Coboundary and Cosystolic Expansion Without Dependence on Dimension or Degree

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Abstract

We give new bounds on the cosystolic expansion constants of several families of high dimensional expanders, and the known coboundary expansion constants of order complexes of homogeneous geometric lattices, including the spherical building of $SL_n(\mathbb{F}_q)$. The improvement applies to the high dimensional expanders constructed by Lubotzky, Samuels and Vishne, and by Kaufman and Oppenheim.

Our new expansion constants do not depend on the degree of the complex nor on its dimension, nor on the group of coefficients. This implies improved bounds on Gromov's topological overlap constant, and on Dinur and Meshulam's cover stability, which may have applications for agreement testing.

In comparison, existing bounds decay exponentially with the ambient dimension (for spherical buildings) and in addition decay linearly with the degree (for all known bounded-degree high dimensional expanders). Our results are based on several new techniques:

- We develop a new "color-restriction" technique which enables proving dimension-free expansion by restricting a multi-partite complex to small random subsets of its color classes.
- We give a new "spectral" proof for Evra and Kaufman's local-to-global theorem, deriving better bounds and getting rid of the dependence on the degree. This theorem bounds the cosystolic expansion of a complex using coboundary expansion and spectral expansion of the links.
- We derive absolute bounds on the coboundary expansion of the spherical building (and any order complex of a homogeneous geometric lattice) by constructing a novel family of very short cones.

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1 Introduction

High dimensional expansion, which is a generalization of graph expansion to higher dimensional objects, is an active topic in recent years. The importance of graph expansion across many areas of computer science and mathematics, suggests that high dimensional expansion may also come to have significant impact. So far we have seen several exciting applications including analysis of convergence of Markov chains [1], and constructions of locally testable codes and quantum LDPC codes [16, 55].

Several notions of expansion that are equivalent in graphs, such as convergence of random walks, spectral expansion, and combinatorial expansion, turn out to diverge into two main notions in higher dimensions.

The first is the notion of local link expansion which has to do with the expansion of the graph underlying each of the links of the complex; where a link is a sub-complex obtained by taking all faces that contain a fixed lower-dimensional face. This notion is qualitatively equivalent to convergence of random walks, it implies agreement testing, and it captures a spectral similarity between a (possibly sparse) high dimensional expander and the dense complete complex. It allows a spectral decomposition of functions on the faces of the complex in the style of Fourier analysis on the Boolean hypercube, see [13, 41, 31, 3, 25].

The second notion is coboundary and cosystolic expansion. Here we look at the complex not only as a combinatorial object but also as a sequence of linear maps, called coboundary maps, defined by the incidence relations of the complex. The i-th coboundary map δ_i maps a function on the i-faces to a function on the $i+1$-faces, $C^0 \overset{\delta_0}{\to} C^1 \overset{\delta_1}{\to} \cdots \overset{\delta_{d-1}}{\to} C^d$ where $C^i = C^i(X, \mathbb{F}_2) = \{f : X(i) \to \mathbb{F}_2\}$ is the space of functions on i faces with coefficients in $\mathbb{F}_2$ (we will consider general groups of coefficients, beyond $\mathbb{F}_2$). These functions are called i-chains. The coboundary map δ_i is defined in a very natural way: the value of $\delta f(s)$ for any $s \in X(i+1)$ is the sum of $f(t)$ for all $s \supset t \in X(i)$ (the precise definition is in Section 2).

Coboundary (or cosystolic[1]) expansion captures how well the coboundary map tests its own kernel, in the sense of property testing. Given $f \in C^i$ such that $\delta f \approx 0$, coboundary expansion guarantees existence of some $g \in \ker \delta_i$ such that $f \approx g$. More precisely, a complex is a β coboundary (or cosystolic) expander if

$$wt(\delta f) \geqslant \beta \cdot \min_{g \in Ker\delta} \mathrm{dist}(f, g)$$

where $wt(\delta f)$ is the hamming weight of δf. We denote by $h^i(X)$ the largest value of β that satisfies the above inequality for all f.

Whereas for $i = 0$ coboundary expansion coincides with the combinatorial definition of edge expansion, for larger i, it may appear at first glance to be quite mysterious. However, this definition is far from being a merely syntactical generalization of the $i = 0$ case and turns out to provide a rich connection between topological and cohomological concepts and between several important concepts in TCS, which we describe briefly below.

The study of coboundary and cosystolic expansion was initiated independently by Linial, Meshulam and Wallach [45], [51] in their study of connectivity of random complexes, and by Gromov [29] in his work on the topological overlapping property. Kaufman and Lubotzky [36] were the first to realize the connection between this definition and property testing. This point of view is important in the recent breakthroughs constructing locally testable codes and quantum LDPC codes [16, 55] (see also earlier works [23]).

Moreover, the coboundary maps come from a natural way to associate a (simplicial) complex to a constraint satisfaction problem. Attach a Boolean variable to each i-face, and view the $(i+1)$-faces as parity constraints. The value that an assignment $f : X(i) \to \mathbb{F}_2$

[1] The difference between coboundary and cosystolic expansion is just whether the cohomology is 0 or not (i.e. whether $Ker\delta_{i+1} = Im\delta_i$). This distinction is not important for this exposition and the expansion inequality is the same in both cases.

gives on $s \in X(i+1)$ is $\delta f(s)$. This connection to CSPs has been harnessed towards showing that the CSPs derived from certain cosystolic expanders are hard to refute for resolution and for the sum of squares hierarchy, [17, 33].

In addition, cosystolic expansion of 1-chains (with non-abelian coefficients) of a complex has been connected to the stability of its topological covers [20]. Informally, a complex is cover-stable if slightly faulty simplicial covers are always "fixable" to valid simplicial covers. Perhaps surprisingly, this is related to agreement testing questions, particularly in the small 1% regime, which is a basic PCP primitive and part of the initial motivation for this work. We discuss agreement testing and its relation to coboundary expansion in more detail further below in this introduction.

In light of all of the above, we believe that cosystolic expansion is a fundamental notion that merits a deeper systematic study. Along with the aim of exploring its various implications, a more concrete research goal would be to give strong bounds, and ultimately nail down exactly, the correct expansion values for the most important and well-studied high dimensional expanders. We mention that to the best of our knowledge even for the simplest cases, such as expansion of k-chains in the n-simplex, exact expansion values are not yet completely determined.

In this work we provide new bounds for the coboundary expansion of the spherical building, and the cosystolic expansion of known bounded-degree high dimensional expanders including the complexes of [49, 48, 42].

Two of the most celebrated results in this area are the works of [35] and [22] showing that the bounded-degree families of Ramanujan complexes of [48] are cosystolic expanders. These works introduce an elegant local-to-global criterion, showing that if the links are coboundary expanders, and further assuming spectral expansion, then the entire complex is a cosystolic expander.

The estimates proven by [35, 22] for the coboundary expansion parameters are roughly $h^k(X) \geqslant \min\left(\frac{1}{Q}, (d!)^{-O(2^k)}\right)$. Here X is a d dimensional LSV complex and Q is the maximal degree of a vertex which is roughly equal to $1/\lambda^{O(d^2)}$ in these complexes, where λ is the spectral bound on the expansion of the links. Subsequent works by Kaufman and Mass [37, 38, 39], improved this bound to

$$h^k(X) \geqslant \min\left(\frac{1}{Q}, (d!)^{-O(k)}\right). \tag{1}$$

We completely get rid of the dependence on the ambient dimension d and on the maximal degree Q, and prove

▶ **Theorem 1.** *For every integer $d > 1$ and every small enough $\lambda > 0$ let X be a d-dimensional LSV complex whose links are λ-one-sided expanders. For every group* [2] *Γ, every small enough $\lambda > 0$ and every integer $k < d-1$, $h^k(X,\Gamma) \geqslant \exp(-O(k^6 \log k))$.*

Our bounds for h^k only depend on the dimension k of the chains, so for $k = 1$ they are absolute constants. For larger k we still suffer an exponential decay. We do not know what the correct bound should be and whether dependence on k is at all necessary.

The case of $k = 1$ is interesting even in complexes whose dimension is $d \gg 1$, because h^1 controls the cover stability of the complex, as shown in [20]. Our bounds also immediately give an improvement for the topological overlap constants, when plugged into the Gromov machinery [30, 21, 22]. We elaborate on both of these applications later below.

[2] The theorem holds for every group Γ for which cohomology is defined, namely, abelian groups for $k > 1$ and any group for $k = 1$.

The result is proven by enhancing the local-to-global criterion of [22], and introducing a variant of the local correction algorithm that makes local fixes only if they are sufficiently cost-effective. This is inspired by and resembles the algorithms in [22, 16, 55].

Our analysis is novel and departs from previous proofs: instead of relying on the so-called "fat machinery" of [22] (and its adaptations [37, 38]), our proof is 100% fat free and relies on the up/down averaging operators on *real-valued functions*. Our main argument is to show that, for a function h that is the indicator of the support of a (locally minimal) k-chain,

$$\|D\cdots Dh\|^2 \gtrsim \cdots \gtrsim \|DDh\|^2 \gtrsim \|Dh\|^2 \gtrsim \|h\|^2,$$

where D is the down averaging operator, and we write $a \gtrsim b$ whenever $a \geqslant \Omega(b)$. From here we easily derive a lower bound on $\|h\|^2$ showing that either the correction algorithm has found a nearby cocycle, or else the coboundary of our function was initially very large to begin with.

This method gives universal bounds on the cosystolic expansion of any complex whose links have both sufficient coboundary-expansion and sufficient local spectral expansion,

▶ **Theorem 2.** *Let $\beta, \lambda > 0$ and let $k > 0$ be an integer. Let X be a d-dimensional simplicial complex for $d \geqslant k+2$ and assume that X is a λ-one-sided local spectral expander. Let Γ be any group. Assume that for every non-empty $r \in X$, X_r is a coboundary expander and that $h^{k+1-|r|}(X_r, \Gamma) \geqslant \beta$. Then*

$$h^k(X,\Gamma) \geqslant \frac{\beta^{k+1}}{(k+2)!\cdot 4} - e\lambda.$$

Here $e \approx 2.71$ is Euler's number.

