A Constant Factor Approximation for Directed Feedback Vertex Set in Graphs of Bounded Genus

Hao Sun
University of Alberta, 116 St & 85 Ave, Edmonton, AB T6G 2R3, Canada

Abstract

The minimum directed feedback vertex set problem consists in finding the minimum set of vertices that should be removed in order to make a directed graph acyclic. This is a well-known NP-hard optimization problem with applications in various fields, such as VLSI chip design, bioinformatics and transaction processing deadlock prevention and node-weighted network design. We show a constant factor approximation for the directed feedback vertex set problem in graphs of bounded genus.

2012 ACM Subject Classification Mathematics of computing → Graphs and surfaces; Mathematics of computing → Approximation algorithms; Mathematics of computing → Graph algorithms; Theory of computation → Rounding techniques; Theory of computation → Packing and covering problems

Keywords and phrases Feedback Vertex Set, Combinatorial Optimization, Approximation Algorithms, min-max relation, linear programming

Digital Object Identifier 10.4230/LIPIcs.APPROX/RANDOM.2024.18

Category APPROX

Related Version *Full Version*: https://arxiv.org/abs/2311.01026

Acknowledgements The author would like to thank Zachary Friggstad for invaluable guidance in the writing process and Jochen Koenemann for suggesting the problem.

1 Introduction

In the directed feedback vertex set problem (DFVS), we are given a (node-weighted) directed graph $G = (V, E)$ with costs c_v $\forall v \in V$ and wish to find a minimum cost set X for which $G\backslash X$ is acyclic. DFVS is one of Karp's original 21 NP-hard problems [16]. The DFVS problem has many applications including deadlock resolution [10], VLSI chip design [19] and program verification [20].

A 2-approximation for (undirected) FVS is given in [2]. DFVS has a 2-approximation in tournaments [21] and bipartite tournaments [24], is polynomial-time solvable on graphs of bounded treewidth, has a 2.4-approximation in planar graphs [3] and has an $O(\log n \log\log n)$-approximation in general graphs [7]. DFVS does not have an $O(1)$-approximation under the unique games conjecture [13]. The genus of a graph is the minimal integer g such that the graph can be drawn without crossing itself on a sphere with g handles.

The following is the natural LP for DFVS and its dual, where $\mathcal{C}$ is the set of directed cycles of our graph.

$$\begin{aligned} \min \quad & c^T x && (\mathrm{P_{DFVS}}) \\ \text{s.t.} \quad & x(C) \geq 1 \quad \forall\, C \in \mathcal{C} && (1) \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & \mathbb{1}^T y && (\mathrm{D}_{DFVS}) \\ \text{s.t.} \quad & \sum_{C \in \mathcal{C}, v \in C} y_C \leq c_v \quad \forall v \in V(G) \\ & y \geq \mathbb{0}\ . \end{aligned}$$

Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2024).
Editors: Amit Kumar and Noga Ron-Zewi; Article No. 18; pp. 18:1–18:20
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Given that constant approximations for DFVS exist for planar graphs, one naturally wonders if DFVS admits constant approximations in bounded genus graphs. We answer this question positively.

▶ **Theorem 1.** *For any fixed genus g, there is a polynomial-time $O(g)$-approximation for DFVS for graphs of genus g. Moreover, the algorithm returns a DFVS with cost $O(g)$ times the optimum solution to* $(\mathrm{P}_{\mathrm{DFVS}})$.

From the proof of Theorem 1, it is clear that the algorithm in Theorem 1 runs in time $O(g)\mathrm{poly}(|V(G)|)$.

For uniform costs, the dual LP (D_{DFVS}) is the natural LP of the *dicycle packing problem*. The dicycle packing problem is the problem of finding the maximum number of vertex disjoint dicycles of a graph. Schlomberg et al. [23] show that the LP gap of the natural LP for dicycle packing is at most $\Omega(\frac{1}{g^2 \log g})$ on any graph of genus g. Our result then also implies that the minimum size of a DFVS is at most $O(g^3 \log g)$ the size of a maximum dicycle packing.

1.1 Our techniques

Informally speaking, for a (directed) graph embedded on a surface where each directed cycle bounds a region homeomorphic to the plane, one can apply the same primal-dual techniques in [12, 3] to obtain a constant factor primal-dual approximation.

In the other case, our algorithm will use the natural LP for DFVS to look for a "separator" [5, 8, 9] $S \subset V$ of cost at most a constant times the optimal DFVS such that $G \backslash S$ is of smaller genus. We obtain a directed cycle C, the removal of which results in a surface of one smaller genus. Traversing along the dicycle, we may define a "left" and "right" side of the dicycle. Like in [7], we solve the DFVS LP and use the LP values as distances.

If there is no short path leaving C from the left and entering C from the right and vice versa then there is a small separator S such that each dicycle of $G \backslash S$ either does not use any "left arc" that is, an arc coming in or leaving C from the left, or does not use any "right arc" that is, an arc coming in or leaving C from the right. G with all left (resp. right) arcs deleted is of genus at least one less so inductively we can solve within a constant factor DFVS on G with all left (resp. right) arcs deleted. These two solutions together with S form a DFVS of constant times more than the optimum. If such short paths exist but all starting points of such paths and ending points of such paths are far apart the analysis is similar.

The final case is where there are short paths P_1, P_2 leaving C from the left and entering C from the right (or vice versa) and the starting point of P_2 is close to the endpoint of P_1. We show that P_1 is "far" from P_2 so to speak (we are using directed distances so this is not the same as P_2 being far from P_1) and compute a suitable separator. We show that the resulting strongly connected components are of smaller genus. We then combine approximations for different components to give an approximation for the original graph.

The presentation in this paper is focused on demonstrating linear dependence on the genus rather than optimizing the constant in Theorem 1.

2 Preliminaries

In our figures, we will use the representation of the torus by taking the unit square $[0,1] \times [0,1]$ and identifying the two pairs of edges $\{0\} \times [0,1]$, $\{1\} \times [0,1]$ and $[0,1] \times \{0\}$, $[0,1] \times \{1\}$, that is, the point $(0,p)$ is identified with $(1,p)$ and the point $(q,0)$ is identified with the point $(q,1)$.

Throughout this paper, all surfaces are orientable and smooth and all curves are piecewise smooth. Distances on surfaces will refer to the geodesic (shortest path on the surface) distance. It is well known (see for instance [22]) that a smooth orientable surface Q is diffeomorphic to the g-genus torus for some g. For $X \subset Q$, denote $\mathrm{cl}(X)$ as the closure of X in Q. We henceforth assume our surface is a g-genus torus for some g.

Let us call a cycle of G, or a closed curve C embedded on a surface Q *facial* if it bounds a region inside(C) of the surface homeomorphic to the plane. If the genus of Q is greater than 0, call inside(C) the *inside region* of $Q \backslash C$. For any set $F \subset V$, define G^F to be the *residual graph*, that is, the subgraph of G induced by those vertices that lie in a dicycle of $G \backslash F$.

3 Hitting the facial cycles of a digraph

In general, given a (node-weighted) directed graph $G = (V, E)$ with costs c_v $\forall v \in V$ a set $\mathcal{C}$ of cycles of a digraph G, we define the $\mathcal{C}$-hitting set problem as the problem of finding a minimum cost set X such that $X \cap C \neq \emptyset$ $\forall C \in \mathcal{C}$. In this section, we are concerned with when $\mathcal{C}$ is the set of facial cycles of our graph.

Given a digraph G embedded on a surface Q, we show how to obtain an $O(g)$-approximation for the problem of finding a minimal hitting set for the set of facial dicycles of a graph embedded on any surface Q of genus g.

▶ **Theorem 2.** *For a graph G embedded on a surface Q of genus g, there is a polynomial-time $O(g)$-approximation for the problem of hitting facial dicycles of G. Moreover, the algorithm returns a DFVS with cost $O(g)$ times the optimum solution to* ($\mathrm{P}_{\mathrm{DFVS}}$) *where $\mathcal{C}$ is the set of facial dicycles.*

Further, if G is embedded in a way such that there exist regions R_1, R_2 of Q homeomorphic to the open disk, such that the inside region of any facial dicycle contains at least one of R_1, R_2, then there is an algorithm that returns the optimum solution to ($\mathrm{P}_{\mathrm{DFVS}}$) *where $\mathcal{C}$ is the set of facial dicycles.*

Proof. We first show that DFVS has an $O(g)$-approximation. If C is a facial cycle bounding a face of G, call C *face minimal*. Note that if G contains a facial cycle, then it must contain a face minimal cycle by the following argument. Let C be a dicycle such that inside(C) is a minimal (by containment) region of Q. Recall that we removed all vertices of G not lying on a dicycle. In particular, any vertex w inside the region inside(C) must lie on a facial dicycle A_w. A_w cannot be contained entirely in cl(inside(C)), as then the region A_w would be strictly contained in inside(C). Thus, inside(A_w) intersects C and there is a dipath P between two nodes u, v of C. If C is not a face, then either there is a vertex w inside the region R_C or there is an edge uv between two nodes u, v of C such that $g(uv) \backslash (g(v) \cup g(u))$ lies in R_C. In either case, there is a dipath P between two nodes u, v of C then P together with either the u-v or v-u dipath in C forms a cycle bounding a smaller region of Q, which is a contradiction.

Our algorithm proceeds as follows. This is a primal-dual algorithm analogous to the technique of [12] for DFVS in planar graphs. Given a feasible dual solution y to (D_{DFVS}), let the *residual cost* of node $v \in V$ be $c_v - \sum_{C \in \mathcal{C}, v \in C} y_C$. For $\hat{S} \subset V(G)$, recall $G^{\hat{S}}$ denotes the subgraph of G induced by those vertices which are in a dicycle of $G \backslash \hat{S}$.

Our primal-dual method begins with a trivial feasible dual solution $y = \mathbb{0}$, and the empty, infeasible hitting set $\hat{S} = \emptyset$.

While $G^{\hat{S}}$ contains a facial cycle, increment the dual variables y_C in $\mathrm{P}_{\mathrm{DFVS}}$ of face minimal cycles C of G. When a node of G becomes tight add it to $\hat{S}$. When $G^{\hat{S}}$ contains no facial cycles apply reverse deletion to $\hat{S}$ with respect to the facial cycles of $G^{\hat{S}}$, that is, we consider

Algorithm 3.1 MinWeightDirectedFVS (G, c).

Input : A digraph $G = (V, E)$ with non-negative node-costs c_v, for each $v \in V$.
Output : A Directed FVS S of G.

$S = \emptyset$
while G^S *contains a facial cycle* **do**
  Increment all dual variables y_C for face minimal cycles of G^S. Add all nodes that became tight to S.
end while
Reverse-Deletion:
  Let $s_1, s_2, .., s_l$ be nodes of S in the order they were added.
  for $t = l$ **downto** 1 **do**
    if $G^{S\setminus\{s_t\}}$ *contains no facial cycle* **then**
      $S \leftarrow S \setminus \{s_t\}$
    end if
  end for

return S

each node v of $\hat{S}$ in the order it was added and if $G \setminus (\hat{S} \setminus \{v\})$ contains no facial cycles, delete v from $\hat{S}$. Denote by $\bar{S}$ the set $\hat{S}$ at the end of the algorithm. In other words, we apply the primal-dual method to solve the problem of hitting all facial dicycles of G.

Clearly, $\bar{S}$ is a feasible hitting set for the set of facial dicycles of G, we claim it has cost $O(g)OPT_{LP}$. To do so we apply that standard analysis of primal-dual methods in [11, 12].

▶ **Theorem 3** ([11]). *Suppose $S \subset V(G)$ and y is a solution to (D_{DFVS}) output by our primal-dual algorithm such that the following holds.*

1. *y is obtained starting with the initial feasible solution $y := 0$ and incrementing some set of dual variables $\{y_C : v \in \mathcal{C}_t\}$ uniformly and maintaining feasibility of y for iterations $t = 1, 2, .., l$ for some $l \in \mathbb{N}$.*
2. *For each iteration $t \in \{1, 2, 3, .., l\}$, the set $\{y_C : C \in \mathcal{C}_t\}$ of incremented dual variables satisfies $\sum_{C \in \mathcal{C}_t} |S \cap C| \leq \beta |\mathcal{C}_t|$.*
3. *$\forall v \in S, \quad \sum_{C \in \mathcal{C} \ v \in C} y_C = c_v$.*

Then S has cost at most $\beta \sum_{C \in \mathcal{C}} y_C$, that is at most β times the LP value.

