010) and Kobayashi and Yin (2012), which focused on graphs with maximum degree at most $t + 1$. Our algorithms are obtained from exploiting the discrete structure of restricted t-matchings and employing an algorithm for the Boolean edge-CSP.

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1 Introduction

The matching problem and its generalizations have been one of the most primary topics in combinatorial optimization, and have been the subject of a large number of studies. A typical generalization of a matching is a *t-matching* for an arbitrary positive integer t: an edge subset M in a graph is a t-matching[1] if each vertex is incident to at most t edges in M.

While the problem of finding a t-matching of maximum cardinality can be solved in polynomial time by a matching algorithm, the problem becomes much more difficult, typically NP-hard, when additional constraints are imposed. The constraints discussed in this paper is

[1] Such an edge set is sometimes called a *simple t-matching* in the literature, but we omit the adjective "simple" because in this article a t-matching is always an edge subset and we never put multiplicities on the edges.

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to exclude certain subgraphs. Let $G = (V, E)$ be a graph and let $\mathcal{K}$ be a family of subgraphs of G. For a subgraph K of G, let $V(K)$ and $E(K)$ denote the vertex set and the edge set of K, respectively.

▶ **Definition 1.** *An edge subset $M \subseteq E$ is* $\mathcal{K}$-free *if $E(K) \not\subseteq M$ for any $K \in \mathcal{K}$.*[2]

The problem formulated below is the central issue in this paper, whose relevance will be described in detail in Section 1.1.

MAXIMUM $\mathcal{K}$-FREE t-MATCHING PROBLEM

Given a graph $G = (V, E)$ and a family $\mathcal{K}$ of subgraphs of G, find a $\mathcal{K}$-free t-matching $M \subseteq E$ of maximum cardinality.

Here, we suppose that $\mathcal{K}$ is explicitly given as a list of its elements; see Remark 7.

Our primary contributions are the following two theorems, showing the polynomial solvability of certain classes of MAXIMUM $\mathcal{K}$-FREE t-MATCHING PROBLEM. The first result concerns the case where $\mathcal{K}$ is an edge-disjoint family of *t-regular complete partite subgraphs* of G. While we defer the definition to Section 2.1, here we remark that a complete graph K_{t+1} and a complete bipartite graph $K_{t,t}$ are examples of a t-regular complete partite graph.

▶ **Theorem 2.** *For a fixed positive integer t,* MAXIMUM $\mathcal{K}$-FREE t-MATCHING PROBLEM *can be solved in polynomial time if all the subgraphs in $\mathcal{K}$ are t-regular complete partite and pairwise edge-disjoint.*

In the second result, instead of the edge-disjointness of the subgraphs in $\mathcal{K}$, we assume that the maximum degree of the input graph G is bounded.

▶ **Theorem 3.** *For a fixed positive integer t,* MAXIMUM $\mathcal{K}$-FREE t-MATCHING PROBLEM *can be solved in polynomial time if all the subgraphs in $\mathcal{K}$ are t-regular complete partite and the maximum degree of G is at most $2t - 1$.*

Theorems 2 and 3 offer larger polynomially solvable classes of MAXIMUM $\mathcal{K}$-FREE t-MATCHING PROBLEM than the previous results summarized in Section 1.1 below. In addition, we will describe the relevance of Theorems 2 and 3 to the literature, together with their extensions and variants in the subsequent sections. Here we just remark that the assumption on the complete partiteness of the forbidden subgraphs in Theorems 2 and 3 is unavoidable, because the problem is NP-hard without this assumption (see Proposition 4 below).

1.1 Previous Work on Restricted t-Matchings

MAXIMUM $\mathcal{K}$-FREE t-MATCHING PROBLEM has its origin in the case where $t = 2$ and $\mathcal{K}$ is composed of short cycles. Let k be a positive integer. If $\mathcal{K}$ is the set of all cycles of length at most k, then a $\mathcal{K}$-free 2-matching is referred to as a *$C_{\leq k}$-free 2-matching*, and MAXIMUM $\mathcal{K}$-FREE 2-MATCHING PROBLEM as the *$C_{\leq k}$-free 2-matching problem.* Similarly, if $\mathcal{K}$ is the set of all cycles of length exactly k, then a $\mathcal{K}$-free 2-matching is referred to as a *C_k-free 2-matching*, and MAXIMUM $\mathcal{K}$-FREE 2-MATCHING PROBLEM as the *C_k-free 2-matching problem.* The $C_{\leq k}$-free and C_k-free 2-matching problems have attracted significant attention because of their relevance to the Hamilton cycle problem; for $k \geq |V|/2$, a $C_{\leq k}$-free 2-matching of cardinality $|V|$ is a Hamilton cycle. When k is small, the $C_{\leq k}$-free 2-matching problem is

[2] Each forbidden subgraph is *not* a subgraph *isomorphic to K*, but a subgraph K itself.

not directly used to find Hamilton cycles, but it can be applied to designing approximation algorithms for related problems such as the graph-TSP and the minimum 2-edge-connected spanning subgraph problem. For example, in a recent paper [16], an approximation algorithm for the minimum 2-edge-connected spanning subgraph problem is provided using a maximum $C_{\leq 3}$-free 2-matching.

The complexity of the $C_{\leq k}$-free 2-matching problem depends on the value of k. It is straightforward to see that this problem can be solved in polynomial time for $k \leq 2$. For $k = 3$, Hartvigsen [8, 10] gave a polynomial-time algorithm for the $C_{\leq 3}$-free 2-matching problem. For $k \geq 5$, Papadimitriou proved the NP-hardness of the $C_{\leq k}$-free 2-matching problem (see [6]).

For the case $k = 4$, it is open whether the $C_{\leq 4}$-free and C_4-free 2-matching problems can be solved in polynomial time, and these problems have rich literature of polynomial-time algorithms for several special cases. First, for subcubic graphs, i.e., graphs with maximum degree at most three, polynomial-time algorithms for the C_4-free and the $C_{\leq 4}$-free 2-matching problems were given by Bérczi and Kobayashi [3] and Bérczi and Végh [4], respectively. Simpler algorithms for both problems in subcubic graphs (and for some of their weighted variants) were designed by Hartvigsen and Li [11] and by Paluch and Wasylkiewicz [23]. It is worth noting that a connection between the C_4-free matching problem and a connectivity augmentation problem is highlighted in [3], underscoring the significance of the C_4-free matching problem. Second, for the graphs in which the cycles of length four are vertex-disjoint, Nam [22] gave a polynomial-time algorithm for the C_4-free 2-matching problem. Finally, for bipartite graphs, min-max theorems [7, 9, 13, 14, 28] and polynomial-time algorithms [2, 9, 24, 29] were devised.

