# Expanderizing Higher Order Random Walks

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## Abstract

We study a variant of the down-up (also known as the Glauber dynamics) and up-down walks over an $n$-partite simplicial complex, which we call *expanderized higher order random walks* – where the sequence of updated coordinates correspond to the sequence of vertices visited by a random walk over an auxiliary expander graph $H$. When $H$ is the clique with self loops on $[n]$, this random walk reduces to the usual down-up walk and when $H$ is the directed cycle on $[n]$, this random walk reduces to the well-known systematic scan Glauber dynamics. We show that whenever the usual higher order random walks satisfy a log-Sobolev inequality or a Poincaré inequality, the expanderized walks satisfy the same inequalities with a loss of quality related to the two-sided expansion of the auxillary graph $H$. Our construction can be thought as a higher order random walk generalization of the derandomized squaring algorithm of Rozenman and Vadhan (RANDOM 2005).

We study the mixing times of our expanderized walks in two example cases: We show that when initiated with an expander graph our expanderized random walks have mixing time (i) $O(n \log n)$ for sampling a uniformly random list colorings of a graph $G$ of maximum degree $\Delta = O(1)$ where each vertex has at least $(11/6 - \varepsilon)\Delta$ and at most $O(\Delta)$ colors, (ii) $O_{\boldsymbol{h}}\left(\frac{n \log n}{(1-\|\mathsf{J}\|_{\mathsf{op}})^2}\right)$ for sampling the Ising model with a PSD interaction matrix $\mathsf{J} \in \mathbb{R}^{n\times n}$ satisfying $\|\mathsf{J}\|_{\mathsf{op}} \le 1$ and the external field $\boldsymbol{h} \in \mathbb{R}^n$– here the $O(\bullet)$ notation hides a constant that depends linearly on the largest entry of $\boldsymbol{h}$. As expander graphs can be very sparse, this decreases the amount of randomness required to simulate the down-up walks by a logarithmic factor.

We also prove some simple results which enable us to argue about log-Sobolev constants of higher order random walks and provide a simple and self-contained analysis of local-to-global $\Phi$-entropy contraction in simplicial complexes – giving simpler proofs for many pre-existing results.

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# 1 Introduction

Let $U_1, \cdots, U_n$ be a collection of finite sets. The *down-up walk* $\mathsf{P}^{\downarrow\uparrow}$ on $\Omega \subset U_1 \times \cdots U_n$ with respect to a given distribution $\pi : \Omega \to \mathbb{R}_{\geq 0}$, also known as the *Glauber dynamics* on $\Omega$ according to $\pi$, is the following simple process: Starting from an arbitrary tuple $\omega^{(0)}$, we obtain the $(t+1)$-st tuple $\omega^{(t+1)}$ visited by this random walk from the $t$-th tuple $\omega^{(t)}$ as follows,

**Update Rule for the Down-Up Walk, $\mathsf{P}^{\downarrow\uparrow}$**

1. sample a uniformly random coordinate $i \sim \mathsf{uni}_{[n]}$,
2. sample a random tuple $\omega^{(t+1)} \sim \pi$ conditional on $\omega_j^{(t+1)} = \omega_j^{(t)}$ for all $j \in [n] \setminus \{i\}$.

The following variant of the down-up walk, called the *systematic scan* $\mathsf{P}_{\mathtt{scan}}$ on $\Omega$ according to $\pi$, is a variant of the down-up walk $\mathsf{P}^{\downarrow\uparrow}$ which uses less randomness and is easier to implement in practice: starting from an arbitrary tuple $\omega^{(0)}$, we obtain the $(t+1)$-st tuple $\omega^{(t+1)}$ visited by this random walk from the $t$-th tuple $\omega^{(t)}$ as follows,

**Update Rule for the Systematic Scan, $\mathsf{P}_{\mathtt{scan}}$**

1. set $i = t + 1 \pmod{n}$,
2. sample a random tuple $\omega^{(t+1)} \sim \pi$ conditional on $\omega_j^{(t+1)} = \omega_j^{(t)}$ for all $j \in [n] \setminus \{i\}$.

In both cases, the coordinate $i$ that is sampled on the first step of the update can be thought as a vertex visited by the simple random walk on a graph. For the down-up walk, this is a random walk on the clique with self-loops, whereas for the systematic scan this is a (deterministic) walk on the directed cycle.

The main object of study in this paper will be the so-called *expanderized down-up walk* $\mathsf{Q}^{\downarrow\uparrow}$ on $\Omega$ with respect to the distribution $\pi : \Omega \to \mathbb{R}_{>0}$ and the $k$-regular graph $H = ([n], E)$ for some constant $k$. Starting this random-walk from an arbitrary coordinate $i^{(0)} \in [n]$ and an arbitrary tuple $\omega^{(0)}$, we obtain the $(t+1)$-st coordinate $i^{(t+1)}$ and tuple $\omega^{(t+1)}$ according to the following update rule,

**Update Rule for the Expanderized Down-Up Walk $\mathsf{Q}^{\downarrow\uparrow}$**

1. sample a random neighbor $s$ of $i^{(t)}$ in $H$,
2. sample a random tuple $\omega^{(t+1)} \sim \pi$ conditional on $\omega_j^{(t+1)} = \omega_j^{(t)}$ for all $j \in [n] \setminus \{s\}$,
3. set $i^{(t+1)}$ to be a random neighbor of $s$ in $H$.

We notice that according to the above update rule when $i^{(0)}$ is sampled uniformly at random and $H$ equals the clique with self-loops on $[n]$ the evolution of $\omega^{(t)}$ is as dictated by the down-up walk $\mathsf{P}^{\downarrow\uparrow}$. Similarly, when $i^{(0)} = 1$ and $H$ is the directed cycle,[1] the evolution of $\omega^{(t)}$ is as dictated by the systematic scan $\mathsf{P}_{\mathtt{scan}}$.

[1] To be more precise, $H$ needs to be a directed cycle of length $2n$, in which the vertices corresponding to the coordinates and *dummy* vertices are interleaved.

The main contribution of this paper is an analysis of the expanderized down-up walk assuming, (i) the graph $H$ is a spectral expander[2] and (ii) the down-up walk $\mathsf{P}^{\downarrow\uparrow}$ satisfies some kind of isoperimetric inequality, e.g. a log-Sobolev inequality or a Poincaré inequality. Indeed our methods allow us to extend our results to all down-up and up-down walks.

**Motivation and Contributions.** The systematic scan $\mathsf{P}_{\mathtt{scan}}$ is a random walk of great practical and theoretical interest. Yet, rapid mixing results for this walk are only known under restricted circumstances [18, 33, 23, 24, 51, 29] and it is very hard to directly relate the rapid mixing of $\mathsf{P}^{\downarrow\uparrow}$ to that of $\mathsf{P}_{\mathtt{scan}}$. A particularly useful framework for establishing rapid mixing for the down-up walk is the method of high-dimensional expansion, in particular the frameworks of spectral independence and entropic independence [2, 9, 16, 17, 14, 7, 6, 8] which led to many breakthrough results in the field of sampling algorithms.

In [3], an attempt was made to study the mixing of the systematic scan[3] using techniques of high-dimensional expansion – while their techniques allowed them to establish rapid mixing results for constant dimensional partite simplicial complexes, their result is too restrictive to take advantage of mixing results obtained through spectral independence or entropic independence. As a step towards directly being able to take advantage of the mixing results for $\mathsf{P}^{\downarrow\uparrow}$, which could potentially be obtained through the high-dimensional expansion framework, we introduce our expanderized down-up walks $\mathsf{Q}^{\downarrow\uparrow}$. As expander graphs have proven themselves very successful at approximating dense objects, we hope – and indeed also prove – that transfering mixing time bounds from the usual down-up walks to our expanderized walks to be an easier task than establishing mixing times for $\mathsf{P}_{\mathtt{scan}}$. As expander graphs can be very sparse, our expanderized walks can be thought as replacing the sparse object used in the definition of the systematic scan $\mathsf{P}_{\mathtt{scan}}$, i.e. the directed cycle, with another sparse yet highly connected object – an expander graph with constant degree.

In spirit, the expanderized walks can be thought as a higher order random walk analogue of the derandomized squaring algorithm introduced in [52]. This algorithm was introduced to simplify the seminal result of [49] concerning the existence of a logspace algorithm for deciding undirected connectivity. The derandomized squaring operation uses an auxiliary $k$-regular expander graph $H$ on the vertex set $[d]$ to approximate the square of a graph $d$-regular graph $G$ on $[n]$. Whereas the actual square $G^2$ is a $d^2$-regular graph, by picking $k = O(1)$ one can ensure that the *derandomized square* is $O(d)$-regular, i.e. a much sparser object. This result rests on the observation that the actual square $G^2$ is obtained from the graph $G$ by attaching a clique to every vertex – replacing this clique with an expander graph suffices to ensure that the resulting *derandomized square* is closed to the actual square. Fortunately, the same intuition also leads to proofs showing that the expanderized walks approximate the standard walks well.

We show that we can use our expanderized walks to have more randomness efficient versions of several Markov chains of interest for sampling list colorings [10, 41] and for sampling from Ising Models [27, 7, 40], under assumptions ensuring bounded marginals. In these settings, our expanderized walks have the same asymptotic mixing time but use fewer random bits. To help us with our goals we also prove some simple estimates for the log-Sobolev constants of higher order random walks and provide a self-contained analysis of local-to-global $\Phi$-entropy contraction.

[2] i.e. all non-trivial eigenvalues of $H$ are bounded away from 1
[3] More formally, $n$ successive steps of the systematic scan, which the authors call the *sequential sweep* $\mathsf{P}_{\mathtt{seq}}$.

**Related Work.** High dimensional expansion has proven itself to be a very successful research program for establishing mixing times for down-up walks. For example [36, 21, 37, 20, 2] use spectral local-to-global arguments for establishing spectral gap bounds for these walks. In conjunction with the spectral independence framework, due to [9, 15, 30], these results paved the way for many new in the field of random sampling: rapid mixing of the down-up walk for the hardcore model in the uniqueness regime [9], rapid mixing of the down-up walk for sampling graph colorings in correlation decay regime [30, 15], optimal mixing for many Markov chains of interest[17, 10, 41]. For more information regarding spectral independence, we refer the reader to the excellent survey [54] and the dissertation [42]. In [6, 8, 17, 32] local-to-global strategies for establishing entropic contraction bounds was studied. In [14] a connection between these local-to-global methods and the stochastic localization framework of [25] was explored. We refer to the works [39, 26, 13, 38, 28, 27] and references therein for applications of the stochastic localization framework. Our inductive strategy for establishing $\Phi$-entropy contraction on simplicial complexes is heavily inspired by the presentation in [14]. In [40] mixing estimates about the walk $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow n-1}$ is used to obtain estimates for $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow \ell}$ for all $\ell < n-1$. The key intuition behind this work is the observation that the down move of the down-up walk is (passively) utilizing an expander, the down-move of the down-up walk of the so-called *Bernoulli-Laplace model*, and that one can use the expansion of this walk to show that once $\ell$ decreases the mixing times estimates get better and better. Morally, this is very similar to our idea of picking the replacement-indices for our expanderized walks via an expander walk as opposed to sampling them uniformly at random. For other classical techniques which can be used to bound mixing times of Markov chains, we refer the reader to the texts [1, 46, 55].

In contrast with down-up walks, results establishing rapid mixing for the random walk $\mathsf{P}_{\texttt{scan}}$ are fewer [18, 33, 23, 51] and mostly rely on estimates on the Dobrushin matrix [22]. [3] studied the mixing time of this random walk using techniques of high dimensional expansion, however their techniques fell short of establishing mixing time bounds under the assumption of spectral independence.

The work of [29] is also related to our work in spirit. In this work, the authors show that under suitable assumptions a wide array of random walks, including the single site systematic scan $\mathsf{P}_{\texttt{scan}}$ and the down-up walk $\mathsf{P}^{\downarrow\uparrow}$, can be derandomized, i.e. they devise efficient deterministic counting algorithms on the basis of rapid mixing results for these chains. It is an interesting question whether one can carefully pick the expander graph $H$, to make this derandomization task more efficient.

As mentioned above our expanderized random walks are heavily inspired by the derandomized squaring algorithm of [52]. This algorithm was initially used to give an alternative and simpler proof of the seminal result of [49] concerning the derandomization of the complexity class **SL** and establishing $\mathbf{SL} = \mathbf{L}$. Concretely, both [49] and the subsequent work of [52] show the existence of a deterministic logspace algorithm deciding undirected graph connectivity. Since then, the derandomized squaring algorithm has also found other uses in derandomization, e.g. [47, 48]. We conclude by noting that the inital algorithm of [49] was based on the zigzag product construction [50], which has also inspired research in the field of high dimensional expansion [35]. For more information on expander graphs, we refer the reader to the excellent survey [34].

# 2 Preliminaries

## 2.1 Linear Algebra

We will denote functions and vectors by bold faces, i.e. $\boldsymbol{f} \in \mathbb{R}^V$. The indicator function of $i \in V$ will be denoted by $\mathbf{1}_i$, i.e. $\mathbf{1}_i(j) = 0$ for all $j \neq i$ and $\mathbf{1}_i(i) = 1$. For $A \subseteq V$, we will write $\mathbf{1}_A = \sum_{a \in A} \mathbf{1}_a$. We will adopt the convention of using $\pi, \nu, \mu : V \to \mathbb{R}_{\geq 0}$ for various probability distributions over $V$.

Let $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^V$ and a measure $\pi : V \to \mathbb{R}_{>0}$ be given. We will use the notations $\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\pi$ and $\|\boldsymbol{f}\|_\pi$ to denote the inner-product and the norm with respect to the distribution $\pi$, i.e.