Using Theorem 3, it suffices to prove that during any iteration t, the face minimal cycles $\mathcal{C}_t$ of G^{S_t}, where S_t is our current hitting set satisfies

$$\sum_{C \in \mathcal{C}_t} |\bar{S} \cap C| \leq O(g)|\mathcal{C}_t|. \tag{2}$$

Again we remove nodes of G that do not lie on any dicycle. Denote $\bar{S}_t$ to be the nodes of $\bar{S}$ that intersect a cycle of $\mathcal{C}_t$. So it suffices to show $\sum_{C \in \mathcal{C}_t} |\bar{S}_t \cap C| \leq O(g)|\mathcal{C}_t|$.

The following definition of crossing cycles was elementary to the approach by Goemans and Williamson [12].

▶ **Definition 4.** *Fix an embedding of a planar graph. Two cycles C_1, C_2* cross *if C_i contains an edge intersecting the interior of the region bounded by C_{3-i}, for $i = 1, 2$. That is, the plane curve corresponding to the embedding of the edge in the plane intersects the interior of the region of the plane bounded by C_{3-i}. A set of cycles $\mathcal{C}$ is* laminar *if no two elements of $\mathcal{C}$ cross.*

Denote $\mathcal{C}'$ the set of facial cycles of $\mathcal{C}$. For a node $v \in \bar{S}$, call a cycle $C \in \mathcal{C}'$ with $C \cap \bar{S} = \{v\}$ a *witness* for v. Since we applied reverse deletion to $\bar{S}$ at the end of the algorithm, each node of $\bar{S}$ has a witness in $\mathcal{C}'$ which is a cycle of G^S.

The following result about the structure of witness cycles was vital to the 3 and 2.4 approximations for DFVS in planar graphs by [12] and [3]. We observe that the proof in [12] which involves iteratively applying an "uncrossing" procedure to two witness cycles that cross yields the same result for facial cycles of graphs on surfaces.

▶ **Lemma 5** ([12]). *There exists a laminar family $\mathcal{A} \subset \mathcal{C}'$ of witness cycles in $G^{S_{\bar{t}}}$ for $\bar{S}_t$.*

The laminar family $\mathcal{A}$ can be represented by a forest where A_1 is an ancestor of A_2 if the inside region of A_1 contains the inside region of A_2. Add a root node r to this forest, make it the parent of every maximal node of the forest and call the resulting tree T.

We assign each cycle C of $\mathcal{C}_t$ to the smallest node of T containing C. Call the set of cycles assigned to $w \in T$, $\mathcal{C}_w$. We assign the nodes that w and the children of w are witnesses of to w and call this set $\bar{S_w}$.

To bound $\sum_{C \in \mathcal{C}_t} |\bar{S}_t \cap C|$, we define the following bipartite graph.

▶ **Definition 6** ([12]). *The* debit graph *for $\mathcal{C}_t$ and S is the bipartite graph $\mathcal{D}_G = (\mathcal{R} \cup S, E)$ with edges $E_{\mathcal{C}_t} = \{(C, s) \in \mathcal{C}_t \times S \mid s \in C\}$.*

Since each $C \in \mathcal{C}_t$ is incident to the vertices of $\bar{S}_t$ on C, $|\bar{S}_t \cap C|$ is the degree of C in $\mathcal{D}_G$. Summing this equality over each $C \in \mathcal{C}_t$ yields $\sum_{C \in \mathcal{C}_t} |\bar{S} \cap C| = E(\mathcal{D}_G)$. By placing the node of the debit graph corresponding to C inside the inside region of C we can see that the debit graph is also embedded on Q.

▶ **Proposition 7** (Corollary of Euler's formula for graphs of genus g). *A (simple) bipartite graph $\bar{G}$ with at least three vertices embedded on a surface of genus g satisfies*

$$E(\bar{G}) \leq (2+g)|V(\bar{G})| - 4$$

if G has two vertices then

$$E(\bar{G}) \leq (2+g)|V(\bar{G})| - 3$$

Proof. Euler's formula (for instance see [17]) for graphs embedded on a surface of genus g yields $2 - 2g = |V(\bar{G})| - |E(\bar{G})| + |F(\bar{G})|$. Following the same method as the proof of Euler's formula for bipartite planar graphs with at least 3 vertices, (for instance see Corollary 4.2.10 of [6]) we observe that each face of $\bar{G}$ having at least 4 edges means $|F(\bar{G})| \leq \frac{1}{2}||$. Thus, for $|V(\bar{G})| \geq 3$, $|E(\bar{G})| \leq 2|V(\bar{G})| - 4 + 4g \leq (2+g)|V(\bar{G})| - 4$. If $|V(\bar{G})| \leq 2$ then $|E(\bar{G})| \leq 1 \leq (2+g)|V(\bar{G})| - 3$. ◀

For a node w of T that is not a leaf or the root, the subgraph of $\mathcal{D}_G$ induced by $\mathcal{C}_w \cup \bar{S}_w$ is embedded on Q and further $|\mathcal{C}_w \cup \bar{S}_w| \geq 3$, thus by Proposition 7,

$$|E(\mathcal{D}_G(\mathcal{C}_w \cup \bar{S}_w))| \leq (2+g)|\mathcal{C}_w| + (2+g)|\bar{S}_w| - 4 = (2+g)|\mathcal{C}_w| + (2+g)(\deg_T(w) - 1) - 4. \quad (3)$$

For a leaf v of T

$$|E(\mathcal{D}_G(\mathcal{C}_v \cup \bar{S}_v))| \leq (2+g)|\mathcal{C}_v| + 2|\bar{S}_v| - 3 = (2+g)|\mathcal{C}_v| + 2(\deg_T(v) - 1) - 3. \quad (4)$$

For the root r of T

$$|E(\mathcal{D}_G(\mathcal{C}_r \cup \bar{S}_r))| \leq (2+g)|\mathcal{C}_r| + 2|\bar{S}_r| = (2+g)|\mathcal{C}_r| + 2(\deg_T(r) - 1). \quad (5)$$

Summing these up we get

$$\begin{aligned}|E(\mathcal{D}_G)| &= \sum_{v\in T}|E(\mathcal{D}_G(\mathcal{C}_v\cup\bar{S}_v))|\\ &\le (2+g)|\mathcal{C}|+\sum_{v\in T}(2+g)\deg_T(v)-4|T|+l+4\\ &\le (2+g)|\mathcal{C}|+2((2+g)|T|-2)-4|T|+l+4\\ &\le (2+g)|\mathcal{C}|+2g|T|+l\\ &\le (3+3g)|\mathcal{C}|\end{aligned}$$

where l is the number of (non-root) leaves of T. Thus, $\sum_{C\in\mathcal{C}_t}|\bar{S}\cap C| = |E(\mathcal{D}_G)| \le (3+3g)|\mathcal{C}|$.

This shows that $\bar{S}$ has cost $O(g)OPT_{LP}$ and hence our algorithm returns a solution of cost $O(g)OPT_{LP}$.

Now let us show that in the case G is embedded in a way such that there exist regions R_1, R_2 of Q homeomorphic to the open disk, such that the inside region of any facial dicycle contains at least one of R_1, R_2, then Algorithm 3.1 is an 8- approximation.

The proof works exactly the same as the general case. The key here is to note that the inside regions of face minimal dicycles do not intersect. Thus, R_1 lies in the inside region of at most one cycle in $\mathcal{C}_t$. Likewise, R_2 lies in the inside region of at most one cycle in $\mathcal{C}_t$. Since inside region of any facial dicycle contains at least one of R_1, R_2, $|\mathcal{C}_t| \le 2$. Again Lemma 5 holds. For a facial dicycle A, denote $\text{inside}_{\mathcal{C}_t}(A)$ the set of cycles of $\mathcal{C}_t$ that lie in the closure of the inside region of A.

▶ **Lemma 8.** *There do not exist distinct $A_1, A_2, A_3 \in \mathcal{A}$ such that $\text{inside}_{\mathcal{C}_t}(A_1) = \text{inside}_{\mathcal{C}_t}(A_2) = \text{inside}_{\mathcal{C}_t}(A_3)$.*

Proof. Suppose such A_1, A_2, A_3 existed. Since they are laminar we may assume w.l.o.g that A_1 is contained in the closure of the inside region of A_2 and A_2 is contained in the closure of the inside region of A_3. Let v_i be the hit node that A_i is the witness of. Note that v_2 does not lie on A_1. Thus, as v_2 lies outside $\text{inside}(A_1)$, it lies outside the closure $\text{cl}(\text{inside}(A_1))$ of $\text{inside}(A_1)$. So v_2 lies in $Q\setminus\text{inside}(A_3)$. Thus, v_2 does not lie on any cycle of $\text{inside}_{\mathcal{C}_t}(A_3)$. Also v_2 does not lie on A_3. Thus, as v_2 lies inside $\text{cl}(\text{inside}(A_3))$, it lies inside $\text{inside}(A_3)$. Thus, v_2 does not lie on any cycle of $\mathcal{C}_t\setminus\text{inside}_{\mathcal{C}_t}(A_3)$.

This implies that v_2 does not lie on any cycle of $\mathcal{C}_t$, which is a contradiction. ◀

This implies that $|\bar{S}_t| = |\mathcal{A}| \le 2(2^{|\mathcal{C}_t|}) \le 8$. Thus, $\sum_{C\in\mathcal{C}_t}|\bar{S}_t\cap C| \le |\bar{S}_t||\mathcal{C}_t| \le 8|\mathcal{C}_t|$. This shows Algorithm 3.1 is an 8-approximation. ◀

4 Solving the case of no facial cycles

We now show the LP gap of the natural LP (P_{DFVS}) for G has integrality gap $O(g)$ in the case G contains no facial cycles. This will allow us to derive an $O(g)$-approximation for the general case by first using Theorem 2 to obtain a hitting set S for the set of facial cycles of cost at most $O(g)OPT$ and then obtaining a hitting set $\bar{S}$ for the remaining dicycles.

▶ **Lemma 9.** *Suppose G is a digraph embedded on a surface Q of some fixed genus g and there is no facial dicycle of G. Then the LP gap of the natural LP (P_{DFVS}) for G has integrality gap $O(g)$.*

Proof. We prove the statement by induction on the genus g. The case $g = 0$ is trivial because all cycles in planar graphs are facial. Suppose the statement is true for $g = g'$. Let Q be a surface of genus g, Let G be a digraph embedded on Q.

First, while the optimal solution $\bar{x}$ to ($\mathrm{P_{DFVS}}$) has a vertex v with value $\bar{x}_v \geq \frac{1}{24}$ add v to our temporary hitting set F. Formally initialize $F = \emptyset$. While the optimal solution $\bar{x}$ to ($\mathrm{P_{DFVS}}$) for G^F contains a value $\bar{x}_v$ which is $\frac{1}{24}$ or more add v to F.

Let F denote the final set obtained. Let $\hat{x}$ be an optimal extreme point solution for the DFVS LP ($\mathrm{P_{DFVS}}$) for G^F, so $\hat{x}_v < \frac{1}{24} \;\; \forall v \in V(G^F)$. Standard results in iterative rounding, see for instance page 14 of [18], show F has cost at most 24 times the optimal value of our LP.

We now seek to define (integral) distances on G^F. By standard LP theory, $\hat{x}$ has rational coordinates. Let $N \in \mathbb{Z}_{>0}$ be such that $N\hat{x}$ and $\frac{1}{12}N$ are integral, call $N\hat{x}_v$, the weight of v. Define the weighted distance of path $P = v_0, v_1, .., v_l$, $\omega(P)$ to be $\omega(P) := \sum_{i=0}^{l-1} N\hat{x}_{v_i}$. For a subgraph H of G^F define the weighted distance $d_{\{\omega,H\}}(u,v)$ from u to v the minimum weight of a u-v path in H. Define $d_\omega := d_{\{\omega,G\}}$. For $U, W \subset V(G)$, define $d_{\omega,H}(U,W) := \min_{u\in U, w\in W} \; d_\omega(u,w)$. Define $d_\omega(U,W) = d_{\omega,G}(U,W)$. Define the weighted distance of a closed walk $P' = v_0, v_1, .., v_l v_0$, $\omega(P')$ to be $\omega(P') := \sum_{i=0}^{l} N\hat{x}_{v_i}$. The results in this paper could also be shown by instead defining the weight of each vertex to be $\hat{x}_v$ and instead defining the layers (see later) to be the vertices at distance a multiple of $1/N$ from a given set of vertices. Since $\hat{x}$ is feasible the following result holds.

