A Logic of East and West for Intervals

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Abstract

This paper proposes a logic of east and west for intervals (*LEWI*), which extends the logic of east and west for points. For intervals in 1D Euclidean space, the logic *LEWI* formalises the qualitative direction relations "east", "west", "definitely east", "definitely west", "partially east", "partially west", etc. To cope with imprecision in geometry representations, the logic *LEWI* is parameterized by a margin of error $\sigma \in \mathbb{R}_{>0}$ and a level of indeterminacy in directions $\tau \in \mathbb{N}_{>1}$. For every τ, we provide an axiomatisation of the logic *LEWI*, and prove that it is sound and complete with respect to 1D Euclidean space.

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1 Introduction

This work is motivated by the problem of matching spatial objects represented in different geospatial datasets, and checking the consistency of *SameAs* matching relations. For objects a and b in different datasets, $SameAs(a, b)$ holds if a and b refer to the same real world object. With the development of semantic web and linked data, many instance matching methods [6, 7, 11] have been developed to automatically generate *SameAs* matching relations. The geospatial datasets and automatically generated *SameAs* matching relations may contain errors. It is desirable to formalise the relations between spatial objects and use automated reasoning to detect contradictions in *SameAs* matching relations.

A logic of east and west for points (*LEW*) has been introduced to represent and reason about the direction relations between spatial objects, which can be used to check *SameAs* matching relations [5]. Intuitively, if in one dataset, a spatial object a is definitely to the east of a spatial object b, then in another dataset, the spatial object corresponding to a cannot be definitely to the west of the spatial object corresponding to b. In *LEW*, a spatial object is interpreted as a single point in 1D Euclidean space $\mathbb{R}$. In geospatial datasets, however, many spatial objects are represented using polygons. Instead of abstracting a polygon as a single point, it is more accurate to use the axis-aligned minimal bounding box of a polygon, which can be projected to a closed interval in 1D Euclidean space, as shown in Figure 1. Therefore, we extend *LEW* to a logic of East and West for Intervals (*LEWI*) which represents and reasons about the qualitative direction relations between closed intervals.

Allen's calculus [1] has been widely used to represent relations between intervals in both spatial and temporal domains. It defines 13 basic relations between intervals, such as "before", "after", "overlaps", etc. Similar to Allen's calculus, *LEWI* also defines relations between intervals based on their endpoints. However, in order to tolerate slight differences

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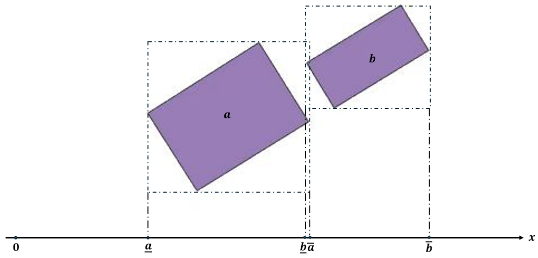

Figure 1 Projection of Polygons onto Closed Intervals.

in geometric representations in different geospatial datasets, following *LEW*, *LEWI* uses a parameter $\sigma \in \mathbb{R}_{>0}$ to denote the margin of error and a parameter $\tau \in \mathbb{N}_{>1}$ to denote the level of indeterminacy in directions.

There are two approaches commonly used to formalise and reason about qualitative spatial or temporal relations, which are referred to as the *relation-algebraic* approach and the *logic-axiomatic approach* [5]. In the relation-algebraic approach, relation algebra is used to represent relations and operations over the relations. There are several qualitative spatial or temporal calculi using this approach, including the point calculus [12], Allen's calculus [1], the rectangle algebra [2], and the cardinal direction relations between regions [9, 10]. In the logic-axiomatic approach, the syntax and semantics of a logic are used to denote the symbols and meanings of relations, respectively. Following the *LEW* [5], the current work also uses the logic-axiomatic approach and investigates the axiomatisations of *LEWI*.

This paper is structured as follows. Section 2 introduces the logic of east and west for intervals (*LEWI*). Section 3 presents an axiomatisations of *LEWI* for every $\tau \in \mathbb{N}_{>1}$. The soundness and completeness of the axiomatisations are proved in Section 4. Section 5 provides a conclusion.

2 A Logic of East and West for Intervals

We introduce a logic of east and west for closed intervals (*LEWI*) in 1D Euclidean space. *LEWI* is an extension of the logic of east and west for points (*LEW*) [5]. *LEWI* includes eight primitive direction relations: east (E), west (W), definitely east (dE), definitely west (dW), partially east (pE), partially west (pW), partially definitely east (pdE), and partially definitely west (pdW).