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\pi = \mathop{\mathbb{E}}_{x \sim \pi} \boldsymbol{f}(x) \boldsymbol{g}(x) = \sum_{x \in V} \pi(x) \cdot \boldsymbol{f}(x) \boldsymbol{g}(x) \quad \text{and} \quad \|\boldsymbol{f}\|_\pi^2 = \langle \boldsymbol{f}, \boldsymbol{f} \rangle_\pi. \tag{1}$$

Given $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^n$ we will write $\langle \boldsymbol{f}, \boldsymbol{g} \rangle_{\ell_2}$ for the inner-product between $\boldsymbol{f}$ and $\boldsymbol{g}$ in the counting measure, i.e. $\langle \boldsymbol{f}, \boldsymbol{g} \rangle_{\ell_2} = \sum_{i=1}^n \boldsymbol{f}(i) \boldsymbol{g}(i)$. We will also write $\|\boldsymbol{f}\|_{\ell_1}, \|\boldsymbol{f}\|_{\ell_2}$, and $\|\boldsymbol{f}\|_{\ell_\infty}$ for the $\ell_1, \ell_2$, and $\ell_\infty$ norms of $\boldsymbol{f}$ respectively. Formally,

$$\|\boldsymbol{f}\|_{\ell_2}^2 = \sum_{i=1}^n \boldsymbol{f}(i)^2 \ ; \ \|\boldsymbol{f}\|_{\ell_1} = \sum_{i=1}^n |\boldsymbol{f}(i)| \ ; \text{and} \ ; \ \|\boldsymbol{f}\|_{\ell_\infty} = \max_{i \in [n]} |\boldsymbol{f}(i)|.$$

### Matrices and Eigenvalues

In this section, we will recall some results concerning eigenvalues and eigenvectors of matrices.

Serif faces will be used to denote matrices, i.e. $\mathsf{A}, \mathsf{B} \in \mathbb{R}^{U \times V}$. We will call a matrix $\mathsf{B} \in \mathbb{R}^{U \times V}$ row stochastic if rows of $\mathsf{B}$ sum up to 1 and $\mathsf{B}$ contains no negative entries. Formally,

$$\text{for all } u \in U, v \in V \quad \mathsf{B}(u, v) \geq 0 \quad \text{and} \quad \mathsf{B}\mathbf{1} = \mathbf{1}. \tag{row stochastic}$$

Let $\mathsf{B} \in \mathbb{R}^{U \times V}$ and distributions $\pi_U : U \to \mathbb{R}_{>0}$ and $\pi_V : V \to \mathbb{R}_{>0}$ be given. The adjoint $\mathsf{B}^*$ of $\mathsf{B}$ with respect to the measures $\pi_U$ and $\pi_V$ is the unique matrix which satisfies the following equation,

$$\langle \boldsymbol{f}, \mathsf{B}\boldsymbol{g} \rangle_{\pi_U} = \langle \mathsf{B}^* \boldsymbol{f}, \boldsymbol{g} \rangle_{\pi_V} \quad \text{for all } \boldsymbol{f} \in \mathbb{R}^U, \boldsymbol{g} \in \mathbb{R}^V. \tag{adjoint}$$

If $U = V$ and $\pi_U = \pi_V$, the operator $\mathsf{B}$ is called self-adjoint when $\mathsf{B}^* = \mathsf{B}$. If $\mathsf{B}$ is a row-stochastic matrix, we will call $\mathsf{B}^*$ the time-reversal of $\mathsf{B}$ with respect to $\pi_U, \pi_V$ and say that $\mathsf{B}$ is reversible if $\mathsf{B} = \mathsf{B}^*$. It is well known that the operator $\mathsf{B}^* \in \mathbb{R}^{V \times U}$ is uniquely determined by the choice of $\mathsf{B} \in \mathbb{R}^{U \times V}$ and the inner-products defined by $\pi_U$ and $\pi_V$ (see e.g. [53, p. 318]),

▶ **Proposition 1.** *Let $\mathsf{B} \in \mathbb{R}^{U \times V}$ be arbitrary. We write $\mathsf{B}^*$ for the adjoint operator to $\mathsf{B}$ with respect to the inner-products defined by the distributions $\pi_U$ and $\pi_V$. Then,*

$$\mathsf{B}^*(y, x) = \mathsf{B}(x, y) \cdot \frac{\pi_U(x)}{\pi_V(y)} \quad \textit{for all} \quad x \in U, y \in V.$$

We also recall the following standard fact which is an immediate consequence of Proposition 1,

▶ **Proposition 2.** *If $\mathsf{B} \in \mathbb{R}^{U,V}$ is a row-stochastic matrix satisfying $\pi_U \mathsf{B} = \pi_V$, then the adjoint matrix $\mathsf{B}^*$ with respect to $\pi_U, \pi_V$ is also row-stochastic and satisfies $\pi_V \mathsf{B}^* = \pi_U$.*

It is well known that a self-adjoint matrix $\mathsf{A} \in \mathbb{R}^{V\times V}$ has $|V|$ real eigenvalues. We will write, $\lambda_1(\mathsf{A}) \geq \lambda_2(\mathsf{A}) \geq \cdots \geq \lambda_{|V|}(\mathsf{A}) := \lambda_{\min}(\mathsf{A})$ for the sequence of eigenvalues of $\mathsf{A}$ sorted in decreasing order. We say that the matrix is positive semi-definite, henceforth PSD, if it is self-adjoint and satisfies $\lambda_{\min}(\mathsf{A}) \geq 0$.

Given a matrix $\mathsf{A} \in \mathbb{R}^{V\times V}$ and a distribution $\mu : V \to \mathbb{R}_{>0}$, we will write $\|\mathsf{A}\|_{\mathsf{op},\mu}$ for the operator norm of $\mathsf{A}$, defined in the following manner

$$\|\mathsf{A}\|_{\mathsf{op},\mu} := \max\left\{ \frac{\|\mathsf{A}\boldsymbol{f}\|_\mu}{\|\boldsymbol{f}\|_\mu} \;\middle|\; \boldsymbol{f} \in \mathbb{R}^V \text{ and } \boldsymbol{f} \neq 0 \right\}. \qquad \text{(operator norm)}$$

If $\mathsf{A}$ is self-adjoint with respect to the measure $\mu$, we have $\|\mathsf{A}\|_{\mathsf{op},\mu} = \max\{\lambda_1(\mathsf{A}), |\lambda_{\min}(\mathsf{A})|\}$.

Similarly when $\mathsf{A} \in \mathbb{R}^{V\times V}$ is a reversible row-stochastic matrix, with stationary measure $\mu$. We will write $\lambda(\mathsf{A})$ for the two-sided expansion of $\mathsf{A}$. Formally,

$$\lambda(\mathsf{A}) = \max\{\lambda_2(\mathsf{A}), |\lambda_{\min}(\mathsf{M})|\}. \qquad \text{(two-sided expansion)}$$

When $\mathsf{A}$ represents the simple random walk over an undirected graph $H = (V, E)$, i.e.

$$\mathsf{A}(i,j) = \frac{\mathbf{1}[\{i,j\} \in E]}{\deg(i)} \quad \text{for all} \quad i, j \in V,$$

we will simply write $\lambda(H)$ instead of $\lambda(\mathsf{A})$. For convenience, we recall

▶ **Observation 3.** *Let $H = (V, E)$ be a $k$-regular graph and suppose $\mathsf{A}$ represents the random walk over $H$. Then, $\mathsf{uni}_V\mathsf{A} = \mathsf{uni}_V$, i.e. the uniform distribution on $V$ is stationary for $\mathsf{A}$.*

We note that there exist infinite families of graphs such that every graph $H$ in the family has constant degree and $\lambda(H)$ bounded above by a constant bounded above by 1 [43, 44]. In this paper, we will consider families that contain graphs on $n$ vertices for *every* sufficiently large $n$. Such constructions were given in [5], and in particular were based on the infinite families from [43, 44]. We refer the reader to the excellent survey [34] for more information on expander graphs.

We will also make use of the following simple result,

▶ **Lemma 4.** *Let a matrix $\mathsf{A} \in \mathbb{R}^{U\times V}$ and measures $\mu_U : U \to \mathbb{R}_{>0}$ and $\mu_V : V \to \mathbb{R}_{>0}$ be given, such that $\mu_U\mathsf{A} = \mu_V$. Assume without loss of generality that $|U| \leq |V|$, then $\lambda_j(\mathsf{A}\mathsf{A}^*) = \lambda_j(\mathsf{A}^*\mathsf{A})$ for all $j = 1, \ldots, |U|$, where $\mathsf{A}^*$ is the adjoint of $\mathsf{A}$ with respect to the measures $\mu_U$ and $\mu_V$.*

## 2.2 Probability Distributions

Throughout the paper, we will assume $\Omega$ (or $X^{(n)}$) to be a set of $n$-tuples for some $n \geq 1$. Given a set $S \subset [n]$, the projection of $\Omega$ on $S$ is denoted by $\Omega[S]$ , i.e.

$$\Omega[S] = \{(\omega_s)_{s\in S} : (\omega_1, \ldots, \omega_n) \in \Omega\}. \qquad \text{(projection)}$$

Let $\mu : \Omega \to \mathbb{R}_{\geq 0}$ be a distribution. For $\omega_S \in \Omega[S]$, the notations $\Omega_{\omega_S}$ and $\mu^{(\omega_S)}$ will be used for the $\omega_S$-pinning of $\Omega$ and $\mu_S$ respectively, where

$$\Omega_{\omega_S} = \{\bar\omega \in \Omega[S^c] : \omega_S \oplus \bar\omega \in \Omega\} \text{ and } \mu^{(\omega_S)}(\bar\omega) = \frac{\mu(\omega_S \oplus \bar\omega)}{\sum_{\tilde\omega\in\Omega[S^c]} \mu(\omega_S \oplus \tilde\omega)}, \qquad (\omega_S\text{-pinning})$$

We recall that the total variation distance $\|\mu - \nu\|_{\mathsf{tv}}$ between two distributions $\mu, \nu : \Omega \to \mathbb{R}_{\geq 0}$ is defined as follows,

$$\|\mu - \nu\|_{\mathsf{tv}} = \frac{1}{2} \cdot \sum_{\omega \in \Omega} |\mu(\omega) - \nu(\omega)| \qquad \text{(total variation distance)}$$

Finally, we talk about some conventions that we will use throughout the paper: (i) We will be using the notation $\mathsf{uni}_A$ to denote the uniform distribution over various finite sets $A$. (ii) When we want to emphasize that the a distribution $\mu : \Omega \to \mathbb{R}_{\geq 0}$ has full support, we will simply write $\mu : \Omega \to \mathbb{R}_{>0}$.

Finally we recall that the product distribution $\mu \otimes \nu \in \triangle_{\Omega \times \Omega'}$, given $\mu \in \triangle_{\Omega}$ and $\nu \in \triangle_{\Omega'}$ is defined by: $(\mu \otimes \nu)(\omega, \omega') = \mu(\omega) \cdot \nu(\omega')$ for all $\omega \in \Omega, \omega' \in \Omega$.

## 2.3 Functional Inequalities, Isoperimetric Constants, and Mixing Times

Given a distribution $\mu \in \triangle_{\Omega}$ and a convex function $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ $\Phi$-entropy functional $\mathrm{Ent}^{\Phi}_{\mu}(\bullet)$ is defined by the equation,

$$\mathrm{Ent}^{\Phi}_{\mu}(\boldsymbol{f}) = \mathop{\mathbb{E}}_{\omega \sim \mu} \Phi(\boldsymbol{f}(\omega)) - \Phi\left(\mathop{\mathbb{E}}_{\omega \sim \pi_n} \boldsymbol{f}(\omega)\right) \quad \text{for all} \quad \boldsymbol{f} \in \mathbb{R}^{\Omega}_{\geq 0} \qquad (\Phi\text{-entropy})$$

We also recall that for the special choices of $\Phi(t) = t \log t$ and $\Phi(t) = t^2$, the $\Phi$-entropy equals the variance functional $\mathrm{Var}_{\mu}(\bullet)$ and entropy functional $\mathrm{Ent}_{\mu}(\bullet)$ respectively.

$$\mathrm{Ent}_{\mu}(\boldsymbol{f}) = \mathop{\mathbb{E}}_{\omega \sim \mu} [\boldsymbol{f}(\omega) \log \boldsymbol{f}(\omega)] - \left(\mathop{\mathbb{E}}_{\omega \sim \mu} \boldsymbol{f}(\omega)\right) \log\left(\mathop{\mathbb{E}}_{\omega \sim \mu} \boldsymbol{f}(\omega)\right), \qquad \text{(entropy)}$$

$$\mathrm{Var}_{\mu}(\boldsymbol{f}) = \mathop{\mathbb{E}}_{\omega \sim \mu} \boldsymbol{f}(\omega)^2 - \left(\mathop{\mathbb{E}}_{\omega \sim \mu} \boldsymbol{f}(\omega)\right)^2. \qquad \text{(variance)}$$

Let $\mathsf{P} \in \mathbb{R}^{\Omega \times \Omega}$ be a reversible Markov chain, with stationary measure of $\pi$. A Poincaré inequality for $\mathsf{P}$ is an inequality of the form,

$$C \cdot \mathrm{Var}_{\pi}(\boldsymbol{f}) \leq \langle \boldsymbol{f}, (\mathsf{I} - \mathsf{P})\boldsymbol{f} \rangle_{\pi} \quad \text{for all} \quad \boldsymbol{f} \in \mathbb{R}^{\Omega}. \qquad \text{(Poincaré inequality)}$$

The largest constant $C > 0$ for which this inequality holds, is called the Poincaré constant or the spectral gap of $\mathsf{P}$ and is denoted by $\mathtt{gap}(\mathsf{P})$. This nomenclature is due to the following well-known consequence of the Courant-Fischer-Weyl Principle,

$$\mathtt{gap}(\mathsf{P}) = \min\left\{ \frac{\langle \boldsymbol{f}, (\mathsf{I} - \mathsf{P})\boldsymbol{f} \rangle_{\pi}}{\mathrm{Var}_{\pi}(\boldsymbol{f})} \;\middle|\; \mathrm{Var}_{\pi}(\boldsymbol{f}) \neq 0 \right\} = 1 - \lambda_2(\mathsf{P}). \qquad \text{(spectral gap)}$$