Armed with an improved local-to-global connection, we derive Theorem 1 from Theorem 2 by further strengthening the coboundary expansion of the links of the LSV complexes, namely spherical buildings. The best previously known bound on coboundary expansion of k-cochains in spherical buildings is due to [30] and [47]. They proved a lower bound of $\left(\binom{d+1}{k+1}(d+2)!\right)^{-1}$. This decays exponentially with the ambient dimension d, and with the cochain level k. We remove the dependence on d by developing a new technique which we call "color-restriction". The d-dimensional spherical buildings are colored, namely, they are $d+1$-partite. For a set of ℓ colors $F \subset [d+1]$, the color restriction X^F is the complex induced on vertices whose color is contained in F. The restriction to the the colors of F reduces the dimension of X from d to $\ell - 1$. We say that a color restriction X^F is a β-local coboundary expander, if X^F is a β-coboundary expander, and the same holds for the intersection of X^F with links (neighbourhoods) of faces whose color is disjoint from F. We show that if a typical color-restriction is a local coboundary expander, then the entire complex is a coboundary expander, and the expansion is independent of the dimension. Namely,

▶ **Theorem 3.** *Let k, ℓ, d be integers so that $k+2 \leqslant \ell \leqslant d$ and let $\beta, p \in (0,1]$. Let X be a $(d+1)$-partite d-dimensional simplicial complex so that*

$$\mathop{\mathbb{P}}_{F\in\binom{[d+1]}{\ell}}\left[X^F \textit{ is a } \beta\textit{-locally coboundary expander}\right] \geqslant p.$$

Then $h^k(X) \geqslant \frac{p\beta^{k+1}}{e(k+2)!}$.

Finally, to prove that the spherical building satisfies the conditions of this theorem, we need to show that a typical random color-restriction is a good coboundary expander. For this we rely on the "cone machinery" developed by Gromov [30], Kozlov and Meshulam [44], and Kaufman and Oppenheim [42]. We construct in the full version of this paper [10], a novel family of short cones, thus proving the following.

▶ **Theorem 4.** *Let $k \geqslant 0$. There is an absolute constant $\beta_k = \exp(-O(k^5 \log k)) \geqslant 0$ so that the following holds. Let X be the $SL_n(\mathbb{F}_q)$-spherical building for any integer $n \geqslant k+1$ and prime power q. Let Γ be any group. Then X is a coboundary expander with constant $h^k(X,\Gamma) \geqslant \beta_k$.*

In fact, we prove a more general version of this theorem, that holds for the order complex of any homogeneous geometric lattice, see the full version of this paper [10].

Most earlier works on cosystolic expansion focus on $\mathbb{F}_2$ coefficients (see [37] and [20] for two exceptions). This is an important case especially in light of Gromov's result connecting $\mathbb{F}_2$-expansion and topological overlap. However, expansion (of 1-chains) with respect to more general coefficients is necessary for results on topological covers and in turn for agreement testing. The theorems stated above show expansion of k-chains with respect to coefficients not only in $\mathbb{F}_2$ but in general abelian groups Γ, and when $k = 1$ also for non abelian groups Γ. In other words, the theorems hold for all groups of coefficients where the cohomology is defined.

Finally, we end with an upper bound. While most of our work is focused on lower bounds for coboundary and cosystolic expansion, we show in the full version of this paper [10] that families of dense simplicial complexes cannot have cosystolic expansion greater than $1 + o(1)$. This implies that high degree, in some weak sense, limits cosystolic expansion. It is interesting to compare this to a result of Kozlov and Meshulam that shows upper bounds on coboundary expansion of complexes with bounded degree [44].

1.1 Applications of cosystolic expansion

We describe two applications of cosystolic expansion for deriving topological properties of simplicial complexes.

Topological overlap

Cosystolic expansion was studied by [30] to give a combinatorial criterion for the topological overlapping property. Let $f : X \to \mathbb{R}^k$ be continuous mapping (with respect to the natural topology on X), i.e. f realizes X in $\mathbb{R}^k$. A point $p \in \mathbb{R}^k$ is called c-heavily covered if

$$\mathop{\mathbb{P}}_{s \in X(k)} [p \in f(s)] \geqslant c.$$

A well known result by [24] showed that for every affine map from the complete 2-dimensional complex to the plane, there exists a $\frac{1}{27}$-heavily covered point. Gromov's greatly generalized this theorem to all *continuous* functions (instead of only affine functions), all dimensions k (instead of $k = 2$) and complexes that are cosystolic expanders (instead of the complete complex), with c that depends on the dimension of the map k, as well as the cosystolic expansion constant. For a precise statement, see the full version of this paper [10].

The motivation for [22] was to show that there exists families of bounded degree simplicial complexes which have this property. They use [48] complexes and achieve a lower bound of $c \geqslant \min(\frac{1}{Q}, (d!)^{-O(2^k)})$, which comes from their bound on cosystolic expansion. This bound has been improved as a corollary of [39] to $\min(\frac{1}{Q}, (d!)^{-O(k)})$. Here again, d is the dimension of X, which may be much larger than k, and Q is the maximal degree of a vertex in X.

Plugging in our bounds into Gromov's theorem gives the bound $c \geqslant \exp(-O(k^7 \log k))$ for the topological overlapping property. This bound is free of the ambient dimension and of the degree.

Cover stability

The second author and Meshulam studied a topological locally testable property called *cover stability* [20]. This property is equivalent to cosystolic expansion of 1-chains. A covering map between two simplicial complexes X, Y is a surjective t-to-1 simplicial map[3] $\rho : Y(0) \to X(0)$ such that for every $\tilde{u} \in Y(0)$ and $\rho(\tilde{u}) = u \in X(0)$, it holds that the links of $\tilde{u}, u$ are isomorphic $Y_{\tilde{u}} \cong X_u$.

Graph covers (also known as lifts) have been quite useful in construction of expander graphs. Bilu and Linial showed that random covers of Ramanujan graphs are almost Ramanujan [6]. A celebrated result by [50] used these techniques to construct bipartite Ramanujan graphs of every degree. Recently, [9] showed that random covers could also be applied for constructing new simplicial complexes that are local spectral expanders.

Dinur and Meshulam [20] show that there exists a test that for any simplicial complex X and an alleged cover given by a simplicial map $\rho : Y \to X$ samples q points $(u_i, \rho(u_i))$ and measures how close ρ is to an actual covering map. The query complexity of the test is $q = 3t$ points. Its soundness is affected by the cosystolic expansion of 1-chains. Using our new bounds on cosystolic expansion, we show that the complexes constructed in [48] or in [42] are cover-stable, i.e. that there exists some universal constant $c > 0$, such that for every $\rho : Y(0) \to X(0)$

$$\mathop{\mathbb{P}}_{(u_i,\rho(u_i))_{i=1}^q} [\text{test fails}] \geqslant c \cdot \min \left\{ \mathrm{dist}(\rho, \psi) \,\middle|\, \psi : Y(0) \to X(0) \text{ is a cover} \right\},$$

where the distance is Hamming distance.

Agreement testing

Coboundary expansion found an exciting new application in agreement testing [28, 11, 5]. An agreement test is a consistency test that originated as a component in low degree testing [58, 2, 56], but has been extensively studied ever since (see e.g. [26, 34, 18]). This test is a crucial component in many PCP constructions [57, 26, 15, 34, 19]. Given a set of partial functions on a set, an agreement test is a way to test whether these functions are correlated with some function that defined on the whole vertex set. The works [28, 11, 5] mentioned above use coboundary expansion to characterize when an agreement test is sound. Via this characterization they analyze agreement tests on high dimensional expanders. Continuing this line of works, [12, 14, 4] use theorems and tools developed in a preliminary version of this paper, to lower bound coboundary expansion of new high dimensional expanders, and with these lower bounds they obtain new agreement tests. These include the first agreement tests where the underlying complex family is bounded degree in the so called "list decoding regime" (the regime that is relevant to high-soundness PCPs such as the parallel repetition PCP [57, 34]).

[3] simplicial means that every i-face in Y is sent to an i-face in X.

1.2 Related work

Coboundary and Cosystolic expansion was defined by Linial, Meshulam and Wallach [45], [51], and indpendently by Gromov [30]. Gromov studied cosystolic expansion as a proxy for showing the topological overlapping property. Linial, Meshulam and Wallach were interested in analyzing high dimensional connectivity of random complexes.

Kaufman, Kazhdan and Lubotzky [35] introduced an elegant local to global argument for proving cosystolic expansion of 1-chains in the *bounded-degree* Ramanujan complexes of [49, 48]. This was significantly extended by Evra and Kaufman [22] to cosystolic expansion in all dimensions, thereby resolving Gromov's conjecture about existence of bounded degree simplicial complexes with the topological overlapping property in all dimensions. Kaufman and Mass [37, 38] generalized the work of Evra and Kaufman from $\mathbb{F}_2$ to all other groups as well, and used this to construct lattices with good distance. The best previously known bound for LSV complexes (1) was shown by Kaufman and Mass in [39].

Following ideas that appeared implicitly in Gromov's work, Lubotzky Mozes and Meshulam analyzed the expansion of many "building like" complexes [47]. Kozlov and Meshulam [44] abstracted the main lower bound in [47] to the definition of cones (which they call chain homotopies), in order to analyze the coboundary expansion of geometric lattices and other complexes. Their work also connects coboundary expansion to other homological notions, and gives an upper bound to the coboundary expansion of bounded degree simplicial complexes. In [42], Kaufman and Oppenheim defined the notion of cones in order to analyze the cosystolic expansion of their high dimensional expanders (see [40]). In addition, they also come up with a criterion for showing that complexes admit short cones. They prove lower bounds on the cosystolic expansion of their complexes for 0- and 1-chains. The case of k-chains with $k \geqslant 2$ is still open.