Let t be an arbitrary positive integer. The C_k-free 2-matching problem is generalized to Maximum $\mathcal{K}$-Free t-Matching Problem for general t in the following way. Let K_t denote the complete graph with t vertices, and $K_{t,t}$ the complete bipartite graph in which each color class has t vertices (see Section 2.1 for a formal definition). Here, note that a cycle of length three is isomorphic to K_3. Thus, the C_3-free 2-matching problem can be naturally generalized to Maximum $\mathcal{K}$-Free t-Matching Problem, where $\mathcal{K}$ is the set of all subgraphs that are isomorphic to K_{t+1}. We refer to this special case of Maximum $\mathcal{K}$-Free t-Matching Problem as the *K_{t+1}-free t-matching problem*. Similarly, by noting that a cycle of length four is isomorphic to $K_{2,2}$, we can generalize the C_4-free 2-matching problem to the *$K_{t,t}$-free t-matching problem*. This is another special class of Maximum $\mathcal{K}$-Free t-Matching Problem, where $\mathcal{K}$ is the set of all subgraphs isomorphic to $K_{t,t}$.

The polynomial solvability of these two problems are open. For certain special cases of Maximum $\mathcal{K}$-Free t-Matching Problem, however, several polynomial-time algorithms are presented, corresponding to those for the $C_{\leq k}$-free and C_k-free 2-matching problems. First, Bérczi and Végh [4] gave a polynomial-time algorithm for Maximum $\mathcal{K}$-Free t-Matching Problem for the case where $\mathcal{K}$ consists of K_{t+1}'s and $K_{t,t}$'s and the input graph G has maximum degree at most $t + 1$. This implies that the $C_{\leq 4}$-free 2-matching problem in subcubic graphs can be solved in polynomial time. Second, Kobayashi and Yin [18] presented a polynomial-time algorithm for Maximum $\mathcal{K}$-Free t-Matching Problem for the case where $\mathcal{K}$ consists of all the subgraphs isomorphic to a fixed t-regular complete partite graph and the input graph G has maximum degree at most $t + 1$. Kobayashi and Yin [18] also proved that this assumption on $\mathcal{K}$ is inevitable.

▶ **Proposition 4** (follows from Kobayashi and Yin [18]). *If H is a connected t-regular graph which is not complete partite and $\mathcal{K}$ is the set of all subgraphs isomorphic to H, then* Maximum $\mathcal{K}$-Free t-Matching Problem *is NP-hard even when the maximum degree of G is at most $t + 1$ and the subgraphs in $\mathcal{K}$ are pairwise edge-disjoint.*[3]

[3] Although the edge-disjointness is not explicitly stated in [18], one can see that their NP-hardness proof

As mentioned above, this NP-hardness explains that the assumption on the complete partiteness of the forbidden subgraphs in Theorems 2 and 3 is unavoidable.

Finally, for the $K_{t,t}$-free t-matching problem in bipartite graphs, polynomial-time algorithms were designed, extending those for the C_4-free 2-matching problem in bipartite graphs (see [27] and references therein).

1.2 Our Contribution

We have seen that the polynomial solvability of the K_{t+1}-free t-matching and $K_{t,t}$-free t-matching problems is unknown. As well as these problems, the polynomial solvability of Maximum $\mathcal{K}$-Free t-Matching Problem in general graphs for $\mathcal{K}$ being an arbitrary family of t-regular complete partite subgraphs is unknown. The main contribution of this paper is polynomial-time algorithms for several classes of this problem, which are described in Theorems 2 and 3 above.

1.2.1 Implications and Extensions of the Main Theorems

Here we present some implications of Theorems 2 and 3. Recall Theorem 2, solving the case where $\mathcal{K}$ is an edge-disjoint family of t-regular complete partite subgraphs of G. By setting $t = 2$, we immediately obtain the following corollary.

▶ **Corollary 5.** Maximum $\mathcal{K}$-Free 2-Matching Problem *can be solved in polynomial time if all the subgraphs in $\mathcal{K}$ are isomorphic to C_3 or C_4, and are pairwise edge-disjoint.*

Corollary 5 extends the result by Nam [22], solving the C_4-free 2-matching problem where the cycles of length four are vertex-disjoint. Namely, Corollary 5 extends vertex-disjointness to edge-disjointness, and allows $\mathcal{K}$ to include not only C_4 but also C_3.

Next, recall Theorem 3, which solves the case where the maximum degree of the input graph is at most $2t - 1$. Theorem 3 expands upon the works of Bérczi and Végh [4] and Kobayashi and Yin [18], which focused graphs with maximum degree at most $t + 1$. That is, Theorem 3 improves the degree bound from $t + 1$ to $2t - 1$, where $2t - 1 > t + 1$ if $t > 2$.

We then present some extensions of Theorems 2 and 3. Below is one extension of Theorem 2, which will be used in our proof for Theorem 3. The pairwise edge-disjointness of the subgraphs in $\mathcal{K}$ is relaxed to the following condition:

(RD) The subgraph family $\mathcal{K}$ is partitioned into subfamilies $\mathcal{K}_1, \dots, \mathcal{K}_\ell$ such that
- for each subfamily $\mathcal{K}_i$ ($i = 1, \dots, \ell$), the number $\left|\bigcup_{K \in \mathcal{K}_i} V(K)\right|$ of its vertices is bounded by a fixed constant (under the assumption that t is a fixed constant), and
- for distinct subfamilies $\mathcal{K}_i$ and $\mathcal{K}_j$ ($i, j \in \{1, \dots, \ell\}$) and for each pair of subgraphs $K \in \mathcal{K}_i$ and $K' \in \mathcal{K}_j$, it holds that K and K' are edge-disjoint.

Here "(RD)" stands for "Relaxed Disjointness."