▶ **Definition 1** (The Language of *LEWI*). *Let Ind be a set of individual names. The language $L(LEWI, Ind)$ (we omit Ind for brevity below) is defined as:*

$$\begin{aligned} \phi,\ \psi \quad := \quad & E(a,b) \mid W(a,b) \mid dE(a,b) \mid dW(a,b) \mid \\ & pE(a,b) \mid pW(a,b) \mid pdE(a,b) \mid pdW(a,b) \mid \neg\phi \mid \phi \wedge \psi \end{aligned}$$

where a, b are in Ind, $\phi \vee \psi =_{def} \neg(\neg\phi \wedge \neg\psi)$, $\phi \to \psi =_{def} \neg\phi \vee \psi$, $\phi \leftrightarrow \psi =_{def} (\phi \to \psi) \wedge (\psi \to \phi)$, and $\bot =_{def} \phi \wedge \neg\phi$.

The lower case letters (e.g., a, b, c, d), sometimes with subscripts or superscripts, are always used to represent individual names in Ind. The language of *LEWI* is a subset of the language of first-order logic [4]. The primitives E, W, dE, and dW also appear in the language of *LEW* [5], while the other relations do not.

We interpret $L(LEWI)$ over models based on 1D Euclidean space, where each individual name is interpreted as a closed interval, rather than a single point.

▶ **Definition 2** (1D Euclidean τ-model). *A 1D Euclidean τ-model M is a structure $(\mathcal{I}, \sigma, \tau)$, where $\mathcal{I}$ is an interpretation function which maps each individual name $a \in Ind$ to a closed interval $[\underline{a}, \overline{a}] \subset \mathbb{R}$ with $\underline{a} \leq \overline{a}$. The parameter $\sigma \in \mathbb{R}_{>0}$ is a margin of error, and $\tau \in \mathbb{N}_{>1}$ refers to the level of indeterminacy in directions.*

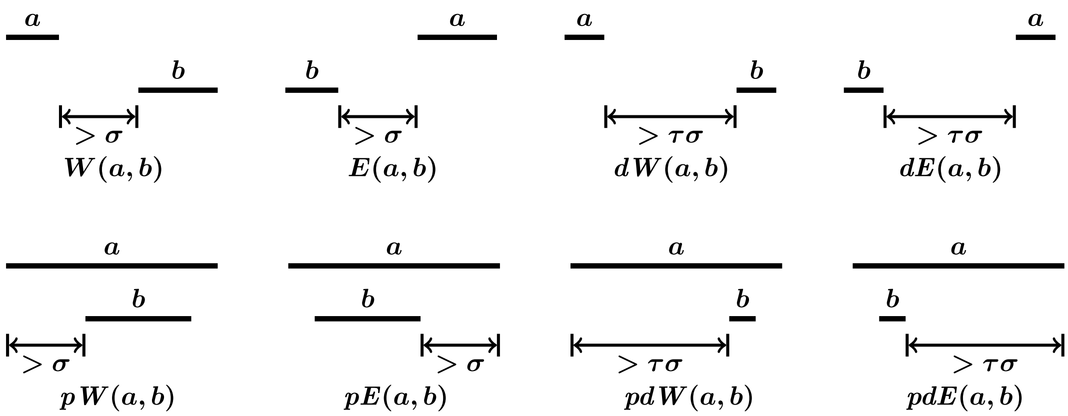

Figure 2 Eight Primitive Relations in *LEWI*.

