Euclidean Capacitated Vehicle Routing in the Random Setting: A 1.55-Approximation Algorithm

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Abstract

We study the unit-demand capacitated vehicle routing problem in the random setting of the Euclidean plane. The objective is to visit n random terminals in a square using a set of tours of minimum total length, such that each tour visits the depot and at most k terminals.

We design an algorithm combining the classical sweep heuristic and the framework for the Euclidean traveling salesman problem due to Arora [J. ACM 1998] and Mitchell [SICOMP 1999]. We show that our algorithm is a polynomial-time approximation of ratio at most 1.55 asymptotically almost surely. This improves on the prior ratio of 1.915 due to Mathieu and Zhou [RSA 2022]. In addition, we conjecture that, for any $\varepsilon > 0$, our algorithm is a $(1+\varepsilon)$-approximation asymptotically almost surely.

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1 Introduction

In the *unit-demand capacitated vehicle routing problem (CVRP)*, we are given a set V of n *terminals* and a *depot* O. The terminals and the depot are located in some metric space. There is an unlimited number of identical vehicles, each of an integer *capacity* k. The tour of a vehicle starts at the depot and returns there after visiting at most k terminals. The objective is to visit every terminal, using a set of tours of minimum total length. Unless explicitly mentioned, for all CVRP instances in this paper, each terminal is assumed to have *unit demand*. Vehicle routing is a basic type of problems in operations research, and several books (see [4, 22, 30, 52] among others) have been written on those problems.

We study the *Euclidean* version of the CVRP, in which all locations (the terminals and the depot) lie in the two-dimensional plane, and the distances are given by the Euclidean metric. The Euclidean CVRP is NP-hard, since it is a generalization of the Euclidean *traveling salesman problem (TSP)*, and the Euclidean TSP is NP-hard [28, 48]. For the Euclidean TSP, Arora [5] and Mitchell [44] gave the first approximation scheme, which is among the most prominent results in theoretical computer science.[1] However, as stated in a survey

[1] For example, the approximation scheme of Arora [5] and Mitchell [44] for the Euclidean TSP won the Gödel Prize in 2010.

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of Arora [6], the Euclidean CVRP resists Arora's framework from [5]. The best-to-date polynomial-time approximation algorithm for the Euclidean CVRP has a ratio of $2+\varepsilon$ for any $\varepsilon > 0$.[2] Whether there is a polynomial-time $(1+\varepsilon)$-approximation for the Euclidean CVRP for any $\varepsilon > 0$ is a fundamental question and remains open regardless of numerous efforts for several decades, e.g., [1, 6, 8, 24, 32, 33, 35, 42].

Given the difficult challenges in the Euclidean CVRP, researchers turned to an analysis beyond worst case, by making some probabilistic assumptions on the distribution of the input instance. In 1985, Haimovich and Rinnooy Kan [32] first studied this problem in the *random setting*, where the terminals are *n independent, identically distributed (i.i.d.)* uniform random points in $[0,1]^2$. An event $\mathcal{E}$ occurs *asymptotically almost surely (a.a.s.)* if $\lim_{n\to\infty} \mathbb{P}[\mathcal{E}] = 1$. It is a long-standing open question whether, in the random setting, there is a polynomial-time $(1+\varepsilon)$-approximation for the Euclidean CVRP a.a.s. for any $\varepsilon > 0$. Haimovich and Rinnooy Kan [32] introduced the classical *iterated tour partitioning (ITP)* algorithm, and they raised the question whether, in the random setting, ITP is a $(1+\varepsilon)$-approximation a.a.s. for any $\varepsilon > 0$. Bompadre, Dror, and Orlin [19] showed that, in the random setting, the approximation ratio of ITP is at most 1.995 a.a.s. Recently, Mathieu and Zhou [41] showed that, in the random setting, the approximation ratio of ITP is at most 1.915 a.a.s. and at least $1+c_0$ a.a.s. for some constant $c_0 > 0$.

In this paper, we design a new algorithm for the Euclidean CVRP (Algorithm 1). Our algorithm is inspired by the classical *sweep heuristic*, one of the most popular heuristics in practice; see Section 1.3. The main result of this paper is to show that, in the random setting, the approximation ratio of Algorithm 1 is at most 1.55 a.a.s. (Theorem 1). Furthermore, we conjecture that, in the random setting, Algorithm 1 is a $(1+\varepsilon)$-approximation a.a.s. for any $\varepsilon > 0$ (Conjecture 8). Note that our analysis can be slightly adapted to show that the approximation ratio of ITP in the random setting is at most 1.55 a.a.s. (Remark 7).

1.1 Our Results

We present a new algorithm for the Euclidean CVRP. See Algorithm 1. For each terminal v, let $\theta(v) \in [0, 2\pi)$ denote the polar angle of v with respect to O. First, we sort all terminals in nondecreasing order of $\theta(v)$. Let $M \geq 1$ be a constant integer parameter. Next, we decompose the sorted sequence into subsequences, each consisting of Mk consecutive terminals, except possibly for the last subsequence containing less terminals. Finally, for the terminals in each subsequence, we compute a solution to the CVRP with a constant number of tours. The last step is achieved near-optimally using the framework for the Euclidean TSP due to Arora [5] and Mitchell [44]; see also Section 1.4.