▶ **Theorem 6.** *For a fixed positive integer t,* Maximum $\mathcal{K}$-Free t-Matching Problem *can be solved in polynomial time if $\mathcal{K}$ is a family of t-regular complete partite subgraphs of G satisfying the condition (RD).*

Further results include extensions from t-matchings to b-matchings (Theorems 13, 16, 17, 18, and 20). For a vector $b \in \mathbb{Z}^V$, a *b-matching* is an edge subset $M \subseteq E$ such that each vertex $v \in V$ is incident to at most $b(v)$ edges in M. Namely, we can deal with inhomogeneous degree constraints.

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uses only disjoint forbidden subgraphs.

1.2.2 Technical Ingredients: Jump Systems and Boolean Edge-CSP

Technically, our algorithms are established by exploiting two important previous results, one is on the discrete structure of $\mathcal{K}$-free t-matchings and the other is on the constraint satisfaction problem (CSP). This is in contrast to the fact that the previous algorithms [4, 18, 22] are based on graph-theoretical methods.

The first result is on *jump systems*, and is outlined as follows. Let $b \in \mathbb{Z}^V$ with $b(v) \leq t$ for each $v \in V$ and let $J \subseteq \mathbb{Z}^V$ be the set of the degree sequences of all $\mathcal{K}$-free b-matchings in G. Kobayashi, Szabó, and Takazawa [17] proved that J forms a *constant-parity jump system* if all the subgraphs in $\mathcal{K}$ are t-regular complete partite (see Theorem 9 below). Here a constant-parity jump system is a subset of $\mathbb{Z}^V$, which offers a discrete structure generalizing matroids; see Section 2.2 for the definition.

The second result is on the polynomial-time solvability of a class of the CSP. The *Boolean edge-CSP* is the problem of finding an edge subset $M \subseteq E$ of a given graph $G = (V, E)$ such that the set of edges in M incident to each vertex $v \in V$ satisfies a certain constraint associated with v; see Section 2.3 for formal description. While the Boolean edge-CSP is NP-hard in general, Kazda, Kolmogorov, and Rolínek [12] showed that this problem can be solved in polynomial time if the constraint associated with v is described by a constant-parity jump system for each $v \in V$ (see Theorem 12 below).

The most distinctive part of this paper is a reduction of Maximum $\mathcal{K}$-Free t-Matching Problem to the Boolean edge-CSP. It appears in the proof of Theorem 13 below, which deals with the problem of finding a $\mathcal{K}$-free b-factor, i.e., a t-matching with specified degree sequence $b \in \mathbb{Z}^V$. Here, on the basis of the relationship between $\mathcal{K}$-free b-matchings and jump systems (Theorem 9), we construct a polynomial reduction of the problem of finding a $\mathcal{K}$-free b-factor to the Boolean edge-CSP with constant-parity jump system constraints.

Theorem 2 is then derived from Theorem 13. In order to prove Theorem 2, we iteratively solve subproblems of finding a $\mathcal{K}$-free b-factor. We remark that constant-parity jump systems play a key role here, as well as the reduction mentioned above. The fact that J is a constant-parity jump system guarantees that the number of iterations is polynomially bounded by the input size (see Lemma 11 below). Theorem 6 is proved in the same manner.

We then derive Theorem 3 from Theorem 6 by constructing a subfamily $\mathcal{K}' \subseteq \mathcal{K}$ such that $\mathcal{K}'$ satisfies (RD), a $\mathcal{K}'$-free t-matching exists in G if and only if a $\mathcal{K}$-free t-matching exists in G, and we can construct a $\mathcal{K}$-free t-matching from a $\mathcal{K}'$-free t-matching in polynomial time.

1.3 Organization

The rest of the paper is organized as follows. In Section 2, we present the basic definitions and results in a formal manner. In Section 3, we solve the problem under the assumption that the subgraphs in $\mathcal{K}$ are pairwise edge-disjoint, and then under the relaxed condition (RD). Section 4 is devoted to a solution to the graphs with maximum degree at most $2t - 1$.

2 Preliminaries

Let $\mathbb{Z}_+$ denote the set of nonnegative integers, and $\mathbf{0}$ (resp. $\mathbf{1}$) denote the all-zero (resp. all-one) vector of appropriate dimension. For a finite set V, its subset $U \subseteq V$, and a vector $x \in \mathbb{Z}^V$, let $x(U) = \sum_{u \in U} x(u)$.

2.1 Basic Definitions on Graphs

Throughout this paper, we assume that graphs have no self-loops to simplify the description, while they may have parallel edges. Let $G = (V, E)$ be a graph. For a subgraph H of G, let $V(H)$ and $E(H)$ denote the vertex set and edge set of H, respectively. For a vertex set $X \subseteq V$, let $G[X]$ denote the subgraph induced by X.

Let $F \subseteq E$ be an edge subset and let $v \in V$ be a vertex. The set of edges in F incident to v is denoted by $\delta_F(v)$. If $F = E(H)$ for some subgraph H of G, then $\delta_{E(H)}(v)$ is often abbreviated as $\delta_H(v)$. When no confusion arises, $\delta_G(v)$ is further abbreviated as $\delta(v)$. The number of edges incident to v, i.e., $|\delta(v)|$, is referred to as the *degree* of v. The *degree sequence* d_F of $F \subseteq E$ is a vector in $\mathbb{Z}_+^V$ defined by $d_F(u) = |\delta_F(u)|$ for each $u \in V$.

For a positive integer t, a graph is called *t-regular* if every vertex has degree t. A graph $G = (V, E)$ is said to be a *complete partite graph* if there exists a partition $\{V_1, \dots, V_p\}$ of V such that $E = \{uv \colon u \in V_i, v \in V_j, i \neq j\}$ for some positive integer p. In other words, a complete partite graph is the complement of the disjoint union of complete graphs. Each V_i is called a *color class* of G.

As defined in Section 1, for a positive integer t, an edge set $M \subseteq E$ is called a *t-matching* if $d_M(v) \leq t$ for every $v \in V$. In particular, if $d_M(v) = t$ holds for every $v \in V$, then M is called a *t-factor*. For a vector $b \in \mathbb{Z}_+^V$, an edge set $M \subseteq E$ is called a *b-matching* (resp. *b-factor*) if $d_M(v) \leq b(v)$ (resp. $d_M(v) = b(v)$) for every $v \in V$.

In what follows, we deal with the following slightly generalized problems.

<u>$\mathcal{K}$-FREE b-FACTOR PROBLEM</u>

Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and a family $\mathcal{K}$ of subgraphs of G, find a $\mathcal{K}$-free b-factor (if one exists).