The log-Sobolev (LSI) inequality for a reversible random walk $\mathsf{P} \in \mathbb{R}^{\Omega \times \Omega}$ with stationary measure $\pi$ is defined to be,

$$C \cdot \mathrm{Ent}_{\pi}(\boldsymbol{f}^2) \leq \langle \boldsymbol{f}, (\mathsf{I} - \mathsf{P})\boldsymbol{f} \rangle_{\pi} \qquad \text{for all} \quad \boldsymbol{f} \in \mathbb{R}^{\Omega}_{\geq 0}. \qquad \text{(LSI)}$$

The largest constants $C \geq 0$ for which LSI holds is called the log-Sobolev constant of $\mathsf{P}$ respectively and is denoted by $\mathtt{ls}(\mathsf{P})$. Formally,

$$\mathtt{ls}(\mathsf{P}) = \inf\left\{ \frac{\langle \boldsymbol{f}, (\mathsf{I} - \mathsf{P})\boldsymbol{f} \rangle_{\pi}}{\mathrm{Ent}_{\pi}(\boldsymbol{f}^2)} \;\middle|\; \mathrm{Ent}_{\pi}(\boldsymbol{f}) \neq 0, \boldsymbol{f} \in \mathbb{R}^{\Omega}_{\geq 0} \right\}. \qquad (2)$$

▶ **Lemma 5** ([19]). *Let $\pi : \Omega \to \mathbb{R}_{>0}$ be a probability distribution and write $\mathsf{J}_\pi = \mathbf{1} \cdot \pi$, i.e. $\mathsf{J}_\pi$ is the walk with stationary measure $\pi$ which mixes in a single step.*

*Then, $\texttt{ls}(\mathsf{J}_\pi) \geq \frac{1-2\pi_\star}{\log(\pi_\star^{-1}-1)}$ if $|\operatorname{supp}(\pi)| > 2$ else $\texttt{ls}(\mathsf{J}_\pi) = 1$. More generally for any reversible Markov chain $\mathsf{M} \in \mathbb{R}^{\Omega\times\Omega}$ and stationary distribution $\pi$, we have $\texttt{ls}(\mathsf{M}) \geq \frac{1-2\pi_\star}{\log(\pi_\star^{-1}-1)} \cdot \texttt{gap}(\mathsf{M})$ if $|\operatorname{supp}(\pi)| > 2$ else $\texttt{ls}(\mathsf{J}_\pi) = \texttt{gap}(\mathsf{M})$.*

For a convex function $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we also define the $\Phi$-entropy contraction constant $\texttt{cf}_\Phi(\mathsf{P})$ of a Markov chain $\mathsf{P} \in \mathbb{R}^{\Omega_1\times\Omega_2}$ satisfying $\pi_1\mathsf{P} = \pi_2$ for some choice of measures $\pi_1 \in \triangle_{\Omega_1}, \pi_2 \in \triangle_{\Omega_2}$, as the solution to the following variational problem,

$$\texttt{cf}_\Phi(\mathsf{P}) = 1 - \sup\left\{ \frac{\operatorname{Ent}^\Phi_{\pi_1}(\mathsf{P}\boldsymbol{f})}{\operatorname{Ent}^\Phi_{\pi_2}(\boldsymbol{f})} \;\middle|\; \boldsymbol{f} \in \mathbb{R}^\Omega_{\geq 0}, \operatorname{Ent}^\Phi_{\pi_2}(\boldsymbol{f}) \neq 0 \right\}. \qquad (\Phi\text{-entropy contraction})$$

We note that $\texttt{cf}_\Phi(\mathsf{P})$ cruicially depends on the choice of distributions $\pi_1, \pi_2$. Since for our purposes the choice of measures $\pi_1$ and $\pi_2$ will always be clear, we will supress this dependency.

It is equivalent to define $\texttt{cf}_\Phi(\mathsf{P})$ as the largest constant $C \in \mathbb{R}_{\geq 0}$ such that the inequality,

$$\operatorname{Ent}^\Phi_{\pi_1}(\mathsf{P}\boldsymbol{f}) \leq (1 - C) \cdot \operatorname{Ent}^\Phi_{\pi_2}(\boldsymbol{f}),$$

is valid for each $\boldsymbol{f} \in \mathbb{R}^{\Omega_2}_{\geq 0}$. When $\Phi(t) = t \log t$, we will simply write $\texttt{ec}(\mathsf{P})$ in place of $\texttt{cf}_\Phi(\mathsf{P})$. Similarly, for the choice of $\Phi(t) = t^2$, it is easy to observe that $\texttt{cf}_\Phi(\mathsf{P}) = \texttt{gap}(\mathsf{P}^*\mathsf{P})$.

We will also need the following consequence of Jensen's inequality.

▶ **Lemma 6** (Data Processing Inequality). *Let $\mathsf{P} \in \mathbb{R}^{\Omega_1\times\Omega_2}$ be a row-stochastic matrix, satisfying $\pi_1\mathsf{P} = \pi_2$ for probability distributions $\pi_1 : \Omega_1 \to \mathbb{R}_{>0}$ and $\pi_2 : \Omega_2 \to \mathbb{R}_{>0}$. Then, for any convex function $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we have: $\operatorname{Ent}^\Phi_{\pi_1}(\mathsf{P}\boldsymbol{f}) \leq \operatorname{Ent}^\Phi_{\pi_2}(\boldsymbol{f})$ for all $\boldsymbol{f} \in \mathbb{R}^{\Omega_2}_{\geq 0}$.*

▶ **Lemma 7** (Proposition 6, [45]). *Let $\mathsf{P} \in \mathbb{R}^{\Omega_1\times\Omega_2}$ satisfying $\mu_1\mathsf{P} = \mu_2$, for distributions $\mu_1 : \Omega_1 \to \mathbb{R}_{>0}$ and $\mu_2 : \Omega \to \mathbb{R}_{>0}$. We have, $\texttt{ec}(\mathsf{P}) \geq \texttt{ls}(\mathsf{P}^*\mathsf{P})$.*

The $\varepsilon$-mixing time $\tau_{\texttt{mix}}(\mathsf{P}, \varepsilon)$ of the random walk is the least time point $t \in \mathbb{N}$, such that the distribution $\mu^{(t)} = \mu^{(0)}\mathsf{P}^t$ of the random walk $\mathsf{P}$ is guaranteed to be $\varepsilon$-close to the stationary distribution $\pi$ in the total variation distance regardless of the initial distribution $\mu^{(0)}$. In particular,

$$\tau_{\texttt{mix}}(\mathsf{P}, \varepsilon) = \min\left\{ t \in \mathbb{N} \mid \left\| \mu^{(t)} - \pi \right\|_{\mathsf{tv}} \leq \varepsilon \text{ for all } \mu^{(0)} \in \triangle_\Omega \right\} \qquad (\varepsilon\text{-mixing time})$$

It is well known that the functional inequalities and the corresponding isoperimetric constants introduced previously can be used to bound mixing times. We recall in particular,

▶ **Theorem 8** ([11]). *There exists a universal constant $C$ such that, for any reversible random walk $\mathsf{P} \in \mathbb{R}^{\Omega\times\Omega}$ with stationary distribution $\pi : \Omega \to \mathbb{R}_{>0}$, i.e. $\pi\mathsf{P} = \pi$. We have*

$$\tau_{\texttt{mix}}(\mathsf{P}, \varepsilon) \leq \frac{C}{\texttt{ec}(\mathsf{P})} \cdot \left( \log\log \frac{1}{\min_{\omega\in\Omega} \pi(\omega)} + \log \varepsilon^{-1} \right).$$

*where the constant $C$ does not depend on the pair $(\mathsf{P}, \pi)$.*

## 2.4 (Partite) Simplicial Complexes

A simplicial complex is a downward closed collection of subsets of a finite set $U$. Formally, $X \subset 2^U$ and whenever $\beta \in X$ for all $\alpha \subset \beta$ we have $\alpha \in X$. The rank of a face $\alpha$ is $|\alpha|$. Given some $j$, we will adopt the notation $X^{(j)}$ to refer to the collection faces of $X$ of rank $j$

and the notation $X^{(\leq j)}$ to refer to the collection of faces of $X$ of rank at most $j$. We say $X$ is a simplicial complex of rank $n$ if the largest rank of any face $\alpha \in X$ is $n$. We note that by definition $X^{(0)} = \{\varnothing\}$.

We say that a simplicial complex $X$ of rank $n$ is pure, if any face $\alpha \in X^{(j)}$ for any $j < n$ is contained in another face $\beta \in X^{(n)}$. Equivalently, in a pure simplicial complex the only inclusion maximal faces are those of maximal rank. In this article, we will only deal with pure simplicial complexes.

A rank-$n$ pure simplicial complex $X$ is called $n$-partite if we can partition $X^{(1)}$ into disjoint sets $X[1], \ldots, X[n]$ such that

$$\text{for all } \beta \in X^{(n)} \text{ and for all } i = 1, \ldots, n \quad \text{we have} \quad |\beta \cap X[i]| = 1. \qquad (n\text{-partiteness})$$

We will call the sets $X[1], \ldots, X[n]$ the sides of the complex $X$. Equivalently, every element of a rank-$n$ face $\beta \in X[n]$ comes from a distinct side $X[i]$. We observe that a bipartite graph is a 2-partite simplicial complex.

To keep our nomenclature simple, we will simply refer to a pure $n$-partite simplicial complex of rank $n$ as an $n$-partite simplicial complex, i.e. we will not consider $n$-partite complexes which are not pure.

For a face $\alpha \in X$ we introduce the notation, $\text{type}(\alpha) = \{i \in [n] : \alpha \cap X[i] \neq \varnothing\}$ for the type of the face $\alpha$, i.e. the collection of sides of $X$ that $\alpha$ intersects.

For any $i \in [n]$ and $\beta \in X^{(n)}$ we will write $\beta_i \in X^{(1)}$ for the unique element of $\beta$ satisfying $\{\beta_i\} = \beta \cap X[i]$. We will refer to $\beta_i$ as the $i$-th coordinate of $\beta$. We will also write $\beta_T = \{\beta_t : t \in T\}$ for all $T \subset [n]$. We extend this notation to arbitrary faces $\alpha \in X$ and $T \subset \text{type}(\alpha)$. In keeping with the view that a face $\alpha \in X$ with $\text{type}(\alpha) = \{t_1, \ldots, t_k\}$ can be represented as a tuple $(a_{t_1}, \cdots, a_{t_k})$, we will favour the notation $\alpha \oplus \alpha'$ to denote the union of two faces $\alpha, \alpha' \in X$ with $\text{type}(\alpha) \cap \text{type}(\alpha') = \varnothing$ over the usual notation $\alpha \cup \alpha'$.

We observe that for facets $\beta \in X^{(n)}$, i.e. faces of maximal rank, we have $\text{type}(\beta) = [n]$. Given, $\alpha \in X$ we recall that the link $X_\alpha$ is defined as,$X_\alpha = \{(\beta \setminus \alpha) \in X : \beta \in X, \beta \supset \alpha\}$.

For $T \subset [n]$, we will also introduce the notation $X[T]$ to refer to all faces of $X$ of type $T$.

## Weighted Simplicial Complexes

A weighted simplicial complex $(X, \pi)$ of rank $n$ is a pure simplicial complex of rank $n$ where $\pi := \pi_n : X^{(n)} \to \mathbb{R}_{\geq 0}$ is a probablity distribution with full support.

For $j \in [0, n-1]$, we inductively define the probability distributions $\pi_j : X^{(j)} \to \mathbb{R}$ as

$$\pi_j(\alpha) = \pi_j(\alpha) = \frac{1}{\binom{n}{j}} \sum_{\substack{\beta \supset \alpha, \\ \beta \in X^{(n)}}} \pi_n(\beta). \tag{3}$$

Similarly, given a face $\alpha \in X^{(j)}$, we define the distribution $\pi^{(\alpha)}$ on $X_\alpha^{(n-j)}$ by conditioning $\pi$ on the containment of $\alpha$. Of particular importance to us will be the link graph $\mathsf{M}_\alpha \in \mathbb{R}^{X_\alpha^{(1)} \times X_\alpha^{(1)}}$ given any $\alpha \in X^{(\leq n-2)}$. We recall that for all distinct pairs of vertices $x, y \in X_\alpha^{(1)}$, we have

$$\mathsf{M}_\alpha(x, y) = \pi^{(\alpha \cup \{x\})}(y) = \frac{\Pr_{\omega \sim \pi_n}[\omega \supset \alpha \cup \{x, y\} \mid \omega \supset \alpha \cup \{x\}]}{n - |\alpha| - 1}, \tag{link}$$

and $\mathsf{M}_\alpha(x, x) = 0$ for all $x \in X_\alpha^{(1)}$.

## 2.5 Higher Order Random Walks on Simplicial Complexes

Let $(X, \pi)$ be a simpicial complex of rank $n$. The up-down walk $\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n} := \texttt{UpDown}_{\ell\leftrightarrow n}(X, \pi)$ between the $\ell$-th and $n$-th levels, $X^{(\ell)}$ and $X^{(n)}$ respectively, is defined as the following random walk on $X^{(\ell)}$: Starting from an arbitrary face $\hat{\omega}^{(0)} \in X^{(\ell)}$ for all $t \geq 1$ move from $\hat{\omega}^{(t-1)}$ to $\hat{\omega}^{(t)}$ according to the following simple rule,

**Update Rule For the Up-Down Walk, $\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}$**

- sample $\omega \sim \pi_n$, conditional on $\omega \supset \hat{\omega}^{(t-1)}$,
- draw a uniformly subset among all the subsets of $\omega$ of size $\ell$, and output it as $\hat{\omega}$.

Similarly, the down-up walk $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow \ell}$ between the $n$-th and $\ell$-th levels, $X^{(n)}$ and $X^{(\ell)}$ respectively, as the following random walk $X^{(n)}$: Starting from an arbitrary $\omega^{(0)} \in X^{(n)}$ and moves from $\omega^{(t-1)}$ to $\omega^{(t)}$ according to the following simple rule,

**Update Rule for the Down-Up Walk, $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow \ell}$**

- draw a subset $\hat{\omega}$ of $\omega$ of size $\ell$, uniformly at random,
- draw a subset $\omega \sim \pi$ conditioned on containing $\hat{\omega}$, and output it as $\omega^{(t)}$.