Our main result shows that, in the random setting, Algorithm 1 has an approximation ratio at most 1.55 a.a.s., see Theorem 1. This improves on previous ratios of 1.995 due to Bompadre, Dror, and Orlin [19] and 1.915 due to Mathieu and Zhou [41].

► **Theorem 1.** *Consider the unit-demand Euclidean CVRP with a set V of n terminals that are i.i.d. uniform random points in $[0,1]^2$, a fixed depot $O \in \mathbb{R}^2$, and a capacity k that takes an arbitrary value in $\{1, 2, \ldots, n\}$. For any constant integer $M \geq 10^5$, Algorithm 1 with parameter M is a polynomial-time approximation of ratio at most* 1.55 *asymptotically almost surely.*

[2] The $(2+\varepsilon)$-approximation algorithm is obtained by first computing a traveling salesman tour using the approximation scheme of Arora [5] and Mitchell [44], and then apply the iterated tour partitioning algorithm [32] on that traveling salesman tour.

Algorithm 1 Algorithm for the CVRP in $\mathbb{R}^2$. Constant integer parameter $M \geq 1$.

Input: set V of n terminals in $\mathbb{R}^2$, depot $O \in \mathbb{R}^2$, capacity $k \in \{1, 2, \ldots, n\}$
Output: set of tours covering all terminals in V

```
Sort the terminals in V into u_1, u_2, ..., u_n such that θ(u_1) ≤ θ(u_2) ≤ ··· ≤ θ(u_n)
for i ← 1 to ⌈n/(Mk)⌉ do
    V_i ← {u_j : (i − 1) · Mk < j ≤ i · Mk}
    Compute a (1 + 1/M)-approximate solution S_i to subproblem (V_i, O, k)   ▷ Lemma 9
return union of all S_i
```

▶ Remark 2. The running time of Algorithm 1 is exponential in the parameter M due to the computation of a $\left(1 + \frac{1}{M}\right)$-approximate solution using the framework of Arora [5] and Mitchell [44]. We let $M \geq 10^5$ in Theorem 1 in order to achieve the claimed ratio of 1.55. Note that there is a tradeoff between the value of M and the approximation ratio, according to (10). Thus we may decrease the value of M at the cost of increasing the approximation ratio. In practice, it is also possible to replace the framework of Arora [5] and Mitchell [44] by some heuristic in order to make Algorithm 1 faster.

1.2 Overview of Techniques

A main contribution in our analysis is the novel concepts of *R-radial cost* and *R-local cost*. These are generalizations of the classical *radial cost* and *local cost* introduced by Haimovich and Rinnooy Kan [32].

▶ **Definition 3** (R-radial cost)**.** *For any $R \in \mathbb{R}^+ \cup \{0, \infty\}$, define the R-*radial cost rad_R *by*

$$\mathrm{rad}_R := \frac{2}{k} \sum_{v \in V} \min\left\{d(O, v), R\right\}.$$

▶ **Definition 4** (R-local cost)**.** *For any $R \in \mathbb{R}^+ \cup \{0, \infty\}$, define the R-*local cost T^*_R *as the minimum cost of a traveling salesman tour on the set of points $\{v \in V : d(O, v) \geq R\}$.*

Using the R-radial cost and the R-local cost, we establish a new lower bound (Theorem 5) on the cost of an optimal solution. This lower bound is a main novelty of the paper. It unites both classical lower bounds from [32]: when $R = 0$, it leads asymptotically to one classical lower bound, which is the local cost; and when $R = \infty$, it leads asymptotically to the other classical lower bound, which is the radial cost. The proof of Theorem 5 is simple and combinatorial; see Section 2.

▶ **Theorem 5.** *Consider the unit-demand Euclidean CVRP with any set V of n terminals in $\mathbb{R}^2$, any depot $O \in \mathbb{R}^2$, and any capacity $k \in \mathbb{N}^+$. Let* opt *denote the cost of an optimal solution. For any $R \in \mathbb{R}^+ \cup \{0, \infty\}$, we have*

$$\mathrm{opt} \geq T^*_R + \mathrm{rad}_R - \frac{3\pi D}{2},$$

where D denotes the diameter of $V \cup \{O\}$.

Next, we establish an upper bound (Theorem 6) on the cost of the solution in Algorithm 1 using the 0-local cost and the ∞-radial cost. To prove Theorem 6, we decompose the plane into regions according to Algorithm 1 and apply a result of Karp [34] to each region; see Section 3.