Several works tried to define quantum LDPC codes as cohomologies of simplicial complexes. Cosystolic expansion is used for analyzing the distance of the quantum code. Works by Evra, Kaufman and Zémor [23] and by Kaufman and Tessler [43] used cosystolic expansion in Ramanujan complexes to construct quantum codes that beat the $\sqrt{n}$-distance barrier. This continued in a sequence of works [54, 32, 7] which culminated in the breakthrough work of [55] that construct quantum LDPC codes with constant rate and distance. This later code is a cohomology of a certain chain complex, albeit not a simplicial complex; and it is analyzed essentially through the cosystolic expansion. Developing new techniques for cosystolic expansion can be potentially useful in this domain as well.

1.3 Open questions

The works by [47], [44] and [42] analyze a variety of symmetric complexes (that support a transitive group action). Could one combine our "color restriction" technique with the cone machinery to get lower bounds independent of degree and dimension on these complexes as well? There are a number of concrete constructions of local spectral high dimensional expanders that have excellent local spectral properties [8, 46, 27, 52, 9]. Are any of them cosystolic expanders?

Another intriguing direction of research is to develop additional techniques for analyzing coboundary or cosystolic expansion. The current techniques are limited to complexes that either have a lot of symmetry, or have excellent local expansion properties. Are there other complexes with these properties?

Our expansion bounds still have a dependence on the level (k) of the chains. In the complete complex, for instance, this is not necessary. The complete complex is a $\beta = 1 + o(1)$ coboundary expander for all k-chains [47]. It is not clear whether a dependence on k is necessary even in the spherical building. Which complexes have coboundary expansion that does not decay with the size of the chains?

Finally, the notion of coboundary and cosystolic expansion is closely related to locally testable codes and quantum LDPC codes. They also have connections to agreement expanders. It is interesting to find additional applications for these expanders.

1.4 Overview of the proof of Theorem 1

We start with a complex X that is a finite quotient of the affine building, as constructed by [48]. Our goal is to lower bound the cosystolic expansion of X. The proof has three components:

- (Theorem 2) A new local-to-global argument that derives cosystolic expansion of the complex from coboundary and spectral expansion of its links.
- (Theorem 3) A general color restriction technique that reduces the task of analyzing the coboundary expansion of a partite complex, to that of analyzing the local coboundary expansion of random color restrictions of it.
- (Theorem 4) Bounds on random color restrictions of (links of) the spherical building. Towards this end we construct a novel family of short cones for the spherical building (not based on apartments as in previous works [47]).

Below we give a short overview of each of these steps. For simplicity we assume in this subsection that $\Gamma = \mathbb{F}_2$, which captures the main ideas.

The local to global argument, Theorem 2

Let X be our simplicial complex. We describe a correction algorithm, that takes as input a k-chain $f : X(k) \to \mathbb{F}_2$, with small coboundary $\mathbb{P}[\delta f \neq 0] = \varepsilon$ and outputs a k-chain $\tilde{f} : X(k) \to \mathbb{F}_2$ close to f that has no coboundary, i.e. $\delta\tilde{f} = 0$. For this overview, we focus on $k = 1$, i.e. f is a function on edges, which already exhibits the main ideas.

Let $\eta > 0$ be some predetermined parameter. Our algorithm locally fixes "stars" of lower dimensional faces, that is, sets $Star_k(r) = \{s \in X(k) \mid s \supseteq r\}$ for $r \in X(j)$ (when $j \leqslant k$). The fix takes place only if it is sufficiently useful: whenever it decreases the weight of δf by at least $\eta\,\mathbb{P}[Star_k(r)]$. In the case at hand, $k = 1$, so r is either a vertex or an edge, so

1. If $r \in X(1)$, $Star_1(r) = \{r\}$ and a fix just means changing the value of $f(r)$.
2. If $r \in X(0)$, $Star_1(r) = \{ru\}_{u\sim r}$ are all edges adjacent to r. Here a fix means changing the values of all $\{f(ru) \mid u \sim r\}$ simultaneously.

▶ **Algorithm 5.**

1. *Set* $f_0 := f$. *Set* $i = 0$.
2. *While there exists a vertex or edge* $r \in X(0) \cup X(1)$ *so that* $Star_k(r)$ *has an assignment that satisfies a* $\eta\,\mathbb{P}[Star_k(r)]$*-fraction of faces more than the current assignment.*
   - *Let* $fix_r : Star_k(r) \to \Gamma$ *be an optimal assignment to* $Star_k(r)$.
   - *Set* $f_{i+1}(s) = \begin{cases} f_i(t) & r \not\subseteq s \\ fix_r(s) & r \subseteq s \end{cases}.$
   - *Set i:=i+1.*
3. *Output the final function* $\tilde{f} := f_i$.

The fact that we correct f locally only if the fix satisfies η fraction more triangles will promise that $\text{dist}(f, \tilde{f}) \leqslant \frac{1}{\eta} wt(\delta f)$. The output of the algorithm, $\tilde{f}$, is *not* necessarily locally minimal in the sense of [35, 22], but it is *"η-locally-minimal"*.

Notation: For functions $g, h : X(\ell) \to \mathbb{R}$ we denote by $\langle g, h\rangle = \mathbb{E}_{r \in X(\ell)}[g(r)h(r)]$ the usual inner product. For $\ell = 1, 2$, denote by D^ℓ the *down operator* that takes $h : X(2) \to \mathbb{R}$ and outputs $D^\ell h : X(2-\ell) \to \mathbb{R}$ via averaging. Namely $D^\ell h(r)$ is the average of $h(s)$ over $s \supseteq r$, $\mathbb{E}_{s \supseteq r}[h(s)]$.

Let $h : X(2) \to \mathbb{R}$ indicate the support of a $\delta\tilde{f}$, so $h(t) = 1$ iff $\delta\tilde{f} \neq 0$. Our main argument is to show

$$\|D^3 h\|^2 \gtrsim \|D^2 h\|^2 \gtrsim \|Dh\|^2 \gtrsim \|h\|^2.$$

Eventually $D^3 h = \mathbb{E}[h]^2$ is just a constant function. This shows that $(\mathbb{E}[h])^2 = \mathit{const} \cdot \mathbb{E}[h]$ which implies that either the algorithm corrected f to a cosystol, i.e. $h = 0$, or that h has large weight, which implies that δf had large weight to begin with.

Let us show for example that $\|D^3h\|^2 \gtrsim \|D^2h\|^2$ given that $\|D^2h\|^2 \gtrsim \|Dh\|^2 \gtrsim \|h\|^2$. To do so, we define an auxiliary averaging operator N based on a random walk from vertices to triangles, and use the fact that in local spectral expanders,

$$\|D^3h\|^2 \approx \langle Nh, D^2h\rangle. \tag{2}$$

The operator $N : \ell_2(X(2)) \to \ell_2(X(0))$ is defined by $Nh(v) = \mathbb{E}_s[h(s)]$, where s is sampled according to the following walk: Given $v \in X(0)$, sample some $t \in X(3)$ such that $v \in t$, and then go to the triangle $s = t \setminus \{v\}$. We mention that the concept of localizing over such a distribution has appeared in [39]. The proof of (2) follows by localizing the expectation to the links and relying on the link expansion as in [53], [13, Claim 8.8] and in [41].

The key lemma in the proof shows that if there are many faces $s' \supseteq v_0$ such that $h(s') = 1$, then there are many s such that $v \notin s, \{v\} \cup s = t \in X(3)$, where $h(s) = 1$. More precisely, we will show that for every $v \in X(0)$ it holds that

$$Nh(v) \gtrsim \beta(D^2h(v) - \eta)^4. \tag{3}$$

This immediately implies that

$$\begin{aligned}
\langle Nh, D^2h\rangle &= \mathop{\mathbb{E}}_v\left[D^2h(v)Nh(v)\right] \\
&\overset{(3)}{\gtrsim} \beta(\mathop{\mathbb{E}}_v\left[(D^2h(v))^2\right] - \eta \mathop{\mathbb{E}}_v\left[D^2h(v)\right]) \\
&\gtrsim \beta\|D^2h\|^2 - \beta\eta\|h\|^2 \\
&\gtrsim \beta\|D^2h\|^2.
\end{aligned}$$

The second inequality follows from $\mathbb{E}_v\left[D^2h(v)\right] = \mathbb{E}_s[h(s)] = \|h\|^2$. The last inequality follows from the assumption that $\|h\|^2 = O(\|D^2h\|^2)$. Combining this with (2) gives us the desired inequality.

Let us understand what is written in (3). On the right-hand side, $D^2h(v) = \mathbb{P}_{xy \in X_v(1)}[h(vxy) = 1]$ is the fraction of triangles vxy containing v, such that $\delta\tilde{f}(vxy) \neq 0$. On the left-hand side, $Nh(v)$ is the fraction of s that complete v to some $t = v \cup s \in X(3)$, so that $\delta\tilde{f}(s) \neq 0$. For such an $s = uxy$,

$$0 = \delta\delta\tilde{f}(vuxy) = \delta\tilde{f}(uxy) + (\delta\tilde{f}(vux) + \delta\tilde{f}(vuy) + \delta\tilde{f}(vxy)). \tag{4}$$

Set $g : X_v(1) \to \mathbb{F}_2$ to be $g(xy) = \delta\tilde{f}(vxy)$, and note that g has the following properties:

1. By (4), $\delta\tilde{f}(uxy) = 1 \iff \delta g(uxy) = 1$.
2. $\mathbb{P}[g \neq 0] = \mathbb{P}_{s \ni v}\left[\delta\tilde{f}(s) \neq 0\right] = D^2h(v)$.
3. η-local-minimality: which is $\mathrm{dist}(g, B^1(X_v)) \geqslant \mathbb{P}[g \neq 0] - \eta$, where $B^1(X_v) = \{\delta\psi \mid \psi : X_v(0) \to \mathbb{F}_2\}$ is the set of coboundaries.