<u>MAXIMUM $\mathcal{K}$-FREE b-MATCHING PROBLEM</u>

Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and a family $\mathcal{K}$ of subgraphs of G, find a $\mathcal{K}$-free b-matching with maximum cardinality.

Note that MAXIMUM $\mathcal{K}$-FREE t-MATCHING PROBLEM is a special case of MAXIMUM $\mathcal{K}$-FREE b-MATCHING PROBLEM, where $b(v) = t$ for each $v \in V$.

▶ Remark 7. In this paper, we only consider the case where $\mathcal{K}$ consists of subgraphs of size bounded by a fixed constant (e.g., t-regular complete partite subgraphs for a fixed integer t, whose vertex set size is at most $2t$). In such a case, since $|\mathcal{K}|$ is polynomially bounded by the size of the input graph, the representation of $\mathcal{K}$ does not affect the polynomial solvability of the problem. Therefore, we suppose that $\mathcal{K}$ is explicitly given as the list of its elements.

▶ Remark 8. Let $G = (V, E)$ be a graph, $b \in \mathbb{Z}_+^V$ with $b(v) \leq t$ for each $v \in V$, and K a connected t-regular subgraph of G. We can see that, if a b-matching $M \subseteq E$ of G contains K, then K forms a connected component of the induced subgraph (V, M) of G by M.

2.2 Jump System

Let V be a finite set. For a subset $U \subseteq V$, let $\chi_U \in \{0, 1\}^V$ denote the characteristic vector of U, that is, $\chi_U(v) = 1$ for $v \in U$ and $\chi_U(v) = 0$ for $v \in V \setminus U$. If $U = \{u\}$ for an element $u \in V$, then $\chi_{\{u\}}$ is simply denoted by χ_u.

For two vectors $x, y \in \mathbb{Z}^V$, a vector $s \in \mathbb{Z}^V$ is called an *(x, y)-increment* if $s = \chi_u$ and $x(u) < y(u)$ for some $u \in V$, or $s = -\chi_u$ and $x(u) > y(u)$ for some $u \in V$. A nonempty set $J \subseteq \mathbb{Z}^V$ is said to be a *jump system* if it satisfies the following exchange axiom (see [5]):

For any $x, y \in J$ and for any (x, y)-increment s with $x + s \notin J$, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$.

In particular, a jump system $J \subseteq \mathbb{Z}^V$ is called a *constant-parity jump system* if $x(V) - y(V)$ is even for any $x, y \in J$.

Constant-parity jump systems include several discrete structures as special classes. First, for a matroid with a basis family $\mathcal{B}$, it follows from the exchange property of matroid bases that $\{\chi_B : B \in \mathcal{B}\}$ is a constant-parity jump system. Second, the characteristic vectors of all the feasible sets of an even delta-matroid form a constant-parity jump system (see [5]). Finally, for a graph $G = (V, E)$, the set $\{d_F : F \subseteq E\}$ of the degree sequences of all the edge subsets is also a constant-parity jump system. See [5, 19, 20] for details on jump systems.

The following theorem shows a relationship between $\mathcal{K}$-free b-matchings and jump systems.

▶ **Theorem 9** (follows from [17, Proposition 3.1]). *Let $G = (V, E)$ be a graph, let t be a positive integer, and let $b \in \mathbb{Z}_+^V$ be a vector such that $b(v) \leq t$ for each $v \in V$. For a family $\mathcal{K}$ of complete partite t-regular subgraphs in G, the degree sequences of all $\mathcal{K}$-free b-matchings in G form a constant-parity jump system.*

▶ Remark 10. Theorem 9 is a modest extension of the original statement [17, Proposition 3.1], in which $b(v) = t$ for each $v \in V$ and $\mathcal{K}$ is the set of all subgraphs in G that are isomorphic to a graph in a given list of complete partite t-regular subgraphs. The same proof, however, works for Theorem 9 as well.

If the degree sequences of all the $\mathcal{K}$-free b-matchings form a constant-parity jump system, then Maximum $\mathcal{K}$-Free b-Matching Problem reduces to $\mathcal{K}$-Free b-Factor Problem which is formally stated as follows.

▶ **Lemma 11.** *Let $G = (V, E)$ be a graph, $\mathcal{K}$ be a family of subgraphs of G, and let $b \in \mathbb{Z}_+^V$. If the degree sequences of all the $\mathcal{K}$-free b-matchings in G form a constant-parity jump system, then a $\mathcal{K}$-free b-matching in G with maximum cardinality can be computed by testing the existence of a $\mathcal{K}$-free b'-factor in G for polynomially many vectors $b' \in \mathbb{Z}_+^V$ with $b' \leq b$.*

Proof. Denote by $J \subseteq \mathbb{Z}^V$ the constant-parity jump system consisting of the degree sequences of all the $\mathcal{K}$-free b-matchings in G. Given an initial vector in J, we can maximize a given linear function over J by using the membership oracle of J at most polynomially many times [1, 5, 26]. Here, the *membership oracle of* J is an oracle that answers whether a given vector is in J or not.

Since an empty edge set is a $\mathcal{K}$-free b-matching, it holds that $\mathbf{0} \in J$. That is, we can take $\mathbf{0}$ as the initial vector in J. Now the lemma follows because accessing the membership oracle of J corresponds to testing the existence of a $\mathcal{K}$-free b'-factor in G. ◀

We here describe a few basic operations on jump systems, which are used in the proofs.

Intersection with a box. A *box* is a set of the form $\{x \in \mathbb{R}^V : \underline{b} \leq x \leq \bar{b}\}$ for some vectors $\underline{b} \in (\mathbb{R} \cup \{-\infty\})^V$ and $\bar{b} \in (\mathbb{R} \cup \{+\infty\})^V$. If $J \subseteq \mathbb{Z}^V$ is a constant-parity jump system, then the intersection

$$J \cap \{x \in \mathbb{R}^V : \underline{b} \leq x \leq \bar{b}\}$$

of J and a box is also a constant-parity jump system unless it is empty.

Minkowski sum. For two sets $J_1, J_2 \subseteq \mathbb{Z}^V$, their *Minkowski sum* $J_1 + J_2$ is a subset of $\mathbb{Z}^V$ defined by

$$J_1 + J_2 = \{x + y \colon x \in J_1,\ y \in J_2\}.$$

It was shown by Bouchet and Cunningham [5] that the Minkowski sum of two constant-parity jump systems is also a constant-parity jump system.