▶ **Theorem 6.** *Consider the unit-demand Euclidean CVRP with any set V of n terminals in $\mathbb{R}^2$, any depot $O \in \mathbb{R}^2$, and any capacity $k \in \mathbb{N}^+$. For any positive integer M, let* $\mathrm{sol}(M)$ *denote the cost of the solution returned by Algorithm 1 with parameter M. Then we have*

$$\mathrm{sol}(M) \leq \left(1 + \frac{1}{M}\right) \left(T_0^* + \mathrm{rad}_\infty + \frac{3\pi D}{2} \left\lceil \frac{n}{Mk} \right\rceil\right),$$

where D denotes the diameter of $V \cup \{O\}$.

Note that both Theorem 5 and Theorem 6 hold for any set of terminals, not only in the random setting, and thus can be of independent interest.

In the rest of this section, we focus on the random setting.

To prove Theorem 1, we let R in Theorem 5 be some well-chosen value so as to minimize the ratio between the upper bound (Theorem 6) and the lower bound (Theorem 5). The proof of Theorem 1 is technical; see Section 4.

▶ Remark 7. In the random setting, the term containing D in the bound in Theorem 5 (resp. Theorem 6) is negligible. The upper bound in Theorem 6 is then asymptotically identical to the upper bound for the ITP algorithm established in [2]. As an immediate consequence, one can prove that the approximation ratio of ITP in the random setting is at most 1.55 a.a.s. using a similar analysis as the proof of Theorem 1.

Note that the upper bound for ITP established in [2] is tight [41, Lemma 7], and the tightness is exploited to show that ITP is at best a $(1 + c_0)$-approximation for some constant $c_0 > 0$ [41]. However, the upper bound for Algorithm 1 established in Theorem 6 is possibly not tight. We conjecture that Algorithm 1 is a $(1 + \varepsilon)$-approximation a.a.s. for any $\varepsilon > 0$.

▶ **Conjecture 8.** *Consider the unit-demand Euclidean CVRP with V, O, and k defined in Theorem 1. For any $\varepsilon > 0$, there exists a positive constant integer M depending on ε, such that Algorithm 1 with parameter M is a polynomial-time $(1+\varepsilon)$-approximation asymptotically almost surely.*

1.3 Related Work

1.3.1 Sweep Heuristic

The classical *sweep heuristic* is well-known and commercially available for vehicle routing problems in the plane. At the beginning, all terminals are sorted according to their polar angles with respect to the depot. For each k consecutive terminals in the sorted sequence, a tour is obtained by computing a traveling salesman tour (exactly or approximately) on those terminals. Some implementations include a post-optimization phase in which vertices in adjacent tours may be exchanged to reduce the overall cost. The first mentions of this type of method are found in a book by Wren [55] and in a paper by Wren and Holliday [56], while the sweep heuristic is commonly attributed to Gillett and Miller [29] who popularized it. See also surveys [21, 37, 38] and the book [52].

Our main contribution is the analysis of a new algorithm (Algorithm 1) inspired by the sweep heuristic. In Algorithm 1, instead of forming groups each of k consecutive terminals, we form groups each of Mk consecutive terminals, for some positive constant integer M. Then for each group, we compute a solution consisting of a constant number of tours using the framework of Arora [5] and Mitchell [44]. We show that Algorithm 1 improves upon the previous best approximation ratio for the Euclidean CVRP in the random setting.

Thanks to the simplicity of Algorithm 1, it could be adapted to other vehicle routing problems, e.g., distance-constrained vehicle routing [25, 27, 40, 46].

1.3.2 Euclidean TSP

The first *polynomial-time approximation scheme (PTAS)* for the Euclidean TSP is due to Arora [5] and Mitchell [44]. That approximation scheme is among the most classical algorithms in textbooks, e.g., [16, 53, 54]. The running time of the approximation scheme was improved to $O(n \log n)$ by Rao and Smith [49] and further to $O(n)$ by Bartal and Gottlieb [9], for any fixed $\varepsilon > 0$. Recently, Kisfaludi-Bak, Nederlof, and Węgrzycki [36] achieved the optimal dependence on ε in the running time of the approximation scheme under the Gap-Exponential Time Hypothesis (Gap-ETH).

1.3.3 Euclidean CVRP

Despite the difficulty of the Euclidean CVRP, there has been progress on several special cases in the deterministic setting. A series of papers designed PTAS's for small k: Haimovich and Rinnooy Kan [32] gave a PTAS when k is constant; Asano et al. [8] extended the techniques in [32] to achieve a PTAS for $k = O(\log n / \log \log n)$; and Adamaszek, Czumaj, and Lingas [1] designed a PTAS for $k \leq 2^{\log^{f(\varepsilon)}(n)}$. For higher dimensional Euclidean metrics, Khachay and Dubinin [35] gave a PTAS for fixed dimension ℓ and $k = O(\log^{\frac{1}{\ell}}(n))$. For arbitrary k and the two-dimensional plane, Das and Mathieu [24] designed a quasi-polynomial time approximation scheme, whose running time was recently improved to $n^{O(\log^6(n)/\varepsilon^5)}$ by Jayaprakash and Salavatipour [33].