We explain the third item. Assume towards contradiction that $\mathrm{dist}(g, B^1(X_v)) < \mathbb{P}[g \neq 0] - \eta$ and let $\delta\psi$ be a coboundary closest to g. Then by changing the values of $\tilde{f}$ on $Star(v)$ to be $\tilde{f}'(vu) := \tilde{f}(vu) + \psi(u)$, we have that whenever $g(xy) = \delta\psi(xy)$, then the fixed function satisfies $\delta\tilde{f}'(vxy) = 0$. I.e.

$$\mathrm{dist}(g, \delta\psi) = \mathop{\mathbb{P}}_{vxy}\left[\delta\tilde{f}'(vxy) = 0\right] < \mathop{\mathbb{P}}_{vxy}\left[\delta\tilde{f}(vxy) = 0\right] - \eta.$$

This is a contradiction to the η-local minimality of $\tilde{f}$ which is guaranteed by the algorithm.

Here is where the coboundary expansion of X_v comes into play. By coboundary expansion, we have that $\mathbb{P}[\delta g(uxy) = 1] \geqslant \beta\,\mathrm{dist}(g, B^1(X_v))$. By combining the above we will get that

$$Nh(v) = \mathop{\mathbb{P}}_{uxy \in X_v(2)}\left[\delta\tilde{f}(uxy) \neq 0\right] \geqslant \beta(\mathop{\mathbb{P}}_{xy \in X_v(1)}[g(xy) \neq 0] - \eta) = \beta(D^2h(v) - \eta).$$

The "color restriction" technique, Theorem 3

For this overview, assume that $k = 2$ The full details are in the full version of this paper [10]. Let Y be a d-dimensional $(d+1)$-partite complex so that a p-fraction of its color restrictions Y^F are β-local-coboundary expanders. We begin with a 2-chain $f : Y(2) \to \mathbb{F}_2$ with small coboundary, namely $\mathbb{P}_{s \in Y(3)}[\delta f(s) \neq 0] = \varepsilon$. We need to find a 1-chain $g : Y(1) \to \mathbb{F}_2$ so that $\mathrm{dist}(f, \delta g) \leqslant O(\frac{\varepsilon}{\beta^3 p})$.

We first select a random color restriction, i.e. a set of colors so that Y^F is a local coboundary expander, that the weight of δf when restricted to triangles whose colors are in F is close to weight of δf on all Y. Averaging arguments guarantee that such F exists. Using this F, we construct g in three steps. In the first step we define g on edges with both endpoints colored in F, $uv \in Y^F$. In the second step we define g on edges with one endpoint colored in F, i.e. $uv \in Y(1)$ where $u \in Y^F$ and $v \notin Y^F$. In the third step we define g on edges $uv \in X(1)$ with neither endpoints colored in F, i.e. where $u, v \notin Y^F$. Every step uses the values of g that were constructed in the step before. For $k > 2$ the $(k-1)$-chain is constructed following a similar sequence of $k+1$ steps.

1. We start with the values of g on edges $vu \in Y^F(1)$. By the choice of F, the weight of δf inside Y^F is roughly ε. Local coboundary expansion implies that there exists a 1-chain g_0 whose coboundary is close to f on Y^F. We set $g(uv) = g_0(uv)$ for all $uv \in Y^F(1)$.
2. Next we define g on edges vu so that $v \notin Y^F$ and $u \in Y^F$. Fix some $v \notin Y^F$. Let $Y_v^F = \left\{s \in Y^F \;\middle|\; s \cup v \in Y\right\}$. This is the color restriction of the link of v. We wish to set values for $g(vu)$ for all edges vu such that $u \in Y_v^F(0)$. We describe a system of equations that we use to set the values of g on the edges vu so as to satisfy a maximal number of equations. For every $u_1u_2 \in Y_v^F(1)$, the triangle vu_1u_2 defines an equation:

$$f(vu_1u_2) + g(u_1u_2) = g(u_1v) + g(u_2v). \tag{5}$$

   Note that the left-hand side of the equation is known since we have the values of f on all triangles, and we already constructed g for edges $u_1u_2 \in Y^F(1)$. So the above is an equation with two unknowns. We set $g(vu)$ simultaneously for all $u \in Y_v^F(1)$ to be an assignment that satisfies the largest fraction of equations (ties broken arbitrarily).
   The idea behind this step is the following. Obviously, we'd like that $f(vu_1u_2) = g(u_1u_2) + g(u_1v) + g(u_2v)$ for as many triangles as possible, so it makes sense to define g to satisfy

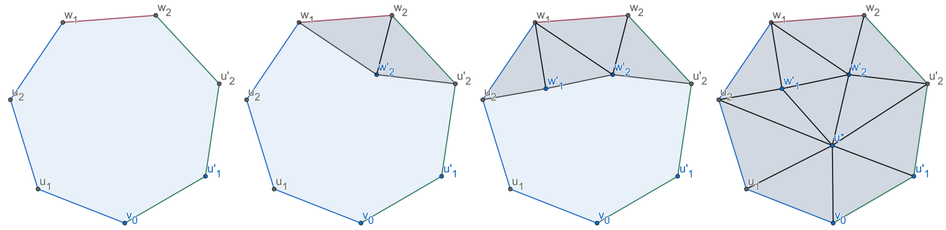

Figure 1 Tiling a cycle.

the largest amount of equations (5). Let $h_v : Y_v^F(1) \to \mathbb{F}_2$ be the left-hand side of (5), i.e. $h_v(u_1u_2) = f(vu_1u_2) + g(u_1u_2)$. We want to find an assignment $g_v : Y_v^F(0) \to \mathbb{F}_2$ so that $h_v(u_1u_2) = g_v(u_1) + g_v(u_2)$ for as many equations (5) as possible (and set $g(vu) = g_v(u)$). Finding a solution $g_v : Y_v^F(0) \to \mathbb{F}_2$ that satisfies (5) is equivalent finding g_v so that $h_v(u_1u_2) = \delta g_v(u_1u_2)$. Hence, to find an assignment that satisfies most of the equations is the same showing that h_v is close to a coboundary. In the analysis we show that $\delta h_v \approx 0$. This together with the local coboundary expansion of Y^F (which says that $h^1(Y_v^F, \mathbb{F}_2) \geqslant \beta$) will show that indeed we can find satisfying $\{g_v\}_{v \notin Y^F}$ so that $f \approx \delta g$ where the distance is over edges uv where $v \notin Y^F, u \in Y^F$.

3. Finally we need to define the values of g on edges vu so that $v, u \notin Y^F$. Let vu be such an edge. Every triangle uvw where $w \in Y_{vu}^F(0)$ defines a constraint on $g(vu)$:

$$f(uvw) + g(uw) + g(vw) = g(uv). \tag{6}$$

As in the previous case, $f(uvw)$ is known, and $g(uw), g(vw)$ were determined in step 2. We set $g(vu) = maj\left\{f(uvw) + g(uw) + g(vw) \,\middle|\, w \in Y_{uv}^F(0)\right\}$. Ties are broken arbitrarily. Here we use the local coboundary expansion of Y^F in a way similar to the previous step, to show that indeed $f \approx \partial g$.

New bounds on color-restrictions of the spherical building via cones, Theorem 4

In order to apply the color restriction technique we need to show that for a d-dimensional spherical building, many color restrictions are coboundary expanders[5]. For this overview we assume that $k = 1$ and $|F| = 5$. Let us see how to bound coboundary expansion by constructing short cones.

It turns out easier to do so when the set of colors is a set of colors that are geometrically increasing (e.g. for $k = 1$ we need colors $F = \{i_1, i_2, ..., i_5\}$ so that $i_j \geqslant 10 i_{j-1}$). The fraction of such sets of colors F is a constant that doesn't depend on d (it may depend on k). For example, there is a constant probability that we select colors F so that for $j = 1, 2, .., 5$, $\frac{d}{10^{16-3j}} \leqslant i_j < \frac{2d}{10^{16-3j}}$, since each of these intervals are a constant fraction of the interval $[1, 2, ..., d]$. When these inequalities hold then $i_j \geqslant 10 i_{j-1}$.

Denote by Y the $SL_d(\mathbb{F}_q)$-spherical building. Let Y^F be a complex induced by the subspaces of dimensions (i.e., colors) $F = \{i_1, i_2, ..., i_5\}$ so that $i_j \geqslant 10 i_{j-1}$). Using the cone technology described in the full version of this paper [10], showing the Y^F is a coboundary expander reduces to showing that there is a short 1-cone on Y^F. A 1-cone consists of three things:

[5] In fact, we need to show that the links of the color restrictions are also coboundary expanders, but we ignore this point in the overview for brevity.

1. A vertex $v \in X(0)$ (sometimes called the apex).
2. For every u, a path p_u from the apex v to u in $Y^F(1)$.
3. For every edge $uw \in$, a tiling by triangles $t_{uw} \subset Y^F(2)$ of the cycle that consists of the path p_u from v to u, the edge uw and the path p_w from w back to v. Denote this cycle by $p_u \circ uw \circ p_w$. Here a tiling is a set of triangles whose boundary is the edges of the cycle.

We give a formal and general definition of cones in the full version of this paper [10]. The radius of a cone is $rad((v, \{p_u\}_{u \in Y^F(0)}, \{t_{uw}\}_{uw \in Y^F(1)})) = \max_{uw \in X(1)} |t_{uw}|$.