▶ **Definition 3** (Truth definition). *Let M be a 1D Euclidean τ-model $(\mathcal{I}, \sigma, \tau)$. For any formula ϕ in $L(LEWI)$, we write $M \models_{LEWI} \phi$ to denote that ϕ is true in M. We define $M \models_{LEWI} \phi$ by structural induction as follows:*
$M \models_{LEWI} \boldsymbol{W(a,b)}$ *iff* $\forall p_a \in \mathcal{I}(a)\ \forall p_b \in \mathcal{I}(b)\ (p_a - p_b < -\sigma)$ *iff* $\overline{a} - \underline{b} < -\sigma$;
$M \models_{LEWI} \boldsymbol{E(a,b)}$ *iff* $\forall p_a \in \mathcal{I}(a)\ \forall p_b \in \mathcal{I}(b)\ (p_a - p_b > \sigma)$ *iff* $\underline{a} - \overline{b} > \sigma$;
$M \models_{LEWI} \boldsymbol{dW(a,b)}$ *iff* $\forall p_a \in \mathcal{I}(a)\ \forall p_b \in \mathcal{I}(b)\ (p_a - p_b < -\tau\sigma)$ *iff* $\overline{a} - \underline{b} < -\tau\sigma$;
$M \models_{LEWI} \boldsymbol{dE(a,b)}$ *iff* $\forall p_a \in \mathcal{I}(a)\ \forall p_b \in \mathcal{I}(b)\ (p_a - p_b > \tau\sigma)$ *iff* $\underline{a} - \overline{b} > \tau\sigma$;
$M \models_{LEWI} \boldsymbol{pW(a,b)}$ *iff* $\exists p_a \in \mathcal{I}(a)\ \forall p_b \in \mathcal{I}(b)\ (p_a - p_b < -\sigma)$ *iff* $\underline{a} - \underline{b} < -\sigma$;
$M \models_{LEWI} \boldsymbol{pE(a,b)}$ *iff* $\exists p_a \in \mathcal{I}(a)\ \forall p_b \in \mathcal{I}(b)\ (p_a - p_b > \sigma)$ *iff* $\overline{a} - \overline{b} > \sigma$;
$M \models_{LEWI} \boldsymbol{pdW(a,b)}$ *iff* $\exists p_a \in \mathcal{I}(a)\ \forall p_b \in \mathcal{I}(b)\ (p_a - p_b < -\tau\sigma)$ *iff* $\underline{a} - \underline{b} < -\tau\sigma$;
$M \models_{LEWI} \boldsymbol{pdE(a,b)}$ *iff* $\exists p_a \in \mathcal{I}(a)\ \forall p_b \in \mathcal{I}(b)\ (p_a - p_b > \tau\sigma)$ *iff* $\overline{a} - \overline{b} > \tau\sigma$;
$M \models_{LEWI} \neg\phi$ *iff* $M \not\models_{LEWI} \phi$;
$M \models_{LEWI} \phi \wedge \psi$ *iff* $M \models_{LEWI} \phi$ *and* $M \models_{LEWI} \psi$,
where $a, b \in Ind$, $\mathcal{I}(a) = [\underline{a}, \overline{a}]$, $\mathcal{I}(b) = [\underline{b}, \overline{b}]$, and ϕ and ψ are formulae in $L(LEWI)$.

The truth definitions of eight primitive relations are shown in Figure 2. For a formula ϕ, if there exists a 1D Euclidean τ-model M such that ϕ is true in M (i.e., $M \models_{LEWI} \phi$), then ϕ is said to be τ-satisfiable; if ϕ is true in every 1D Euclidean τ-model M (equivalently, if the negation of ϕ is not τ-satisfiable), then ϕ is said to be τ-valid and written as $\models_{LEWI} \phi$. For every $\tau \in \mathbb{N}_{>1}$, *LEWI* is the set of all τ-valid formulas in $L(LEWI)$.

By Definition 3, as $pW(a,b)$ is not equivalent to $pE(b,a)$ and $pdW(a,b)$ is not equivalent to $pdE(b,a)$, the following inverse relations are introduced as syntactic sugar. These inverse relations are used in following sections for clearer expression of the axiomatisations.

▶ **Definition 4** (Inverse Relation). *The inverse relations are defined as follows:*
- ***partially west inverse:*** $pWi(a,b) =_{def} pW(b,a)$
- ***partially definitely west inverse:*** $pdWi(a,b) =_{def} pdW(b,a)$
- ***partially east inverse:*** $pEi(a,b) =_{def} pE(b,a)$
- ***partially definitely east inverse:*** $pdEi(a,b) =_{def} pdE(b,a)$

3 Axiomatisations

We present, for every $\tau \in \mathbb{N}_{>1}$, a calculus $LEWI^{\tau}$, which will be shown (in Section 4) to be sound and complete for $LEWI$. Here a and b, sometimes with subscripts, are *meta* variables which may be instantiated by any individual name in Ind. An instance of an axiom is a formula in $L(LEWI)$ obtained by instantiating every meta variable in the axiom by an individual name in Ind. For example, by Axiom 3, for every pair of individual names a, b in Ind, the formula $W(a,b) \leftrightarrow E(b,a)$ is an instance of Axiom 3 and it is τ-valid. AS 11 is an axiom schema, where n is the number of conjuncts in the antecedent of the axiom, and $number(\alpha)$ is the number of occurrences of α in $\{R_1, \dots, R_n\}$. It is worth noting that $number(\alpha)$ is a meta-language notation, not in $L(LEWI)$.