It is well known, [2, 21, 20], that the random walks $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow \ell}$ and $\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}$ can be decomposed as a product of random down- and up-movements on $X$. Formally, for $0 \leq \ell \leq k \leq n$, we define the up-walk $\mathsf{P}^{\uparrow}_{\ell\to k} := \texttt{Up}_{\ell\to k}(X, \pi)$ and the down-walk $\mathsf{P}^{\downarrow}_{k\to \ell} := \texttt{Down}_{k\to\ell}(X, \pi)$ as the following random walks,

$$\mathsf{P}^{\uparrow}_{\ell\to k}(\hat{\omega}, \omega) = \pi^{(\hat{\omega})}_{k-\ell}(\omega) = \frac{\mathbf{1}[\omega \supset \hat{\omega}] \cdot \Pr_{\widetilde{\omega}\sim\pi_n}[\widetilde{\omega} \supset \omega \mid \widetilde{\omega} \supset \hat{\omega}]}{\binom{n-\ell}{k-\ell}}, \tag{up-walk}$$

$$\mathsf{P}^{\downarrow}_{k\to \ell}(\omega, \hat{\omega}) = \frac{\mathbf{1}[\hat{\omega} \subset \omega]}{\binom{k}{\ell}}. \tag{down-walk}$$

▶ **Proposition 9** (Folklore). *Let $(X, \pi)$ be a simplical complex of rank $n$, then writing $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow \ell} := \texttt{DownUp}_{n\leftrightarrow\ell}(X, \pi), \mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n} := \mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}(X, \pi)$, $\mathsf{P}^{\uparrow}_{\ell\to n} = \texttt{Up}_{\ell\to n}(X, \pi)$, and $\mathsf{P}^{\downarrow}_{n\to\ell} = \texttt{Down}_{n\to\ell}(X, \pi)$ for the down-up, up-down, up- and down-walks between the $n$-th and $\ell$-th levels of $X$ respectively, we have*

1. $\left(\mathsf{P}^{\uparrow}_{\ell\to n}\right)^* = \mathsf{P}^{\downarrow}_{n\to\ell}$, *i.e. the operators $\mathsf{P}^{\uparrow}_{\ell\to n}$ and $\mathsf{P}^{\downarrow}_{n\to\ell}$ are adjoint operators with respect to the measures $\pi_n$ and $\pi_\ell$,*
2. $\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n} = \mathsf{P}^{\uparrow}_{\ell\to n}\mathsf{P}^{\downarrow}_{n\to\ell}$ *– in particular the operator $\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}$ is PSD,*
3. $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow \ell} = \mathsf{P}^{\downarrow}_{n\to\ell}\mathsf{P}^{\uparrow}_{\ell\to n}$ *– in particular the operator $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow \ell}$ is PSD.*

For any $\hat{\omega} \in X$ and any $0 \leq \ell \leq n' = n - |\hat{\omega}|$, we will write $\mathsf{P}^{\uparrow}_{\hat{\omega},\ell\to n'}, \mathsf{P}^{\downarrow}_{\hat{\omega},n'\to\ell}, \mathsf{P}^{\uparrow\downarrow}_{\hat{\omega},\ell\leftrightarrow n'}$, and $\mathsf{P}^{\downarrow\uparrow}_{\hat{\omega},n'\leftrightarrow\ell}$ for the corresponding up, down, up-down, and down-up walks in the complex $(X_{\hat{\omega}}, \pi^{\hat{\omega}})$.

## 2.6 Local to Global Analysis

Given a simplicial complex $(X, \pi)$ of rank $n$, we define the local $\Phi$-entropy contraction factor $\texttt{lc}_\Phi(\hat{\omega})$ for any $\hat{\omega} \in X^{(\leq r-2)}$ as follows,

$$\texttt{lc}_\Phi(\hat{\omega}) := \sup\left\{ \frac{\mathrm{Ent}^{\Phi}_{\pi_1^{(\hat{\omega})}}(\mathsf{P}^{\uparrow}_{\hat{\omega},1\to n'}\boldsymbol{g})}{\mathrm{Ent}^{\Phi}_{\pi_{n'}^{(\hat{\omega})}}(\boldsymbol{g})} \;\middle|\; \boldsymbol{g} \in \mathbb{R}^{X_{\hat{\omega}}^{(n')}} \text{ and } n' = n - |\hat{\omega}|. \right\} \tag{4}$$

Equivalently, $\texttt{lc}_\Phi(\hat{\omega}) \in \mathbb{R}_{>0}$ is the smallest constant satisfying the equality

$$\mathrm{Ent}^{\Phi}_{\pi_1^{\hat{\omega}}}(\mathsf{P}^{\uparrow}_{\hat{\omega},1\to n'}\boldsymbol{g}) \leq \texttt{lc}_\Phi(\hat{\omega}) \cdot \mathrm{Ent}^{\Phi}_{\pi^{(\hat{\omega})}_{n'}}(\boldsymbol{g}) \quad \text{for all} \quad \boldsymbol{g} \in \mathbb{R}^{X_{\hat{\omega}}^{(n')}} \quad \text{where} \quad n' = n - |\hat{\omega}|.$$

When $\Phi(t) = t\log t$, we will simply write $\texttt{lec}(\hat{\omega})$ in place of $\texttt{lc}_\Phi(\hat{\omega})$. We also make the following observation for the special case $\Phi(t) = t^2$, i.e. when $\mathrm{Ent}^{\Phi}_{\bullet}(\bullet)$ equals the variance functional $\mathrm{Var}_{\bullet}(\bullet)$. The following proposition is well understood,

▶ **Proposition 10.** [4] *Let $(X,\pi)$ be a simplicial complex of rank $n$. Then, for the choice of $\Phi(t) = t^2$, for any $\hat{\omega} \in X^{(\leq n-2)}$ we have*

$$\texttt{lc}_\Phi(\hat{\omega}) = \frac{1}{n - |\hat{\omega}|} + \frac{n - |\hat{\omega}| - 1}{n - |\hat{\omega}|} \cdot \lambda_2(\mathsf{M}_{\hat{\omega}}),$$

*where $\mathsf{M}_{\hat{\omega}}$ is the link graph of $\hat{\omega}$.*

A crucial tool we will be using in Section 4 is the so called Garland method, due to [31]. To this end, we define the localization $\boldsymbol{f}|_{\hat{\omega}} \in \mathbb{R}^{X_{\hat{\omega}}(k-j)}$ of a function $\boldsymbol{f} \in \mathbb{R}^{X^{(k)}}$ on a link $\hat{\omega} \in X^{(j)}$ for $j \leq k$ as the following function,

$$\boldsymbol{f}|_{\hat{\omega}}(\alpha) = \boldsymbol{f}(\hat{\omega} \sqcup \alpha) \quad \text{for all} \quad \alpha \in X_{\hat{\omega}}^{(k-j)}. \qquad \text{(localization)}$$

We first observe that by appealing to the chain rule for the $\Phi$-entropy, one can obtain a convenient expression for it in terms of localizations.

▶ **Lemma 11** (Chain Rule for $\Phi$-Entropy). [5] *Let $(X,\pi)$ be a simplicial complex of rank $n$. For all $0 \leq \ell \leq r \leq n$ and non-negative $\boldsymbol{f} \in \mathbb{R}^{X^{(r)}}_{\geq 0}$, we have*

$$\mathrm{Ent}^{\Phi}_{\pi_r}(\boldsymbol{f}) = \mathop{\mathbb{E}}_{\hat{\omega}\sim\pi_\ell} \mathrm{Ent}^{\Phi}_{\pi^{(\hat{\omega})}_{r-\ell}}(\boldsymbol{f}|_{\hat{\omega}}) + \mathrm{Ent}_{\pi_\ell}\Big(\mathsf{P}^{\uparrow}_{\ell\to r}\boldsymbol{f}\Big),$$

*where $\mathsf{P}^{\uparrow}_{\ell\to r} := \texttt{Up}_{\ell\to r}(X,\pi)$ is the up-walk on $X$*

We also recall the following identities,

▶ **Lemma 12.** *Let $(X,\pi)$ be a simplicial complex of rank $n$. Writing $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow r} = \texttt{DownUp}_{n\leftrightarrow r}(X,\pi)$, $\mathsf{P}^{\uparrow\downarrow}_{n-1} = \texttt{UpDown}_{n-1\leftrightarrow n}(X,\pi)$, and $\mathsf{M}_{\hat{\omega}}$ for the link of the face $\hat{\omega} \in X^{(\leq n-2)}$, for all $\boldsymbol{f} \in \mathbb{R}^{X^{(n)}}$ and $\ell \leq r \leq n$, we have*

1. $\langle \boldsymbol{f}, \boldsymbol{f}\rangle_{\pi_n} = \mathbb{E}_{\hat{\omega}\sim\pi_\ell}\langle \boldsymbol{f}|_{\hat{\omega}}, \boldsymbol{f}|_{\hat{\omega}}\rangle_{\pi^{(\hat{\omega})}_{n-\ell}}$,
2. $\big\langle \boldsymbol{f}, \mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow r}\boldsymbol{f}\big\rangle_{\pi_n} = \mathbb{E}_{\hat{\omega}\sim\pi_\ell}\Big\langle \boldsymbol{f}|_{\hat{\omega}}, \mathsf{P}^{\downarrow\uparrow}_{\hat{\omega},n-\ell\leftrightarrow r-\ell}\boldsymbol{f}|_{\hat{\omega}}\Big\rangle_{\pi^{(\hat{\omega})}_{n-\ell}}$,
3. $\Big\langle \boldsymbol{f}, \mathsf{P}^{\uparrow\downarrow}_{n-1}\boldsymbol{f}\Big\rangle_{\pi_n} = \mathbb{E}_{\hat{\omega}\sim\pi_{n-2}}\Big(\big\langle \boldsymbol{f}|_{\hat{\omega}}, \big(\tfrac{1}{n} + \tfrac{n-1}{n}\cdot\mathsf{M}_{\hat{\omega}}\big)\boldsymbol{f}|_{\hat{\omega}}\big\rangle_{\pi_1^{(\hat{\omega})}}\Big)$

We recall the following result due to [40],

▶ **Lemma 13** (Theorem 3.5, [40]). *Let $(X,\pi)$ be an $n$-partite simplicial complex and let $0 \leq k < n$. If $\texttt{ec}(\mathsf{P}^{\uparrow}_{n-1\to n}) \geq (Cn)^{-1}$ for some $C \in \mathbb{R}_{>0}$, then $\texttt{ec}\Big(\mathsf{P}^{\uparrow}_{k\to k+1}\Big) \geq \frac{1}{(k+1)(C+1)}$.*

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[4] We provide a proof for this statement in the full version of our paper, [4]
[5] A proof is supplied in the full version of our paper, [4]

# 3 Expanderized Random Walks

Let $(X, \pi)$ be an $n$-partite simplicial complex. For any $\ell \leq n$, the up-down walk $\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n} := \mathtt{UpDown}_{\ell\leftrightarrow n}(X, \pi)$ on the $\ell$-th level $X^{(\ell)}$ of $X$ introduced in Section 2.5 admits the following alternative description: Starting from an arbitrary face $\hat{\omega}^{(0)} \in X^{(\ell)}$ move from $\hat{\omega}^{(t-1)}$ to $\hat{\omega}^{(t)}$ according to the following simple rule,

**Update Rule for the Up-Down Walk, $\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}$**

- sample $\omega \sim \pi$, conditional on $\omega \supset \hat{\omega}^{(t-1)}$,
- sample $S \sim \mathsf{uni}_{\binom{[n]}{\ell}}$,
- output $\hat{\omega}^{(t)} = \omega_S$.

We will expanderize the up-down walk $\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}$ in the following manner: Given a $k$-regular labelled graph $H$ on the vertex set $\binom{[n]}{\ell}$, we will denote the $a$-th neighbor of vertex $v$ by $\mathtt{Out}_H(v, a)$. We define the expanderized up-down walk $\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n} = \mathtt{UpDown}_{\ell\leftrightarrow n}(X, \pi, H)$ as the walk which starts from an arbitrary face $\hat{\omega}^{(0)}$ and moves from $\hat{\omega}^{(t-1)}$ to $\hat{\omega}^{(t)}$ according to the following simple rule,

**Update Rule for the Expanderized Up-Down Walk, $\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n}$**

- sample $\omega \sim \pi$, conditional on $\omega \supset \hat{\omega}^{(t-1)}$,
- sample $a \sim \mathsf{uni}_{[k]}$ and set $S = \mathtt{Out}_H(\mathrm{type}(\hat{\omega}^{(t-1)}), a)$,[a]
- output $\hat{\omega}^{(t)} = \omega_S$.

[a] Where we recall that the type of $\hat{\omega}$ is the sides of the simplicial complex that $\hat{\omega}$ intersects

Similarly, the down-up walk $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow\ell}$ between the $n$-th level $X^{(n)}$ and the $\ell$-th level $X^{(\ell)}$ of an $n$-partite simplicial complex $(X, \pi)$ introduced in Section 2.5 admits the following alternative description: Start from $\omega^{(0)} \in X^{(n)}$ and move from $\omega^{(t-1)}$ to $\omega^{(t)}$ according to the following simple rule,

**Update Rule for the Down-Up Walk, $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow\ell}$**

- sample $S \sim \mathsf{uni}_{\binom{[n]}{\ell}}$ uniformly at random,
- set $\hat{\omega} = \omega_S$,
- set $\omega^{(t)}$ to be a random face drawn from $\pi$, conditional on containing $\hat{\omega}$.