We start by choosing an apex $v = v_0$ of dimension i_1 arbitrarily. Next we choose our paths to be as short as possible, and to consist of subspaces of dimension as low as possible. Explicitly we do the following.

1. For u adjacent to v_0, set $p_u = (v_0, u)$.
2. For u of the same dimension as v_0 we find some w of dimension i_2 so that w is a neighbour of v_0 and u, and set $p_u = (v_0, w, u)$. This is always possible since the dimension of $u + v_0$ is at most $2i_1$, so we can take any w of dimension $i_2 \geqslant 2i_1$ that contains the sum of spaces. (Notice how the fact that dimensions are geometrically increasing is important here).
3. For other $u \in Y^F(0)$, we first take some $w_2 \subseteq u$ of dimension i_1. Then we find some w_1 who is a neighbour of v_0 and of w_2 and we set $p_u = (v_0, w_1, w_2, u)$.

Constructing $t_{w_1 w_2}$ requires more care. Let us first consider the easier case. If $dim(w_1), dim(w_2) \leqslant i_4$ then the cycle $p_{w_1} \circ w_1 w_2 \circ p_{w_2}$ contains at most 7 vertices, all of dimension $\leqslant i_4$. In particular, the sum of all the vertices/subspaces is of dimension at most $7i_4 \leqslant i_5$, so there is a vertex u^* of dimension i_5 that contains all the vertices in the cycle. The set of triangles $u^* xy$ for all edges xy in the cycles is indeed a tiling of the cycle.

In the general case, it could be that the dimension of (say) w_1 is i_5. For example, assume that $dim(w_1) = i_5, dim(w_2) = i_4$ (in particular $w_2 \subseteq w_1$. It is useful to read this description while looking at Figure 1. In this case, we first find a tiling that "shifts" the cycle to a cycle of low dimension vertices. More explicitly, we find some $w_2' \subseteq w_2$ of dimension i_3, that is also connected to w's neighbours in the cycle. These neighbours are w_1 (and any subspace of w_2 is connected to it), and some u_2' of dimension $\leqslant i_2$, so we can indeed find some w_2' that is connected to u and u_2' of dimension i_3. We tile the cycle with $w_2 w_2' u_2', w_2 w_2' w_1$. This exchanges w_2 with w_2' in the untiled cycle. We perform a similar vertex-switch, for w_1 as well, finding some w_1' of dimension i_4 that is connected to w_1 neighbours in the untiled cycle. After these two steps, we can find a u^* that is connected to all the (now low-dimensional) cycle as in the previous case.

1.5 Organization of this paper and the full version

Section 2 contains preliminaries. We prove Theorem 2 that connects coboundary expansion in links to cosystolic expansion in Section 3 via the local correction algorithm. We develop the "color restriction" technique and prove Theorem 3 in the full version of this paper [10]. We analyze the expansion of the spherical building and other homogeneous geometric lattices in the full version of this paper [10]. In the full version [10], we also tie everything up and prove Theorem 1, as well as present the aforementioned applications of this bound. We also give there an upper bound on the cosystolic expansion of dense complexes.

2 Preliminaries and notation

For a more thorough preliminary section, see the full version of the paper [10].

Simplicial complexes

A pure d-dimensional simplicial complex X is a set system (or hypergraph) consisting of an arbitrary collection of sets of size $d+1$ together with all their subsets. The sets of size $i+1$ in X are denoted by $X(i)$, and in particular, the vertices of X are denoted by $X(0)$. We will sometimes omit set brackets and write for example $uvw \in X(2)$ instead of $\{u,v,w\} \in X(2)$. As convention $X(-1) = \{\emptyset\}$. Unless it is otherwise stated, we always assume that X is finite. Let X be a d-dimensional simplicial complex. Let $k \leqslant d$. We denote the set of oriented k-faces in X by $\vec{X}(k) = \{(v_0, v_1, ..., v_k) \mid \{v_0, v_1, ..., v_k\} \in X(k)\}$. For $s = (v_0, v_1, ..., v_k) \in \vec{X}(k)$ we denote $set(s) = \{v_i\}_{i=0}^k$, but when its clear from context we abuse notation and write s for its underlying set instead of $set(s)$. For an oriented face $s \in \vec{X}(k)$ and an index $i \in \{0, 1, ..., k\}$, we denote by s_i the face obtained by removing the i-th vertex of s.

Finally, Let $s = (v_0, ..., v_i)$, and $t = (u_0, ..., u_j)$. We denote by the concatenation $s \circ t = (v_0, v_1, ..., v_i, u_0, u_1, ..., u_j)$.

Probability over simplicial complexes

Let X be a simplicial complex and let $\mathbb{P}_d : X(d) \to (0,1]$ be a density function on $X(d)$ (i.e. $\sum_{s \in X(d)} \mathbb{P}_d(s) = 1$). This density function induces densities on lower level faces $\mathbb{P}_k : X(k) \to (0,1]$ by $\mathbb{P}_k(t) = \frac{1}{\binom{d+1}{k+1}} \sum_{s \in X(d), s \supset t} \mathbb{P}_d(s)$. We can also define a probability over directed faces, where we choose an ordering uniformly at random. Namely, for $s \in \vec{X}(k)$, $\mathbb{P}_k(s) = \frac{1}{(k+1)!} \mathbb{P}_k(set(s))$. When it's clear from the context, we omit the level of the faces, and just write $\mathbb{P}[T]$ or $\mathbb{P}_{t \in X(k)}[T]$ for a set $T \subseteq X(k)$.

2.1 Coboundary and cosystolic expansion

Asymmetric functions

Let X be a d-dimensional simplicial complex. Let $-1 \leqslant k \leqslant d$ be an integer. Let Γ be a group. A function $f : \vec{X}(k) \to \Gamma$ is *asymmetric* if for every $(v_0, v_1, ..., v_k) \in \vec{X}(k)$, and every permutation $\pi : [k] \to [k]$ it holds that

$$f(v_0, v_1, ..., v_k) = f(v_{\pi(0)}, v_{\pi(1)}, ..., v_{\pi(k)})^{sign(\pi)}.$$

We denote the set of these functions by $C^k(X, \Gamma)$. We note that by fixing some order to the vertices $X(0) = \{v_0, v_1, ..., v_n\}$, there is a bijection between functions $f : X(k) \to \Gamma$ and asymmetric functions $\vec{f} : \vec{X}(k) \to \Gamma$. Given $f : X(k) \to \Gamma$ and a set $s = \{v_{i_0}, v_{i_1}, ..., v_{i_k}\}$ so that $i_0 < i_1 < ... < i_k$, we set $\vec{f}(v_{\pi(i_0)}, v_{\pi(i_1)}, ..., v_{\pi(i_k)}) = f(s)^{\text{sign}(\pi)}$.

Let $f : \vec{X}(k) \to \Gamma$. The weight of f is $wt(f) = \mathbb{P}_{t \in X(k)}[f(t) \neq 0]$. For two functions $f, g : \vec{X}(k) \to \Gamma$ the distance between f and g is $\text{dist}(f,g) = wt(f-g) = \mathbb{P}_{t \in X(k)}[f(t) \neq g(t)]$.

Cohomology

Let Γ be an abelian group. The coboundary operator $\delta_k : C^k(X, \Gamma) \to C^{k+1}(X, \Gamma)$ is defined by $\delta_k f(s) = \sum_{i=0}^k (-1)^i f(s_i)$. It is a direct calculation to verify that for any asymmetric function $f \in C^k$ the function $\delta_k f$ is indeed an asymmetric function, and that $\delta_{k+1} \circ \delta_k = 0$.

Let $B^k(X,\Gamma) = \mathrm{I}m(\delta_{k-1})$ be the space of coboundaries. Let $Z^k(X,\Gamma) = \mathrm{K}er(\delta_k)$ be the space of cosystols. As $\delta_{k+1} \circ \delta_k = 0$, it holds that $B^k(X,\Gamma) \subseteq Z^k(X,\Gamma)$. The k-cohomology is $H^k(X,\Gamma) = Z^k(X,\Gamma)/B^k(X,\Gamma)$.

Coboundary expansion

For a function $f : \vec{X}(k) \to \Gamma$ let $\mathrm{dist}(f, B^k) = \min_{g \in C^{k-1}} \mathrm{dist}(f, \delta g)$, be the minimal distance between f and a coboundary. The k-th coboundary constant of a complex X (with respect to an abelian group Γ) is $h^k(X,\Gamma) = \min_{f \in C^k \setminus B^k} \frac{wt(\delta f)}{\mathrm{dist}(f,B^k)}$ where $B^k = B^k(X,\Gamma)$. Note that $h^k(X,\Gamma) > 0$ if and only if $H^k = 0$.

Cosystolic expansion

A very related high dimensional notion of expansion is cosystolic expansion. The k-th cosystolic expansion constant of X (with respect to an abelian group Γ) is

$$h^k(X,\Gamma) = \min_{f \in C^k \setminus Z^k} \frac{wt(\delta f)}{\mathrm{dist}(f, Z^k)},$$

where $Z^k = Z^k(X,\Gamma)$. Notice that when $B^k(X,\Gamma) = Z^k(X,\Gamma)$, namely, when $H^k = 0$, this coincides with the definition of coboundary expansion, and this justifies using the same notation h^k, where the term coboundary expansion (as opposed to cosystolic expansion) is taken to indicate $H^k = 0$.

Another useful way to understand the constant is the following. $h^k(X,\Gamma) \geqslant \beta$ if and only if for every $f : \vec{X}(k) \to \Gamma$ there is some $h \in Z^k(X,\Gamma)$ so that $\beta\,\mathrm{dist}(f,h) \leqslant wt(\delta f)$. We note that in the work of [22] cosystolic expanders were also required to have no small weight $f \in Z^k(X,\Gamma) \setminus B^k(X,\Gamma)$. We don't focus on this notion in our work.

