A Simple Computability Theorem for Colorless Tasks in Submodels of the Iterated Immediate Snapshot

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Abstract

The Iterated Immediate Snapshot model (IIS) is a central model in distributed computing. We present our work in the message adversary setting. We consider general message adversaries whose executions are arbitrary subsets of executions $\mathcal{M}$ of the IIS message adversary. We present a complete and explicit characterization of solvable colorless tasks given any submodel of IIS.

Based upon the geometrization mapping *geo* introduced in [8] to investigate set-agreement in general submodels, we give a simple necessary and sufficient condition for computability. The geometrization *geo* associates to any execution a point in $\mathbb{R}^N$. A colorless task $(\mathcal{I}, \mathcal{O}, \Delta)$ is solvable under $\mathcal{M}$ if and only if there is a continuous function $f : geo(skel^n(\mathcal{I}) \times \mathcal{M}) \longrightarrow |\mathcal{O}|$ carried by Δ.

This necessary and sufficient condition for colorless tasks was already known for full models like the Iterated Immediate Snapshot model [14, Th. 4.3.1] so our result is an extension of the characterization to any arbitrary submodels. It also shows the notion of continuity that is relevant for distributed computability of submodels is not the one from abstract simplicial complexes but the standard one from $\mathbb{R}^N$. As an example of its effectiveness, we can now derive the characterization of the computability of set-agreement on submodels from [8] by a direct application of the No-Retraction theorem of standard topology textbook. We also give a new fully geometric proof of the known characterization of computable colorless tasks for $t-$resilient layered snapshot model by using cross-sections of fiber bundles, a standard tool in algebraic topology.

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1 Introduction

Distributed computability is the general investigation of which tasks could be solved in which distributed models. It is known since [12], [2] and [9] that some distributed tasks could have no algorithmic solution valid in all scenarios. Following the seminal works of Herlihy and Shavit [18], Borowsky and Gafni [5], Saks and Zaharoglou [28], using topological methods has proved very fruitful for distributed computing and for distributed computability in particular.

A distributed model that is widely used is the Iterated Immediate Snapshot (IIS) model, which is known to have the same task-computability power as the standard asynchronous read write wait-free model. In the setting of message adversaries, we consider general submodels $\mathcal{M}$ of the IIS model. These submodels correspond to arbitrary subsets of executions of IIS. We work on a subclass of distributed tasks, the *colorless tasks* : intuitively, any process can replace his input (resp. output) with the input (resp. output) of other processes while still correctly solving the task. Many important tasks like Consensus, $k-$set agreement, are colorless tasks. The ones needing to “break symmetry”, like Election or Renaming, are not.

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1.1 Related Work in Distributed Computability for IIS and Submodels

Distributed computability is a long time subject in distributed computing and topological methods are involved since introduced in [18, 28], see also the textbook [14] of Herlihy, Kozlov and Rajsbaum. We focus on this section on the previous results directly linked to our investigation, that is distributed computability for arbitrary submodels of the IIS model.

Models, Submodels and Message Adversaries. Topological methods have been first applied to the wait-free model, to be extended to some computability equivalent layered models, where these methods are more directly applicable. The Iterated Immediate Snapshot model appears now as a central model for distributed computing, either as a shared memory model or as a message adversary model.

We consider arbitrary subsets of executions of IIS, which is a setting that captures a wide range of models. Numerous submodels of the IIS model have already been investigated, they are usually called adversaries, see [14, Chap. 5] for results about colorless tasks for specific adversaries. Another category of adversaries are the affine adversaries of Kuznetsov and Rieutord [27, 22].

Our result is one of the few general results that can be applied to any submodel of the IIS model, which subsumes all models cited above. In this line of research, there have been works of Gafni, Kuznetsov and Manolescu in [10], and a recent extension by Attiya, Castañeda and Nowak in [3], following the work of Godard and Perdereau [11] for only two processes. In [3], a general computability theorem is presented for all submodels of the IIS model. Their results address both colorless and colored tasks, it is expressed using special infinite complexes called terminating subdivisions. This particular object capture non-uniform termination (to in part deal with non-compact sub-models) with infinite simplicial complexes. Since in this article we get rid of terminating subdivision we use the geometrization topology of [3].