1.3.4 Probabilistic Analyses

The random setting in which the terminals are i.i.d. uniform random points is perhaps the most natural probabilistic setting. The Euclidean CVRP in the random setting has been studied in several special cases. In one special case when the capacity is infinite, Karp [34] gave a polynomial-time $(1+\varepsilon)$-approximation a.a.s. for any $\varepsilon > 0$. In another special case when k is fixed, Rhee [50] and Daganzo [23] analyzed the value of an optimal solution.

1.3.5 CVRP in Other Metrics

On general metrics, the CVRP has been extensively studied [2, 17, 18, 32, 39]. The best-to-date approximation ratio on general metrics is $1+\alpha-\varepsilon$ due to Blauth, Traub, and Vygen [17], where α is the approximation ratio of a TSP algorithm and $\varepsilon > 0$ is a constant depending on α. For planar graphs, when the tour capacity is bounded, Becker, Klein, and Saulpic [12] gave a QPTAS, which was improved to a PTAS by Becker, Klein and Schild [14]. The CVRP has also been studied on trees and bounded treewidth [7, 11, 15, 33, 42], bounded-genus graphs [12, 20], graphic metrics [45], graphs of bounded highway dimension [13], and minor-free graphs [20].

1.3.6 CVRP with Arbitrary Demands

A natural way to generalize the unit demand version of the CVRP is to allow terminals to have arbitrary unsplittable demands, which is called the *unsplittable* version of the CVRP. On general metrics, the approximation of this problem was first studied by Altinkemer and Gavish [3]. Recently, the approximation ratio was improved by Blauth, Traub, and Vygen [17], and further by Friggstad, Mousavi, Rahgoshay, and Salavatipour [26]. This problem has also been studied in the Euclidean plane [31] and on trees [43].

1.4 Notations and Preliminaries

For any two points u and v in $\mathbb{R}^2$, let $d(u, v)$ denote the *distance* between u and v in $\mathbb{R}^2$. For any curve s in $\mathbb{R}^2$, let $\|s\|$ denote the length of s; and for any set S of curves in $\mathbb{R}^2$, let $\|S\| := \sum_{s\in S} \|s\|$. For any set U of points in $\mathbb{R}^2$, the *convex hull* of U is the minimal convex set in $\mathbb{R}^2$ containing U.

Asano et al. [8] observed that the PTAS of Arora [5] and Mitchell [44] for the Euclidean TSP implies a PTAS for the Euclidean CVRP when the capacity k is at least a constant fraction of the number of terminals, see [8, Section 6]. Lemma 9 is a reformulation of [8] by setting $M = 1/\varepsilon$.

▶ **Lemma 9** ([8]). *Let $M \geq 1$ be an integer constant. There exists a polynomial-time $\left(1 + \frac{1}{M}\right)$-approximation algorithm for the unit-demand Euclidean capacitated vehicle routing problem with any finite set U of terminals in $\mathbb{R}^2$, any depot $O \in \mathbb{R}^2$, and any capacity k such that $k = \Omega(|U|)$.*

2 Proof of Theorem 5

Let OPT denote an optimal solution to the CVRP. Let C denote the circle centered at O and with radius R. Let $t \in \mathbb{N}$ be such that the union of the tours in OPT intersects C at $2t$ points. Let $y_1, y_2, \ldots, y_{2t}$ be those intersection points in clockwise order. For notational convenience, we let $y_{2t+1} := y_1$. Let $C_1, C_2, \ldots, C_t$ be t continuous curves that correspond to the intersection between OPT and the closure of the exterior of C. See Figure 1.

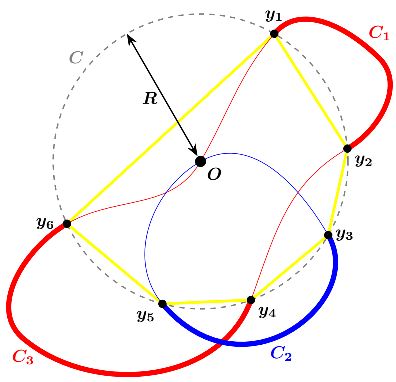
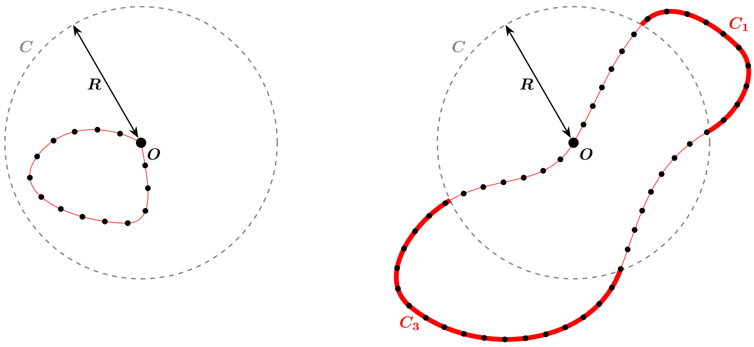


Figure 1 The circle C is dashed. In this example, OPT consists of two tours, in red and in blue, respectively. The two tours intersect the circle C at $y_1, \ldots, y_6$. The segments $\{y_i y_{i+1}\}_{1\leq i\leq 6}$ are in yellow. C_1, C_2, and C_3 are the thick curves.