Non abelian coboundary and cosystolic expansion

For $k = 0, 1$ we can define the cohomology with respect to non abelian groups as well. Let Γ be a non abelian group. As before, for every k we can define $C^k(X,\Gamma)$. We define the coboundary operators as follows:

1. $\delta_{-1} : C^{-1}(X,\Gamma) \to C^0(X,\Gamma)$ is $\delta_{-1}h(v) = h(\emptyset)$.
2. $\delta_0 : C^0(X,\Gamma) \to C^1(X,\Gamma)$ is $\delta_0 h(vu) = h(v)h(u)^{-1}$.
3. $\delta_1 : C^1(X,\Gamma) \to C^2(X,\Gamma)$ is $\delta_1 h(vuw) = h(vu)h(uw)h(wv)$.

It is easy to check that $\delta_{k+1} \circ \delta_k f = e$ where $e \in \Gamma$ is the unit. The definitions for $h^k(X,\Gamma)$ and coboundary expansion are the same as in the abelian case for $k = 0, 1$.

2.2 Local properties of simplicial complexes

Links of faces

Let X be a d-dimensional simplicial complex. Let $k < d$ and $s \in X(k)$. The link of s is a $d-k-1$-dimensional simplicial complex defined by $X_s = \{t \setminus s \mid \ t \in X, t \supseteq s\}$. We point out that the link of the empty set is $X_\emptyset = X$. Let $s \in X(k)$ for some $k \leqslant d$. The density function $\mathbb{P}_d$ on X induces on the link is $\mathbb{P}^s_{d-k-1} : X(d-k-1) \to (0,1]$ where $\mathbb{P}^s_{d-k-1}[t] = \frac{\mathbb{P}[t \cup s]}{\mathbb{P}[s]\binom{d+1}{k+1}}$. We usually omit s in the probability, and for $T \subseteq X_s(k)$ we write $\mathbb{P}_{t \in X_s(k)}[T]$ instead.

High dimensional local spectral expanders

Let X be a d-dimensional simplicial complex. Let $k \leqslant d$. The k-skeleton of X is $X^{\leqslant k} = \bigcup_{j=-1}^{k} X(j)$. In particular, the 1-skeleton of X is a graph.

▶ **Definition 6** (Spectral expander). *Let $G = (V, E)$ be a graph (that is, a 1-dimensional simplicial complex). Let A be its normalized adjacency operator, i.e. for every $f : V \to \mathbb{R}$, $Af : V \to \mathbb{R}$ is the function $Af(v) = \mathbb{E}_{uv\in E}[f(u)]$. Let $1 = \lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_{|V|} \geqslant -1$ be the eigenvalues of A.*

Let $\lambda \geqslant 0$. We say that G is a λ-one sided spectral expander if $\lambda_2 \leqslant \lambda$. We say that G is a λ-two sided spectral expander if $\lambda_2 \leqslant \lambda$ and $\lambda_{|V|} \geqslant -\lambda$.

▶ **Definition 7** (high dimensional local spectral expander). *Let X be a d-dimensional simplicial complex. Let $\lambda \geqslant 0$. We say that X is a λ-one sided (two sided) local spectral expander if for every $s \in X^{\leqslant d-2}$, the 1-skeleton of X_s is a λ-one sided (two sided) spectral expander.*

3 Cosystolic expansion

In this section we prove that local spectral expanders whose links are coboundary expanders are cosystolic expanders, that is, Theorem 2.

In fact, we prove a slightly more general statement, allowing for different coboundary expansion in every level.

▶ **Theorem 8.** *Let $k > 0$ be an integer and let $\beta_0, \beta_1, \beta_2, ..., \beta_k \in (0, 1]$ and $\lambda > 0$. Let X be a d-dimensional simplicial complex for $d \geqslant k + 2$ and assume that X is a λ-one-sided local spectral expander. Let Γ be any group. Assume that for every $0 \leqslant \ell \leqslant k$ and $r \in X(\ell)$, X_r is a coboundary expander and that $h^{k-\ell}(X_r, \Gamma) \geqslant \beta_{k-\ell}$. Then $h^k(X, \Gamma) \geqslant \frac{\prod_{\ell=0}^{k}\beta_\ell}{(k+2)!\cdot 4} - e\lambda$.*

Here $e \approx 2.71$ is Euler's number. Obviously, Theorem 2 follows from Theorem 8 by setting $\beta_\ell = \beta$ for every $\ell = 0, 1, 2, ..., k$. The following proposition, that is important for the topological overlapping property will also be proven via similar arguments.

▶ **Proposition 9.** *Let $k > 0$ be an integer and let $\beta_0, \beta_1, \beta_2, ..., \beta_{k-1} \in (0, 1]$ and $\lambda > 0$. Let X be a d-dimensional simplicial complex for $d \geqslant k + 1$ and assume that X is a λ-one-sided local spectral expander. Let Γ be any group. Assume that for every $0 \leqslant \ell \leqslant k - 1$ and $r \in X(\ell)$, X_r is a coboundary expander and that $h^{k-\ell}(X_r, \Gamma) \geqslant \beta_{k-\ell-1}$. Then every $g \in Z^k(X, \Gamma) \setminus B^k(X, \Gamma)$, has $wt(g) \geqslant \frac{\prod_{\ell=0}^{k-1}\beta_\ell}{(k+1)!} - e\lambda$.*

We remark that the when Γ is non abelian, these statements make sense only when $k = 1$. Turning back to Theorem 8, we present a correction algorithm. We will show that when $f \in C^k(X, \Gamma)$ has a small coboundary, then the algorithm below returns some $\tilde{f} \in Z^k(X, \Gamma)$ that is close to f.

▶ **Algorithm 10.** *Input: A function $f : \vec{X}(k) \to \Gamma$, a parameter $\eta \leqslant 1$. Output: A function $\tilde{f} : \vec{X}(k) \to \Gamma$.*

1. *Set $f_0 := f$. Set $i = 0$.*
2. *While there exists $\ell \leqslant k$, and a face $r \in \vec{X}(\ell)$ so that $Star_k(r) = \{s \in X(k) \mid r \subseteq s\}$ has an assignment that satisfies a $\eta\mathbb{P}[Star_k(r)]$-fraction of faces more than the current assignment, do:*
   - *Let $fix_r : Star_k(r) \to \Gamma$ be an optimal assignment to $Star_k(r)$, satisfying the maximal number of $k + 1$-faces containing r.*
   - *Set $f_{i+1}(s) = \begin{cases} f_i(s) & r \not\subseteq s \\ fix_r(s) & r \subseteq s \end{cases}$.*
   - *Set i:=i+1.*
3. *Output $\tilde{f} := f_i$.*

3.1 Properties of Algorithm 10

Before proving Theorem 8 we record some properties of Algorithm 10.

▷ Claim 11. Algorithm 10 halts on every input.

▷ Claim 12. Let $f : \vec{X}(k) \to \Gamma$ and let $\eta \leqslant 1$. Let $\tilde{f} : \vec{X}(k) \to \Gamma$ be the output of Algorithm 10 on (f, η). Then $\eta \operatorname{dist}(f, \tilde{f}) \leqslant wt(\delta f)$.

These claims are elementary, they proven in full in the full version of this paper [10].

3.2 Local minimality

▶ **Definition 13** (Restriction). *Let $g \in C^k(X, \Gamma)$ and let $r \in X(\ell)$ for some $0 \leqslant \ell \leqslant k-1$. The restriction of g to r is the function $g_r \in C^{k-\ell-1}(X_r, \Gamma)$ is defined by $g_r(p) = g(r \circ p)$.*

▶ **Definition 14** (Local minimality). *Let $\eta \geqslant 0$ and let $g \in C^k(X, \Gamma)$. We say that g is η-locally minimal, if for every $0 \leqslant \ell \leqslant k-1$, every $r \in X(\ell)$, and every $h \in C^{k-\ell-2}(X_r, \Gamma)$ it holds that $wt(g_r) \leqslant wt(g_r + \delta h) + \eta$.*

▶ **Definition 15** (Non abelian local minimality). *If Γ is non-abelian we need the correct analogy to adding coboundaries. The definition of η-minimality is as follows. If $k = 1$, we say that g is η-locally minimal if for every $v \in X(0)$, and every $\gamma \in \Gamma$, it holds that $wt(g_v) \leqslant wt(\gamma \cdot g_v) + \eta$. If $k = 2$, we say that g is η-locally minimal if:*

1. *For every edge uv and every $\gamma \in \Gamma$, it holds that $wt(g_{uv}) \leqslant wt(\gamma \cdot g_{uv}) + \eta$.*
2. *For every vertex v and every function $h : X_v(0) \to \Gamma$, it holds that $wt(g_v) \leqslant wt(g_v^h) + \eta$, where $g_v^h(uw) = h^{-1}(u) g_v(uw) h(w)$.*

The following claim is standard and is proven in the full version of this paper [10]

▷ Claim 16. Let $f : \vec{X}(k) \to \Gamma$ and let $\eta \leqslant 1$. Let $\tilde{f} : \vec{X}(k) \to \Gamma$ be the output of Algorithm 10 on (f, η). Then $\delta \tilde{f}$ is η-locally minimal.

3.3 Locally minimal cosystols are heavy

The following lemma states that non-zero functions that are locally minimal must have large weight.