Geometrization Topology. To include non-compact submodel we use the geometrization topology introduced by Godard and Perdereau in [11] for only two processes. It was later generalized by Coutouly and Godard in [8] to any number of processes in the case of general submodels of the Iterated Immediate Snapshot model. The geometrization mapping *geo* associates to any execution of IIS a point in $\mathbb{R}^N$. This induces a topology on IIS by considering as open sets the pre-images of open sets of $\mathbb{R}^N$. In [8], Coutouly and Godard only investigated the set-agreement task, not general colorless tasks. Moreover, the geometrization topology is mostly used in a descriptive way, not as a topology *per se*. This means that the geometrization topology is introduced in [8], but it is not actually used except as a way to provide some intuition to the combinatorial descriptions of some classes of executions. The main result of this paper, summarized in the following section, infers that there was actually more than an intuition, since we directly use this topology on sub model of IIS to express our computability results.

1.2 Our Contributions

A Generalized Computability Result. We build upon the *geo* mapping introduced by [8] and use it to express a new and simple colorless computability characterization. The *geo* mapping associates to any execution of IIS a point in $\mathbb{R}^N$ by considering an encoding by geometrical simplicial complexes. We adapt it to the colorless setting and a geometric universal colorless algorithm is presented, that is coined the Colorless Chromatic Average algorithm. The characterization is as follows in Theorem 11: a message adversary $\mathcal{M} \subset IIS_n$ solves a colorless task $(\mathcal{I}, \mathcal{O}, \Delta)$ if and only if there exists a continuous function $f : geo(skel^n(\mathcal{I}) \times \mathcal{M}) \longrightarrow |\mathcal{O}|$ carried by Δ.

This characterization was already known when $\mathcal{M} = IIS_n$, see [14, Thm. 4.3.1] and note that $geo(\mathcal{I} \times IIS_n) = |\mathcal{I}|$. Our result is therefore a wide extension of this simple topological characterization to any arbitrary submodel of the IIS model.

We discuss now the relevance of Theorem 11. The continuous function involved in this Theorem is continuous in the classical sense, i.e. for the topology of $\mathbb{R}^N$. In [3], a very general theorem is shown that relates computability to continuity of some function. However, this continuity is defined for a well chosen, but quite abstract and involved, topology on the set of executions. Here, thanks to the geometrization topology, we have basically to only deal with the standard continuity of the functions of $\mathbb{R}^N$, which actually appears more convenient, like in the Set-Agreement case, detailed below.

To compare to [3, Th 4.1] we focus less general task (only the colorless one) to remove the need from terminating subdivision. Both article have result on general adversaries of the IIS model. In this setting, a question of computability can be transformed to the existence of a continuous function between two classical topological spaces. For instance, we can directly use results from topology textbook as the No Retraction theorem [13, Cor. 2.15] to obtain a characterization for the Set-Agreement task. We also give a fully topological proof of the known characterization of the computable colorless tasks for the $t-$resilient layered snapshot protocol model by using cross-sections of fiber bundles.

These applications and their associated simple proofs, fully justifies, in our opinion, the move from abstract complexes to a fully geometric description of distributed systems by geometric simplicial complexes embedded in the ambient topology of $\mathbb{R}^N$. That is, the relevant simplicial complexes for distributed computability of colorless tasks are geometric simplicial complexes, seen as subspaces of $\mathbb{R}^N$ with its classical topology. It is known that abstract simplicial complex and geometric simplicial complex coincide when the complexes are of finite size. However, since dealing with general submodels, in particular so-called non-compact models, implies to associate a complex of possibly infinite size to distributed executions, we believe this is necessary.

Applications. As illustrations of our main computability theorem, we give new simple topological proofs of two known results : the characterization of submodels for which set-agreement is solvable (as already given in [8]) and the computability of colorless tasks against adversary. Our application are simple in the sense that we only use textbook theorems for "classical" topological spaces (like the standard Euclidian space $\mathbb{R}^N$ or the standard ball $\mathbb{S}^n$).

We investigate colorless tasks for so-called *adversary models*. These are sub-models of IIS where the failures can be not homogeneous: there is an arbitrary list F of sets of processes that can fail simultaneously, F is assumed to be inclusion closed.