▶ **Proposition 10.** *The weighted distance of any (directed) closed walk P' is at least N.*

Since $\hat{x}$ is optimal, there exists a dicycle $C^1 := v_1, v_2, .., v_{l'}$ such that $\sum_{v\in C^1} \hat{x}_v = 1$. The motivation of our definition of weighted distance comes from [7]. In [7], they also scale the LP values of ($\mathrm{P_{DFVS}}$) so that the resulting values are integer. For any vertex v with $\hat{x}_v = 0$, they "bypass" the vertex, that is, for each out neighbour u of v and in neighbour w of v, they add the edge wu to the graph and when they have done this for all neighbours, they delete v from the graph. For any vertex v with $N\hat{x}_v >$ 1' they replace v by a "chain" of $N\hat{x}_v >$ 1' vertices $v_1 \to v_2 \to .. \to v_{N\hat{x}_v}$, that is, for $i = 1, 2, .., N\hat{x}_v - 1$, $v_i v_{i+1}$ is an edge. wv_1 and $v_{N\hat{x}_v}u$ are edges for each in neighbour w and out neighbour u. Call this graph H.

For any $W \subset V(H)$ they define "layers" $L_i = \{v \in H : \;\; d_H(W,v) = i\}$ the nodes at distance i from W. They show that the cost of all layers $L_0, L_1, ...$ is $\sum_{v\in V(G)} N\hat{x}_v$. This is very useful for us as we will use this to show that one layer in $L_1, ...L_m$ has cost at most $\frac{1}{m}\sum_{v\in V} N\hat{x}_v$. However, the bypassing operation and replacing a node with a chain operation of [7] do not preserve the genus of the graph. We instead define the notion of weighted distance d_ω. Denote the i-th layer from W as $L_i := \{v \in V : i \geq d_\omega(W,v) > i - \omega(v)\}$ the set of nodes for which the distance from W to v is at most i, but for which the distance plus the weight of v is more than i. One can see that v lies in $N\hat{x}_v$ different L_i, which is analogous to how H defined in [7] contains $N\hat{x}_v$ copies of v each lying in different layers as well. In particular, a node of weight 0 does not lie in any L_i, which is analogous to how a vertex of weight 0 is bypassed in [7].

Consider the embedding of C^1 on our surface. Given a subgraph W of G, denote by $g(W)$ the subset of our surface occupied by a vertex or edge of W. We want to define a "small" neighbourhood around $g(C^1)$, not containing any vertices outside C^1, which we divide up into "left" of $g(C^1)$ and "right" of $g(C^1)$, which we do using the following propositions. These are slightly informal statements of the exact propositions we require, the precise statements appear in Section 5.

▶ **Proposition 11** (Informal statement of Proposition 23 and Proposition 24)**.** *Given a closed continuous non-self-intersecting curve C' embedded on an orientable surface Q, we may partition a small open neighbourhood about C into a "left" L and "right" R. For any curve $f : [0,1] \to Q$ disjoint from C' except at $f(1)$ the partition allows us to say that f "reaches" C' from either the left or right.*

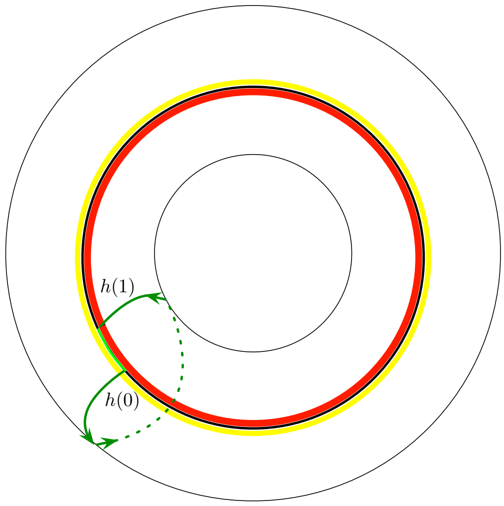

Figure 1 L and R from Proposition 11 in yellow and red respectively curve C' depicted in black. The curve h leaving C' from the left and entering from the right is depicted in dark green. The closed curve formed by h and the subcurve of C' between $h(0)$ and $h(1)$ depicted in light green forms a non-facial closed curve.

▶ **Proposition 12** (Informal statement of corollary of Proposition 26). *Let C' be a non-facial closed curve. If a curve $h : [0, 1] \to Q$ "leaves" C' at a point $h(0) \in C'$ from the left and reaches C' at a point $h(1) \in C'$ from the right, then $h([0, 1])$ together with a subcurve of C' from $h(0)$ to $h(1)$ is a non-facial closed curve.*

We defer the proofs of Proposition 11 and Proposition 12 for now. We apply Proposition 11 to $g(C^1)$. Let L, R be as in Proposition 11 so that each $g(e)$ for $e \in E(G)\backslash E(C^1)$ is disjoint from at least one of L, R and for each $e \in E(G\backslash C^1)$, $g(e)$ is disjoint from both L, R. For each arc uv of G^F with exactly one endpoint v on C^1, $g(uv)$ can be parameterized by a (continuous) curve $f : [0, 1] \to g(uv)$ with $f(0) = g(u)$, $f(1) = g(v)$. If f reaches $g(C^1)$ from the left we say that uv reaches C^1 from the left, otherwise, we say uv reaches C^1 from the right.

Let $u'_{i,1}, u'_{i,2}, .., u'_{i,l_i}$ be the out neighbours of v_i such that the edges $u'_{i,t'}v_i$ reach v_i from the left, that is, the arc obtained from reversing the arc $v_iu'_{i,t'}$ of our graph reaches v_i from the left. Let $w'_{i,1}, w'_{i,2}, .., w'_{i,z_i}$ be the in neighbours of v_i such that the edges $w'_{i,t'}v_i$ reach v_i from the right. Subdivide each edge $v_iu'_{i,t}$ into a path $v_iu_{i,t}u'_{i,t}$ and each edge $w'_{j,t'}v_j$ into a path $w'_{j,t'}w_{j,t'}v_j$ and give the new vertices $w_{j,t'}, u_{i,t}$ infinite cost. There is a natural embedding of our new graph on our surface by placing each $u_{i,t}$ where the midpoint of the curve $g(v_iu'_{i,t})$ was embedded and likewise for $w_{j,t'}$. By abuse of notation, we continue to call our graph G and define $\hat{x}_{u_{i,t}} = \hat{x}_{w_{j,t'}} = 0$ for all $u_{i,t}$, $w_{j,t'}$. Denote $U := \cup_{i=1}^{l'}\{u_{i,1}, u_{i,2}, .., u_{i,l_i}\}$ and $W := \cup_{i=1}^{l'}\{w_{i,1}, w_{i,2}, .., w_{i,z_i}\}$. For $X \subset [l']$, denote $U_X := \cup_{i\in X}\{u_{i,1}, u_{i,2}, .., u_{i,l_i}\}$, $V_X = \{v_i : \ i \in X\}$ and $W_X := \cup_{i\in X}\{w_{i,1}, w_{i,2}, .., w_{i,z_i}\}$.

Let $\tau_- := \{i \in [l'] : \exists w_{i,t'} \in W, \ \ \exists u_{j,t} \in U : d_{\omega,G^F\backslash C^1}(u_{j,t}, w_{i,t'}) < \frac{1}{12}N\}$ the first indices of the set of vertices of W of weighted distance at most $\frac{1}{12}N$ from U in $G^F\backslash C^1$. Let $\tau_+ := \{j \in [l'] : \exists u_{j,t} \in U, \ \ \exists w_{i,t'} \in W : d_{\omega,G^F\backslash C^1}(u_{j,t}, w_{i,t'}) < \frac{1}{12}N\}$ the first indices of the set of vertices of U that can reach W in $G^F\backslash C^1$ with a path of weighted distance at most $\frac{1}{12}N$.

▷ Claim 13. If $d_\omega(V_{\tau_-}, V_{\tau_+}) > \frac{1}{12}N$, then we can find $S \subset V$, $c(S) = O(1)OPT_{LP}$, where $OPT_{LP} := \sum_{v\in V} c_vx_v$ is the value of the optimal fractional solution, such that any strongly connected component of $G^F\backslash S$ does not contain a directed path from U to W in $G\backslash C^1$.

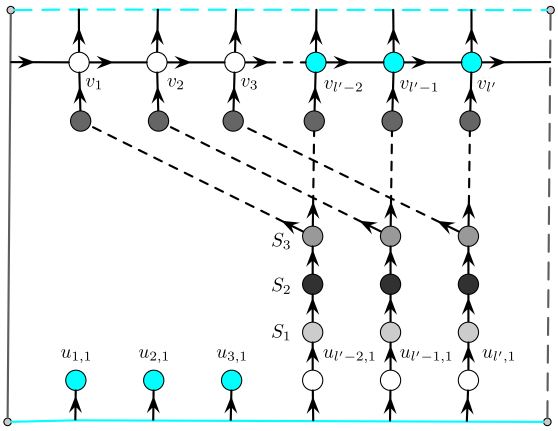

Figure 2 Nodes of U_{τ_+} and V_{τ_-} shown in blue.

If $d_\omega(V_{\tau_-}, V_{\tau_+}) \leq \frac{1}{12}N$, then the LP gap of the natural LP ($\mathrm{P_{DFVS}}$) for G^F has integrality gap $O(g)$.

Proof. Suppose $d_{\omega,G\backslash C^1}(V_{\tau_-}, V_{\tau_+}) > \frac{1}{12}N$. For $i = 0,..,\frac{1}{12}N$ let $S_i := \{v \in V : i \geq d_{\omega,G^F\backslash C^1}(U\backslash U_{\tau_+}, v) > i - \omega(v)\}$ denote the set of vertices of V that are at weighted distance i from $U\backslash U_{\tau_+}$ in $G^F\backslash C^1$. (see Figure 2). Since $d_{\omega,G^F\backslash C^1}(U\backslash U_{\tau_+}, W) > \frac{1}{12}N$, for $i = 0,..,\frac{1}{12}N$, $W \cap S_i = \emptyset$ and W is not reachable from $U\backslash U_{\tau_+}$ in $(G^F\backslash C^1)\backslash S_i$ for any i.

Since each v can lie in at most $\omega(v)$ S_i, $\sum_{i=0}^{\frac{1}{12}N} c(S_i) \leq N \cdot OPT_{LP}$. Let S' be the S_i of minimum cost. For $i = 0,..,\frac{1}{12}N$ let $T_i := \{v \in V : i > d_{\omega,G^F\backslash C^1}(v, W\backslash W_{\tau_-}) - \omega(v),\quad d_{\omega,G^F\backslash C^1}(v, W\backslash W_{\tau_-}) \geq i\}$. Since for $v \in U$, $d_{\omega,G^F\backslash C^1}(v, W\backslash W_{\tau_-}) - \omega(v) = d_\omega(v, W\backslash W_{\tau_-}) > \frac{1}{12}N$, $U \cap T_i = \emptyset$ for $i = 0,..,\frac{1}{12}N$. Hence $W\backslash W_{\tau_-}$ is not reachable from U in $(G^F\backslash C^1)\backslash T_i$ for any i. Let T' be the T_i of minimum cost.

Finally, let $Y_i := \{v \in V : \ i \geq d_\omega(V_{\tau_-}, v) > i - \omega(v)\}$ the set of vertices of weighted distance i from V_{τ_-}. By assumption $d_\omega(V_{\tau_-}, V_{\tau_+}) > \frac{1}{12}N$ and hence V_{τ_+} is not reachable from V_{τ_-} in $G^F\backslash Y_i$ for any $i = 1, 2, .., \frac{1}{12}N$. Let Y' be the Y_i of minimum cost.

Let $S := S' \cup T' \cup Y'$. We claim no strongly connected component K' of $G^F\backslash S$ contains a directed path from U to W in $G^F\backslash C^1$. Suppose for a contradiction that some strongly connected component K' of $G^F\backslash S$ contains a directed path from some $u_{i,t} \in U$ to some $w_{j,t'} \in W$.

If $j \notin \tau_-$, then $w_{j,t'}$ is not reachable from U in $G^F\backslash S$. If $i \notin \tau_+$, then W is not reachable from $u_{i,t}$ in $G^F\backslash S$. Thus, if either $j \notin \tau_-$ or $i \notin \tau_+$ then there is no path from $u_{i,t}$ to $w_{j,t'}$ in $G^F\backslash S$. Thus, $j \in \tau_-$ and $i \in \tau_+$. As K' is strongly connected, this implies that $G^F\backslash S$ contains a path from V_{τ_-} to V_{τ_+}, which is not possible.

Now suppose that $d_\omega(V_{\tau_-}, V_{\tau_+}) \leq \frac{1}{12}N$. Let $i \in \tau_-$ and $j \in \tau_+$ be such that $d_\omega(v_i, v_j) \leq \frac{1}{12}N$. Let P_1, P_2, P_3 be $u_{a,t}$-v_i, $u_{j,t'}$-v_b and v_i-v_j paths of weight at most $\frac{1}{12}N$, with the second last vertices of P_1, P_2 being in W, for some a, b. Such paths exist as $i \in \tau_-$ and $j \in \tau_+$. If $a = i$, then $P_1 v_i u_{a,t}$ is a cycle for which $\sum_{v \in P_1 v_i u_{a,t}} \hat{x}_v < 1$ which is a contradiction. So $a \neq i$, likewise $b \neq j$.