▶ **Lemma 17.** *Let $\beta_0, \ldots, \beta_{k-1}$ and λ be as in Theorem 8. Let X be such that for every $0 \leqslant \ell \leqslant k-1$ and every $s \in X(\ell)$ it holds that X_s is a coboundary expander and $h^{k-\ell-1}(X_s, \Gamma) \geqslant \beta_{k-\ell-1}$. Assume further that X is a λ-local spectral expander. Let $g \in Z^k(X, \Gamma)$ be non-zero and η-locally minimal. Then*

$$wt(g) \geqslant \frac{\prod_{\ell=0}^{k-1} \beta_\ell}{(k+1)!} - e(\eta + \lambda). \tag{7}$$

Additionally, for the case of non-abelian Γ, when $k = 2$, (7) holds for η-locally minimal and non-zero $g = \delta f$, for any $f \in C^1(X, \Gamma)$.

The last remark regarding $k = 2$ is needed since $Z^2(X, \Gamma)$ is not defined for non-abelian groups Γ. This lemma implies Theorem 2 and Proposition 9 directly.

Proof of Theorem 8, given Lemma 17. Fix $\eta = \frac{\prod_{\ell=0}^{k}\beta_\ell}{4((k+2)!)}$. Let $\tilde{f}$ be the output of Algorithm 10 for some function f and η. If $wt(\delta f) \geqslant \frac{\prod_{\ell=0}^{k}\beta_\ell}{4(k+2)!} - e\lambda$ there is nothing to prove, so we assume that $wt(\delta f) < \frac{\prod_{\ell=0}^{k}\beta_\ell}{4(k+2)!} - e\lambda$. Then $\delta\tilde{f} \in Z^{k+1}(X,\Gamma)$ is an η-locally minimal function so that $wt(\delta\tilde{f}) \leqslant wt(\delta f)$. Hence by Lemma 17 (applied with $k+1$ instead of k), $\delta\tilde{f} = 0$ and $\tilde{f}$ is a cosystol. By Claim 12, $\eta\,\mathrm{dist}(f,\tilde{f}) \leqslant wt(\delta f)$, and we are done. ◀

Proof of Proposition 9, given Lemma 17. For every $r \in X(j)$ and $h \in C^{k-j-1}(X_r,\Gamma)$, we define $h^{\uparrow} : X(k) \to \Gamma$ by

$$h^{\uparrow}(s) = \begin{cases} h(p) & s = r \circ p \\ 0 & r \not\subseteq s. \end{cases}.$$

It is easy to see that $g_r + \delta h = (g + \delta h^{\uparrow})_r$.

Now let $0 \neq g \in Z^k(X,\Gamma) \setminus B^k(X,\Gamma)$ be minimal among all $Z^k(X,\Gamma) \setminus B^k(X,\Gamma)$. By the above, g is also 0-locally minimal (since otherwise we could have found some non-zero coboundary $\delta h^{\uparrow}$ to add to g and decrease its weight). Thus $wt(g) \geqslant \frac{\prod_{\ell=0}^{k-1}\beta_\ell}{(k+1)!} - e\lambda$ as required.

We remark that the case where Γ is non-abelian and $k = 1$ is similar. Given $g \in Z^1(X,\Gamma) \setminus B^1(X,\Gamma)$ that is non-zero and has minimal weight over all such functions. First we establish that it is locally minimal. Indeed, assume towards contradiction that there is some vertex $v \in X(0)$ and $\gamma \in \Gamma$ so that $wt(g_v) < wt(\gamma g_v)$. Then the function

$$g'(xy) = \begin{cases} \gamma g(xy) & x = v \\ g(xy)\gamma^{-1} & y = v \\ g(xy) & otherwise \end{cases}.$$

is also a cosystol. Taking some triangle $vuw \in X(2)$ that contains v, the value of

$$\delta g'(vuw) = \gamma \delta g(vuw)\gamma^{-1} = e$$

(the identity in Γ). For any triangle uwx that doesn't contain v we have that $\delta g'(uwx) = \delta g(uwx) = e$. On the other hand, $wt(g') < wt(g)$ so g' is trivial, which implies that $g = \delta h$ where $h(v) = \gamma$ and $h(u) = e$. A contradiction to the fact that $g \notin B^1(X,\Gamma)$. ◀

The remainder of this section is devoted to proving Lemma 17. For this we need to define averaging operators that play a crucial role in the theory behind local-spectral expanders. We will only define what we need so for a more thorough exposition see e.g. [13]. Let $\ell_2(X(j))$ be the Hilbert space of all functions $f : X(j) \to \mathbb{R}$ where the inner product is $\langle f, g\rangle = \mathbb{E}_{r\in X(j)}[f(r)g(r)]$. Let $D_k : \ell_2(X(k)) \to \ell_2(X(k-1))$ be the following operator

$$D_k f(s) = \mathop{\mathbb{E}}_{t \supseteq s}[f(t)].$$

This operator's adjoint is $U_{k-1} : \ell_2(X(k-1)) \to \ell_2(X(k))$ that is defined by

$$U_{k-1} f(t) = \mathop{\mathbb{E}}_{s \subseteq t}[f(s)].$$

As a shorthand we write $D_k^{\ell} = D_{k-\ell+1}D_{k-\ell+2}...D_k$ for $\ell \geqslant 1$ (and the same for U). For $\ell = 0$ $D_k^0 = U_k^0 = Id$. We record that $D_k^{\ell} f$ is a function whose domain is $X(k-\ell)$, and that $U_k^{\ell} f$ is a function whose domain is $X(k+\ell)$.

Let $j \leqslant k < d$. The operator $N_{k\to j} : \ell_2(X(k)) \to \ell_2(X(j))$ is defined by

$$N_{k\to j} f(r) = \mathop{\mathbb{E}}_{t\in X(k+1), t\supseteq r}\left[\mathop{\mathbb{E}}_{s\subseteq t, r\not\subseteq s}[f(s)]\right].$$

Let us spell out this expression. We average over $f(s)$ where s is chosen according to the following rule. We first sample some $t \supseteq r$ in $X(k+1)$, and then we sample $s \subseteq t$ given that it does not contain r.

When j, k is clear from the context we simply write D, U, N.

The following is an operator norm inequality that is similar to [13], but for the one-sided case. We prove it in the full version of this paper [10].

▷ Claim 18. Let X be a λ-one-sided local spectral expander. Then $U_j^{k-j} N_{k\to j} \preceq U_{j-1}^{k-j+1} D_k^{k-j+1} + \lambda Id$ for every $j \leqslant k$.

Here $A \preceq B$ for self adjoint operators A, B means that $B - A$ is positive semi-definite, that is, $\langle (B-A)h, h\rangle \geqslant 0$ for every function h in the domain of A, B.

Proof of Lemma 17. Let $h = \mathbf{1}_{g\neq 0}$. We will prove that $wt(g) = \mathbb{E}[h] \geqslant \frac{\prod_{\ell=0}^{k-1}\beta_\ell}{(k+1)!} - e(\eta+\lambda)$. We do this by showing that

1. $\|D_k h\|^2 \geqslant \frac{1}{k+1}\|h\|^2 - \lambda\|h\|^2$.
2. For $0 \leqslant j < k$, $\|D_k^{k-j+1} h\|^2 \geqslant \frac{\beta_{k-j-1}}{j+1} \cdot \|D_k^{k-j} h\|^2 - \left(\frac{\beta_{k-j-1}\eta}{j+1} + \lambda\right)\|h\|^2$.

We note that $D^{k+1}h$ is a constant (as λ-local spectral expansion says in particular that the complex is connected) - the average of h on all faces. Hence $\|D^{k+1}h\|^2 = \mathbb{E}[h]^2$. By iteratively applying these inequalities we get that

$$\begin{aligned}
\mathbb{E}[h]^2 &= \|D^{k+1}h\|^2 \\
&\geqslant \beta_{k-1}\|D^k h\|^2 - (\beta_{k-1}\eta + \lambda)\|h\|^2 \\
&\geqslant \frac{\beta_{k-1}\beta_{k-2}}{2}\|D^{k-1}h\|^2 - \beta_{k-1}\left(\frac{\beta_{k-2}\eta}{2} + \lambda\right)\|h\|^2 - (\beta_{k-1}\eta + \lambda)\|h\|^2 \\
&\dots \\
&\geqslant \|h\|^2 \cdot \left(\frac{\prod_{\ell=0}^{k-1}\beta_\ell}{(k+1)!} - \eta\sum_{j=0}^{k-1}\frac{\beta_j}{(k-j+1)!} - \lambda\left(1 + \sum_{j=0}^{k-1}\frac{\beta_j}{(k-j+1)!}\right)\right).
\end{aligned}$$

By assuming $\beta_j \leqslant 1$, we upper bound $\sum_{j=0}^{k-1}\frac{\beta_j}{(k-j+1)!} \leqslant \sum_{j=0}^{\infty}\frac{1}{j!} = e$, and get $\mathbb{E}[h]^2 \geqslant \|h\|^2 \cdot \frac{\prod_{\ell=0}^{k-1}\beta_\ell}{(k+1)!} - e(\eta+\lambda)$. As $\|h\|^2 = \mathbb{E}[h]$ the lemma follows.

Let us begin with the first item. we call $s \in X(k)$ *active* if $h(s) = 1$. By assumption, $g \in Z^k(X, \Gamma)$, i.e.

$$\delta g(t) = \sum_{i=0}^{k+1}(-1)^i g(t_i) = 0.$$

Thus if $t \in X(k+1)$ contains an active $s = t_{i_1}$, then it must also contain a second active

$s' = t_{i_2}$[6]. This implies that $N_{k\to k}h(s) \geqslant \frac{1}{k+1}h(s)$, and so

$$\langle h, N_{k\to k}h\rangle = \mathop{\mathbb{E}}_t[h(t)N_{k\to k}h(t)] \geqslant \frac{1}{k+1}\|h\|^2.$$

By Claim 18 $N^{k\to k} \preceq UD + \lambda Id$, so

$$\frac{1}{k+1}\|h\|^2 \leqslant \langle N_{k\to k}h, h\rangle \leqslant \langle UDh, h\rangle + \lambda\|h\|^2 = \|Dh\|^2 + \lambda\|h\|^2$$

so the first item is proven.