PL A finite sound and complete axiomatisation of classical propositional logic
Axiom 1 $\neg pW(a,a)$
Axiom 2 $\neg pE(a,a)$
Axiom 3 $W(a,b) \leftrightarrow E(b,a)$
Axiom 4 $dW(a,b) \leftrightarrow dE(b,a)$
Axiom 5 $W(a,b) \to pW(a,b)$
Axiom 6 $E(a,b) \to pE(a,b)$
Axiom 7 $dW(a,b) \to pdW(a,b)$
Axiom 8 $dE(a,b) \to pdE(a,b)$
Axiom 9 $pdW(a,b) \to pW(a,b)$
Axiom 10 $pdE(a,b) \to pE(a,b)$
AS 11 For all $n \in \mathbb{N}_{>1}$, if for every i in $\{1, \dots, n\}$, R_i is in $\{W, dW, \neg E, \neg dE, pW, pdW, \neg pE, \neg pdE, pEi, pdEi, \neg pWi, \neg pdWi\}$, and for every R_i in $\{\neg pE, \neg pdE, pEi, pdEi, \neg E, \neg dE\}$, R_{i+1} is in $\{\neg pE, \neg pdE, pEi, pdEi, W, dW\}$ with $R_{n+1} =_{def} R_1$, and $number(W) + number(pW) + number(pEi) + \tau * (number(dW) + number(pdW) + number(pdEi)) \geq number(\neg E) + number(\neg pE) + number(\neg pWi) + \tau * (number(\neg dE) + number(\neg pdE) + number(\neg pdWi))$, then $R_1(a_0, a_1) \wedge \dots \wedge R_n(a_{n-1}, a_0) \to \bot$ is an axiom.
MP Modus ponens: $\phi,\ \phi \to \psi\ \vdash\ \psi$

The notion of τ-derivability $\Gamma \vdash_{LEWI^{\tau}} \phi$ in the $LEWI^{\tau}$ calculus is standard. A formula ϕ in $L(LEWI)$ is called τ-derivable if $\vdash_{LEWI^{\tau}} \phi$. Γ is said to be τ-inconsistent if for some formula ϕ, both ϕ and $\neg\phi$ are τ-derivable (otherwise, it is said to be τ-consistent).

In AS 11, for every i in $\{1, \dots, n\}$, by Definition 3, the truth condition of $W(a_{i-1}, a_i)$ or $dW(a_{i-1}, a_i)$ is an inequality between $\overline{a_{i-1}}$ and $\underline{a_i}$, the truth condition of $\neg E(a_{i-1}, a_i)$ or $\neg dE(a_{i-1}, a_i)$ is an inequality between $\underline{a_{i-1}}$ and $\overline{a_i}$, the truth condition of $pW(a_{i-1}, a_i)$, $pdW(a_{i-1}, a_i)$, $\neg pWi(a_{i-1}, a_i)$, or $\neg pdWi(a_{i-1}, a_i)$ is an inequality between $\underline{a_{i-1}}$ and $\underline{a_i}$, the truth condition of $pEi(a_{i-1}, a_i)$, $pdEi(a_{i-1}, a_i)$, $\neg pE(a_{i-1}, a_i)$, $\neg pdE(a_{i-1}, a_i)$ is an inequality between $\overline{a_{i-1}}$ and $\overline{a_i}$, where $a_n = a_0$. In addition, for every i in $1, \dots, n$, by Definition 2, an inequality exists between $\underline{a_i}$ and $\overline{a_i}$ (i.e., $\underline{a_i} - \overline{a_i} \leq 0$). For every $R_i \in \{\neg pE, \neg pdE, pEi, pdEi, \neg E, \neg dE\}$, the truth condition of $R_i(a_{i-1}, a_i)$ is an inequality with $\overline{a_i}$ as its second variable, hence, AS 11 requires that R_{i+1} is in $\{\neg pE, \neg pdE, pEi, pdEi, W, dW\}$ so that the truth condition of $R_{i+1}(a_i, a_{i+1})$ is an inequality with $\overline{a_i}$ as its first variable.