▶ **Lemma 10.** *We have*

$$\sum_{i=1}^{t} \|C_i\| \geq T_R^* - \frac{3\pi D}{2}.$$

Proof. Let Z denote the set of segments $y_i y_{i+1}$ for all $1 \leq i \leq 2t$. Let Z_{odd} (resp. Z_{even}) denote the set of segments $y_i y_{i+1}$ for all $1 \leq i \leq 2t$ such that i is odd (resp. even). Let Z^* be one of Z_{odd} and Z_{even} that has a smaller total length, breaking ties arbitrarily. Let W denote the union of the curves $C_1, C_2, \ldots, C_t$, the segments in Z, and the segments in Z^*. Then W is a connected graph with no odd degree vertices. So W has an Eulerian path. Since



(a) Case 1: when U_s is empty.

(b) Case 2: when U_s is nonempty. In this example, $U_s = \{1, 3\}$.

Figure 2 The tour s is in red; the dots represent the points in V_s.

W visits all vertices $v \in V$ such that $d(O, v) \geq R$, the total length of W is at least T_R^* by Definition 4. Hence

$$\|Z\| + \|Z^*\| + \sum_{i=1}^{t} \|C_i\| \geq T_R^*.$$

Because the distance between any two points y_i and y_j $(1 \leq i < j \leq 2t)$ is at most D, the perimeter of the convex hull of $\{y_i : 1 \leq i \leq 2t\}$ is at most πD by [51, Section 2]. Thus $\|Z\| \leq \pi D$. Since $\|Z^*\| \leq \frac{1}{2}\|Z\|$, we have

$$\|Z\| + \|Z^*\| \leq \frac{3\|Z\|}{2} \leq \frac{3\pi D}{2}.$$

The claim follows. ◀

The next lemma (Lemma 11) is the key to the proof of Theorem 5.

▶ **Lemma 11.** *Let s be any tour in* OPT*. Let $V_s \subseteq V$ denote the set of points in V that are visited by s. Let $U_s \subseteq \{1, 2, \ldots, t\}$ denote the set of indices i such that C_i is part of s. We have*

$$\|s\| \geq \sum_{i \in U_s} \|C_i\| + \frac{2}{k} \sum_{v \in V_s} \min\{d(O, v), R\}. \tag{1}$$

Proof. We consider the following two cases.

Case 1: when U_s is empty (see Figure 2a). Then we have

$$\|s\| \geq 2 \max_{v \in V_s} d(O, v) \geq \frac{2}{|V_s|} \sum_{v \in V_s} d(O, v) \geq \frac{2}{|V_s|} \sum_{v \in V_s} \min\{d(O, v), R\}.$$

The claim follows since $|V_s| \leq k$.

Case 2: when U_s is nonempty (see Figure 2b). Then the tour s must first travel through a path to a point on C, paying at least R, then visit all curves C_i for $i \in U_s$, and finally, travel from a point on C back to the depot, paying at least R. Thus we have

$$\|s\| \geq 2R + \sum_{i \in U_s} \|C_i\|.$$

Observe that

$$2R = \frac{2}{|V_s|} \sum_{v\in V_s} R \geq \frac{2}{|V_s|} \sum_{v\in V_s} \min\{d(O,v),R\}.$$

The claim follows since $|V_s| \leq k$. ◀

Summing (1) over all tours $s \in \mathrm{OPT}$, we have

$$\begin{aligned}\mathrm{opt} &= \sum_{s\in\mathrm{OPT}} \|s\| \\ &\geq \sum_{s\in\mathrm{OPT}} \sum_{i\in U_s} \|C_i\| + \frac{2}{k} \sum_{s\in\mathrm{OPT}} \sum_{v\in V_s} \min\{d(O,v),R\} \\ &= \sum_{i=1}^{t} \|C_i\| + \frac{2}{k} \sum_{v\in V} \min\{d(O,v),R\} \\ &\geq T_R^* - \frac{3\pi D}{2} + \mathrm{rad}_R,\end{aligned}$$

where the last inequality follows from Lemma 10 and the definition of R-radial cost (Definition 3).

Therefore, the claim in Theorem 5 follows.