Splitting. Let $\{U_v \colon v \in V\}$ be a family of nonempty disjoint finite sets indexed by $v \in V$, and let $U = \bigcup_{v \in V} U_v$. For a set $J \subseteq \mathbb{Z}^V$, we define the *splitting* of J to U as

$$J' = \{x' \in \mathbb{Z}^U \colon x'(U_v) = x(v) \text{ for each } v \in V \text{ for some } x \in J\}.$$

The splitting of a constant-parity jump system is also a constant-parity jump system [15, 21].

2.3 Boolean Edge-CSP

The *constraint satisfaction problem* (*CSP*) is a fundamental topic in theoretical computer science and has been intensively studied in various fields (see, e.g., [25]).

Let Γ denote a collection of subsets of $\{0, 1\}^n$ for positive integers n, where a subset of $\{0, 1\}^n$ is referred to as a *relation*. In this paper, we focus on the *Boolean edge-CSP* with respect to Γ, which is formulated as follows.

BOOLEAN EDGE-CSP($\mathbf{\Gamma}$)

Given a graph $G = (V, E)$ and an edge subset family $\mathcal{F}_v \subseteq 2^{\delta(v)}$ whose corresponding relation $\{\chi_F \colon F \in \mathcal{F}_v\}$ belongs to Γ for each vertex $v \in V$, find an edge set $M \subseteq E$ such that $\delta_M(v) \in \mathcal{F}_v$ for each $v \in V$ (if one exists).

We remark that the relation $\mathcal{F}_v \subseteq 2^{\delta(v)}$ $(v \in V)$ is not given by membership oracles but by the list of the edge subsets, and hence the input size is $O(|V| + |E| + \sum_{v \in V} \sum_{F \in \mathcal{F}_v} |F|)$.

Kazda, Kolmogorov, and Rolínek [12] proved that BOOLEAN EDGE-CSP(Γ) belongs to class P if every relation in Γ is an even delta-matroid. For the unity of terminology, hereafter we refer to an even delta-matroid as a constant-parity jump system, since an even delta-matroid can be identified with a constant-parity jump system with each coordinate being in $\{0, 1\}$. Let $\Gamma_{\text{cp-jump}}$ denote the set of all constant-parity jump systems over the Boolean domain.

▶ **Theorem 12** (Kazda, Kolmogorov, and Rolínek [12]). BOOLEAN EDGE-CSP($\Gamma_{\text{cp-jump}}$) *can be solved in polynomial time.*

3 Edge-Disjoint Forbidden Subgraphs

In this section, we consider the case when $\mathcal{K}$ is an edge-disjoint family of t-regular complete partite subgraphs. We first give a polynomial-time algorithm for $\mathcal{K}$-FREE b-FACTOR PROBLEM by reducing the problem to BOOLEAN EDGE-CSP($\Gamma_{\text{cp-jump}}$) in Theorem 13. Then, by using this algorithm as a subroutine, we present a polynomial-time algorithm for MAXIMUM $\mathcal{K}$-FREE b-MATCHING PROBLEM (Theorem 16), which implies Theorem 2. Finally, we prove the polynomial solvability under the condition (RD) in Theorem 17, which will be used in the next section.

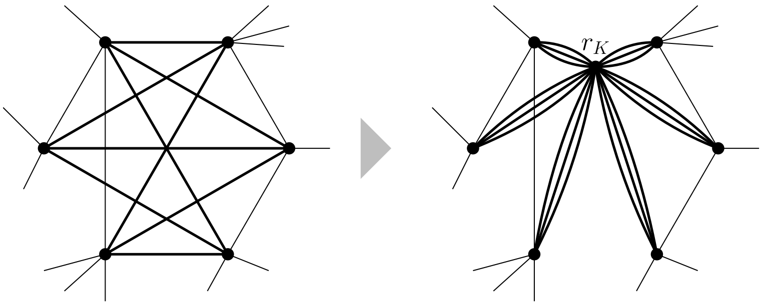

Figure 1 The graph on the left shows the edge set $E(K)$ of the t-regular complete partite graph K by the thick edges, while the thin edges belong to $E \setminus E(K)$. In this example, K is a 3-regular complete bipartite graph. The thick edges in the graph on the right depict the newly added three parallel edges between r_K and each vertex $v \in V(K)$.

▶ **Theorem 13.** *For a fixed positive integer t, $\mathcal{K}$-Free b-Factor Problem can be solved in polynomial time if $b(v) \le t$ for each $v \in V$ and all the subgraphs in $\mathcal{K}$ are t-regular complete partite and pairwise edge-disjoint.*

Proof. We prove the theorem by constructing a polynomial reduction to Boolean Edge-CSP($\Gamma_{\text{cp-jump}}$). Let $(G, b, \mathcal{K})$ be an instance of $\mathcal{K}$-Free b-Factor Problem, where $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and $\mathcal{K}$ is a family of subgraphs in G.

Recall that an input of the Boolean edge-CSP consists of a graph and a constraint on each vertex. Our input graph $G' = (V', E')$ of the Boolean edge-CSP is constructed as follows (see also Figure 1):

- Introduce a new vertex r_K for each $K \in \mathcal{K}$, and define the vertex set V' by

$$V' = V \cup \{r_K : K \in \mathcal{K}\}.$$

- For each $K \in \mathcal{K}$ and $v \in V(\mathcal{K})$, introduce t new parallel edges between r_K and v, and let $E'_{v,K}$ denote the set of these t new parallel edges. Define the edge set E' by

$$E' = \left(E \cup \bigcup_{K \in \mathcal{K}} \bigcup_{v \in V(K)} E'_{v,K} \right) \setminus \bigcup_{K \in \mathcal{K}} E(K).$$

Our input constraint $\mathcal{F}_v \subseteq 2^{\delta_{G'}(v)}$ $(v \in V')$ is constructed as follows:

- For each subgraph $K \in \mathcal{K}$, compute a set $D_K \subseteq \mathbb{Z}_+^{V(K)}$ of the degree sequences in the K-free b-matchings in K, i.e.,

$$\begin{aligned} D_K &= \left\{ d_F \in \mathbb{Z}_+^{V(K)} : F \text{ is a } K\text{-free } b\text{-matching in } K \right\} \\ &= \left\{ d_F \in \mathbb{Z}_+^{V(K)} : F \text{ is a } b\text{-matching in } K \right\} \setminus \{(t, \dots, t)\}. \end{aligned}$$

Then, for each vertex $v \in V'$, define $\mathcal{F}_v \subseteq 2^{\delta_{G'}(v)}$ by

$$\mathcal{F}_v = \begin{cases} \{F' \subseteq \delta_{G'}(v) : |F'| = b(v)\} & \text{if } v \in V, \\ \{F' \subseteq \delta_{G'}(v) : (d_{F'}(u))_{u \in V(K)} \in D_K\} & \text{if } v = r_K \text{ for some } K \in \mathcal{K}. \end{cases} \tag{1}$$

Note that each D_K and each $\mathcal{F}_v$ can be computed efficiently in a brute force way: $|V(K)| = O(t)$ and hence D_K has $t^{O(t)}$ elements for the fixed integer t; and $\mathcal{F}_v$ has a polynomial size.