For every $\tau \in \mathbb{N}_{>1}$, according to AS 11, if each of the relations pW, dW, $\neg pdE$, and $\neg pWi$ presents exactly once, then $pW(a_0, a_1) \wedge \neg pdE(a_1, a_2) \wedge dW(a_2, a_3) \wedge \neg pWi(a_3, a_0) \to \bot$ is an axiom, as shown in Figure 3.

Notably, $pW(a_0, a_1) \land \neg pdE(a_1, a_2) \land dW(a_2, a_3) \land \neg pE(a_3, a_0) \to \bot$ is not an axiom in AS 11, since the relation after $\neg pE$ is pW that is not in $\{\neg pE, \neg pdE, pEi, pdEi, W, dW\}$, as shown in Figure 4.

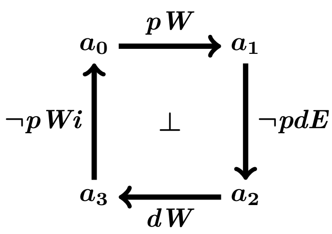

Figure 3 an example axiom in AS 11.

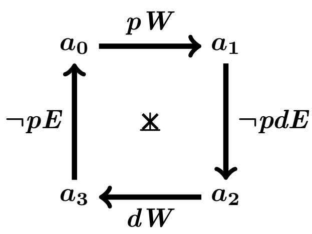

Figure 4 a counterexample of AS 11.

4 Soundness and Completeness

In this section, we prove that the $LEWI^\tau$ calculus is sound and complete, i.e., every derivable formula is valid and every valid formula is derivable:

▶ **Theorem 5.** *For every $\tau \in \mathbb{N}_{>1}$, the $LEWI^\tau$ calculus is sound and complete for 1D Euclidean τ-models.*

Soundness. For all $\tau \in \mathbb{N}_{>1}$, to prove the soundness of the $LEWI^\tau$ calculus, we show that every τ-derivable formula ϕ in $L(LEWI)$ is τ-valid. This can be carried out by a straightforward induction on the length of the derivation of ϕ. By Definition 3, every axiom is τ-valid; and modus ponens preserves validity.

Completeness. For the rest of this section, we focus on the proof of completeness. As is common in mathematical logic (see also [5, Section 4.2]) we prove the following equivalent statement: *for every $\tau \in \mathbb{N}_{>1}$, if a finite set of $L(LEWI)$ formulae Σ is τ-consistent, then there is a 1D Euclidean τ-model satisfying it.* For the proof, we use some basic concepts from graph theory and linear inequalities.

We represent a graph G by the pair (V, E), in which V and E denote the sets of vertices and edges, respectively. We consider *basic* linear inequalities, i.e., those that take the form $x - y \leq c$ or $x - y < c$, where x and y are variables, and c is a real number constant. For a set S of basic linear inequalities, we construct a graph $G = (V, E)$, in which V is the set of variables appearing in S, and for each basic inequality $(x - y) \sim c$ in S, we add to E an edge from x to y and label it with the inequality $(x - y) \sim c$, where $\sim$ is $<$ or $\leq$. A path $P = v_1, \ldots, v_n$ through G is said to be a loop (or cycle) if $v_1 = v_n$. A loop is called simple if its intermediate vertices are all distinct.

▶ **Definition 6** (infeasible loop). *Assume that $P = v_1, \ldots, v_n$ is a simple loop, in which for all $1 \leq i < n$, $(v_i - v_{i+1}) \sim c_i$, $(v_n - v_1) \sim c_n$, where $\sim$ is $<$ or $\leq$, and $c_1, \ldots, c_n$ are real numbers. If every $\sim$ is $\leq$, then P is said to be infeasible iff $(\sum_{1 \leq i \leq n} c_i) < 0$; otherwise, P is called infeasible iff $(\sum_{1 \leq i \leq n} c_i) \leq 0$.*

▶ **Theorem 7** ([5, 8]). *Let G be the graph constructed for a set S of linear inequalities of the form $(x - y) \sim c$, where x, y are real variables, $\sim$ is $\leq$ or $<$, and c is a real number. Then, S is satisfiable iff G has no infeasible simple loop.*

▶ **Definition 8** (τ-σ-translation). *The "τ-σ-translation" function $tr(\tau,\sigma)$ is defined as follows:*