3 Proof of Theorem 6

Let i be any integer with $1 \leq i \leq \lceil \frac{n}{Mk} \rceil$. Let the point set V_i and the solution S_i be defined in Algorithm 1. Let S_i^* denote an optimal solution to the subproblem (V_i, O, k). Since S_i is a $\left(1+\frac{1}{M}\right)$-approximate solution, we have $\|S_i\| \leq \left(1+\frac{1}{M}\right) \cdot \|S_i^*\|$. Let TSP_i denote the minimum cost of a traveling salesman tour on the set of points $V_i \cup \{O\}$. By [2, Lemma 2], we have

$$\|S_i^*\| \leq \mathrm{TSP}_i + \frac{2}{k} \sum_{v\in V_i} d(O,v).$$

Thus

$$\|S_i\| \leq \left(1+\frac{1}{M}\right) \left(\mathrm{TSP}_i + \frac{2}{k} \sum_{v\in V_i} d(O,v)\right). \tag{2}$$

Let t^* be an optimal traveling salesman tour on the set of points V. If the polar angles of points in V_i have a span of at most π, let Y_i be the interior of the convex hull of $V_i \cup \{O\}$; otherwise, let Y_i be the exterior of the convex hull of $(V \setminus V_i) \cup \{O\}$. By a result of Karp [34, Theorem 3], we have[3]

$$\mathrm{TSP}_i - \|t^* \cap Y_i\| \leq \frac{3}{2}\,\mathrm{per}(Y_i),$$

where $\mathrm{per}(Y_i)$ denotes the perimeter of Y_i. Since either Y_i or the complement of Y_i is convex with diameter at most D, the perimeter of Y_i is at most πD by [51, Section 2]. Thus

$$\mathrm{TSP}_i \leq \|t^* \cap Y_i\| + \frac{3\pi D}{2}. \tag{3}$$

In order to bound $\sum_i \|t^* \cap Y_i\|$, we need the following lemma.

[3] Note that the setting in [34] is a rectangle. However, the proof of Theorem 3 in [34], which is based on duplicating the boundary, holds in the more general setting of a polygon or the exterior of a polygon.

▶ **Lemma 12.** *For any i and j with $1 \le i < j \le \lceil \frac{n}{Mk} \rceil$, Y_i and Y_j do not intersect.*

Proof. For each $v \in V$, let $\theta(v) \in [0, 2\pi)$ denote the polar angle of v respect to O. By the definition of V_i and V_j, we have

$$0 \le \min_{v \in V_i} \theta(v) \le \max_{v \in V_i} \theta(v) \le \min_{v \in V_j} \theta(v) \le \max_{v \in V_j} \theta(v) < 2\pi.$$

Hence

$$\max_{v \in V_i} \theta(v) - \min_{v \in V_i} \theta(v) \le \pi \tag{4}$$

or

$$\max_{v \in V_j} \theta(v) - \min_{v \in V_j} \theta(v) \le \pi. \tag{5}$$

If (4) holds and (5) does not hold, then by definition, Y_i is the interior of the convex hull of $V_i \cup \{O\}$, which is contained in the interior of the convex hull of $(V \setminus V_j) \cup \{O\}$. Thus Y_i and Y_j do not intersect. If (5) holds and (4) does not hold, the argument is similar.

It remains to consider the case when both (4) and (5) hold. Let Z_i be the set

$$Z_i := \left\{ x \in \mathbb{R}^2 : \min_{v \in V_i} \theta(v) < \theta(x) < \max_{v \in V_i} \theta(v) \right\},$$

and Z_j be the set

$$Z_j := \left\{ x \in \mathbb{R}^2 : \min_{v \in V_j} \theta(v) < \theta(x) < \max_{v \in V_j} \theta(v) \right\}.$$

By (4) and (5), Z_i and Z_j are convex sets. By the definition of Y_i and Y_j, we have $Y_i \subset Z_i$ and $Y_j \subset Z_j$. Since Z_i and Z_j do not intersect, Y_i and Y_j do not intersect. ◀

Therefore, we have

$$\begin{aligned} \mathrm{sol}(M) &= \sum_{i=1}^{\lceil \frac{n}{Mk} \rceil} \|S_i\| \\ &\le \left(1 + \frac{1}{M}\right) \left(\sum_{i=1}^{\lceil \frac{n}{Mk} \rceil} \mathrm{TSP}_i + \frac{2}{k} \sum_{v \in V} d(O, v) \right) \\ &\le \left(1 + \frac{1}{M}\right) \left(\sum_{i=1}^{\lceil \frac{n}{Mk} \rceil} \|t^* \cap Y_i\| + \frac{3\pi D}{2} \left\lceil \frac{n}{Mk} \right\rceil + \mathrm{rad}_\infty \right), \end{aligned}$$

where the first inequality follows from (2) and the fact that $\bigcup_i V_i = V$, and the last inequality follows from (3) and the definition of ∞-radial cost (Definition 3). Using Lemma 12 and the definition of 0-local cost (Definition 4), we have

$$\sum_{i=1}^{\lceil \frac{n}{Mk} \rceil} \|t^* \cap Y_i\| \le \|t^*\| = T_0^*.$$

The claim in Theorem 6 follows.

4 Proof of Theorem 1

In this section, we prove a strong law for the approximation ratio of Algorithm 1, as presented in Theorem 13. Since almost sure convergence implies convergence in probability, Theorem 13 implies Theorem 1.