For $i', j' \in \{1, 2, .., l'\}$, let $C^1_{(v_{i'}, v_{j'})} := v_{i'}, v_{i'+1}, v_{i'+2}, .., v_{j'-1} v_{j'}$ (where $v_t = v_{t \pmod{l'}}$) denote the directed path in C^1 from v_i to v_j. Note that $d_\omega(v_{i'}, v_{j'}) = \omega(C^1_{(v_{i'}, v_{j'})})$, for otherwise there is a $v_{i'}$-$v_{j'}$ path P' of weight less than $d_\omega(v_{i'}, v_{j'})$. Then $C^1_{(v_{j'}, v_{i'})} \cup P'$ is a directed closed walk of weight $\omega(C^1) - \omega(C^1_{(v_{i'}, v_{j'})}) + d_\omega(v_{i'}, v_{j'}) < \omega(C^1)$. Noting that the weighted distance of a cycle is equal to $N \sum_{v \in C^1} \hat{x}_v$, we obtain $N \sum_{v \in C^1_{(v_{j'}, v_{i'})} \cup P'} \hat{x}_v < \omega(C^1) = N$, from which it follows the sum of the $\hat{x}_v$ values along the closed walk $C^1_{(v_{j'}, v_{i'})} \cup P'$, $\sum_{v \in C^1_{(v_{j'}, v_{i'})} \cup P'} \hat{x}_v$ is strictly less than 1, which contradicts the feasibility of $\hat{x}$.

We claim $\omega(C^1_{(v_a, v_i)}), \omega(C^1_{(v_j, v_b)}) \le \frac{1}{12}N$. Suppose for a contradiction that $\omega(C^1_{(v_a, v_i)}) > \frac{1}{12}N$. Since $\omega(C^1) = N$, this implies that $\omega(C^1_{(v_{i+1}, v_{a-1})}) < N - \frac{1}{12}N$. Then the cycle $P_1 C^1_{(v_i, v_a)} v_a u_{a,t}$ satisfies $\sum_{v \in V(P_1 C^1_{(v_i, v_a)} v_a u_{a,t})} \hat{x}_v < 1 - \frac{1}{12} + \frac{1}{12} = 1$ which is a contradiction. Likewise, $\omega(v_j, v_b) \le \frac{1}{12}N$.

Let us show $C^1_{(v_a, v_i)} \cap C^1_{(v_j, v_b)} = \{v_i\} \cap \{v_j\}$, that is the paths $C^1_{(v_a, v_i)}$ and $C^1_{(v_j, v_b)}$ are disjoint except in the case $i = j$ when their intersection is v_i. First, let us address the case $i \ne j$. Suppose for a contradiction that $C^1_{(v_a, v_i)} \cap C^1_{(v_j, v_b)} \ne \emptyset$. Let $v \in C^1_{(v_a, v_i)} \cap C^1_{(v_j, v_b)}$. Note $v \ne v_i, v_j$ for otherwise $C^1_{(v_j, v_i)} P_3$ is a closed walk of weight less than N. Let Q_1 be a path from v to v_i in $C^1_{(v_a, v_i)}$ and Q_2 a path from v_j to v in $C^1_{(v_j, v_b)}$. Then $Q_2 Q_1 P_3$ is a closed walk of weighted distance at most $\frac{1}{4}N$ which is a contradiction.

Now suppose that $i = j$. Suppose for a contradiction that $C^1_{(v_a, v_i)} \cap C^1_{(v_j, v_b)} \ne \{v_i\}$. Let $v \in (C^1_{(v_a, v_i)} \cap C^1_{(v_j, v_b)}) \backslash v_i$. Let Q_1 be a path from v to v_i in $C^1_{(v_a, v_i)}$ and Q_2 a path from v_i to v in $C^1_{(v_i, v_b)}$. Then $Q_1 Q_2$ is a closed walk of weighted distance at most $\frac{1}{6}N$, which is a contradiction.

▷ Claim 14. $d_\omega(P_2 \cup C^1_{(v_{j+1}, v_b)}, P_1 \cup C^1_{(v_a, v_i)}) \ge \frac{1}{6}N$.

Proof. Suppose for a contradiction that $d_\omega(P_2 \cup C^1_{(v_{j+1}, v_b)}, P_1 \cup C^1_{(v_a, v_i)}) < \frac{1}{6}N$. Let $s \in P_2 \cup C^1_{(v_{j+1}, v_b)}$ and $q \in P_1 \cup C^1_{(v_a, v_i)}$ be such that $d_\omega(s, q) < \frac{1}{6}N$. Let P_1' be the directed path in $P_1 \cup C^1_{(v_a, v_i)}$ from q to v_i. P_2' the directed path in $P_2 \cup C^1_{(v_j, v_b)}$ from v_j to s and Q a path of weight at most $\frac{1}{6}N$ from s to q. Then $\bar{C} := v_j P_2' Q P_1' P_3$ is a closed walk such that $\sum_{v \in \bar{C}} \hat{x}_v < 1$ which is a contradiction (see Figure 3).

Thus, $d_\omega(P_2 \cup C^1_{(v_i, v_b)}, P_1 \cup C^1_{(v_a, v_i)}) \ge \frac{1}{6}N$. Since $d_\omega(u, v_i) \le \frac{1}{12}N$ for any $u \in C^1_{(v_a, v_i)}$, $d_\omega(P_2, C^1_{(v_a, v_i)}) \ge \frac{1}{12}N$ ◁

For $i = 0, 1, .., \frac{1}{12}N$, define $R_i := \{v \in V : i \ge d_\omega(P_2 \cup C^1_{(v_{j+1}, v_b)}, v) > i - \omega(v)\}$. Each vertex $v \in V$ lies in at most $\omega(v)$ R_i. Let R' be the R_i of the smallest cost, so $c(R') \le 12 OPT_{LP}$. Since $d_\omega(P_2 \cup C^1_{(v_{j+1}, v_b)}, P_1 \cup C^1_{(v_a, v_i)}) \ge \frac{1}{12}N$, it follows that $(P_1 \cup C^1_{(v_a, v_i)}) \backslash R_i$ is not reachable from $(P_2 \cup C^1_{(v_{j+1}, v_b)}) \backslash R_i$ in $G \backslash R_i$ for any i. Thus, $(P_1 \cup C^1_{(v_a, v_i)}) \backslash R'$ is not reachable from $(P_2 \cup C^1_{(v_{j+1}, v_b)}) \backslash R'$ in $G \backslash R'$.

Thus, any strongly connected component of $G \backslash R'$ is either contained in $G^F \backslash (P_1 \cup C^1_{(v_a, v_i)})$ or $G^F \backslash (P_2 \cup C^1_{(v_{j+1}, v_b)})$. For $i = 1, 2, .., \frac{1}{12}N$ let $K_i^+ := \{v \in V : \ i \ge d_\omega(v_j, v) > i - \omega(v)\}$ be the vertices of weighted distance i from v_j. Let K'^+ denote the K_i^+ of minimum cost.

▷ Claim 15. v_j is not contained in a cycle in $G^F \backslash (R' \cup K'^+)$.

Proof. Suppose that there is a cycle $a_1, a_2, .., a_p v_j a_1$ in $G^F \backslash (R' \cup K'^+)$. If $d_\omega(v_j, a_p) > \frac{1}{12}N$, then a_p is not reachable from v_j in $G^F \backslash (R' \cup K'^+)$. Thus, there is a path P_a of weighted distance at most $\frac{1}{12}N$ from v_j to a_p. Then the closed walk $v_j p_a a_p v_j$ has weighted distance at most $\omega(P_a) + \omega(a_p) \le \frac{1}{12}N + \frac{1}{12}N < N$ which is a contradiction. ◁

Recall that any dicycle of $G^F \backslash R'$ is contained in either $G^F \backslash (P_1 \cup C^1_{(v_a, v_i)})$ or $G^F \backslash (P_2 \cup C^1_{(v_{j+1}, v_b)})$. Since v_j is not contained in any dicycle of $G^F \backslash (R' \cup K'^+)$ it follows that any dicycle of $G^F \backslash (R' \cup K'^+)$ is either contained in $G \backslash (P_1 \cup C^1_{(v_a, v_i)})$ or in $G \backslash (P_2 \cup C^1_{(v_j, v_b)})$. By Proposition 12, $g(P_1 \cup C^1_{(v_a, v_i)})$ and $g(P_2 \cup C^1_{(v_j, v_b)})$ are nonfacial.

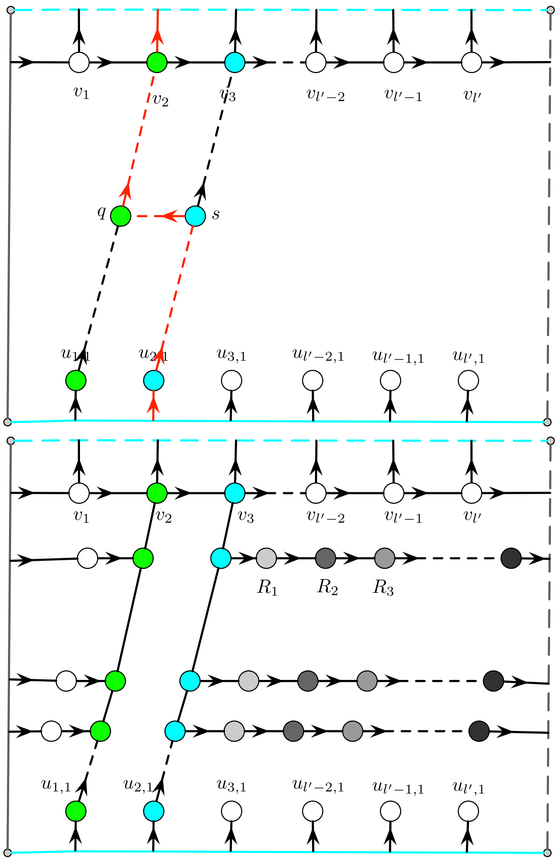

Figure 3 On the left, there are $u_{1,1}$-v_2 and $u_{2,1}$-v_3 paths (green and blue vertices respectively) of weight at most $\frac{1}{12}N$ and s-q path of length at most $\frac{1}{12}N$. The red cycle would then have weight at most N, which is a contradiction. On the right are the sets R_i, vertices at distance i from $P_2 \cup C^1_{(v_{j+1},v_b)}$.

▶ **Definition 16** ([1, 14]). *Given a simple closed curve f on a surface without boundary Q, not dividing the surface into 2 regions, we say Q' is obtained by doing* surgery *along f if Q' is obtained as follows. "Thicken" f to obtain a cylinder and remove this cylinder from Q, call this resulting surface Q''. The boundary of Q'' consists of 2 circles we "glue" two cones N_1, N_2 along these circles and call this final surface Q'.*

▶ **Theorem 17** ([1] p.162). *For a surface without boundary Q of genus g', Q' obtained by Definition 16 is a surface without boundary of genus at most $g' - 1$.*

We apply the surgery of Definition 16 to $g(P_1 \cup C^1_{(v_a,v_i)})$ to obtain a surface Q' of genus one less than Q. Let N'_1, N'_2 denote the two cones glued to Q'. We also apply the surgery of Definition 16 to $g(P_2 \cup C^1_{(v_j,v_b)})$ to obtain a surface $\hat{Q}$ of genus one less than Q. Let $\hat{N}_1, \hat{N}_2$ denote the two cones glued to $\hat{Q}$.

▶ **Lemma 18.** *Let G be a graph embedded on a surface Q with no dicycles. Let h be a non-facial curve of $Q \backslash G$. Let Q' be the surface obtained by applying the surgery of Definition 16 to with respect to the curve h and surface Q. There is a natural embedding of G on Q' (by leaving each node of G where it was in Q). Let N_1, N_2 denote the two cones glued to Q' during the surgery process. Then each facial cycle of G with respect to its embedding in Q' contains either N_1 or N_2 in its inside region.*

Proof. Let C be a facial cycle of $G^F\backslash(P_1 \cup C^1_{(v_a,v_i)})$ with respect to its embedding in Q'. If neither of the cones N_1, N_2 are contained in the inside region of C, then C is a facial cycle of G with respect to its embedding in Q, which is a contradiction. ◀

Thus, any facial cycle of $G^F\backslash(P_1 \cup C^1_{(v_a,v_i)})$ contains either N_1' or N_2' in its inside region.