- $tr(\tau,\sigma)(W(a,b)) = (\overline{a} - \underline{b} < -\sigma)$;
- $tr(\tau,\sigma)(E(a,b)) = (\overline{b} - \underline{a} < -\sigma)$;
- $tr(\tau,\sigma)(dW(a,b)) = (\overline{a} - \underline{b} < -\tau\sigma)$;
- $tr(\tau,\sigma)(dE(a,b)) = (\overline{b} - \underline{a} < -\tau\sigma)$;
- $tr(\tau,\sigma)(pW(a,b)) = (\underline{a} - \underline{b} < -\sigma)$;
- $tr(\tau,\sigma)(pE(a,b)) = (\overline{b} - \overline{a} < -\sigma)$;
- $tr(\tau,\sigma)(pdW(a,b)) = (\underline{a} - \underline{b} < -\tau\sigma)$;
- $tr(\tau,\sigma)(pdE(a,b)) = (\overline{b} - \overline{a} < -\tau\sigma)$;
- $tr(\tau,\sigma)(\neg\phi) = \neg(tr(\tau,\sigma)(\phi))$, *where* $\neg(z_1 - z_2 < c) = (z_2 - z_1 \leq -c)$.

Now, we present the proof of completeness of $LEWI^{\tau}$ for every $\tau \in \mathbb{N}_{>1}$.

Proof (Theorem 5). Take an arbitrary $\tau \in \mathbb{N}_{>1}$. Suppose a finite set of $L(LEWI)$ formulae Σ is τ-consistent. Let $names(\Sigma)$ be the set of individual names that appear in Σ. We rewrite the set Σ as a formula ϕ that is the conjunction of all the formulae in Σ. We rewrite ϕ in disjunctive normal form $\phi_1 \vee \cdots \vee \phi_m$, where $m > 0$ and every literal is of one of the forms: $W(a,b)$, $E(a,b)$, $dW(a,b)$, $dE(a,b)$, $pW(a,b)$, $pE(a,b)$, $pdW(a,b)$, $pdE(a,b)$, and their negations. To show that there is a 1D Euclidean τ-model satisfying Σ, it is sufficient to show that there exists a disjunct ϕ_i which is τ-satisfiable.

We prove this by contradiction. Suppose that every disjunct ϕ_i is not τ-satisfiable. Take an arbitrary disjunct ϕ_i. We obtain a set of inequalities S_i by translating every literal in a disjunct ϕ_i according to Definition 8 and adding $\underline{a} - \overline{a} \leq 0$ for every individual name a in $names(\Sigma)$. The inequalities in S_i are of the form $(x_a - x_b) \sim c$, where x_a and x_b are real variables, $\sim$ is $\leq$ or $<$, and c is a real number. By Theorem 7, the formula ϕ_i is not τ-satisfiable iff the graph G_i of S_i has an infeasible simple loop P. By Definitions 6, the sum of the constants around P is no greater than zero. There are two cases, as discussed below.

Case 1. The loop P contains vertices from at least two distinct individual names. Let us assume that P is over a sequence of vertices $v_0, v_1, \ldots, v_{m-1}$, and the linear inequalities in P are of the form $(v_0 - v_1) \sim c_1$, $(v_1 - v_2) \sim c_2$, $\ldots$, $(v_{m-1} - v_0) \sim c_m$, where m is the length of the loop satisfying $m > 1$ and $\sim$ is $\leq$ or $<$.

By Definition 8, each linear inequality in P is of one of the following forms: $\overline{a} - \underline{b} < -\sigma$ (from the form $W(a,b)$ or $E(b,a)$), $\overline{a} - \underline{b} < -\tau\sigma$ (from the form $dW(a,b)$ or $dE(b,a)$), $\underline{a} - \overline{b} \leq \sigma$ (from the form $\neg E(a,b)$ or $\neg W(b,a)$), $\underline{a} - \overline{b} \leq \tau\sigma$ (from the form $\neg dE(a,b)$ or $\neg dW(b,a)$), $\underline{a} - \underline{b} < -\sigma$ (from the form $pW(a,b)$), $\underline{a} - \underline{b} < -\tau\sigma$ (from the form $pdW(a,b)$), $\underline{a} - \underline{b} \leq \sigma$ (from the form $\neg pW(b,a)$), $\underline{a} - \underline{b} \leq \tau\sigma$ (from the form $\neg pdW(b,a)$), $\overline{a} - \overline{b} \leq \sigma$ (from the form $\neg pE(a,b)$), $\overline{a} - \overline{b} \leq \tau\sigma$ (from the form $\neg pdE(a,b)$), $\overline{a} - \overline{b} < -\sigma$ (from the form $pE(b,a)$), $\overline{a} - \overline{b} < -\tau\sigma$ (from the form $pdE(b,a)$) and $\underline{a} - \overline{a} \leq 0$.