Similarly, we define the expanderized down-up walk $\mathsf{Q}^{\downarrow\uparrow}_{n\leftrightarrow\ell} = \mathtt{DownUp}_{n\leftrightarrow\ell}(X, \pi, H)$ to be the random walk on $X^{(n)} \times \binom{[n]}{\ell}$, starting from an arbitrary face-subset pair $(\omega^{(0)}, S^{(0)})$ and move from $(\omega^{(t-1)}, S^{(t-1)})$ to $(\omega^{(t)}, S^{(t)})$ according to the following simple rule,

**Update Rule for the Expanderized Down-Up Walk, $\mathsf{Q}^{\downarrow\uparrow}_{n\leftrightarrow\ell}$**

- sample $a \sim \mathsf{uni}_{[k]}$ and set $S' = \mathtt{Out}_H(S^{(t-1)}, a)$,
- set $\hat{\omega} = \omega_{S'}$,
- set $\omega^{(t)} \sim \pi$ to be a random face conditional on containing $\hat{\omega}$,
- sample $b \sim \mathsf{uni}_{[k]}$ and set $S^{(t)} = \mathtt{Out}_H(S^{(t-1)}, b)$,
- output $(\omega^{(t)}, S^{(t)})$.

For convenience we also define the expanderized down- and up-walks given a degree regular labelled graph $H = (\binom{[n]}{\ell}, E)$

$$\mathsf{Q}^{\downarrow}_{n\to\ell} = \mathtt{Down}_{n\to\ell}(X,\pi,H) \in \mathbb{R}^{(X^{(n)}\times\binom{[n]}{\ell})\times X^{(\ell)}} \quad\text{and}\quad \mathsf{Q}^{\uparrow}_{\ell\to n} = \mathtt{Up}_{\ell\to[n]}(X,\pi,H) \in \mathbb{R}^{X^{(\ell)}\times(X^{(n)}\times\binom{[n]}{\ell})},$$

as follows,

$$\mathsf{Q}^{\downarrow}_{n\to\ell}((\omega,S),\hat{\omega}) = \frac{\mathbf{1}[S\sim_H \mathrm{type}(\hat{\omega})]}{k}\cdot\mathbf{1}[\omega\supset\hat{\omega}] \ \text{ for all }\ \omega\in X^{(n)},\hat{\omega}\in X^{(\ell)}, S\in\binom{[n]}{\ell},$$

and $\mathsf{Q}^{\uparrow}_{\ell\to n} = \left(\mathsf{Q}^{\downarrow}_{n\to\ell}\right)^*$ where the adjoint is taken with respect to the distributions $\pi_n\otimes\mathsf{uni}_{\binom{[n]}{\ell}}$ and $\pi_\ell$, i.e.

$$\mathsf{Q}^{\uparrow}_{\ell\to n}(\hat{\omega},(\omega,S)) = \frac{\mathbf{1}[S\sim_H \mathrm{type}(\hat{\omega})]}{k}\cdot\Pr_{\widetilde{\omega}\sim\pi_n}[\widetilde{\omega}=\omega\mid\omega\supset\hat{\omega}] \ \text{ for all }\ \omega\in X^{(n)},\hat{\omega}\in X^{(\ell)}, S\in\binom{[n]}{\ell},$$

and the notation $T\sim_H S$ is used to denote the adjacency relation in the graph $H$, i.e. $\{S,T\}\in E(H)$.

We summarize the random movements described by the expanderized up- and down-walks in words as follows: The expanderized down-walk $\mathsf{Q}^{\downarrow}_{n\to\ell}$ first samples a random neighbor of $T$ of $S$ in $\binom{[n]}{\ell}$, and then restricts the coordinates of $\omega$ to $T$, i.e. moves to $\omega_T$. The expanderized up-walk $\mathsf{Q}^{\uparrow}_{\ell\to n}$ on the other hand first samples a facet $\omega\in X^{(n)}$ from $\pi$ conditional on containing $\hat{\omega}$ and after picking a random neighbor $S$ of type$(\hat{\omega})$ in $H$ moves to $(\omega,S)$.

▶ **Proposition 14.** *For any $n$-partite pure simplicial complex $(X,\pi)$ and a $k$-regular labelled graph $H = (\binom{[n]}{\ell}, E)$, writing $\mathsf{Q}^{\downarrow}_{n\to\ell} = \mathtt{Down}_{n\to\ell}(X,\pi,H)$ and $\mathsf{Q}^{\uparrow}_{\ell\to n} = \mathtt{Up}_{\ell\to n}(X,\pi,H)$ we have,*

$$\left(\pi_n\otimes\mathsf{uni}_{\binom{[n]}{\ell}}\right)\mathsf{Q}^{\downarrow}_{n\to\ell} = \pi_\ell \quad\textit{and}\quad \pi_\ell\mathsf{Q}^{\uparrow}_{\ell\to n} = \pi_n\otimes\mathsf{uni}_{\binom{[n]}{\ell}}.$$

**Proof.** Let $(\omega,S)\sim\pi_n\otimes\mathsf{uni}_{\binom{[n]}{\ell}}$ be a random sample. Notice that a random neighbor of $S$ in $H$ is still distributed uniformly at random as the uniform distribution stationary for the random walk over a $k$-regular graph, q.v. Observation 3. Thus, conditional on $\omega$, a single step of $\mathsf{Q}^{\downarrow}_{n}$ ends up restricting $\omega$ to a random set of coordinates $S$ – this precisely yields the distribution $\pi_\ell$, q.v. Equation (3).

The second statement follows since $\mathsf{Q}^{\uparrow}_{\ell\to n}$ is the adjoint operator, q.v. Proposition 1. ◀

The following is easy to verify,

▶ **Corollary 15.** *For any $n$-partite simplicial complex $(X,\pi)$ and $k$-regular labelled graph $H = \left((\binom{[n]}{\ell}), E)\right)$,*

- $\mathtt{UpDown}_{\ell\leftrightarrow n}(X,\pi,H^2) = \mathtt{Up}_{\ell\to n}(X,\pi,H)\cdot\mathtt{Down}_{n\to\ell}(X,\pi,H)$,
- $\mathtt{DownUp}_{n\leftrightarrow\ell}(X,\pi,H) = \mathtt{Down}_{n\to\ell}(X,\pi,H)\cdot\mathtt{Up}_{\ell\to n}(X,\pi,H)$.

We now summarize several useful properties of the expanderized up- and down-walks,

▶ **Corollary 16.** *Let $(X,\pi)$ be an $n$-partite complex and $H = ((\binom{[n]}{\ell}), E)$ a $k$-regular graph. For any $\ell\le n$, writing $\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n} = \mathtt{UpDown}_{\ell\leftrightarrow n}(X,\pi,H^2)$, $\mathsf{Q}^{\downarrow\uparrow}_{n\leftrightarrow\ell} = \mathtt{DownUp}_{n\leftrightarrow\ell}(X,\pi,H)$ $\mathsf{Q}^{\downarrow}_{n\to\ell} = \mathtt{Down}_{n\to\ell}(X,\pi,H)$ and $\mathsf{Q}^{\uparrow}_{\ell\to n} = \mathtt{Up}_{\ell\to n}(X,\pi,H)$ we have,*

1. *$(\pi_n\otimes\mathsf{uni}_{\binom{[n]}{\ell}})\mathsf{Q}^{\downarrow\uparrow}_{n\leftrightarrow\ell} = \pi_n\otimes\mathsf{uni}_{\binom{[n]}{\ell}}$, i.e. $\pi_n\otimes\mathsf{uni}_{\binom{[n]}{\ell}}$ is the stationary distribution of $\mathsf{Q}^{\downarrow\uparrow}_{n\leftrightarrow\ell}$,*
2. *$\pi_\ell\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n} = \pi_\ell$, i.e. $\pi_\ell$ is the stationary distribution of $\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n}$.*

3. $\mathsf{Q}^{\downarrow\uparrow}_{n\leftrightarrow\ell}$ *and* $\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n}$ *are PSD operators.*
4. $\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n}$ *and* $\mathsf{Q}^{\downarrow\uparrow}_{n\leftrightarrow\ell}$ *are self-adjoint operators.*

Since in our proofs it will be more convenient to use $\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n} := \mathtt{UpDown}_{\ell\leftrightarrow n}(X,\pi,H)$ directly, initialized with $H$ and not $H^2$, we also note the following.

▶ **Proposition 17.** [6] *Let $(X,\pi)$ be an $n$-partite complex and $H = (\binom{[n]}{\ell}, E)$ a $k$-regular graph. Then, the expanderized up-down walk $\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n} := \mathtt{UpDown}_{\ell\leftrightarrow n}(X,\pi,H)$ has the stationary distribution $\pi_\ell$ and is reversible.*

Now, we present the results we prove for expanderized random walks. Our first result shows that the expanderized up-down walk approximates the usual up-down walk in the operator norm,

▶ **Theorem 18.** *Let $(X,\pi)$ be an $n$-partite simplicial complex and let $H$ be a $k$-regular labelled graph on the vertex set $\binom{[n]}{\ell}$. Writing $\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n} := \mathtt{UpDown}(X,\pi,H)$ and $\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n} := \mathtt{UpDown}(X,\pi)$ for the expanderized- and the regular up-down walks on $X^{(\ell)}$, we have*

$$\left\| \mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n} - (1-\lambda(H)) \cdot \mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n} \right\|_{\mathsf{op},\pi_\ell} \le \lambda(H).$$

We present the proof of Theorem 18 in Section 3.1. Theorem 18 immediately implies the following bounds for the spectral gap of expanderized walks,

▶ **Corollary 19.** [7] *Let $(X,\pi)$ be an $n$-partite simplicial complex and let $H$ be a $k$-regular labelled graph on the vertex set $\binom{[n]}{\ell}$. Writing $\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n} := \mathtt{UpDown}(X,\pi,H)$, $\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n} = \mathtt{UpDown}_{\ell\leftrightarrow n}(X,\pi)$, $\mathsf{Q}^{\downarrow\uparrow}_{n\leftrightarrow\ell} = \mathtt{DownUp}(X,\pi,H)$, and $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow\ell} = \mathtt{DownUp}_{n\leftrightarrow\ell}(X,\pi)$, we have*

$$\begin{aligned} \mathtt{gap}\Big(\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n}\Big) &\ge \mathtt{gap}\Big(\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}\Big) \cdot \mathtt{gap}^\star(H), \\ \mathtt{gap}\Big(\mathsf{Q}^{\downarrow\uparrow}_{n\leftrightarrow\ell}\Big) &\ge \mathtt{gap}\Big(\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow\ell}\Big) \cdot \mathtt{gap}^\star(H^2), \end{aligned}$$

*where $\mathtt{gap}^\star(G) = 1-\lambda(G)$ and $\lambda(G)$ denotes the two-sided expansion of the graph $G$.*

Unfortunately, a bound on the spectral gap is in many settings not enough to obtain optimal mixing time bounds. We show however, that Theorem 18 allows us to transfer log-Sobolev inequalities (LSI) for the usual up-down walks to the expanderized up-down walks,

▶ **Corollary 20.** *Let $(X,\pi)$ be an $n$-partite simplicial complex and let $H$ be a $k$-regular labelled graph on the vertex set $\binom{[n]}{\ell}$. Writing $\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n} := \mathtt{UpDown}_{\ell\leftrightarrow n}(X,\pi,H)$ and $\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n} := \mathtt{UpDown}_{\ell\leftrightarrow n}(X,\pi)$ for the up-down walk on $X^{(\ell)}$, we have*

$$\mathtt{ls}\Big(\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n}\Big) \ge \mathtt{ls}\Big(\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}\Big) \cdot \mathtt{gap}^\star(H),$$

*where $\mathtt{gap}^\star(H) = 1-\lambda(H)$ and $\lambda(H)$ denotes the two-sided expansion of the graph $H$.*

We will prove Corollary 20 in Section 3.2. We state a convenient corollary of Corollary 20 which immediately follows from Lemma 7, Corollary 16, and the data processing inequality Lemma 6,

[6] As Corollary 16 reaches the same end by replacing $H$ with $H^2$ we will omit this proof and refer the reader to the full version of our paper [4]

[7] Since this result does not get used in our applications, we refer the reader to the full-version of our paper [4] for a proof.

▶ **Corollary 21.** *Let $(X,\pi)$ be an $n$-partite simplicial complex and let $H$ be a $k$-regular labelled graph on the vertex set $\binom{[n]}{\ell}$. Then, writing $\mathsf{Q}^{\downarrow}_{n\leftrightarrow\ell} = \mathtt{Down}_{n\to\ell}(X,\pi,H)$ and $\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n} = \mathtt{UpDown}_{\ell\leftrightarrow n}(X,\pi)$ we have,*

$$\mathtt{ec}(\mathsf{Q}^{\downarrow}_{n\leftrightarrow\ell}) \geq \mathtt{ls}(\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}) \cdot \mathtt{gap}^{\star}(H^2).$$

*In particular, we have for $\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n} := \mathtt{UpDown}_{\ell\leftrightarrow n}(X,\pi,H^2)$ and $\mathsf{Q}^{\downarrow\uparrow}_{n\leftrightarrow\ell} = \mathtt{DownUp}_{n\leftrightarrow\ell}(X,\pi,H)$,*

$$\mathtt{ec}(\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n}) \geq \mathtt{ls}(\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}) \cdot \mathtt{gap}^{\star}(H^2) \quad \textit{and} \quad \mathtt{ec}(\mathsf{Q}^{\downarrow\uparrow}_{n\leftrightarrow\ell}) \geq \mathtt{ls}(\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}) \cdot \mathtt{gap}^{\star}(H^2),$$

*where $\mathtt{gap}^{\star}(H^2) = 1 - \lambda(H^2)$ and $\lambda(H^2)$ denotes the two-sided expansion of the graph $H^2$.*

As we will see in Section 5, Corollary 21 will indeed allow us to prove optimal mixing time bounds for the expanderized walks in many cases of interest.

## 3.1 Closeness in Operator Norm: Proof of Theorem 18

**Proof of Theorem 18 .** For convenience, we will write $\mathsf{Q}^{\uparrow\downarrow} := \mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n}$ and $\mathsf{P}^{\uparrow\downarrow} := \mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}$. Let $\mathsf{M}$ denote the random-walk matrix of the graph $H$ where each transition occurs with the probability $1/k$ and $\mathsf{J}$ the random-walk matrix of the clique over $\binom{[n]}{\ell}$ with self-loops, i.e. $\mathsf{J} = \mathbf{1}\mathbf{1}^{\top}/\binom{[n]}{\ell}$. We will write, $\lambda := \lambda(\mathsf{M})$.