Now let $G_1, G_2, .., G_l$ be the strongly connected components of $G^F\backslash(R' \cup K'^+)$. Since any closed walk of of $G^F\backslash(R' \cup K'^+)$ is either contained in $G\backslash(P_1 \cup C^1_{(v_a,v_i)})$ or in $G\backslash(P_2 \cup C^1_{(v_j,v_b)})$ each strongly connected component is either contained in $G\backslash(P_1 \cup C^1_{(v_a,v_i)})$ or in $G\backslash(P_2 \cup C^1_{(v_j,v_b)})$. If G_i is contained in $G\backslash(P_1 \cup C^1_{(v_a,v_i)})$, then there is a natural embedding of G_i in Q' (obtained by leaving all nodes and edges where they are in the surgery for Definition 16). Likewise, if G_i is contained in $G\backslash(P_1 \cup C^1_{(v_a,v_i)})$, then there is a natural embedding of G_i in $\hat{Q}$. Thus, for any G_i contained in $G\backslash(P_1 \cup C^1_{(v_a,v_i)})$ by Theorem 2, there is an 8-approximation for the problem of hitting the facial cycles of G_i (with respect to the natural embedding in Q'). Likewise, for any G_i contained in $G\backslash(P_2 \cup C^1_{(v_j,v_b)})$ there is an 8-approximation for the problem of hitting the facial cycles of G_i. Let Z_i be a solution for the problem of hitting facial cycles of G_i of cost at most $8OPT_{LP(G_i)}$ as guaranteed by Theorem 2.

Then each $G_i\backslash Z_i$ is embedded in a surface of smaller genus with no facial cycles.

By induction, there are solutions A_i to $G_i\backslash Z_i$ of cost $c_{g-1}OPT_{LP(G_i\backslash Z_i)}$, where c_g is the integrality gap of the DFVS LP for graphs of genus g.

Define $\hat{x}^{G_i\backslash Z_i} \in \mathbb{R}^{V(G_i\backslash Z_i)}$ as $\hat{x}^{G_i}(v) = \hat{x}_v$, where $\hat{x}$ is as in the proof of Lemma 9. Since graphs G_i are vertex disjoint, $G_i\backslash Z_i$ are vertex disjoint, so $\sum_{i=1}^{l} OPT_{LP(G_i)} \leq \sum_{i=1}^{l} \sum_{v\in V(G_i)} \hat{x}^G_i(v) \leq \sum_{v\in V(G)} \hat{x} = OPT_{LP(G)}$. Now $F \cup R' \cup K'^+ \cup (\cup_{i=1}^{l} A_i) \cup (\cup_{i=1}^{l} Z_i)$ is a DFVS of cost $(O(1)+c_{g-1})OPT_{LP(G)} = (O(1)+O(g-1))OPT_{LP(G)} = (O(g))OPT_{LP(G)}$. ◁

Note that the argument in Claim 13 is symmetric with respect to left and right and we may swap right and left to get the following result. Let $b'_{i,1}, b'_{i,2}, .., b'_{i,l'_i}$ be the in neighbours of v_i such that each edge $b'_{i,t}v_i$ reaches v_i from the left and $d'_{i,1}, d'_{i,2}, .., d'_{i,t'_i}$ be the out neighbours of v_i such that the edge $d'_{i,t'}v_i$ reaches v_i from the right. Subdivide each edge $b'_{j,t'}v_j$ into a path $b'_{j,t'}b_{j,t'}v_j$ and each edge $v_id'_{i,t}$ into a path $v_id_{i,t}d'_{i,t}$ and give the new vertices $d_{j,t'}, b_{i,t}$ infinite cost. There is a natural embedding of our new graph on our surface by placing each $b_{i,t}$ where the midpoint of the curve $g(v_ib'_{j,t'})$ was embedded and likewise for $d_{j,t'}$. By abuse of notation, we continue to call our graph G and define $\hat{x}_{b_{i,t}} = \hat{x}_{d_{j,t'}} = 0$ for all $b_{i,t}$, $d_{j,t'}$.

Denote $B := \cup_{i=1}^{l'}\{b_{i,1}, b_{i,2}, .., b_{i,l'_i}\}$, $D = \cup_{i=1}^{l'}\{d'_{i,1}, d'_{i,2}, .., d'_{i,t'_i}\}$. Let $\kappa_- : \{i \in [l'] : \exists b_{i,t'} \in D,\ \ \exists d_{j,t} \in U : d_{\omega,G^F\backslash C^1}(d_{j,t}, b_{i,t'}) < \frac{1}{12}N\}$ the first indices of the set of vertices of B of weighted distance at most $\frac{1}{12}N$ from D in $G^F\backslash C^1$. Let $\kappa_+ := \{j \in [l'] : \exists d_{j,t} \in U,\ \ \exists b_{i,t'} \in W : d_{\omega,G^F\backslash C^1}(d_{j,t}, b_{i,t'}) < \frac{1}{12}N\}$ the first indices of the set of vertices of D that can reach B with a path of weighted distance at most $\frac{1}{12}N$ in $G^F\backslash C^1$. Similarly to how we proved Claim 13, we can show the following:

▷ Claim 19. If $d_\omega(V_{\kappa_-}, V_{\kappa_+}) > \frac{1}{12}N$, then we can find $T \subset V$, $c(T) = O(1)OPT_{LP}$, (recall $OPT_{LP} := \sum_{v\in V} c_vx_v$ is the value of the optimal fractional solution), such that any strongly connected component of $G^F\backslash T$ does not contain a directed path from D to B in $G\backslash C^1$.

If $d_\omega(V_{\kappa_-}, V_{\kappa_+}) \leq \frac{1}{12}N$, then the LP gap of the natural LP ($\mathrm{P_{DFVS}}$) for G has integrality gap $O(1)$.

We now construct a DFVS of cost at most $O(g)OPT_{LP}$. If either $d_\omega(V_{\kappa_-}, V_{\kappa_+}) \leq \frac{1}{12}N$ or $d_\omega(V_{\tau_-}, V_{\tau_+}) \leq \frac{1}{12}N$. Then Claim 13 or Claim 19 respectively shows that that the LP gap of the natural LP ($\mathrm{P_{DFVS}}$) for G has integrality gap $O(g)$.

Now assume both $d_\omega(V_{\tau_-}, V_{\tau_+}), d_\omega(V_{\kappa_-}, V_{\kappa_+}) > \frac{1}{12}N$. Then by Claim 13 and Claim 19, there are sets S, T such that any strongly connected component of $G^F \backslash (S \cup T)$ does not contain a path from U to W or a path from D to B in $G^F \backslash C^1$.

For any digraph H define $\mathrm{un}(H)$ to be the underlying (undirected) graph of H. Let K be any strongly connected component of $G^F \backslash (S \cup T)$. We will prove $\mathrm{un}(K)$ does not contain any path from $U \cup B$ to $W \cup D$ in $\mathrm{un}(K) \backslash C^1$.

▶ **Proposition 20.** *If there is a (undirected) path $P = u_{i,t} q_1, q_2, .., q_t$ from some $u_{i,t} \in U$ (resp $u_{i,t} \in D$) in $\mathrm{un}(K) \backslash C^1$, then there is a directed path from U (resp D) to q_j in $G^F \backslash (S \cup T \cup C^1)$ for any $j = 1, 2, .., t$.*

If there is a (undirected) path $P = q_1, q_2, .., q_t w_{i,t}$ from some $w_{i,t} \in W$ (resp $b_{i,t} \in B$) in $\mathrm{un}(K) \backslash C^1$, then there is a directed path from q_j to W (resp B) in $G^F \backslash (S \cup T \cup C^1)$ for any $j = 1, 2, .., t$.

Proof. Let $P = u_{i,r} q_1, q_2, .., q_t$ be a path in $\mathrm{un}(K) \backslash C^1$ from some $u_{i,r} \in U$ (resp $u_{i,r} \in D$). We prove by induction t' on that there is a directed path from U to q_j in $G^F \backslash (S \cup T \cup C^1)$ for any $j = 1, 2, .., t'$. The case $t' = 1$ is clear as each $u_{i,r} \in U$ (resp $u_{i,r} \in D$) only has out neighbours so the undirected edge $\{u_{i,r}, q_1\}$ in $\mathrm{un}(K)$ is directed from $u_{i,r}$ to q_1.

Now assume the statement true for $t' = t''$. For $t' = t'' + 1$, if the undirected edge $\{q_{t'}, q_{t'+1}\}$ is directed from $q_{t'}$ to $q_{t'+1}$, then there is a directed path from $u_{i,r}$ to $q_{t'+1}$ in $G^F \backslash (S \cup T \cup C^1)$.

Otherwise $\{q_{t'}, q_{t'+1}\}$ is directed from $q_{t'+1}$ to $q_{t'}$. By strong connectedness of K, there is a directed path P' from $q_{t'}$ to $q_{t'+1}$ in $K \backslash (S \cup T)$. If P' does not intersect C then there is a directed path from $u_{i,r}$ to $q_{t'+1}$ in $G^F \backslash (S \cup T \cup C^1)$. So, assume P' intersects W or B. Let P'' denote the subpath of P' from $q_{t'}$ to when P' first intersects U or B. By construction P'' lies in $G^F \backslash (S \cup T \cup C^1)$. As $u_{i,r}$ lies in U (resp D) P'' does not intersect W (resp. B), as then we would have a U-W (resp. D-B) path in $G^F \backslash (S \cup T \cup C^1)$. Thus, P'' is a $q_{t'}$-B (resp. $q_{t'}$-W) path. Consider the subpath Q of the reversal of P' starting from $q_{t'+1}$ to when the reversal of P' first intersects D or U. Let $\mathrm{rev}(Q)$ denote the reversal of Q. Note $\mathrm{rev}(Q)$ lies in $G^F \backslash (S \cup T \cup C^1)$. If the starting vertex of $\mathrm{rev}(Q)$ is in D (resp. U), then $\mathrm{rev}(Q) \cup \{q_{t'+1} q_{t'}\} \cup P''$ is a D-B (resp. U-W) path in $G^F \backslash (S \cup T \cup C^1)$. This contradicts Claim 19. Thus, the starting vertex of of $\mathrm{rev}(Q)$ is in U (resp. D). This implies there is a path from U (resp. D) to $q_{t'+1}$ completing the induction. The proof of the second part is similar. ◀

▶ **Proposition 21.** *There is no (undirected) path from $W \cup D$ to $U \cup B$ in $\mathrm{un}(K) \backslash C^1$.*

Proof. If we have a U-W path $P = u_{i,t} q_1, q_2, .., q_t w_{j,t'}$ in $\mathrm{un}(K) \backslash C^1$, then by Proposition 20, there are directed U-q_1 and q_1-W paths P_1 and P_2 in $\mathrm{un}(K) \backslash C^1$. Then $P_1 \cup P_2$ is a directed U-W path in $K \backslash C^1$ which contradicts Claim 19. Thus, we do not have a U-W path $P = u_{i,t} q_1, q_2, .., q_t w_{j,t'}$ in $K \backslash C^1$. Likewise, we do not have a D-B path $P = u_{i,t} q_1, q_2, .., q_t w_{j,t'}$ in $K \backslash C^1$.