We translate the linear inequalities in P back to formulae as follows. We translate every linear inequality of the form $\overline{a} - \underline{b} < -\sigma$ to $W(a,b)$; every $\overline{a} - \underline{b} < -\tau\sigma$ to $dW(a,b)$; every $\underline{a} - \overline{b} \leq \sigma$ to $\neg E(a,b)$; every $\underline{a} - \overline{b} \leq \tau\sigma$ to $\neg dE(a,b)$; every $\underline{a} - \underline{b} < -\sigma$ to $pW(a,b)$; every $\underline{a} - \underline{b} < -\tau\sigma$ to $pdW(a,b)$; every $\underline{a} - \underline{b} \leq \sigma$ to $\neg pWi(a,b)$; every $\underline{a} - \underline{b} \leq \tau\sigma$ to $\neg pdWi(a,b)$; every $\overline{a} - \overline{b} \leq \sigma$ to $\neg pE(a,b)$; every $\overline{a} - \overline{b} \leq \tau\sigma$ to $\neg pdE(a,b)$; every $\overline{a} - \overline{b} < -\sigma$ to $pEi(a,b)$; and every $\overline{a} - \overline{b} < -\tau\sigma$ to $pdEi(a,b)$. Only the form $\underline{a} - \overline{a} \leq 0$ does not get translated.

In this way, from P we obtain a sequence of formulae of the form $R_1(a_0, a_1)$, $\ldots$, $R_n(a_{n-1}, a_0)$, where n is the number of distinct individual names involved in P. For every integer i such that $1 \leq i \leq n$, R_i is one of $W, dW, \neg E, \neg dE, pW, pdW, \neg pE, \neg pdE, pEi, pdEi$, $\neg pWi$, $\neg pdWi$, and if R_i is in $\{\neg pE, \neg pdE, pEi, pdEi, \neg E, \neg dE\}$, it must be translated from the form $(\overline{u}-\overline{v}) \sim c_1$ or $(\underline{u}-\overline{v}) \sim c_1$. Thus, R_{i+1} must be translated from the form $(\overline{v}-\overline{w}) \sim c_2$ or $(\overline{v}-\underline{w}) \sim c_2$, where $R_{n+1} =_{def} R_1$, $\sim$ is $\leq$ or $<$, and c_1, c_2 are in $\{-\sigma, \sigma, -\tau\sigma, \tau\sigma\}$. Therefore, R_{i+1} is in $\{\neg pE, \neg pdE, pEi, pdEi, W, dW\}$. Since the sum of the constants around P is non-positive, we have $number(W) + number(pW) + number(pEi) + \tau * (number(dW) + number(pdW) + number(pdEi)) \geq number(\neg E) + number(\neg pE) + number(\neg pWi) + \tau * (number(\neg dE) + number(\neg pdE) + number(\neg pdWi))$. By AS 11, we have $R_1(a_0, a_1) \wedge \cdots \wedge R_n(a_{n-1}, a_0) \rightarrow \bot$.

By Definition 8 and Definition 4, for every occurrence of $W(a,b)$ in $R_1(a_0,a_1) \wedge \cdots \wedge R_n(a_{n-1},a_0)$, the formula $W(a,b)$ or $E(b,a)$ is a conjunct in ϕ_i; for every occurrence of $dW(a,b)$, the formula $dW(a,b)$ or $dE(b,a)$ is a conjunct in ϕ_i; for every occurrence of $\neg E(a,b)$, the formula $\neg E(a,b)$ or $\neg W(b,a)$ is a conjunct in ϕ_i; for every occurrence of $\neg dE(a,b)$, the formula $\neg dE(a,b)$ or $\neg dW(b,a)$ is a conjunct in ϕ_i; for every occurrence of $pW(a,b)$, the formula $pW(a,b)$ is a conjunct in ϕ_i; for every occurrence of $pdW(a,b)$, the formula $pdW(a,b)$ is a conjunct in ϕ_i; for every occurrence of $\neg pE(a,b)$, the formula $\neg pE(a,b)$ is a conjunct in ϕ_i; for every occurrence of $\neg pdE(a,b)$, the formula $\neg pdE(a,b)$ is a conjunct in ϕ_i; for every occurrence of $pEi(a,b)$, the formula $pE(b,a)$ is a conjunct in ϕ_i; for every occurrence of $pdEi(a,b)$, the formula $pdE(b,a)$ is a conjunct in ϕ_i; for every occurrence of $\neg pWi(a,b)$, the formula $\neg pW(b,a)$ is a conjunct in ϕ_i; for every occurrence of $\neg pdWi(a,b)$, the formula $\neg pdW(b,a)$ is a conjunct in ϕ_i. By Axiom 3, we have $W(a,b) \leftrightarrow E(b,a)$. By Axiom 4, we have $dW(a,b) \leftrightarrow dE(b,a)$. Therefore, $\bot$ is τ-derivable from ϕ_i.