Let $S \in \binom{[n]}{\ell}$ be arbitrary and suppose some $\bar{\omega} \in X^{(\ell)}$ is given such that $\mathrm{type}(\bar{\omega}) = S$.

For all $\boldsymbol{f} \in \mathbb{R}^{X^{(\ell)}}$, we have

$$[\mathsf{Q}^{\uparrow\downarrow}\boldsymbol{f}](\bar{\omega}) = \sum_{\hat{\omega}\in X[S^c]} \Pr_{\omega\sim\pi}[\omega_{S^c} = \hat{\omega} \mid \omega_S = \bar{\omega}] \cdot \sum_{a\in[k]} \frac{\boldsymbol{f}\big((\bar{\omega}\oplus\hat{\omega})_{\mathtt{Out}_H(S,a)}\big)}{k}.$$

Similarly, we have

$$[\mathsf{P}^{\uparrow\downarrow}\boldsymbol{f}](\bar{\omega}) = \sum_{\hat{\omega}\in X[S^c]} \Pr_{\omega\sim\pi}[\omega_{S^c} = \hat{\omega} \mid \omega_S = \bar{\omega}] \sum_{T\in\binom{[n]}{\ell}} \frac{\boldsymbol{f}((\bar{\omega}\oplus\hat{\omega})_T)}{\binom{n}{\ell}}.$$

For any given facet $\omega \in X^{(n)}$ we define the function $\boldsymbol{g}_{\omega} \in \mathbb{R}^{\binom{[n]}{\ell}}$ as, $\boldsymbol{g}_{\omega}(T) = \boldsymbol{f}(\omega_T)$.

We have,

$$[\mathsf{M}\boldsymbol{g}_{\omega}](T) = \sum_{a\in[k]} \frac{\boldsymbol{f}\big(\omega_{\mathtt{Out}_H(T,a)}\big)}{k} \quad \text{and} \quad [\mathsf{J}\boldsymbol{g}_{\omega}](i) = \sum_{T\in\binom{[n]}{\ell}} \frac{\boldsymbol{f}(\omega_T)}{\binom{n}{\ell}}.$$

Thus, we have

$$[\mathsf{Q}^{\uparrow\downarrow}\boldsymbol{f}](\bar{\omega}) = \sum_{\hat{\omega}\in X[S^c]} \Pr_{\omega\sim\pi}[\omega_{S^c} = \hat{\omega} \mid \omega_S = \bar{\omega}] \cdot [\mathsf{M}\boldsymbol{g}_{\bar{\omega}\oplus\hat{\omega}}](S), \tag{5}$$

$$[\mathsf{P}^{\uparrow\downarrow}\boldsymbol{f}](\bar{\omega}) = \sum_{\hat{\omega}\in X[S^c]} \Pr_{\omega\sim\pi}[\omega_{S^c} = \hat{\omega} \mid \omega_S = \bar{\omega}] \cdot [\mathsf{J}\boldsymbol{g}_{\bar{\omega}\oplus\hat{\omega}}](S). \tag{6}$$

In particular combining Equation (5) and Equation (6) and noticing that for $\omega \sim \pi$ the law of $\omega_{S^c}$ conditional on $\omega_S = \bar{\omega}$ is given by $\pi^{(\hat{\omega})}_{n-\ell}$,

$$\left\|(\mathsf{Q}^{\uparrow\downarrow} - (1-\lambda)\mathsf{P}^{\uparrow\downarrow})\boldsymbol{f}\right\|_{\pi_\ell}^2 = \mathop{\mathbb{E}}_{\bar{\omega}\sim\pi_\ell}\left(\mathop{\mathbb{E}}_{\hat{\omega}\sim\pi_{n-\ell}^{(\bar{\omega})}}\left[[\mathsf{M}\boldsymbol{g}_{\bar{\omega}\oplus\hat{\omega}}](\mathrm{type}(\bar{\omega})) - (1-\lambda)[\mathsf{J}\boldsymbol{g}_{\bar{\omega}\oplus\hat{\omega}}](\mathrm{type}(\bar{\omega}))\right]\right)^2,$$
$$\leq \mathop{\mathbb{E}}_{\bar{\omega}\sim\pi_\ell}\mathop{\mathbb{E}}_{\hat{\omega}\sim\pi_{n-\ell}^{(\bar{\omega})}}\left[\left([\mathsf{M}\boldsymbol{g}_{\bar{\omega}\oplus\hat{\omega}}](\mathrm{type}(\bar{\omega})) - (1-\lambda)[\mathsf{J}\boldsymbol{g}_{\bar{\omega}\oplus\hat{\omega}}](\mathrm{type}(\bar{\omega}))\right)^2\right].$$

where the last inequality is obtained by appealing to Jensen's inequality and the convexity of $t \to t^2$.

Now, we observe the law of $\bar{\omega} \oplus \hat{\omega}$ obtained by first sampling $\bar{\omega} \sim \pi_\ell$ and then $\hat{\omega} \sim \pi^{(\bar{\omega})}$ is given by $\pi$. Furthermore, any $\omega \in X^{(n)}$ occurs exactly $\binom{[n]}{\ell}$ times in the expectation above – once for each $S \in \binom{[n]}{\ell}$ acting as type($\bar{\omega}$), which happens with probability $\binom{n}{\ell}^{-1}$. Thus,

$$\left\|(\mathsf{Q}^{\uparrow\downarrow} - (1-\lambda)\mathsf{P}^{\uparrow\downarrow})\boldsymbol{f}\right\|_{\pi_\ell}^2 \leq \mathop{\mathbb{E}}_{\omega\sim\pi_n}\left[\mathop{\mathbb{E}}_{S\sim\mathsf{uni}_{\binom{[n]}{\ell}}}\left([(\mathsf{M}-(1-\lambda)\mathsf{J})\boldsymbol{g}_\omega](S)\right)^2\right],$$
$$\leq \mathop{\mathbb{E}}_{\omega\sim\pi_n}\left[\lambda^2\cdot\|\boldsymbol{g}_\omega\|^2_{\mathsf{uni}_{\binom{[n]}{\ell}}}\right]$$

where the last inequality is due to $\|\mathsf{M}-(1-\lambda)\mathsf{J}\|_{\mathsf{op},\mathsf{uni}_{\binom{[n]}{\ell}}} \leq \lambda$ as $\lambda(\mathsf{M}) \leq \lambda$!

Now, we finally note

$$\mathop{\mathbb{E}}_{\omega\sim\pi}\left[\|\boldsymbol{g}_\omega\|^2_{\mathsf{uni}_{\binom{[n]}{\ell}}}\right] = \mathop{\mathbb{E}}_{\omega\sim\pi}\left[\frac{1}{\binom{n}{\ell}}\sum_{S\in\binom{[n]}{\ell}}\boldsymbol{f}(\omega_S)^2\right] = \mathop{\mathbb{E}}_{\bar{\omega}\sim\pi_\ell}\boldsymbol{f}(\bar{\omega})^2 = \|\boldsymbol{f}\|^2_{\pi_\ell},$$

The last equality is due to the observation that first sampling $\omega \sim \pi$ and then outputting $\omega_S$ for $S \sim \mathsf{uni}_{\binom{[n]}{\ell}}$ picked uniformly at random amounts to simply sampling $\bar{\omega} \sim \pi_\ell$, q.v. Equation (3).

In particular,

$$\left\|(\mathsf{Q}^{\uparrow\downarrow} - (1-\lambda)\mathsf{P}^{\uparrow\downarrow})\boldsymbol{f}\right\|_{\pi_\ell} \leq \lambda\cdot\|\boldsymbol{f}\|_{\pi_\ell}.$$

As $\boldsymbol{f}$ was picked arbitrarily, this allows us to conclude the proof of our theorem by appealing to the definition of the operator norm. ◀

## 3.2 Log-Sobolev Bound: Proof of Corollary 20

**Proof of Corollary 20.** Let $\boldsymbol{f} \in \mathbb{R}_{\geq 0}^{X^{(\ell)}}$ be an arbitrary function satisfying $\mathrm{Ent}_{\pi_\ell}(\boldsymbol{f}^2) \neq 0$. We have,

$$\frac{\left\langle\boldsymbol{f},\left(\mathsf{I}-\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n}\right)\boldsymbol{f}\right\rangle_{\pi_\ell}}{\mathrm{Ent}_{\pi_\ell}(\boldsymbol{f})^2} = \frac{\left\langle\boldsymbol{f},\left(\mathsf{I}-(1-\lambda(H))\cdot\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}\right)\boldsymbol{f}\right\rangle_{\pi_\ell}}{\mathrm{Ent}_{\pi_\ell}(\boldsymbol{f}^2)} + \frac{\left\langle\boldsymbol{f},\left((1-\lambda(H))\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}-\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n}\right)\boldsymbol{f}\right\rangle_{\pi_\ell}}{\mathrm{Ent}_{\pi_\ell}(\boldsymbol{f}^2)}.$$

Notice that by Theorem 18, we should have

$$\left\langle\boldsymbol{f},\left((1-\lambda(H))\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}-\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n}\right)\boldsymbol{f}\right\rangle_{\pi_\ell} \geq -\lambda(H)\cdot\langle\boldsymbol{f},\mathsf{I}\boldsymbol{f}\rangle_{\pi_\ell}.$$

Thus,

$$\frac{\left\langle\boldsymbol{f},\left(\mathsf{I}-\mathsf{Q}^{\uparrow\downarrow}_{\ell\leftrightarrow n}\right)\boldsymbol{f}\right\rangle_{\pi_\ell}}{\mathrm{Ent}_{\pi_\ell}(\boldsymbol{f})^2} \geq \frac{\left\langle\boldsymbol{f},\left(\mathsf{I}-\left((1-\lambda(H))\cdot\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}+\lambda(H)\cdot\mathsf{I}\right)\right)\boldsymbol{f}\right\rangle_{\pi_\ell}}{\mathrm{Ent}_{\pi_\ell}(\boldsymbol{f}^2)},$$
$$\geq \mathtt{ls}\left((1-\lambda(H))\cdot\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n}+\lambda(H)\cdot\mathsf{I}\right),$$
$$= \mathtt{ls}(\mathsf{P}^{\uparrow\downarrow}_{\ell\leftrightarrow n})\cdot\mathtt{gap}^\star(H),$$

where the last inequality is obtained by noticing:

$$\langle \boldsymbol{f}, (\mathsf{I} - (a \cdot \mathsf{I} + (1-a)\mathsf{P}))\boldsymbol{f}\rangle_\mu = (1-a) \cdot \langle \boldsymbol{f}, (\mathsf{I} - \mathsf{P})\boldsymbol{f}\rangle_\mu,$$

and the variational formula for the log-Sobolev constant (Equation (2)) . Appealing to the definition of the log-Sobolev inequality (LSI) once again yields the result. ◀

## 4 Functional Inequalities on Simplical Complexes

In this section, we will prove several functional inequalities involving the down-up walk $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow\ell}$. For convenience we define the set $\mathcal{C}_\ell(X)$ as the set of $\ell$-chains in $X$, i.e. the collection of sequences

$$\varnothing := \omega^{(0)} \subsetneq \omega^{(1)} \subsetneq \cdots \subsetneq \omega^{(\ell)} \in X^{(\ell)},$$

such that $\omega^{(i)} \in X^{(i)}$ for all $i = 0, \dots, \ell$. Similarly, for $x \in X^{(1)}$ we define $\mathcal{C}_\ell(x)$ as the set of $\ell$-chains in $X$ starting from $x \in X^{(1)}$, i.e. the collection of sequences

$$x =: \omega^{(1)} \subsetneq \omega^{(2)} \subsetneq \cdots \subsetneq \cdots \subsetneq \omega^{(\ell)},$$

such that $\omega^{(i)} \in X^{(i)}$ for all $i = 0, \dots, \ell$.