Suppose we have a U-D path $P = u_{i,t} q_1, q_2, .., q_t d_{j,t'}$ in $\mathrm{un}(K) \backslash C^1$. By Proposition 20, there are directed U-q_1 and D-q_1 paths P_1 and P_2 in $K \backslash C^1$. Recall U has no in-neighbours of in $G \backslash C^1$, so the edge $\{u_{i,t}, q_1\}$ in K is directed from $u_{i,t}$ to q_1. By 2 connectedness of K, there is a path P_3 from q_1 to $u_{i,t}$. The only in-neighbours of $u_{i,t}$ are in C^1, thus P_3 intersects $W \cup B$. Let P_3' be the subpath of P_3 from q_1 to when it the path first intersects $W \cup B$. If the endpoint of P_3' is in W, then $P_1 \cup P_3'$ is a U-W path in $K \backslash C^1$. Otherwise, if the endpoint of P_3' is in B, then $P_2 \cup P_3'$ is a D-B path in $K \backslash C^1$. Either way this contradicts Claim 19. ◀

▶ **Proposition 22** ([15, 25]). *Suppose G is a graph embedded on a surface Q. Let C be a cycle of G that does not divide Q into two separate regions such that there is no edge between vertices of C that is not part of C. Define a "left" and "right" as in Proposition 11. Let $\hat{L}$, $\hat{R}$ denote the neighbours of C that are "left" or "right" of C. Suppose each connected component of $G \backslash C$ only contains nodes of $\hat{L}$ or $\hat{R}$ but not both. There is a non-facial closed curve h in $Q \backslash G$.*

Applying Proposition 21, we get that G^F satisfies Proposition 22 with respect to C^1. Thus, there is a non-facial closed curve h in $Q \backslash G^F$. We apply the surgery of Definition 16 with respect to the closed curve h and surface Q to obtain a surface Q' of lower genus. Let $G_1, G_2, ., G_l$ be the strongly connected components of $G^F \backslash (S \cup T)$, so each G_i is embeddable on Q'. By Lemma 18 each facial dicycle of G_i contains one of the cones of Q'. Hence there is an algorithm that returns a hitting set Z_i to the set of facial cycles of G_i of cost at most $8OPT_{LP(G_i)}$. By induction, there are solutions A_i to $G_i \backslash Z_i$ of cost $c_{g-1}OPT_{LP(G_i)}$, where c_g is the integrality gap of the DFVS LP for graphs of genus g. Define $\hat{x}_i^G \in \mathbb{R}^{V(G_i)}$ as $\hat{x}_i^G(v) = \hat{x}_v$, where $\hat{x}$ is as in the proof of Lemma 9. Since graphs G_i are vertex disjoint, $\sum_{i=1}^{l} OPT_{LP(G_i)} \leq \sum_{i=1}^{l} \sum_{v \in V(G_i)} \hat{x}_i^G(v) \leq \sum_{v \in V(G)} \hat{x} = OPT_{LP(G)}$. Then $S \cup T \cup F \cup (\cup_{i=1}^{l} A_i) \cup (\cup_{i=1}^{l} Z_i)$ is a DFVS of cost $(O(1)+c_{g-1})OPT_{LP(G)} = (O(1)+O(g-1))OPT_{LP(G)} = (O(g))OPT_{LP(G)}$. ◀

As observed in [7], ($\mathrm{P_{DFVS}}$) can be solved in polynomial-time via the ellipsoid method. Hence Lemma 9 yields a polynomial time $O(g)$-approximation algorithm for DFVS in graphs of genus g with no facial cycle.

5 Statement and proofs of topological results we use

First let us prove Proposition 22.

Proof. Suppose each connected component of $G \backslash C$ only contains nodes of $\hat{L}$ or $\hat{R}$ but not both. Let G_L and G_R be the unions of the components of $G - C$ that only contain nodes from$\hat{L}$ and $\hat{R}$ respectively. Assume that $Q \backslash G$ contains no non-facial curve h.

Case 1: At least one of G_L or G_R is empty.

Suppose, without loss of generality, that G_L is empty. Consider the face of G that contains C and intersects the left of C. But, C is not contractible (else it would separate the surface Q into two components). Hence, a small leftward shift of C which will lie in the face f will produce a non-facial curve h.

Case 2: Both G_L and G_R are nonempty.

We claim that if a face contains vertices of G_L, G_R and of C then there is a non-facial curve in $Q \backslash G$. Let f be such a face of degree d. Let $\partial f = v_0 \cdots v_{d-1}$ be the boundary cycle of f, where $i \in \mathbb{Z}/d\mathbb{Z}$. Without loss of generality, assume that $v_0 \in \hat{L}$ and for some q $v_1, v_2, .., v_q \in C$, and $v_{q+1} \in \hat{R}$. There is are points p_l on the edge $v_0 v_1$ in the interior of L and p_R on the edge $v_q v_{q+r}$ in the interior of R. Let $h : [0, 1] \to Q$ be a non-self-intersecting curve in f from p_R to p_L. Let $r_L, r_R > 0$ be such that $B_Q(p_l, r_L) \subset L$, $B_Q(p_R, r_R) \subset R$. Let h^L, h^R be non-self-intersecting curves in $B_Q(p_l, r_L)$ and $B_Q(p_R, r_R)$ from p_l to v_1 and p_R to v_q respectively not intersecting h. Then $h \cup h^L \cup h^R$ satisfies the conditions of Proposition 12. Thus, $h \cup h^L \cup h^R \cup g(p_l v_0, v_1, ..., v_{q+1} p_R)$ does not bound a region of the closure of f. As this curve lies in the closure of f, this implies that f is not homeomorphic to an open disk. By the classification theorem for orientable surfaces (see for instance page 87 of [17]), cl(f) is homeomorphic to a m-torus T_m with a finite number of open disks removed. Since f is not homeomorphic to an open disk, f contains a non-facial closed curve h in its interior.

If there is a face f of G whose boundary contains vertices of G_L and of G_R (but not of C), then as there is no edge between G_L and G_R, the boundary of f is not connected and so f is not homeomorphic to an open disk. Just as before this implies f contains a non-facial closed curve h.

Now, consider the subsets Q_L and Q_R of Q obtained by taking the union of all the vertices, edges and faces induced by $G_L \cup C$ and $G_R \cup C$, respectively. By assumption 3, every component of $G - C$ is in G_L or G_R, so every vertex and edge of G belongs to Q_L or Q_R. By the subcases eliminated above under Case 2, every face of G also belongs to Q_L or Q_R (but not both). Then, $Q = Q_L \cup Q_R = (Q_L - C) \sqcup (Q_R - C) \sqcup C$. This means that C separates Q into two components, which contradicts assumption 1. ◀

It is well known (see for instance [1] page 15) that smooth surfaces Q have the property that for each $v \in Q$ there is an open ball $B_Q(v, r_0)$ of some small radius $r_0 > 0$ in Q and a diffeomorphism ψ from $B_Q(v, r_0)$ to the open disk $B_{\mathbb{R}^2}(0, r_0)$ of radius r_0 about the origin in the two-dimensional plane such that $\psi(v) = (0, 0)$ and ψ preserves distances from v, that is $\text{dist}_Q(v, x) = \|\psi(v) - \psi(x)\|$, where $\text{dist}_Q(v, x)$ is the geodesic distance from v to x in Q. For $p \in Q$, $r > 0$, denote by $B(p, r)$ the open ball of radius r about p. We now formally state and prove what Proposition 11 and Proposition 12 informally say.

▶ **Proposition 23.** *Given a closed continuous non-self-intersecting curve C' embedded on an orientable surface Q. There exist some radius $r > 0$ and disjoint subsets L, R "on each side" of C' such that the set $\{B(v, r) : v \in C'\}$ (where $B(v, r)$ is the open ball around v of radius r in Q) is contained in the union $L \cup R \cup C'$, and for each $v \in C'$, $r' \le r$, $L \cap B(v, r')$ and $R \cap B(v, r')$ are the two connected components of $B(v, r') \setminus C'$. There is a diffeomorphism ϕ from $L \cup C' \cup R$ to a connected open neighbourhood of $C' \times \{0\}$ in $C' \times \mathbb{R}$ and small $q > 0$ with $C' \times (-q, 0) \subset \phi(L) \subset C' \times (-\infty, 0)$, $C' \times (0, q) \subset \phi(R) \subset C' \times (0, \infty)$ and $\phi(C') = C' \times \{0\}$. Further for any (piecewise smooth) curve $f : [0, 1] \to Q$ such that $f(x) \notin C'$ for any $x \in [0, 1)$, $f(1) \in C'$ satisfies that for some $\beta \in (0, 1)$, either $f((\beta, 1)) \in L$, that is the curve "reaches C from the left" L or $f((\beta, 1)) \in R$, that is the curve "reaches C' from the right" R.*

▶ **Proposition 24.** *For a finite set of curves $f_1, f_2, f_3, ..f_{l'}, h_1, h_2, .., h_{t'} : [0, 1] \to Q$ such that for each i, $f_i(x) \notin C'$ for any $x \in [0, 1)$ and $h_i(x) \notin C'$ for any $x \in [0, 1]$ we may choose L, R, r above so that each curve $f_i([0, 1))$ is disjoint from at least one of L, R and each curve $h_i([0, 1])$ is disjoint from both L, R. We refer to L and R as the left and right of C' respectively.*

Further, there are curves $f_L : [0, 1] \to L$, $f_R : [0, 1] \to R$ which are homotopic to C'. Informally speaking, these are obtained by "slightly shifting" f "left" and "right" respectively.

▶ **Proposition 25.** *Lastly let $h : [0, 1] \to Q$ be any curve that reaches C' from the right at a point $c_2 = h(1)$ on C', leaves C' from the left at $c_1 = h(0)$, that is the curve $\bar{\psi}(t) = h(1 - t)$ reaches C' at c_1 from the left and h is otherwise disjoint from C'. Assume $c_1 \neq c_2$ and let C'_{c_1, c_2} be a subcurve of C' with endpoints c_1 and c_2. Then there is a curve $\hat{h} : [0, 2] \to Q$ that reaches C' from the right at $c_1 = \hat{h}(1)$ and leaves C' at a point $c_1 = \hat{h}(0)$ $\hat{h}$ is otherwise disjoint from C', and a there is a homeomorphism of Q that maps $\hat{h}$ to the concatenation of h and C'_{c_1, c_2}.*

▶ **Proposition 26.** *Let Q be an orientable surface and $\phi : [0, 1] \to Q$ a closed curve not dividing Q into 2 regions with disjoint subsets L, R "on each side" of ϕ as in Proposition 23. Let $c_1, c_2 \in [0, 1)$, with $c_2 \ge c_1$. Suppose that $\phi_1 : [0, 1] \to Q$ is a curve with $\phi_1(0) = \phi(c_1)$ $\phi_1(1) = \phi(c_2)$, $\phi_1((0, 1))$ is disjoint from $\phi([0, 1])$ and the curve $\phi_1([0, 0.5])$ approaches $\phi([0, 1])$ from the left L and $\phi_1([0.5, 1])$ approaches $\phi([0, 1])$ from the right R.*

Then the curve $\phi_2 : [0,1] \to Q$, $\phi_2(x) = \phi(c_2 - x)$ *if* $x \leq c_2 - c_1$ *and* $\phi_2(x) = \phi_1(\frac{1}{c_2-c_1}(x - c_2 + c_1))$ *for* $x > c_2 - c_1$, *that is the curve obtained by joining the portion of* ϕ *from* c_1 *to* c_2 *to* $\phi_1([0,1])$ *does not divide* Q *into 2 regions.*

Proof. We prove Proposition 23, Proposition 24, and Proposition 25. We use the following corollary of the tubular neighbourhood theorem (see for instance [4]).

▶ **Proposition 27** (corollary of tubular neighbourhood theorem [4]). *Given a curve* C' *embedded in a surface* Q *there is an open neighbourhood* U *of* C *and an open set* V *in* $C' \times \mathbb{R}$ *such that there is a diffeomorphism* $\phi : U \to V$ *with* $\phi(C') = C' \times \{0\}$.

Let $w : [0,1] \to C'$ with $w(0) = w(1)$, $w(0.5) = c_2$ be a parameterization of C'. We define distance on $C' \times \mathbb{R}$ by $\mathrm{dist}((a_1, b_1), (a_2, b_2)) := (\mathrm{dist}_Q(a_1, a_2)^2 + |b_1 - b_2|^2)^{\frac{1}{2}}$, where $\mathrm{dist}_Q(a_1, a_2)$ is the geodesic distance between a_1 and a_2 in Q. For each $v \in C'$ let q_v be the minimum of 1 and $\sup\{q' : B(v, q') \subset V\}$. q_v is continuous in v and $q_v > 0$ $\forall v \in C'$. By compactness of C', $q := \min_{v \in C'} q_v$ exists and is positive. Now define $L' = \phi^{-1}(C' \times (-q, 0))$, $R' = \phi^{-1}(C' \times (0, q)$.

Define $U' = L' \cup C' \cup R'$. Let $f : [0,1] \to Q$ be a curve with $f(x) \notin C'$ for any $x \in [0,1)$. By continuity of f there is some $\beta \in (0,1)$ for which $f((\beta, 1]) \subset U'$. We claim $f((\beta, 1)) \subset L'$ or $f((\beta, 1]) \subset R'$. If $\phi(f((\beta, 1)))$ contains a point in $C' \times (-\infty, 0)$ and a point in $\phi^{-1}(C' \times (0, \infty)$ then by continuity $\phi(f((\beta, 1)))$ contains a point in $C' \times \{0\}$ and hence $f((\beta, 1))$ contains a point in C' which is a contradiction. Thus, either $\phi(f((\beta, 1))) \subset C' \times (-\infty, 0)$ or $\phi(f((\beta, 1))) \subset C' \times (0, \infty)$. If $\phi(f((\beta, 1))) \subset C' \times (-\infty, 0)$, then $f((\beta, 1)) \subset L'$. If $\phi(f((\beta, 1))) \subset C' \times (0, \infty)$, then $f((\beta, 1)) \subset R'$.