▶ **Theorem 13.** *Let $v_1, v_2, \ldots$ be an infinite sequence of i.i.d. uniform random points in $[0,1]^2$. Let O be a point in $\mathbb{R}^2$. Let $k_1, k_2, \ldots$ be an infinite sequence of positive integers. Let $M \geq 10^5$ be a positive integer. For each positive integer n, consider the unit-demand Euclidean CVRP with the set of terminals $V = \{v_1, \ldots, v_n\}$, the depot O, and the capacity k_n. Let* opt *denote the cost of an optimal solution, and* sol(M) *denote the cost of the solution returned by Algorithm 1 with parameter M. Then we have*

$$\limsup_{n\to\infty} \frac{\text{sol}(M)}{\text{opt}} < 1.55$$

almost surely.

In the rest of this section, we prove Theorem 13.

Let $R = \frac{3}{4}\,\mathbb{E}\left(d(O,v)\right)$, where v is a uniform random point in $[0,1]^2$. Let D denote the diameter of $[0,1]^2 \cup \{O\}$. Let T_0^* and T_R^* denote the 0-local and R-local costs respectively. Let rad_∞ and rad_R denote the ∞-radial and R-radial costs respectively. By Theorem 5 and Theorem 6, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\text{sol}(M)}{\text{opt}} \\
\leq &\limsup_{n\to\infty} \frac{\left(1+\frac{1}{M}\right)\left(T_0^* + \text{rad}_\infty + \frac{3\pi D}{2}\left(\frac{n}{Mk_n}+1\right)\right)}{T_R^* + \text{rad}_R - \frac{3\pi D}{2}} \\
\leq &\limsup_{n\to\infty} \max\left\{\frac{\left(1+\frac{1}{M}\right)\left(T_0^*+\frac{3\pi D}{2}\right)}{T_R^* - \frac{3\pi D}{2}}, \frac{\left(1+\frac{1}{M}\right)\left(\text{rad}_\infty + \frac{3\pi Dn}{2Mk_n}\right)}{\text{rad}_R}\right\} \\
= &\left(1+\frac{1}{M}\right)\max\left\{\limsup_{n\to\infty}\frac{T_0^*+\frac{3\pi D}{2}}{T_R^*-\frac{3\pi D}{2}}, \limsup_{n\to\infty}\frac{\text{rad}_\infty+\frac{3\pi Dn}{2Mk_n}}{\text{rad}_R}\right\} \qquad (6)
\end{aligned}$$

almost surely.

The upper bounds on both of the limit superiors in (6) are established in Lemma 14 and Lemma 15, respectively.

▶ **Lemma 14.** *We have*

$$\limsup_{n\to\infty} \frac{T_0^* + \frac{3\pi D}{2}}{T_R^* - \frac{3\pi D}{2}} \leq \frac{48}{31}$$

almost surely.

▶ **Lemma 15.** *We have*

$$\limsup_{n\to\infty} \frac{\text{rad}_\infty + \frac{3\pi Dn}{2Mk_n}}{\text{rad}_R} \leq \frac{48}{31}\left(1+\frac{15\pi}{4M}\right)$$

almost surely.

The proof of Lemma 14 is given in Section 4.1. The proof of Lemma 15 is given in Section 4.2. Finally, in Section 4.3, we prove Theorem 13 using Lemma 14 and Lemma 15.

4.1 Proof of Lemma 14

Let λ_R denote the measure of the set $\left\{x \in [0,1]^2 : d(O,x) > R\right\}$. Let $S_R(n)$ denote the size of the set $\{1 \le i \le n : d(O,v_i) > R\}$. By the strong law of large numbers, we have

$$\lim_{n\to\infty} \frac{S_R(n)}{n} = \lambda_R \tag{7}$$

almost surely.

▶ **Lemma 16.** *We have* $\lambda_R \ge \frac{31}{48}$.

The proof of Lemma 16 is available in the full version [47] of the paper.

By Lemma 16, $S_R(n) \to \infty$ as $n \to \infty$. By applying the main result of [10] to the infinite sequence $v_1, v_2, \ldots$ and its intersection with the set $\left\{x \in [0,1]^2 : d(O,x) > R\right\}$, we have

$$\lim_{n\to\infty} \frac{T_0^*}{\sqrt{n}} = \lim_{n\to\infty} \frac{T_R^*}{\sqrt{\lambda_R S_R(n)}} > 0 \tag{8}$$