▶ **Theorem 22.** *For all $n$-partite simplicial complexes $(X, \pi)$ and convex $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we have:*

$$\mathtt{cf}_\Phi(\mathsf{P}^{\uparrow}_{\ell\to n}) \geq \min\left\{ \prod_{j=0}^{\ell-1}(1 - \mathtt{lc}_\Phi(\omega^{(j)})) \,\middle|\, \varnothing =: \omega^{(0)} \subsetneq \omega^{(1)} \subsetneq \cdots \subsetneq \omega^{(\ell)} \in \mathcal{C}_\ell(X) \right\}. \tag{7}$$

*In particular, writing $\mathtt{lc}^{(i)}_\Phi(X,\pi) = \max_{\hat\omega \in X^{(i)}} \mathtt{lc}_\Phi(\hat\omega)$, we have*

$$\mathtt{cf}_\Phi\left(\mathsf{P}^{\uparrow}_{\ell\to n}\right) \geq \prod_{j=0}^{\ell-1}\left(1 - \mathtt{lc}^{(j)}_\Phi(X,\pi)\right).$$

As mentioned before our proof is inspired by the exposition in [14] and follows the Garland method, [31]. After submitting our results to arxiv, it came to our attention that the same proof technique for proving a weaker version of Theorem 22 already appeared in [42] in the context of variance contraction. We will list a few immediate consequences of Theorem 22. The following bound is immediate given Proposition 10 and Theorem 22,

▶ **Corollary 23** (Spectral Gap Bound). *Let $(X, \pi)$ be a simplicial complex of rank $n$. We have,*

$$\mathtt{gap}(\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow\ell}) \geq \frac{n-\ell}{n} \cdot \min\left\{ \prod_{i=0}^{\ell-1} \mathtt{gap}(\mathsf{M}_{z_i}) \,\middle|\, \varnothing =: \omega^{(0)} \subsetneq \omega^{(1)} \subsetneq \cdots \subsetneq \omega^{(\ell)} \in \mathcal{C}_{\ell-1}(X) \right\}. \tag{8}$$

*In particular, writing $\mathtt{gap}_k(X,\pi) := \min_{x\in X^{(k)}} \mathtt{gap}(\mathsf{M}_x)$ we have*

$$\mathtt{gap}(\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow\ell}) \geq \frac{n-\ell}{n} \cdot \prod_{i=0}^{\ell-1} \mathtt{gap}_i(X,\pi).$$

We also prove a useful lemma that shows we can directly relate the entropy contraction constant to the log-Sobolev constant of the down-up walk,

▶ **Lemma 24.** *Let $(X,\pi)$ be a simplicial complex of rank $n$. For any $\hat\omega \in X$, we set*

$$\pi^\star_{\hat\omega,k} = \min_{\widetilde\omega \in X^{(k)}_{\hat\omega}} \pi^{(\hat\omega)}_k(\widetilde\omega), \quad \mathtt{gap}_{n-2}(X,\pi) = \min_{\hat\omega\in X^{(n-2)}} \mathtt{gap}(\mathsf{M}_{\hat\omega}), \quad \textit{and}$$

$$C_{\hat\omega,k} = \begin{cases} 1 & \pi^\star_{\hat\omega,k} > 1/2, \\ \frac{1-2\pi^\star_{\hat\omega,k}}{\log\left(\left(\pi^\star_{\hat\omega,k}\right)^{-1}-1\right)} & \textit{otherwise.}\end{cases}$$

*where $\mathsf{M}_{\hat\omega}$ is the link of $\hat\omega$ and $\mathtt{gap}(\bullet)$ denotes the spectral gap. We have,*

$$\mathtt{ls}(\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow\ell}) \;\geq\; \min\left\{ C_{\omega^{(\ell)},n-\ell} \;\middle|\; \omega^{(\ell)} \in X^{(\ell)}\right\}\cdot \mathtt{ec}\left(\mathsf{P}^{\uparrow}_{\ell\to n}\right),$$

$$\mathtt{ls}(\mathsf{P}^{\uparrow\downarrow}_{n-1}) \;\geq\; \frac{n-1}{n}\cdot\min\left\{ C_{\omega^{(n-2)},1} \;\middle|\; \omega^{(n-2)} \in X^{(n-2)}\right\}\cdot \mathtt{gap}_{n-2}(X,\pi)\cdot \mathtt{ec}(\mathsf{P}^{\uparrow}_{n-2\to n-1}).$$

*In particular, writing $\mathtt{lec}_i(X,\pi) := \min_{\hat\omega\in X^{(i)}} \mathtt{lec}(\hat\omega)$ and $C_{\ell,k} = \min_{\hat\omega\in X^{(\ell)}} C_{\hat\omega,k}$,*

$$\mathtt{ls}(\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow\ell}) \;\geq\; C_{\ell,n-\ell}\cdot\prod_{i=0}^{\ell-1}(1-\mathtt{lec}_i(X,\pi))$$

We will prove this result in Section 4.2.

## 4.1 Proof of Φ-Entropy Contraction Bounds, Theorem 22

**Proof of Theorem 22.** For $\ell = 0$, the LHS is equal to 0, thus we see the product in Equation (7) is taken over an empty set and equals 1. Thus, equality holds in this case with $\mathtt{cf}_\Phi(\mathsf{P}^\uparrow_{0\to n}) = 1$. We proceed by induction on the rank of the simplicial complex. We have by the chain rule for Φ-entropy (Lemma 11),

$$\mathrm{Ent}^\Phi_{\pi_\ell}(\mathsf{P}^\uparrow_{\ell\to n}\boldsymbol{f}) = \mathop{\mathbb{E}}_{x\sim\pi_1} \mathrm{Ent}^\Phi_{\pi^{(x)}_{\ell-1}}(\mathsf{P}^\uparrow_{x,\ell-1\to n-1}\boldsymbol{f}|_x) + \mathrm{Ent}^\Phi_{\pi_1}(\mathsf{P}^\uparrow_{1\to n}\boldsymbol{f}),$$

where we have used (i) $(\mathsf{P}^\uparrow_{\ell\to n}\boldsymbol{f})_x(\omega) = [\mathsf{P}^\uparrow_{x;\ell-1\to n-1}\boldsymbol{f}|_x](\omega\setminus x)$ and (ii) that $\mathsf{P}^\uparrow_{\ell\to n}$ is row-stochastic, i.e. $\mathbb{E}(\mathsf{P}^\uparrow_{\ell\to n}\boldsymbol{f}) = \mathbb{E}\,\boldsymbol{f}$. Let $c := \min_{x\sim X^{(1)}} \mathtt{cf}_\Phi(\mathsf{P}^\uparrow_{x,\ell-1\to n-1})$. By the induction hypothesis,

$$c \geq \min\left\{ \prod_{j=1}^{\ell-1}(1-\mathtt{lc}_\Phi(\omega^{(j)})) \;\middle|\; x\in X^{(1)}, x =: \omega^{(1)} \subsetneq \omega^{(2)} \subsetneq \cdots \subsetneq \omega^{(\ell-1)} \in \mathcal{C}_{\ell-1}(x)\right\}. \quad (9)$$

Hence, we obtain,

$$\mathrm{Ent}^\Phi_{\pi_\ell}(\mathsf{P}^\uparrow_{\ell\to n}\boldsymbol{f}) \;\leq\; (1-c)\mathop{\mathbb{E}}_{x\sim\pi_1}\mathrm{Ent}^\Phi_{\pi^{(x)}_{n-1}}(\boldsymbol{f}|_x) + \mathrm{Ent}^\Phi_{\pi_1}(\mathsf{P}^\uparrow_{1\to n}\boldsymbol{f}).$$

Now, using the chain-rule (Lemma 11) for Φ-entropy once more, we have $\mathbb{E}_{x\sim\pi_1}\mathrm{Ent}^\Phi_{\pi^{(x)}_{n-1}}(\boldsymbol{f}|_x) = \mathrm{Ent}^\Phi_{\pi_n}(\boldsymbol{f}) - \mathrm{Ent}_{\pi_1}(\mathsf{P}^\uparrow_{1\to n}\boldsymbol{f})$. Substituting this in the inequality above, we obtain:

$$\begin{aligned}\mathrm{Ent}^\Phi_{\pi_\ell}(\mathsf{P}^\uparrow_{\ell\to n}\boldsymbol{f}) \;&\leq\; (1-c)\cdot\left(\mathrm{Ent}^\Phi_{\pi_n}(\boldsymbol{f}) - \mathrm{Ent}^\Phi_{\pi_1}(\mathsf{P}^\uparrow_{1\to n}\boldsymbol{f})\right) + \mathrm{Ent}^\Phi_{\pi_1}(\mathsf{P}^\uparrow_{1\to n}\boldsymbol{f}),\\ &=\; (1-c)\cdot\mathrm{Ent}^\Phi_{\pi_n}(\boldsymbol{f}) + c\cdot\mathrm{Ent}^\Phi_{\pi_1}(\mathsf{P}^\uparrow_{1\to n}\boldsymbol{f}).\end{aligned}$$

Now, using $\mathrm{Ent}^\Phi_{\pi_1}(\mathsf{P}^\uparrow_{1\to n}\boldsymbol{f}) \leq \mathtt{lc}_\Phi(\varnothing)\cdot\mathrm{Ent}^\Phi_{\pi_n}(\boldsymbol{f})$ we obtain

$$\mathrm{Ent}^\Phi_{\pi_\ell}(\mathsf{P}^\uparrow_{\ell\to n}\boldsymbol{f}) \leq (1-c\cdot(1-\mathtt{lc}_\Phi(\varnothing)))\cdot\mathrm{Ent}^\Phi_{\pi_n}(\boldsymbol{f}).$$

The statement now follows from Equation (9) and the definition of the Φ-entropy contraction factor. ◀

## 4.2 Proof of the log-Sobolev Inequality, Lemma 24

**Proof of Lemma 24.** We follow a similar strategy to what we have followed to establish Theorem 22. We have,

$$\begin{aligned}\left\langle \boldsymbol{f}, \left(\mathsf{I} - \mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow\ell}\right)\boldsymbol{f}\right\rangle_{\pi_n} &= \langle \boldsymbol{f}, \boldsymbol{f}\rangle_{\pi_n} - \left\langle \mathsf{P}^{\uparrow}_{\ell\to n}\boldsymbol{f}, \mathsf{P}^{\uparrow}_{\ell\to n}\boldsymbol{f}\right\rangle_{\pi_\ell},\\ &= \mathop{\mathbb{E}}_{\hat\omega\sim\pi_\ell}\left[\langle \boldsymbol{f}|_{\hat\omega}, \boldsymbol{f}|_{\hat\omega}\rangle_{\pi^{(\hat\omega)}_{n-\ell}} - \mathop{\mathbb{E}}_{\hat\omega\sim\pi_\ell}\left\langle \mathsf{P}^{\uparrow}_{\hat\omega,0\to n-\ell}\boldsymbol{f}|_{\hat\omega}, \mathsf{P}^{\uparrow}_{\hat\omega,0\to n-\ell}\boldsymbol{f}|_{\hat\omega}\right\rangle_{\pi^{(\hat\omega)}_{n-\ell}}\right],\\ &= \mathop{\mathbb{E}}_{\hat\omega\sim\pi_\ell}\left[\left\langle \boldsymbol{f}|_{\hat\omega}, \left(\mathsf{I} - \mathsf{P}^{\downarrow\uparrow}_{\hat\omega,n-\ell\leftrightarrow 0}\right)\boldsymbol{f}|_{\hat\omega}\right\rangle_{\pi^{(\hat\omega)}_{n-\ell}}\right],\\ &= \mathop{\mathbb{E}}_{\hat\omega\sim\pi_\ell}\left[\left\langle \boldsymbol{f}|_{\hat\omega}, \left(\mathsf{I} - \mathsf{J}_{\pi^{(\hat\omega)}_{n-\ell}}\right)\boldsymbol{f}|_{\hat\omega}\right\rangle_{\pi^{(\hat\omega)}_{n-\ell}}\right],\end{aligned}$$

where we have used Items (1) and (2) of Lemma 12 to obtain the second equality and have written $\mathsf{J}_\mu = \mathbf{1}\mu$ for the clique with respect to $\mu$.

Now, by Lemma 5, we have $\mathtt{ls}\left(\mathsf{J}_{\pi^{(\hat\omega)}_{n-\ell}}\right) \geq C_{\hat\omega,n-\ell}$ – where $C_{\hat\omega,n-\ell}$ is defined as in the statement of Lemma 24. Thus, writing $C_{\ell,n-\ell} := \min_{\hat\omega\in X^{(\ell)}} C_{\hat\omega,n-\ell}$, we have

$$\left\langle \boldsymbol{f}, \left(\mathsf{I} - \mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow\ell}\right)\boldsymbol{f}\right\rangle_{\pi_n} \geq C_{\ell,n-\ell}\cdot \mathop{\mathbb{E}}_{\hat\omega\sim\pi_\ell} \mathrm{Ent}_{\pi^{(\hat\omega)}_{n-\ell}}\left(\boldsymbol{f}^2|_{\hat\omega}\right), \;\geq C_{\ell,n-\ell}\cdot\left(\mathrm{Ent}_{\pi_n}(\boldsymbol{f}^2) - \mathrm{Ent}_{\pi_\ell}\left(\mathsf{P}^{\uparrow}_{\ell\to n}\boldsymbol{f}^2\right)\right)$$

where we have used the chain rule for entropy, Lemma 11, to obtain the last statement.

Now, using the definition of $\Phi$-entropy contraction, i.e. writing for $\Phi(t) = t\cdot\log t$,

$$\mathrm{Ent}_{\pi_\ell}\left(\mathsf{P}^{\uparrow}_{\ell\to n}\boldsymbol{f}^2\right) \leq \left(1 - \mathsf{ec}\left(\mathsf{P}^{\uparrow}_{\ell\to n}\right)\right)\cdot \mathrm{Ent}_{\pi_n}(\boldsymbol{f}^2).$$

Thus,

$$\left\langle \boldsymbol{f}, \left(\mathsf{I} - \mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow\ell}\right)\boldsymbol{f}\right\rangle_{\pi_n} \geq C_{\ell,n-\ell}\cdot \mathsf{ec}\left(\mathsf{P}^{\uparrow}_{\ell\to n}\right)\cdot \mathrm{Ent}_{\pi_n}(\boldsymbol{f}^2).$$

Now, the first statement follows by appealing to the definition of the log-Sobolev inequality (LSI) and the log-Sobolev constant (Equation (2)). The second statement concerning $\mathsf{P}^{\downarrow\uparrow}_{n\leftrightarrow\ell}$ now immediately follows from Theorem 22.