For each f_i there exists β_i for which $f((\beta_i, 1]) \in L'$ or $f((\beta_i, 1]) \in R'$. For each $x \in C'$ define r'_x to be the supremum of all radius r''_x for which the ball $B(x, r''_x)$ of radius r''_x is entirely contained in U' and for which $B(x, r''_x)$ is disjoint from $f_1([0, \beta_1)), f_2([0, \beta_s)), f_3([0, \beta_3)), ..f_{l'}([0, \beta_l)),\ h_1([0,1]), h_2([0,1]), .., h_{t'}([0,1])$. If the supremum does not exist, set $r'_x = \infty$. Define $r_x = \min\{1, r'_x\}$. Again $r_x > 0$ for all $x \in C'$ and is continuous in x. Since C' is a compact set, $r := \min_{x \in C'} r_x$ exists and is positive. Note each curve $f_i([0,1))$ is disjoint from at least one of $L' \cap \{B(x, r) : x \in C\}, R' \cap \{B(x, r) : x \in C\}$ and each curve $h_i([0,1])$ is disjoint from both $L' \cap \{B(x, r) : x \in C\}, R' \cap \{B(x, r) : x \in C\}$.

Let us show that by making r smaller if necessary $B(v, r) \backslash C'$ contains two connected components.

▶ **Proposition 28.** *Given a (piece-wise smooth non-self-intersecting) curve* $f : [0,1] \to \mathbb{R}^2$ *with* $t_0 \in (0,1)$ *there exists* $r > 0$ *for which* $B(f(t_0), r) \backslash f([0,1])$ *contains exactly two components.*

Proof. Let $[t_0, t_1]$ be an interval in which f is smooth.

Let $f(t) = f(t_0) + (t - t_0)\nabla f(t_0) + g(t - t_0)$. By smoothness of f, $\nabla g(t - t_0)$ is bounded for $t \in [t_0, t_1]$ and $g(t - t_0) = o(t - t_0)$. Differentiating $\|f(t) - f(t_0)\|^2 = \|(t - t_0)\nabla f(t_0) + g(t - t_0)\|^2$ we obtain

$$\begin{aligned}
\frac{d}{dt}\|f(t) - f(t_0)\|^2 &= 2(\nabla f(t_0) + \nabla g(t - t_0))^t((t - t_0)\nabla f(t_0) + g(t - t_0)) \\
&= 2(\nabla f(t_0) + o(1))^t((t - t_0)\nabla f(t_0) + o(t - t_0)) \\
&= 2(\nabla f(t_0) + o(1))^t(t - t_0)\nabla f(t_0) + o(t - t_0)
\end{aligned}$$

For t close enough to t_0 the last line is positive. This implies that for some $t_2 > t_0$, $\|f(t) - f(t_0)\|^2$ is increasing on $[t_0, t_2]$. Likewise, for some $t_3 < t_0$, $\|f(t) - f(t_0)\|^2$ is decreasing on $[t_3, t_0]$.

Let $r > 0$ be such that $\|f(t_0) - f(t)\| \geq 2r$ for all $t \in [0,1] \setminus \mathrm{frac}([t_3, t_2])$, where $\mathrm{frac}(x)$ is the fractional part of x. Then $\|f(t_0) - f(t_2)\|, \|f(t_0) - f(t_3)\| \geq 2r$. Then since $\|f(t_0) - f(t)\|$ is increasing on $[t_0, t_2]$ there is exactly one $t_4 \in [t_0, t_2]$ with $\|f(t_0) - f(t_4)\| = r$. Likewise, there is exactly one $t_5 \in [t_3, t_0]$ with $\|f(t_0) - f(t_5)\| = r$. So $f([t_5, t_4])$ forms a simple curve in the closed ball $\bar{B}(f(t_0), r)$ with endpoints on the boundary and $f((t_5, t_4))$ lying in the interior. It follows from the Jordan curve theorem that $B(f(t_0), r) \setminus f([0,1]) = B(f(t_0), r) \setminus f((t_5, t_4))$ contains exactly two connected components. ◀

Let $v \in C'$. For small enough r_0, $B(v, r_0)$ is diffeomorphic to the open disk $B(0, r_0)$ in $\mathbb{R}^2$ via some diffeomorphism ψ with $\psi(v) = (0,0)$. Let w be a parameterization of C' with $w(0.5) = v$. Let $t_1 := \inf\{t : \ w([t, 0.5]) \subset B(v, r_0)\}$ and $t_2 := \sup\{t : \ w([0.5, t]) \subset B(v, r_0)\}$ that is $[t_1, t_2]$ is a maximal interval for which $w([t_1, t_2]) \subset B(v, r_0)$ by continuity $t_1 < 0.5 < t_2$. Choose $r_1 > 0$ less than the distance from v to $C' \setminus w((t_1, t_2))$ and $r_1 < \mathrm{dist}_Q(v, w(t_1)), \mathrm{dist}_Q(v, w(t_2))$. From the previous proposition there exists $r_2 > 0$ such that $B_{\mathbb{R}^2}(\psi(v), r_2) \setminus \psi(w([t_1, t_2]))$ contains exactly two connected components. By making r smaller than r_1 and r_2 if necessary we get that for any $0 < \hat{r} \leq r$, $B_{\mathbb{R}^2}(\psi(v), \hat{r}) \setminus \psi(C')$ contains exactly two connected components. Thus, $B(v, \hat{r}) \setminus C'$ contains exactly two connected components.

Define $L = L' \cap \{B(x, r) : \ x \in C\}$, $R = R' \cap \{B(x, r) : \ x \in C\}$. Each curve $f : [0,1] \to Q$ with $f(x) \notin C'$ for any $x \in [0, 1)$ satisfies $f((\beta, 1)) \subset L$ or $f((\beta, 1]) \subset R$. Each curve $f_i([0,1))$ is disjoint from at least one of L, R and each curve $h_i([0,1])$ is disjoint from both L, R.

Since $\phi(v) = \{v\} \times \{0\}$, $\phi(B(v, \hat{r}))$ intersects both $\{v\} \times (-\infty, 0)$ and $\{v\} \times (0, \infty)$. Recall $B(v, \hat{r}) \subset L \cup R \cup C'$, so $B(v, \hat{r})$ intersects both L and R. Since there is no path from L to R in $L \cup R \cup C'$ one component of $B(v, \hat{r}) \setminus C'$ is contained in L and the other is contained in R.

For each $v \in C'$ let y_v be the supremum of $\{y'_v \geq 0 : \ \{v\} \times (-y'_v, y'_v) \subset \phi(B(v, r))\}$. Again y_v is positive and continuous in v so $y := \min_{v \in C'} y_v$ exists and is positive. Then $C' \times (-y, 0) \subset \phi(L)$ and $C' \times (0, y) \subset \phi(R)$. Define the curves f_L, f_R to be parameterizations of $\phi^{-1}(C' \times \{\frac{-y}{2}\})$ and $\phi^{-1}(C' \times \{\frac{y}{2}\})$ respectively.

Lastly given a curve $h : [0,1] \to Q$ be any curve that reaches C' from the right at a point $c_2 = h(1)$ on C', leaves C' from the left at $c_1 = h(0)$ and C'_{c_1, c_2} be a subcurve of C' with endpoints c_1 and c_2. Let $j : [0,1] \to C'_{c_1, c_2}$ be a parameterization of C'_{c_1, c_2} and denote $\bar{j} : [0, 2] \to Q$ by $\bar{j}(t) = h(t)$ for $t \in [0,1]$ and $\bar{j}(t) = j(t-1)$ for $t > 1$. Informally speaking, we "slightly shift" all points in $L \cup C' \cup R$ to the right while keeping $h([0,1]) \cap L$ to the left of C'. Let $\phi(x) = (\phi_1(x), \phi_2(x))$, $\phi^{-1}(x) = (\phi_1^{-1}(x), \phi_2^{-1}(x))$.

Let $\gamma : C' \to (-y, 0)$ be any continuous function such that $\gamma(c_2) = 0$ and for any $t \in [0,1]$ for which $\phi(h(t)) \in C' \times (-y, 0)$, $(\phi(h(t))_2) < \gamma(\phi(h(t))_1) < 0$. Informally γ is a curve lying to the right of $\phi(h([0,1])) \cap C' \times (-y, 0)$ and to the left of $[0,1] \times \{0\}$. Such γ exists for instance define $-\gamma(t)$ to be half of the minimum of the distance from the point $(t, 0)$ to $\phi(h([0,1])) \cap C' \times (-y, 0)$ and y.

Define $\bar{\gamma} : C' \times (-y, y) \to C' \times (-y, y)$ as follows. For $(a, b) \in C' \times (-y, y)$ if $b < \frac{\gamma(a)}{4}$ define $\bar{\gamma}(a, b) = (a, b)$. If $\frac{\gamma(a)}{4} \leq b < 0$, define $\bar{\gamma}(a, b) = (a, \frac{\gamma(a)}{4} + 2(b - \frac{\gamma(a)}{4}))$. If $0 \leq b \leq \frac{-\gamma(a)}{2}$, define $\bar{\gamma}(a, b) = (a, \frac{-\gamma(a)}{4} + \frac{b}{2})$. If $b \geq \frac{-\gamma(a)}{2}$ $\bar{\gamma}(a, b) = (a, b)$. Informally, $\bar{\gamma}$ shifts $C' \times (-y, y)$ to the right while keeping $\phi(h([0,1])) \cap C' \times (-y, 0)$ left of $C' \times \{0\}$.

Define $\hat{\gamma} : Q \to Q$ by γ by $\hat{\gamma}(v) = v$ if $v \notin \phi^{-1}(C' \times (-y, y))$ and $\hat{\gamma}(v) = \phi^{-1}\bar{\gamma}(\phi(v))$ if $v \in \phi^{-1}(C' \times (-y, y))$. Note $\hat{\gamma}$ is a homeomorphism from $\phi^{-1}(C' \times (-y, y))$ to $\phi^{-1}(C' \times (-y, y))$ and from $Q \setminus \phi^{-1}(C' \times (-y, y))$ to $Q \setminus \phi^{-1}(C' \times (-y, y))$. Further, $\hat{\gamma}$ agrees on the boundary of $\phi^{-1}(C' \times (-y, y))$ and $Q \setminus \phi^{-1}(C' \times (-y, y))$, that is for sequence $\{a_i\}_{i=1}^{\infty}$ converging to a boundary point a of $\phi^{-1}(C' \times (-y, y))$ (resp $Q \setminus \phi^{-1}(C' \times (-y, y))$) the sequence $\{\hat{\gamma}(a_i)\}_{i=1}^{\infty}$ converges to $\hat{\gamma}(a)$. Thus, $\hat{\gamma}$ is a homeomorphism on Q, in fact, it turns out to be a continuous deformation.

Define $\hat{h} : [0,2] \to Q$ by $\hat{h}(t) = \hat{\gamma}(\bar{j}(t))$. Then $\hat{\gamma}$ is a homeomorphism mapping $\hat{h}([0,2])$ to $\bar{j}([0,2])$. ◀

The actual statement of the tubular neighbourhood theorem involves first defining the normal fibre N_x as the quotient T_xQ/T_xC where T_xQ and T_xC are the tangent plane and tangent to the curve C at x and the normal bundle NX as $\{(x,v) : \ x \in C \ v \in N_x\}$.

▶ **Theorem 29** (tubular neighbourhood theorem)**.** *There are open sets U in Q containing C and V in NX such that there is a diffeomorphism $\gamma : U \to V$.*

Let us quickly show how the version of the tubular neighbourhood theorem in Proposition 27 follows from the tubular neighbourhood theorem.

By orientability each point $x \in Q$ has a normal vector $n(x)$ and n is continuous. We may parameterize C' with a function $\psi : [0,\beta] \to C'$ with derivative $\psi'(x) = 1$ for some β. Define $v(x)$ to be of unit norm and positively orthogonal to $n(x), \phi'(x)$ that is $n(x)^t v(x) = (\phi'(x))^t v(x) = 0$ and $\begin{bmatrix} n(x) & \psi'(x) & v(x) \end{bmatrix}$ has determinant 1. By the inverse function theorem, $v(x)$ is continuous. $n(x), \psi'(x), v(x)$ is a basis for $\mathbb{R}^3$ known as the Darboux frame. Since $v(x)$ is orthogonal to $n(x)$ it lies in the tangent plant T_xQ since $v(x)$ is orthogonal to $\phi'(x)$, $v(x), \phi'(x)$ is a basis for T_xQ. Thus, $N_x = T_xQ/T_xC$ is diffeomorphic to $\{av(x) : \ a \in \mathbb{R}\}$ which is diffeomorphic to $\mathbb{R}$. Thus, NX is diffeomorphic to $C' \times \mathbb{R}$.