Now we have constructed an instance of the Boolean edge-CSP consisting of $G' = (V', E')$ and $(\mathcal{F}_v)_{v \in V'}$. We first show the following claim, which implies that this instance actually belongs to Boolean Edge-CSP($\Gamma_{\text{cp-jump}}$).

▷ Claim 14. For each $v \in V'$, the set $\{\chi_{F'} \in \mathbb{Z}^{\delta_{G'}(v)} \colon F' \in \mathcal{F}_v\}$ of the characteristic vectors of the edge sets in $\mathcal{F}_v$ is a constant-parity jump system.

Proof of Claim 14. If $v \in V$, then the claim follows from the fact that $\mathcal{F}_v$ is the basis family of a uniform matroid. Suppose that $v = r_K$ for $K \in \mathcal{K}$. By applying Theorem 9 with $G = K$ and $\mathcal{K} = \{K\}$, we obtain that D_K is a constant-parity jump system. Now, $\{\chi_{F'} \in \mathbb{Z}^{\delta_{G'}(v)} \colon F' \in \mathcal{F}_v\}$ is obtained from splitting D_K to $\bigcup_{u \in V(K)} E'_{u,K}$ and then taking the intersection with a box $\{x \in \mathbb{R}^{\delta_{G'}(v)} \colon \mathbf{0} \le x \le \mathbf{1}\}$, and thus is a constant-parity jump system; recall Section 2.2. ◁

It follows from Claim 14 and Theorem 12 that the instance $(G', (\mathcal{F}_v)_{v \in V'})$ belongs to Boolean Edge-CSP($\Gamma_{\text{cp-jump}}$) and can be solved in polynomial time, respectively. Namely, we can find an edge set $M' \subseteq E'$ such that

$$\delta_{M'}(v) \in \mathcal{F}_v \text{ for each } v \in V' \tag{2}$$

or conclude that such M' does not exist in polynomial time. In what follows, we show that the existence of such an edge set $M' \subseteq E'$ is equivalent to the existence of a $\mathcal{K}$-free b-factor in the original graph G.

▷ Claim 15. The graph G' has an edge set $M' \subseteq E'$ satisfying (2) if and only if the original graph G has a $\mathcal{K}$-free b-factor $M \subseteq E$.

Proof of Claim 15. We first show the sufficiency ("if" part). Let $M \subseteq E$ be a $\mathcal{K}$-free b-factor in G. We construct an edge set $M' \subseteq E'$ satisfying (2) in the following way. For each subgraph $K \in \mathcal{K}$, let $F_K \subseteq \delta_{G'}(r_K)$ be an edge set in G' composed of exactly $d_{M \cap E(K)}(u)$ parallel edges between u and r_K for each vertex $u \in V(K)$. Note that such an edge set F_K must exist, because M is a b-factor, $b(u) \le t$, and G' has t parallel edges between u and r_K. Now define $M' \subseteq E'$ by

$$M' = \left(M \setminus \bigcup_{K \in \mathcal{K}} E(K)\right) \cup \bigcup_{K \in \mathcal{K}} F_K.$$

Here we show that this edge set M' satisfies (2). If $v \in V$, it holds that $\delta_{M'}(v) \in \mathcal{F}_v$, since $|\delta_{M'}(v)| = |\delta_M(v)| = b(v)$. Let $K \in \mathcal{K}$ and $v = r_K$. The fact that M is $\mathcal{K}$-free implies $(d_{M \cap E(K)}(u))_{u \in V(K)} \in D_K$. Since $d_{F_K}(u) = d_{M \cap E(K)}(u)$ for each vertex $u \in V(K)$, it follows from the definition (1) of $\mathcal{F}_{r_K}$ that $F_K \in \mathcal{F}_{r_K}$, and hence $\delta_{M'}(r_K) = F_K \in \mathcal{F}_{r_K}$. We thus conclude that M' satisfies (2).

We next show the necessity ("only if" part). Let $M' \subseteq E'$ be an edge set satisfying (2). We construct a $\mathcal{K}$-free b-factor M in G in the following manner. For each subgraph $K \in \mathcal{K}$, let $F_K := \delta_{M'}(r_K)$. It follows from (2) that $F_K \in \mathcal{F}_{r_K}$, namely, there exists a b-matching $N_K \subsetneq E(K)$ such that $d_{N_K}(u) = d_{F_K}(u)$ for each vertex $u \in V(K)$. Now define $M \subseteq E$ by

$$M = \left(M' \setminus \bigcup_{K \in \mathcal{K}} F_K\right) \cup \bigcup_{K \in \mathcal{K}} N_K.$$

We complete the proof by showing that M is a $\mathcal{K}$-free b-factor in G. Let $v \in V$ be an arbitrary vertex in G. Since $d_{F_K}(u) = d_{N_K}(u)$ for each $K \in \mathcal{K}$ and each $u \in V(K)$, it holds that $d_M(v) = d_{M'}(v) = b(v)$, where the last equality follows from $\delta_{M'}(v) \in \mathcal{F}_v$. We thus have that M is a b-factor. Furthermore, since $N_K \subsetneq E(K)$ for each $K \in \mathcal{K}$, we conclude that M is $\mathcal{K}$-free. ◁

The proof of Claim 15 provides a polynomial-time construction of a $\mathcal{K}$-free b-factor M in G from an edge set $M' \subseteq E'$ satisfying (2). We thus conclude that the original instance $(G, b, \mathcal{K})$ of $\mathcal{K}$-Free b-Factor Problem can be solved in polynomial time. ◀

By using Theorem 13, we can give a polynomial-time algorithm for Maximum $\mathcal{K}$-Free b-Matching Problem under the same assumptions.