To obtain the log-Sobolev inequality for $\mathsf{P}^{\uparrow\downarrow}_{n-1}$, we make use of Items (1) and (3) in Lemma 12 and proceed as above. We have,

$$\left\langle \boldsymbol{f}, \left(\mathsf{I} - \mathsf{P}^{\uparrow\downarrow}_{n-1}\right)\boldsymbol{f}\right\rangle_{\pi_{n-1}} = \frac{n-1}{n}\cdot \mathop{\mathbb{E}}_{\hat\omega\sim\pi_{n-2}}\left[\langle \boldsymbol{f}|_{\hat\omega}, (\mathsf{I} - \mathsf{M}_{\hat\omega})\boldsymbol{f}|_{\hat\omega}\rangle_{\pi^{(\hat\omega)}_1}\right]$$

Now, appealing to Lemma 5, we obtain $\mathtt{ls}(\mathsf{M}_{\hat\omega}) \geq \mathsf{gap}(\mathsf{M}_{\hat\omega})\cdot C_{\hat\omega,1}$ for all $\hat\omega \in X^{(n-2)}$. Thus,

$$\begin{aligned}\left\langle \boldsymbol{f}, \left(\mathsf{I} - \mathsf{P}^{\uparrow\downarrow}_{n-1}\right)\boldsymbol{f}\right\rangle_{\pi_{n-1}} &\geq \frac{n-1}{n}\cdot C_{n-2,1}\cdot \mathsf{gap}_{n-2}(X,\pi)\cdot \mathop{\mathbb{E}}_{\hat\omega\sim\pi_{n-2}} \mathrm{Ent}_{\pi^{(\hat\omega)}_1}(\boldsymbol{f}^2|_{\hat\omega}),\\ &= \frac{n-1}{n}\cdot C_{n-2,1}\cdot \mathsf{gap}_{n-2}(X,\pi)\cdot\left(\mathrm{Ent}_{\pi_{n-1}}(\boldsymbol{f}^2) - \mathrm{Ent}_{\pi_\ell}\left(\mathsf{P}^{\uparrow}_{n-2\to n-1}\boldsymbol{f}^2\right)\right),\\ &= \frac{n-1}{n}\cdot C_{n-2,1}\cdot \mathsf{gap}_{n-2}(X,\pi)\cdot \mathsf{ec}\left(\mathsf{P}^{\uparrow}_{n-2\to n-1}\right)\cdot \mathrm{Ent}_{\pi_{n-1}}(\boldsymbol{f}^2), \qquad (10)\end{aligned}$$

where we have appealed to the chain rule for entropy, Lemma 11, to obtain the first equality. ◀

## 5 Application: Sampling Using the Expanderized Walks

In the present section, present our results concerning the rapid mixing of the expanderized walks for sampling (i) list-colorings and (ii) Ising models with bounded interaction matrix. First, we describe the random sampling problems we are interested in mention the state of the art sampling results we are interested in expanderizing, and state our results and present a proof of the list coloring chain. Due to space considerations, we only present the proof of our list-coloring result here in Section 5.1 and refer the reader to the full version of our paper in [4] for the proof of our mixing time estimate for the Ising Model.

A list coloring instance $(G, \mathcal{L})$ consists of a graph $G = (V, E)$ and a collection of colours $\mathcal{L} = (L(v))_v$ for every vertex. A valid list coloring of $(G, \mathcal{L})$ is then a set of pairs $\{(v, c(v))\}_{v \in V}$ satisfying the following two conditions,

1. $c(v) \in L(v)$ for all vertices $v \in L$,
2. $c(u) \neq c(v)$ for all edges $\{u, v\} \in E$.

We will write $(X^{(G,\mathcal{L})}, \mathsf{uni}^{(G,\mathcal{L})})$ for the simplicial complex of proper list coloring of $(G, \mathcal{L})$ weighted by the uniform distribution $(G, \mathcal{L})$ on all list colorings, i.e.

$$X^{(G,\mathcal{L})} = \left\{ \alpha \subset \bigsqcup_{v \in V} \{v\} \times L(v) \;\middle|\; \text{there is a proper list coloring } \chi \text{ of } (G, \mathcal{L}) \text{ such that } \alpha \subset \chi \right\}.$$

We will show that the expanderized walks rapidly mix when sampling list colorings of bounded degree graphs. Further, the lower bound in the number of colors matches with the state of the art – see [12, 41, 10].

▶ **Theorem 25.** *Let $(G, \mathcal{L})$ be a list-coloring instance where $G = (V, E)$ is a graph on $n$ vertices of maximum degree $\Delta \leq O(1)$ and $H_n$ be a labelled graph on $[n]$ of constant two-sided expansion $\lambda(H_n)$ bounded away from 1. Then, for some absolute constant $\varepsilon \approx 10^{-5}$,[8] and any $K = O(1)$, if $(11/6 + K)\Delta \geq |L(v)| \geq (11/6 - \varepsilon) \cdot \Delta$ for all vertices $v \in V$, the mixing time of the expanderized walks $\mathsf{Q}^{\uparrow\downarrow}_{n-1} = \mathtt{UpDown}_{\ell \leftrightarrow n}(X^{(G,\mathcal{L})}, \mathsf{uni}^{(G,\mathcal{L})}, H^2)$ and $\mathsf{Q}^{\downarrow\uparrow}_{n} = \mathtt{DownUp}_{n \leftrightarrow \ell}(X^{(G,\mathcal{L})}, \mathsf{uni}^{(G,\mathcal{L})}, H)$ satisfies,*

$$\tau_{\mathtt{mix}}(\mathsf{Q}^{\uparrow\downarrow}_{n-1}, \varepsilon) \leq C_1 \cdot n\big(\log n + \log \varepsilon^{-1}\big) \quad \textit{and} \quad \tau_{\mathtt{mix}}(\mathsf{Q}^{\downarrow\uparrow}_{n}, \varepsilon) \leq C_2 \cdot n\big(\log n + \log\log \varepsilon^{-1}\big),$$

*where $C_1$ and $C_2$ are universal constants not depending on $n$ but on $\Delta$.*

▶ Remark 26. By [5], we can pick a constant degree graph as the graph $H_n$ in the statement of Theorem 25. Thus, a single step of the random walk can be implemented using $O(1)$-random bits – making the total number of random bits used in the random walk $O(n \log n)$. In contrast, the standard down-up walk or the up-down walk requires $O(\log n)$ random bits to perform a single step, and $O(n \log^2 n)$ random bits in total.

We recall that the Ising model $\mu_{\mathsf{J},\boldsymbol{h}} : \{+1, -1\}^n \to \mathbb{R}_{\geq 0}$ with interaction matrix $\mathsf{J} \in \mathbb{R}^{n \times n}$ and external field $\boldsymbol{h} \in \mathbb{R}^n$ from statistical physics is a probability distribution on the hypercube satisfying,

$$\mu_{\mathsf{J},\boldsymbol{h}}(\boldsymbol{x}) = \frac{\exp\big(\frac{1}{2}\langle \boldsymbol{x}, \mathsf{J}\boldsymbol{x}\rangle_{\ell_2} + \langle \boldsymbol{h}, \boldsymbol{x}\rangle_{\ell_2}\big)}{Z(\mathsf{J}, \boldsymbol{h})} \quad \text{where } Z(\mathsf{J}, \boldsymbol{h}) = \sum_{\boldsymbol{x} \in \{+1,-1\}^n} \exp\Big(\frac{1}{2}\langle \boldsymbol{x}, \mathsf{J}\boldsymbol{x}\rangle_{\ell_2} + \langle \boldsymbol{h}, \boldsymbol{x}\rangle_{\ell_2}\Big) \tag{11}$$

We notice that we can identify any $\boldsymbol{x} \in \{+1, -1\}^n$ with a value by using the encoding,

$$\boldsymbol{x}^{\pm} = \{(i, \boldsymbol{x}(i)) \mid i \in [n]\}.$$

Thus, we define the simplicial complex $(X^{(\mathsf{J},\boldsymbol{h})}, \mu_{\mathsf{J},\boldsymbol{h}})$, where

$$X^{(\mathsf{J},\boldsymbol{h})} = \{\alpha \subset [n] \times \{\pm 1\} \mid \text{ for each } i \in [n], \alpha \text{ contains at most one element } (i, x)\}.$$

We show that our expanderize walks mix rapidly assuming that the external field $\boldsymbol{h} \in \mathbb{R}^n$ is well-behaved, i.e. $\|\boldsymbol{h}\|_{\ell_\infty}$ does not grow with $n$,

[8] See [12].

▶ **Theorem 27.** *Let $(X^{(\mathsf{J},\boldsymbol{h})}, \mu_{\mathsf{J},\boldsymbol{h}})$ be the simplicial complex defined above corresponding to the Ising model defined by the interaction matrix $\mathsf{J} \in \mathbb{R}^{n\times n}$ and external field $\boldsymbol{h} \in \mathbb{R}^n$ and $H_n$ a constant degree graph whose two-sided expansion is a constant bounded away from 1. Under the assumption that $\mathsf{J}$ is PSD and satisfies $\|\mathsf{J}\|_{\mathsf{op}} \le 1$, the following hold,*

$$\tau_{\mathtt{mix}}(\mathsf{Q}^{\uparrow\downarrow}_{n-1}, \varepsilon) \le \frac{O(\|\boldsymbol{h}\|_{\ell_\infty}) \cdot n}{(1-\|\mathsf{J}\|_{\mathsf{op}})^2}\left(\log(n + \|\boldsymbol{h}\|_{\ell_1}) + \log \varepsilon^{-1}\right) \quad \textit{and}$$

$$\tau_{\mathtt{mix}}(\mathsf{Q}^{\downarrow\uparrow}_{n}, \varepsilon) \le \frac{O(\|\boldsymbol{h}\|_{\ell_\infty}) \cdot n}{(1-\|\mathsf{J}\|_{\mathsf{op}})^2}\left(\log(n + \|\boldsymbol{h}\|_{\ell_1}) + \log \varepsilon^{-1}\right),$$

*where the $O(\bullet)$ notation hides a universal constant not depending on $n, \mathsf{J}$, or $\boldsymbol{h}$. Furthermore, the term $(1-\|\mathsf{J}\|_{\mathsf{op}})^2$ in the denominator can be replaced with $(1-\|\mathsf{J}\|_{\mathsf{op}})(1-\theta)$ if the maximum operator norm of any two-by two principal submatrix of $\mathsf{J}$ is $\theta$.*

▶ Remark 28. By [5], we can pick a constant degree graph as the graph $H_n$ in the statement of Theorem 27. Thus, ignoring numerical difficulties in simulating biased coins, a single step of the random walk can be implemented using $O(1)$-random bits – making the total number of random bits used in the random walk $O(n \log n)$ when $\|\boldsymbol{h}\|_{\ell_\infty} = O(1)$. In contrast, the standard down-up walk or the up-down walk requires $O(\log n)$ random bits to perform a single step and $O(n \log^2 n)$ random bits in total.

## 5.1 List Coloring of Bounded Degree Graphs

We make the following observations about the complex associated to proper list colorings,

▶ **Proposition 29** (Folklore). [9] *Let $(G = (V, E), \mathcal{L})$ be a list-coloring instance. Let $K_+, K_- \in \mathbb{N}$ be parameters satisfying $\deg(v) + K_+ \ge |L(v)| \ge \deg(v) + K_-$ for all $v \in V$. Then, writing $(Y, \pi) := (X^{(G,\mathcal{L})}, \mathsf{uni}^{(G,\mathcal{L})})$ we have,*

$$\lambda_2(\mathsf{M}_{\hat{\chi}}) \le \frac{1}{K_-} \quad \textit{for all} \quad \hat{\chi} \in Y^{(n-2)},$$

*where $\mathsf{M}_{\hat{\chi}}$ is the link of the face $\hat{\chi}$.*

*Similarly, for any $\hat{\chi} \in Y^{(n-2)}$, we have $\min_{(u,c)\in Y^{(1)}_{\hat{\chi}}} \pi_1^{(\hat{\chi})}(u, c) \ge \frac{K_-}{(\Delta+K_+)^2}$ where $\Delta = \max_{v\in V} \deg(v)$.*

We recall the following result of [41, 10],

▶ **Theorem 30** (Theorem 1.2, [41]). *Let $(G, \mathcal{L})$ be a list-coloring instance where $G = (V, E)$ is a graph on $n$ vertices of maximum degree $\Delta \le O(1)$. Then, for some absolute constant $\varepsilon \approx 10^{-5}$,*[10] *if $|L(v)| \ge (11/6 - \varepsilon) \cdot \Delta$ for all vertices $v \in V$, then the spectral gap and the log-Sobolev constants (Equation (2)) of the down-up walk $\mathsf{P}^{\downarrow\uparrow}_n = \mathtt{DownUp}_{n\leftrightarrow n-1}(X^{(G,\mathcal{L})}, \mathsf{uni}^{(G,\mathcal{L})})$ on the collection of proper list colorings is $\Omega(n^{-1})$.*

Then, the following corollary immediately follows by Lemma 7 and Proposition 9,

▶ **Corollary 31.** *Let $(G, \mathcal{L})$ be a list-coloring instance where $G = (V, E)$ is a graph on $n$ vertices of maximum degree $\Delta \le O(1)$. Then, for some absolute constant $\varepsilon \approx 10^{-5}$, if $|L(v)| \ge (11/6 - \varepsilon) \cdot \Delta$ for all vertices $v \in V$, then the up-operator $\mathsf{P}^{\uparrow}_{n-1\to n} = \mathtt{Up}_{n-1\to n}\left(X^{(G,\mathcal{L})}, \mathsf{uni}^{(G,\mathcal{L})}\right)$ on the collection of proper list colorings of $(G, \mathcal{L})$ satisfies $\mathsf{ec}(\mathsf{P}^{\uparrow}_{n-1\to n}) \ge \Omega(n^{-1})$.*

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[9] See the full version of this paper [4] for a proof.
[10] See [12]

**Proof of Theorem 25.** Notice that by Lemma 13, Corollary 31 implies that $\mathtt{ec}(\mathsf{P}^{\uparrow}_{n-2\to n-1}) \geq \Omega(n^{-1})$ since by Proposition 29 when $\Delta = O(1)$, $C_{n-2} = \Omega(1)$, we have $\mathtt{gap}_{n-2}(X^{(G,\mathcal{L})}, \mathsf{uni}^{(G,\mathcal{L})}) = \Omega(1)$, by invoking Lemma 24 we obtain that the up-down walk $\mathsf{P}^{\uparrow\downarrow}_{n-1} = \mathtt{UpDown}_{n-1\leftrightarrow n}(X^{(G,\mathcal{L})}, \mathsf{uni}^{(G,\mathcal{L})})$ satisfies, $\mathtt{ls}(\mathsf{P}^{\uparrow\downarrow}_{n-1}) \geq \Omega(n^{-1})$. Then, by Corollary 21 and the assumption that the two-sided expansion $\lambda(H_n)$ is a constant bounded away from 1, we obtain $\mathtt{ec}(\mathsf{Q}^{\uparrow\downarrow}_{n}), \mathtt{ec}(\mathsf{Q}^{\downarrow\uparrow}_{n-1}) \geq \Omega(n^{-1})$. The result, concerning mixing times follows using Theorem 8 and the observation that the state space for both walks is of size at most $n \cdot ((K + 11/6) \cdot \Delta)^n$. ◀

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