To prove Proposition 26, we first prove the following special case.

▶ **Proposition 30.** *Let Q be an orientable surface and $\phi : [0,1] \to Q$ a closed curve not dividing Q into 2 regions with disjoint subsets L, R "on each side" of ϕ as in Proposition 23. Let $c \in [0,1)$, with $c_2 \geq c_1$. Suppose that $\phi_1 : [0,1] \to Q$ is a curve with $\phi_1(0) = \phi(c)$ $\phi_1(1) = \phi(c)$, $\phi((0,1))$ is disjoint from $\phi([0,1])$ and the curve $\phi_1([0,0.5])$ approaches $\phi([0,1])$ from the left L and $\phi_1([0.5,1])$ approaches $\phi([0,1])$ from the right R.*

Then ϕ_1 does not divide Q into 2 regions.

Proof. Suppose for a contradiction that ϕ_1 divides Q into 2 regions. It's a well-known result that one of the regions Q_1 must be homeomorphic to an open disk. Let Q_2 be the other region.

There exists a small radius r_1 for which the ball $B(\phi(c), r)$ of radius r_1 about $\phi(c)$ such that $B(\phi(c), r_1)$ is homeomorphic to an open disk. Let r be as in Proposition 23 and define $r_2 = \min\{r, r_1\}$.

▶ **Definition 31.** *Given 2 curves $f_1, f_2 : [0,1] \to Q$ on a surface Q, with $f_1(0.5) = f_2(0.5)$ and $f_1(x) \neq f_2(y)$ $\forall x, y \in ([0,1]\backslash\{0.5\})$, that is they intersect only at $f_1(0,5)$, we say that f_1* crosses *f_2 at $f_1(0.5)$ if there exists $r_0 > 0$ such that for all $r \leq r_0$ such that f_2 intersects both regions of $B(f_1(0.5), r)\backslash f_1([0,1])$, where $B(p,r)$ is the open ball around p of radius r.*

▶ **Lemma 32.** *For two curves f_1, f_2 on a surface Q and p a point on both curves, f_1 crosses f_2 at p if and only if f_2 crosses f_1 at p.*

Proof. Let $f_1(b_1) = p = f_2(b_2)$. Let L_1, R_1 and r_1 (resp L_2, R_2 and r_2) be the left, right and radius respectively for f_1 (resp. f_2) as guaranteed by Proposition 23. Define $r = \min\{r_1, r_2\}$.

▷ Claim 33. f_1 crosses f_2 at p if and only if for all $r \geq r_0 > 0$ none of $L_1 \cap B(p, r_0)$, $R_1 \cap B(p, r_0)$, $L_2 \cap B(p, r_0)$, $R_2 \cap B(p, r_0)$ is contained in another.

Proof. Suppose that f_1 crosses f_2. Let $r \geq r_0 > 0$. Let $t_L, t_R \in \mathbb{R}$ be such that $f_2(t_L) \in L_1 \cap B(p, r_0)$, $f_2(t_R) \in R_1 \cap B(p, r_0)$. Since $R_1 \cap B(p, r_0), L_1 \cap B(p, r_0)$ are open, there exists r_L, r_R be such that $B(f_2(t_L), r_L) \subset L_1 \cap B(p, r_0)$ and $B(f_2(t_R), r_R) \subset R_1 \cap B(p, r_0)$. Since $B(f_2(t_L), r_L) \cap L_2, B(f_2(t_L), r_L) \cap R_2, B(f_2(t_R), r_R) \cap L_2, B(f_2(t_R), r_R) \cap R_2 \neq \emptyset$, none of $L_1 \cap B(p, r_0)$, $R_1 \cap B(p, r_0)$, $L_2 \cap B(p, r_0)$, $R_2 \cap B(p, r_0)$ is contained in another.

Conversely, suppose that for any $0 < r_0 \leq r$ none of $L_1 \cap B(p, r_0)$, $R_1 \cap B(p, r_0)$, $L_2 \cap B(p, r_0)$, $R_2 \cap B(p, r_0)$ is contained in another. Then for any $0 < r_0 \leq r$, if f_2 does not intersect $L_1 \cap B(p, r_0)$, then $L_1 \cap B(p, r_0)$ is connected in $B(p, r_0) \backslash f_2$, that is $L_1 \cap B(p, r_0)$ is contained in one of the two components $L_2 \cap B(p, r_0)$, $R_2 \cap B(p, r_0)$ of $B(p, r_0) \backslash f_2$. Hence f_2 intersects $L_1 \cap B(p, r_0)$, likewise f_2 intersects $R_1 \cap B(p, r_0)$. Hence f_2 crosses f_1. ◁

From the previous claim it's clear that crossing is a symmetric relation. ◀

Define $\phi_2(x) = \phi(\text{frac}(x - 0.5 + c))$, where $\text{frac}(x) = x - \lfloor x \rfloor$ is the fractional value of x and $\phi_3(x) = \phi_1(\text{frac}(x - 0.5))$, that is, ϕ_2, ϕ_3 are reparameterized versions of ϕ and ϕ_1. For some $0 < \beta_1 < \beta_2 < 1$ $\phi_1(x) \in L$ for $x \in (0, \beta_1)$ and $\phi_1(x) \in R$ for $x \in (\beta_2, 1)$.

Define $\beta_1' = \min\{\beta_1, 0.5\} + 0.5$ $\beta_2' = \max\{\beta_2, 0.5\}$. Then $\phi_3((0.5, \beta_1')) \subset L$ and $\phi_3((\beta_2', 0.5)) \subset R$. Since $L \cup R$ covers $\bar{B}(\phi(c), r) \backslash \phi([0, 1])$, this implies that ϕ_3 crosses ϕ_2.

By Lemma 32 ϕ_2 crosses ϕ_3. Let L_{ϕ_3}, R_{ϕ_3} be the left and right of ϕ_3 as in Proposition 23. Since L_{ϕ_3}, R_{ϕ_3} are connected, L_{ϕ_3}, R_{ϕ_3} belong to different regions of $Q \backslash \phi_3$. Since ϕ_2 crosses ϕ_3, there exists t_0 for which $\phi(t_0) \in Q_1$ and t_1 for which $\phi(t_1) \in Q_2$. Let $t_2 = \inf\{a \in [0, 1] : \exists b \in (a, 1] \text{ s.t. } \phi((a, b)) \subset Q_1\}$, $t_3 = \sup\{b \in [t_2, 1] : \text{ s.t.} \phi((t_2, b)) \subset Q_1\}$, that is $[t_2, t_3]$ is a maximal interval for which $\phi([t_2, t_3]) \subset Q_1$. It follows $\phi(t_2), \phi(t_3)$ lie on the boundary of Q_1, that is on ϕ_1. This implies $t_2, t_3 \in \{0, 1\}$. Since $t_2 < t_3$ $t_2 = 0$ and $t_3 = 1$. This implies that $\phi([0, 1]))$ lies in $Q_1 \cup \{\phi(0)\}$ contradicting that there exists t_1 for which $\phi(t_1) \in Q_2$. ◀

Proof. (of Proposition 26) Let $f : [0, 1] \to C'$ be a parameterization of C' and let $c = f^{-1}(c_1)$. Note that by Proposition 25 the curve ϕ_2 is homeomorphic to a curve ϕ_3 that enters ϕ from the right and leaves ϕ from the left at the same point $f(c)$. By Proposition 30, ϕ_3 does not divide Q into 2 regions. Thus, ϕ_2 does not divide Q into 2 regions. ◀

References

1 M.A. Armstrong. *Basic Topology*. Undergraduate Texts in Mathematics. Springer New York, 2013. URL: `https://books.google.ca/books?id=NJbuBwAAQBAJ`.

2 Vineet Bafna, Piotr Berman, and Toshihiro Fujito. A 2-approximation algorithm for the undirected feedback vertex set problem. *SIAM J. Discrete Math.*, 12:289–297, 1999.

3 Piotr Berman and Grigory Yaroslavtsev. Primal-dual approximation algorithms for node-weighted network design in planar graphs. In Anupam Gupta, Klaus Jansen, José Rolim, and Rocco Servedio, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, pages 50–60, Berlin, Heidelberg, 2012.

4 A.C. da Silva and LINK (Online service). *Lectures on Symplectic Geometry*. Number no. 1764 in Lecture Notes in Mathematics. Springer, 2001. URL: `https://books.google.ca/books?id=r9i6pXc7GEQC`.

5 E. D. Demaine and M. Hajiaghayi. The bidimensionality theory and its algorithmic applications. *The Computer Journal*, 51(3):292–302, 2008.

6 Reinhard Diestel. *Graph Theory*. Springer Publishing Company, Incorporated, 5th edition, 2017.

7 G. Even, J. (Seffi) Naor, B. Schieber, and M. Sudan. Approximating minimum feedback sets and multicuts in directed graphs. *Algorithmica*, 20(2):151–174, 1998.

8 Fedor V. Fomin, Daniel Lokshtanov, Venkatesh Raman, and Saket Saurabh. Bidimensionality and eptas. In *Proceedings of the Twenty-second Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '11, pages 748–759, Philadelphia, PA, USA, 2011. Society for Industrial and Applied Mathematics. URL: http://dl.acm.org/citation.cfm?id=2133036.2133095.

9 Fedor V. Fomin, Daniel Lokshtanov, Saket Saurabh, and Dimitrios M. Thilikos. Bidimensionality and kernels. *CoRR*, 2016.

10 Georges Gardarin and Stefano Spaccapietra. Integrity of data bases: A general lockout algorithm with deadlock avoidance. In G. M. Nijssen, editor, *Modelling in Data Base Management Systems, Proceeding of the IFIP Working Conference on Modelling in Data Base Management Systems, Freudenstadt, Germany, January 5-8, 1976*, pages 395–412. North-Holland, 1976.

11 Michel X. Goemans and David P. Williamson. *The Primal-Dual Method for Approximation Algorithms and Its Application to Network Design Problems*, pages 144–191. PWS Publishing Co., USA, 1996.

12 Michel X. Goemans and David P. Williamson. Primal-dual approximation algorithms for feedback problems in planar graphs. *Combinatorica*, 18(1):37–59, 1998.

13 Venkatesan Guruswami and Euiwoong Lee. Simple proof of hardness of feedback vertex set. *Theory of Computing*, 12(6):1–11, 2016. doi:10.4086/toc.2016.v012a006.

14 Allen Hatcher. Notes on basic 3-manifold topology, 2007. URL: https://pi.math.cornell.edu/~hatcher/3M/3M.pdf.

15 The Amplitwist (https://mathoverflow.net/users/183188/the amplitwist). If a graph embedded on a surface is divided by a curve into a right and left that do not intersect can it be embedded on a surface of smaller genus? MathOverflow. (version: 2023-11-19). URL: https://mathoverflow.net/q/458733.

16 Richard M. Karp. *Reducibility among Combinatorial Problems*, pages 85–103. Springer US, Boston, MA, 1972. doi:10.1007/978-1-4684-2001-2_9.

17 L.C. Kinsey. *Topology of Surfaces*. Undergraduate Texts in Mathematics. Springer New York, 1997. URL: https://books.google.gl/books?id=AKghdMm5W-YC.

18 Lap Chi Lau, R. Ravi, and Mohit Singh. *Iterative Methods in Combinatorial Optimization*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2011. doi:10.1017/CBO9780511977152.

19 Charles E. Leiserson and James B. Saxe. Retiming synchronous circuitry. *Algorithmica*, 6(1):5–35, 1991. doi:10.1007/BF01759032.

20 Orna Lichtenstein and Amir Pnueli. Checking that finite state concurrent programs satisfy their linear specification. In *Proceedings of the 12th ACM SIGACT-SIGPLAN Symposium on Principles of Programming Languages*, POPL '85, pages 97–107, New York, NY, USA, 1985. Association for Computing Machinery. doi:10.1145/318593.318622.

21 Daniel Lokshtanov, Pranabendu Misra, Joydeep Mukherjee, Fahad Panolan, Geevarghese Philip, and Saket Saurabh. 2-approximating feedback vertex set in tournaments. *ACM Trans. Algorithms*, 17(2), April 2021. doi:10.1145/3446969.

22 J. Rydholm. Classification of compact orientable surfaces using morse theory, 2016.

23 Niklas Schlomberg, Hanjo Thiele, and Jens Vygen. Packing cycles in planar and bounded-genus graphs, 2023. arXiv:2207.00450.

24 Anke van Zuylen. Linear programming based approximation algorithms for feedback set problems in bipartite tournaments. In *TAMC*, 2009.

25 J. W. T. Youngs. Minimal imbeddings and the genus of a graph. *J. Math. Mech.*, 12:303–315, 1963.