▶ **Theorem 16.** *For a fixed positive integer t,* Maximum $\mathcal{K}$-Free b-Matching Problem *can be solved in polynomial time if $b(v) \leq t$ for each $v \in V$ and all the subgraphs in $\mathcal{K}$ are t-regular complete partite and pairwise edge-disjoint.*

Proof. It follows from Theorem 9 that the set of the degree sequences of all $\mathcal{K}$-free b-matchings in G is a constant-parity jump system. Therefore, by Lemma 11 and Theorem 13, we can solve Maximum $\mathcal{K}$-Free b-Matching Problem in polynomial time. ◀

We remark that Theorem 2 is immediately derived from Theorem 16 by setting $b(v) = t$ for every $v \in V$.

As described in Section 1, the edge-disjointness of the subgraphs in $\mathcal{K}$ is relaxed to the condition (RD).

▶ **Theorem 17.** *For a fixed positive integer t,* $\mathcal{K}$-Free b-Factor Problem *and* Maximum $\mathcal{K}$-Free b-Matching Problem *can be solved in polynomial time if $b(v) \leq t$ for each $v \in V$ and $\mathcal{K}$ is a family of t-regular complete partite subgraphs of G and satisfies the condition (RD).*

Proof. It follows from Theorem 9 and Lemma 11 that Maximum $\mathcal{K}$-Free b-Matching Problem can also be solved in polynomial time if $\mathcal{K}$-Free b-Factor Problem can. Hence, below we prove that $\mathcal{K}$-Free b-Factor Problem can be solved in polynomial time in a similar way to Theorem 13.

Let $(G, b, \mathcal{K})$ be an instance of $\mathcal{K}$-Free b-Factor Problem, where $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and $\mathcal{K}$ is a family of subgraphs in G satisfying the condition (RD). Let $\mathcal{K}_1, \ldots, \mathcal{K}_\ell$ be the partition of $\mathcal{K}$ in the condition (RD).

For each $i \in \{1, \ldots, \ell\}$, execute the following procedure. Let H_i be the graph defined as the union of all $K \in \mathcal{K}_i$, i.e.,

$$H_i := \left(\bigcup_{K \in \mathcal{K}_i} V(K), \bigcup_{K \in \mathcal{K}_i} E(K) \right).$$

Then,

- add a new vertex r_i and t parallel edges between r_i and v for each $v \in V(H_i)$, and remove the original edges in $E(H_i)$; and
- compute a set $D_{H_i} \subseteq \mathbb{Z}_+^{V(H_i)}$ of the degree sequences in the $\mathcal{K}_i$-free b-matchings in H_i, i.e.,

$$D_{H_i} = \left\{ d_F \in \mathbb{Z}_+^{V(H_i)} \colon F \text{ is a } \mathcal{K}_i\text{-free } b\text{-matching in } H_i \right\}.$$

For each $i \in \{1, \ldots, \ell\}$, it follows from Theorem 9 that the set D_{H_i} is a constant-parity jump system. We also remark that D_{H_i} can be computed efficiently in a brute force way, since $|V(H_i)|$ and t are bounded by a fixed constant.

Now, by the same argument as in the proof of Theorem 13, we can solve $\mathcal{K}$-Free b-Factor Problem in polynomial-time with the aid of Theorem 12. ◀

4 Degree Bounded Graphs

In this section, we consider the case where the maximum degree of G is at most $2t-1$.

▶ **Theorem 18.** *For a fixed positive integer t, $\mathcal{K}$-Free b-Factor Problem can be solved in polynomial time if the maximum degree of G is at most $2t-1$, $b(v) \leq t$ for each $v \in V$, and all the subgraphs in $\mathcal{K}$ are t-regular complete partite.*

Proof. If $t = 1$, then the problem is trivial, because the maximum degree is one and a t-regular complete partite subgraph must be composed of a single edge. Therefore, it suffices to consider the case where $t \geq 2$.

Without loss of generality, we may assume that each subgraph $K \in \mathcal{K}$ satisfies

$$b(v) = t \text{ for each vertex } v \in V(K), \tag{3}$$

since otherwise we can remove K from $\mathcal{K}$.

Define a vertex subset family $\mathcal{X} \subseteq 2^V$ by $\mathcal{X} = \{V(K) \colon K \in \mathcal{K}\}$. Construct a subfamily $\mathcal{X}^* \subseteq \mathcal{X}$ of disjoint vertex subsets in $\mathcal{X}$ in the following manner: start with $\mathcal{X}^* = \emptyset$; and while there exists a set in $\mathcal{X}$ disjoint from every set in $\mathcal{X}^*$, add an inclusionwise maximal one to $\mathcal{X}^*$. We denote $\mathcal{X}^* = \{X_1, X_2, \ldots, X_\ell\}$. It follows from the construction that $\mathcal{X}^* \subseteq \mathcal{X}$ satisfies the following property:

$$\text{for each } X \in \mathcal{X} \setminus \mathcal{X}^*, \text{ there exists } X_i \in \mathcal{X}^* \text{ such that } X \cap X_i \neq \emptyset \text{ and } X_i \not\subseteq X. \tag{4}$$

For each $X_i \in \mathcal{X}^*$, let $\mathcal{K}_i = \{K \in \mathcal{K} \colon V(K) \subseteq X_i\}$ and let H_i be the union of all subgraphs in $\mathcal{K}_i$, i.e.,

$$H_i = \left(X_i, \bigcup_{K \in \mathcal{K}_i} E(K) \right).$$

Let $\mathcal{K}^* = \bigcup_{i=1}^{\ell} \mathcal{K}_i$. Note that $\mathcal{K}_1, \ldots, \mathcal{K}_\ell$ form a partition of $\mathcal{K}^*$, and they satisfy the condition (RD).

By using Theorem 17, in polynomial time, we can find a $\mathcal{K}^*$-free b-factor M in G or conclude that G has no $\mathcal{K}^*$-free b-factor. In the latter case, we can conclude that G has no $\mathcal{K}$-free b-factor, because $\mathcal{K}^*$ is a subfamily of $\mathcal{K}$. In the former case, we transform M into a $\mathcal{K}$-free b-factor as shown in the following claim.