Case 2. The vertices in P are only from one individual name. Then there are at most two vertices in P. If there is only one vertex in P, as P is a simple infeasible loop, by Definition 8, the only one linear inequality in P is of one of the following forms: $\underline{a} - \underline{a} < -\sigma$ which is translated from $pW(a,a)$, $\underline{a} - \underline{a} < -\tau\sigma$ which is translated from $pdW(a,a)$, $\overline{a} - \overline{a} < -\sigma$ which is translated from $pE(a,a)$, and $\overline{a} - \overline{a} < -\tau\sigma$ which is translated from $pdE(a,a)$. By Axiom 1, Axioms 1 and 9, Axiom 2, Axioms 2 and 10, respectively, $\bot$ is τ-derivable.

If there are two vertices in the loop P, as P is simple, there are two linear inequalities of the form $(\overline{a} - \underline{a}) \sim c_1$ and $(\underline{a} - \overline{a}) \sim c_2$ in P, where $\sim$ is $\leq$ or $<$. By Definition 8, the linear inequality of the first form should be of one of the two possible forms: $\overline{a} - \underline{a} < -\sigma$ which is translated from $W(a,a)$ or $E(a,a)$, or $\overline{a} - \underline{a} < -\tau\sigma$ which is translated from $dW(a,a)$ or $dE(a,a)$. By Axioms 5 and 1, Axioms 6 and 2, Axioms 7, 9 and 1, Axioms 8, 10 and 2, respectively, $\bot$ is τ-derivable.

Therefore, in each case, $\bot$ is τ-derivable from ϕ_i for arbitrary i, hence from ϕ. A contradiction. ◀

By Theorem 5, in $L(LEWI)$, for every $\tau \in \mathbb{N}_{>1}$, a formula is τ-satisfiable iff it is τ-consistent. Therefore, a satisfiability checking procedure can check the consistency of a finite set of $L(LEWI)$ formulae. Given a finite set of $L(LEWI)$ formulae Σ, we rewrite Σ in disjunctive normal form $\phi_1 \vee \cdots \vee \phi_m$, where $m \geq 1$ and every literal is of one of the following forms: $E(a,b)$, $W(a,b)$, $dE(a,b)$, $dW(a,b)$, $pE(a,b)$, $pW(a,b)$, $pdE(a,b)$, $pdW(a,b)$, and their negations. As such, Σ is τ-satisfiable iff there exists a disjunct ϕ_i, where $1 \leq i \leq m$, such that ϕ_i is τ-satisfiable. From each ϕ_i, where $1 \leq i \leq m$, we obtain a set of linear inequalities S_i by translating every literal in ϕ_i using the τ-σ-translation function $tr(\tau, \sigma)$ of Definition 8. By Definition 3, ϕ_i is τ-satisfiable iff S_i is satisfiable, i.e., iff S_i has a solution in real numbers. Hence, Σ is τ-satisfiable iff S_i is satisfiable, for some $1 \leq i \leq m$. Some solvers (e.g., Z3 satisfiability modulo theories solver [3]) can be used to check the satisfiability of each S_i.

5 Conclusion

In this paper, we have extended the logic of east and west for points (*LEW*) to a logic of east and west for closed intervals (*LEWI*) in 1D Euclidean space. The logic *LEWI* contains a parameter $\sigma \in \mathbb{R}_{>0}$ that represents the margin of error and a parameter $\tau \in \mathbb{N}_{>1}$ that denotes the level of indeterminacy. For every $\tau \in \mathbb{N}_{>1}$, the $LEWI^{\tau}$ calculus is shown to be sound and complete. As future work, we will develop a reasoner based on *LEWI* and apply it to reason with real-world geospatial datasets.

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