Next, we will prove the second item. As before, we will show that

$$\langle U^{k-j}N_{k\to j}h, h\rangle \geqslant \frac{\beta_{k-j-1}}{j+1}\cdot(\|D^{k-j}h\|^2 - \eta\|h\|^2). \tag{8}$$

Then we rely on Claim 18

$$\|D^{k-j+1}h\|^2 \geqslant \langle U^{k-j}N_{k\to j}h, h\rangle - \lambda\|h\|^2. \tag{9}$$

Combining these inequalities completes the proof.

We now state the following claim, which is proven using the coboundary expansion of X_r where r is a j-face.

▶ **Lemma 19** (Key lemma). *Let $r \in X(j)$. Then*

$$N_{k\to j}h(r) \geqslant \frac{\beta_{k-j-1}}{j+1}(D^{k-j}h(r) - \eta).$$

From this pointwise inequality, (8) follows easily:

$$\begin{aligned}\langle U^{k-j}N_{k\to j}h, h\rangle = \langle N_{k\to j}h, D^{k-j}h\rangle &\geqslant \mathop{\mathbb{E}}_r\left[D^{k-j}h(r)\cdot\frac{\beta_{k-j-1}}{j+1}\cdot(D^{k-j}h(r) - \eta)\right]\\ &= \frac{\beta_{k-j-1}}{j+1}\cdot(\|D^{k-j}h\|^2 - \eta\|h\|^2)\end{aligned} \tag{10}$$

◀

We will prove Lemma 19 under the assumption that Γ is abelian since additive notation is more convenient. For non-abelian groups, see Remark 20.

Proof of Lemma 19. First, let us understand the meaning of the inequality in Lemma 19. Recall that $N_{k\to j}h(r)$ is an average of $h(s)$ over faces $s \in X(k)$ so that $r, s \subseteq t$ for some $t \in X(k+1)$ and $r \not\subseteq s$. As h is an indicator function this is the same as writing

$$N_{k\to j}h(r) = \mathop{\mathbb{P}}_{t,s}[h(s) = 1],$$

where t, s are as above. On the other side of the inequality there is $D^{k-j}h(r) = \mathbb{P}_{s\supseteq r}[h(s) = 1]$. Hence, we need to show that if there are many active faces that contain r, there must also be many active faces that “complete” r to a $(k+1)$-face.

We first note that

$$N_{k\to j}h(r) = \mathop{\mathbb{P}}_{t,s}[h(s) = 1] \geqslant \frac{1}{j+1}\mathop{\mathbb{P}}_t[\exists s \subseteq t\ h(s) = 1 \text{ and } r \not\subseteq s], \tag{11}$$

so we shall actually lower bound $\mathbb{P}_t[\exists s \subseteq t\ h(s) = 1 \text{ and } r \not\subseteq s]$.

[6] in the case where Γ is non-abelian and $g = \delta f \in C^2(X,\Gamma)$, even though $\delta g(abcd)$ is not defined, one still observes that $\delta f(abc) = \delta f(acd) = \delta f(abd) = e$ implies that $\delta f(bcd) = 0$ so the same conclusion holds.

As $g \in Z^k(X,\Gamma)$, for every $t = r \circ p \in X(k+1)$

$$0 = \delta g(r \circ p) = \sum_{i=0}^{j}(-1)^i g(r_i \circ p) + (-1)^j \sum_{i=0}^{k-j}(-1)^i g(r \circ p_i). \tag{12}$$

And in particular

$$\sum_{i=0}^{k-j}(-1)^i g(r \circ p_i) \neq 0 \iff \sum_{i=0}^{j}(-1)^i g(r_i \circ p) \neq 0. \tag{13}$$

Recall that the restriction of g is $g_r : X_r(k-j-1) \to \Gamma$, defined by $g_r(p) = g(r \circ p)$. As we can see, $\delta g_r(p)$ is the left-hand side of (13). Thus

$$\mathop{\mathbb{P}}_{t}\left[\exists s \subseteq t\ h(s) = 1 \text{ and } r \not\subseteq s\right] \geqslant \mathop{\mathbb{P}}_{t=r\circ p}\left[\sum_{i=0}^{k-j}(-1)^i g(r \circ p_i) \neq 0\right] \tag{14}$$

$$= \mathop{\mathbb{P}}_{p \in X_r(k-j)}\left[\delta g_r(p) \neq 0\right]. \tag{15}$$

By assumption X_r is a β_{k-j-1}-coboundary expander, this is at least $\beta_{k-j-1} \cdot \mathrm{dist}(g_r, B^{k-j-1}(X_r,\Gamma))$.

To conclude we need to show that

$$\mathrm{dist}(g_r, B^{k-j-1}(X_r,\Gamma)) \geqslant \mathop{\mathbb{P}}_{s \supseteq r}\left[g(s) \neq 0\right] - \eta. \tag{16}$$

But

$$\mathrm{dist}(g_r, B^{k-j-1}(X_r,\Gamma)) = \min_{f \in C^{k-j-2}(X_r,\Gamma)}\{wt(g_r + \delta f)\} \geqslant wt(g_r) - \eta. \tag{17}$$

where the inequality follows from η-minimality of g. As $wt(g_r) = \mathbb{P}_{s\supseteq r}\left[h(s) = 1\right]$ we have proven

$$N_{k\to j}h(r) \geqslant \frac{\beta_{k-j-1}}{j+1}\,\mathrm{dist}(g_r, B^{k-j-1}(X_r,\Gamma)) \geqslant \frac{\beta_{k-j-1}}{j+1}\left(\mathop{\mathbb{P}}_{s\supseteq r}\left[h(s)=1\right] - \eta\right). \quad ◀$$

▶ Remark 20 (The non-abelian case). The first place where we need to accommodate for the non-commutativity is in the derivation of (14). Let us understand how to substitute (12) which implies (13), for non-abelian groups.

If for example, if $r \in X(0)$ and $g \in Z^1(X,\Gamma)$, and $ruw \in X(2)$ we can write

$$e = \delta g(ruw) = g(ru)g(uw)g(wr)$$

instead of (12). This implies that

$$g(uw) = g(ur)g(rw) \tag{18}$$

or

$$g(rw)g(uw)g(wr) = g(rw)\cdot(g(ur)g(rw))\cdot g(wr) = g(rw)g(ru)^{-1} = g_r(w)g_r(u)^{-1} = \delta g_r(wu)$$

where in the first equality we plugged in the first part of (18) and in the second to last equality we plugged in the second part of (18). Since the left hand side is a conjugation of $g(uw)$, we deduce that $g(uw) \neq e \iff \delta g_r(uw) \neq e$. This is the same conclusion as we get in (12). The case where $r \in X(1)$ is similar.

If $k=2$ we cannot even define (12) since the coboundary map is not defined. Still, let us see that a similar conclusion to (13) holds. Let $g = \delta f \in C^2(X,\Gamma)$. Let $r = ab \in X(1)$ and $t = abcd \in X(3)$. Denote by $\gamma = f(ab)f(bc)f(cd)$. Then

$$\begin{aligned}\gamma^{-1}\delta g_r(cd)\gamma &= \gamma^{-1} g(rc) g(rd)^{-1}\gamma \\ &= \gamma^{-1} g(abc) g(adb)\gamma \\ &= f(dc)\cdot (f(cb)f(ba)f(ab)f(bc))\cdot f(ca)f(ad)f(db)\cdot (f(ba)f(ab))\cdot f(bc)f(cd) \\ &= (f(dc)f(ca)f(ad))(f(db)f(bc)f(cd)) \\ &= \delta f(dca)\delta f(dbc) \\ &= g(dca)g(dcb)^{-1}.\end{aligned}$$

In particular, we deduce that $g(dca)g(dcb)^{-1} \neq e \iff \delta g_r(cd) \neq e$, and (14) now becomes

$$\mathop{\mathbb{P}}_{t}[\exists s \subseteq t\ h(s) = 1 \text{ and } r \not\subseteq s] \geqslant \mathop{\mathbb{P}}_{t = r\circ cd}\left[g(dca)g(dcb)^{-1} \neq 0\right] = \mathop{\mathbb{P}}_{cd \in X_r(1)}\left[\delta g_r(cd) \neq 0\right]. \quad (19)$$

Similarly, when $r = a \in X(0)$ and $t = abcd$ we observe similarly that

$$\begin{aligned}f(ab)g(bcd)f(ba) &= f(ab)(f(bc)f(cd)f(db))f(ba) \\ &= f(ab)f(bc)\cdot (f(ca)f(ac))\cdot f(cd)\cdot (f(da)f(ad))\cdot f(db)f(ba) \\ &= \delta f(abc)\delta f(acd)\delta f(adb) \\ &= g_a(bc)g_a(cd)g_a(db) \\ &= \delta g_a(bcd),\end{aligned}$$

and in particular

$$\mathop{\mathbb{P}}_{t}[\exists s \subseteq t\ h(s) = 1 \text{ and } r \not\subseteq s] \geqslant \mathop{\mathbb{P}}_{t = a\circ bcd}[g(bcd) \neq e] = \mathop{\mathbb{P}}_{bcd \in X_a(2)}[\delta g_a(bcd) \neq e]. \quad (20)$$

The second equality we need to modify is (17). For example, take an η-locally minimal $g \in C^2(X,\Gamma)$, a vertex $r \in X(0)$, and $\delta h \in B^1(X_r,\Gamma)$ that is a closest coboundary to $g_r \in C^1(X_r,\Gamma)$. Then

$$\mathrm{dist}(g_r,\delta h) = \mathbb{P}\left[g_r(vu) \neq h(v)h(u)^{-1}\right] = wt(g_r^h) \geqslant wt(g_r) - \eta.$$

The case where $r \in X(1)$ is similar.

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