almost surely. Thus

$$\begin{aligned}
&\lim_{n\to\infty} \frac{T_0^* + \frac{3\pi D}{2}}{T_R^* - \frac{3\pi D}{2}} \\
=&\lim_{n\to\infty} \frac{\frac{T_0^*}{\sqrt{n}} + \frac{3\pi D}{2\sqrt{n}}}{\lambda_R \sqrt{\frac{S_R(n)}{\lambda_R\, n}} \frac{T_R^*}{\sqrt{\lambda_R S_R(n)}} - \frac{3\pi D}{2\sqrt{n}}} \\
=&\frac{\lim_{n\to\infty} \frac{T_0^*}{\sqrt{n}} + \lim_{n\to\infty} \frac{3\pi D}{2\sqrt{n}}}{\lambda_R \lim_{n\to\infty} \sqrt{\frac{S_R(n)}{\lambda_R\, n}} \lim_{n\to\infty} \frac{T_R^*}{\sqrt{\lambda_R S_R(n)}} - \lim_{n\to\infty} \frac{3\pi D}{2\sqrt{n}}} \\
=&\frac{1}{\lambda_R}
\end{aligned}$$

almost surely, where the last equality follows from Equation (7) and Equation (8). Since $\lambda_R \ge \frac{31}{48}$ (Lemma 16), the claim in Lemma 14 follows.

4.2 Proof of Lemma 15

By the strong law of large numbers, we have

$$\lim_{n\to\infty} \frac{k_n \operatorname{rad}_\infty}{2n} = \mathbb{E}\left(d(O,v)\right)$$

and

$$\lim_{n\to\infty} \frac{k_n \operatorname{rad}_R}{2n} = \mathbb{E}\left(\min\left\{d(O,v), R\right\}\right)$$

almost surely. Thus

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\operatorname{rad}_\infty + \frac{3\pi D n}{2M k_n}}{\operatorname{rad}_R} \\
=&\lim_{n\to\infty} \frac{\frac{k_n \operatorname{rad}_\infty}{2n} + \frac{3\pi D}{4M}}{\frac{k_n \operatorname{rad}_R}{2n}} \\
=&\frac{\mathbb{E}\left(d(O,v)\right) + \frac{3\pi D}{4M}}{\mathbb{E}\left(\min\left\{d(O,v), R\right\}\right)}
\end{aligned}$$

almost surely.

▶ **Lemma 17.** *For $R = \frac{3}{4}\,\mathbb{E}\left(d(O,v)\right)$, we have*

$$\mathbb{E}\left(\min\left\{d(O,v), R\right\}\right) \geq \frac{31}{48}\,\mathbb{E}\left(d(O,v)\right).$$

The proof of Lemma 17 is available in the full version [47] of the paper.

From Lemma 17, we obtain

$$\lim_{n\to\infty} \frac{\mathrm{rad}_\infty + \frac{3\pi D n}{2Mk_n}}{\mathrm{rad}_R} \leq \frac{48\left(\mathbb{E}\left(d(O,v)\right) + \frac{3\pi D}{4M}\right)}{31\,\mathbb{E}\left(d(O,v)\right)} \tag{9}$$

almost surely.

▶ **Lemma 18.** *We have*

$$D \leq 5\,\mathbb{E}\left(d(O,v)\right).$$

Proof. Let $O_c = \left(\frac{1}{2}, \frac{1}{2}\right) \in \mathbb{R}^2$ denote the center of the square $[0,1]^2$. Let $\overline{v}$ denote the reflection of v across the point O_c. Then we have

$$\frac{d(O,v) + d(O,\overline{v})}{2} \geq \frac{d(v,\overline{v})}{2} = d(O_c, v).$$

Because v and $\overline{v}$ have the same distribution, we have

$$\mathbb{E}\left(d(O,v)\right) \geq \mathbb{E}\left(d(O_c,v)\right).$$

We use a closed-form formula of $\mathbb{E}\left(d(O_c,v)\right)$ established in the full version [47] of the paper to obtain

$$\mathbb{E}\left(d(O_c,v)\right) = \frac{\sqrt{2} + \log\left(1+\sqrt{2}\right)}{6} \geq \frac{\sqrt{2}}{4}.$$

Therefore, by the definition of D, we have

$$D \leq \sqrt{2} + \mathbb{E}\left(d(O,v)\right) \leq 4\,\mathbb{E}\left(d(O_c,v)\right) + \mathbb{E}\left(d(O,v)\right) \leq 5\,\mathbb{E}\left(d(O,v)\right).$$ ◀

Lemma 15 follows from Equation (9) and Lemma 18.

4.3 Proof of Theorem 13

From (6), we have, almost surely,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\mathrm{sol}(M)}{\mathrm{opt}} \\
&\leq \left(1 + \frac{1}{M}\right) \max\left\{\limsup_{n\to\infty} \frac{T_0^* + \frac{3\pi D}{2}}{T_R^* - \frac{3\pi D}{2}}, \limsup_{n\to\infty} \frac{\mathrm{rad}_\infty + \frac{3\pi D n}{2Mk_n}}{\mathrm{rad}_R}\right\} \\
&\leq \frac{48}{31}\left(1 + \frac{1}{M}\right)\left(1 + \frac{15\pi}{4M}\right) \\
&< 1.55,
\end{aligned} \tag{10}$$

where the second inequality follows from Lemma 14 and Lemma 15, and the last inequality follows from the assumption $M \geq 10^5$.

This completes the proof of Theorem 13.

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