▷ Claim 19. Given a $\mathcal{K}^*$-free b-factor M in G, we can construct a $\mathcal{K}$-free b-factor in polynomial time.

Proof of Claim 19. For a b-factor M in G, define a subgraph family $\mathcal{K}(M)$ by

$$\mathcal{K}(M) = \{K \in \mathcal{K} \colon E(K) \subseteq M\},$$

the set of forbidden subgraphs included in M. Obviously, M is $\mathcal{K}$-free if and only if $\mathcal{K}(M) = \emptyset$. In what follows, given a $\mathcal{K}^*$-free b-factor M, we modify M so that $\mathcal{K}(M)$ becomes smaller.

Let M be a $\mathcal{K}^*$-free b-factor and suppose that $\mathcal{K}(M) \neq \emptyset$. Then, there exists a subgraph $K \in \mathcal{K} \setminus \mathcal{K}^*$ such that $K \in \mathcal{K}(M)$, i.e., $E(K) \subseteq M$. It follows from $K \in \mathcal{K} \setminus \mathcal{K}^*$ that $V(K) \in \mathcal{X} \setminus \mathcal{X}^*$. Then, (4) implies that there exists $X_i \in \mathcal{X}^*$ such that $V(K) \cap X_i \neq \emptyset$ and $X_i \not\subseteq V(K)$. It holds that $X_i = V(K^*)$ for some $K^* \in \mathcal{K}_i$, which follows from the construction of $\mathcal{X}^*$ and the definition of $\mathcal{K}_i$. We thus obtain $V(K) \cap V(K^*) \neq \emptyset$ and $V(K^*) \not\subseteq V(K)$.

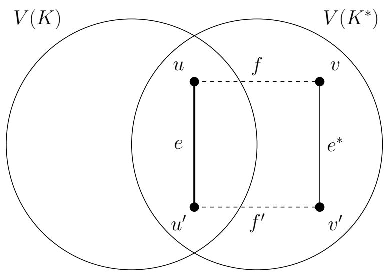

Figure 2 All of the edges are in $E(K^*)$ and, particularly, all of the solid edges are in M. The solid bold edge is in $E(K^*) \cap E(K)$ and the other thin edge is in $E(K^*) \setminus E(K)$.

Take a vertex u in $V(K) \cap V(K^*)$. Since $|\delta_K(u)| = |\delta_{K^*}(u)| = t$ and $|\delta_G(u)| \leq 2t - 1$, there exists an edge $e \in \delta_K(u) \cap \delta_{K^*}(u)$, in particular $e \in E(K) \cap E(K^*)$. We denote $e = uu'$. Note that $e \in M$ since $E(K) \subseteq M$.

Since $V(K^*) \not\subseteq V(K)$, there exists a vertex $v \in V(K^*) \setminus V(K)$. From (3) and $K^* \in \mathcal{K}$, we obtain $|\delta_M(v)| = b(v) = t$. It then follows from $|\delta_G(v)| \leq 2t - 1$ and $|\delta_{K^*}(v)| = t$ that $\delta_M(v) \cap \delta_{K^*}(v) \neq \emptyset$, that is, there exists an edge $e^* \in \delta_{K^*}(v)$ contained in M. We denote $e^* = vv'$. Since K is a connected component of the subgraph induced by M (see Remark 8), it holds that $v' \in V(K^*) \setminus V(K)$; see Figure 2.

Since $e, e^* \in E(K^*)$ and K^* is a complete partite graph, u and u' are contained in different color classes of K^*, and so are v and v'. This shows that K^* contains two edges: uv and $u'v'$; or uv' and $u'v$. Without loss of generality, assume that $f = uv$ and $f' = u'v'$ are contained in K^*; see Figure 2 again. Note that f and f' are not contained in M, because $\delta_M(u) = \delta_K(u)$ and $\delta_M(u') = \delta_K(u')$ hold.

Define $M' = (M \setminus \{e, e^*\}) \cup \{f, f'\}$, which is also a b-factor. In what follows, we prove that M' is the desired $\mathcal{K}^*$-free b-factor, i.e., $\mathcal{K}(M') \subsetneq \mathcal{K}(M)$. Since $K \notin \mathcal{K}(M')$, it suffices to show that $\mathcal{K}(M') \subseteq \mathcal{K}(M)$.

Assume to the contrary that there exists a subgraph $K' \in \mathcal{K}(M') \setminus \mathcal{K}(M)$. Then, K' must contain at least one of f and f', and without loss of generality assume that $f \in E(K')$. Since $K - e$ is connected by $t \geq 2$ and M' contains $(E(K) \setminus \{e\}) \cup \{f\}$ by $e^* \notin E(K)$, it follows from Remark 8 that $V(K) \cup \{v\}$ is contained in K', in particular $u, u', v \in V(K')$.

Since all the edges in $\delta_M(u')$ are contained in K and $v \notin V(K)$, M has no edge connecting u' and v, and neither does M'. It then follows from $K' \in \mathcal{K}(M')$, i.e., $E(K') \subseteq M'$, that $u'v \notin E(K')$. Since e is the only edge in M connecting u and u', we have $uu' \notin M'$, which implies that $uu' \notin E(K')$. It now follows from $u'v, uu' \notin E(K')$ that u, u' and v are contained in the same color class of K', since K' is complete partite. This contradicts the fact that K' contains $f = uv$, and thus we conclude that $\mathcal{K}(M') \subsetneq \mathcal{K}(M)$.

By repeating the above procedure, we obtain a b-factor M with $\mathcal{K}(M) = \emptyset$, i.e., M is $\mathcal{K}$-free. It is straightforward to see that this procedure can be executed in polynomial time, which completes the proof. ◁

Therefore, we conclude that $\mathcal{K}$-Free b-Factor Problem can be solved in polynomial time. ◀

From Theorem 18, we can derive the following theorem by applying the same argument as Theorem 16.

▶ **Theorem 20.** *For a fixed positive integer t,* Maximum $\mathcal{K}$-Free b-Matching Problem *can be solved in polynomial time if the maximum degree of G is at most $2t - 1$, $b(v) \leq t$ for each $v \in V$, and all the subgraphs in $\mathcal{K}$ are t-regular complete partite.*

From Theorem 20, we immediately obtain Theorem 3 by setting $b(v) = t$ for every $v \in V$.

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