A common example of adversary submodels is the $t-$faulty submodel which is a setting where at most t processes will eventually crash. In our IIS setting, this corresponds to the $t-$resilient layered snapshot protocol model. This is a well studied model, since [17] it is known such model cannot solve the $t-$set agreement. Nowadays, we have a nice topological understanding thanks to [15], a task is solvable in a $t-$resilient layered snapshot model if and only there is a continuous map from $|skel^t(\mathcal{I})| \to |\mathcal{O}|$ carried by Δ. From this, it can be deduced that t-resilient model cannot solve the k-set-Agreement task unless $k > t$. This result was also obtained in [4] using an algorithmic construction. A nice overview can be found in [21], and a comprehensive investigation in [14, Chap. 5]. We present here a new topological proof of these results that exploits a new topological and geometric interpretation of the reduction between models by using fiber bundles and cross sections, which are standard notion of topological spaces.

2 Models of Computation and Definitions

2.1 Models of Computation

We introduce our notations. Let $n \in \mathbb{N}$, we consider systems with $n+1$ processes. We denote $\Pi_n = [0, .., n]$ the set of processes. Since sending a message is an asymmetric operation, we will work with directed graphs. We use standard directed graph (or digraph) notations : given G, $V(G)$ is the set of vertices, $A(G) \subset V(G) \times V(G)$ is the set of arcs.

▶ **Definition 1.** *We denote by $\mathcal{G}_n$ the set of directed graphs with vertices in Π_n.*

A dynamic graph $\mathbf{G}$ *is a sequence $G_1, G_2, \cdots, G_r, \cdots$ where G_r is a directed graph with vertices in Π_n. We also denote by $\mathbf{G}(r)$ the digraph G_r. A* message adversary *is a set of dynamic graphs. Since that n will be mostly fixed through the paper, we use Π for the set of processes and $\mathcal{G}$ for the set of graphs with vertices Π when there is no ambiguity.*

Intuitively, the graph at position r of the sequence describes whether there will be, or not, transmission of some messages sent at round r. A formal definition of an execution under a dynamic graph will be given in Section 2.3. We will use the standard following notations in order to describe more easily our message adversaries [25].

A dynamic graph is seen as a infinite word over the alphabet $\mathcal{G}$.Given $U \subset \mathcal{G}$, U^* is the set of all finite sequences of elements of U, U^ω is the set of all infinite ones and $U^\infty = U^* \cup U^\omega$.

Given $\mathbf{G} \in \mathcal{G}^\omega$, if $\mathbf{G} = \mathbf{HK}$, with $\mathbf{H} \in \mathcal{G}^*, \mathbf{K} \in \mathcal{G}^\omega$, we say that $\mathbf{H}$ is *a prefix* of $\mathbf{G}$, and $\mathbf{K}$ *a suffix*. $Pref(\mathbf{G})$ denotes the set of prefixes of $\mathbf{G}$. A message adversary of the form U^ω, with $U \subset \mathcal{G}$, is called an *oblivious adversary* or an *iterated adversary*. A word in $\mathcal{M} \subset \mathcal{G}^\omega$ is called a *communication scenario* (or *scenario* for short) of message adversary $\mathcal{M}$. Given a word $\mathbf{H} \in \mathcal{G}^*$, it is called a *partial scenario* and $len(\mathbf{H})$ is the length of this word. The prefix of $\mathbf{G}$ of length r is denoted $\mathbf{G}_{|r}$ (not to be confused with $\mathbf{G}(r)$ which is the r-th letter of $\mathbf{G}$, it is the digraph at time r).

We show how standard fault environments are conveniently described in our framework.Consider a synchronous system of two processes $\circ$ and $\bullet$ where at most one of the processes can crash, the associated adversary is the following (using rational expression): $C_1 = \{\circ\!\leftrightarrow\!\bullet^\omega\} \cup \{\circ\!\leftrightarrow\!\bullet\}^*(\{\circ\!\leftarrow\!\bullet^\omega, \circ\!\rightarrow\!\bullet^\omega\})$. In the system of two processes $\circ$ and $\bullet$ where, at each round, only one message can be lost, the associated message adversary is $\{\circ\!\leftrightarrow\!\bullet, \circ\!\leftarrow\!\bullet, \circ\!\rightarrow\!\bullet\}^\omega$.

2.2 Iterated Immediate Snapshot Message Adversary

The previous example is IIS_1, we now detail the main message adversary we consider. Given a graph G, we denote by $In_G(a) = \{b \in V(G) \mid (b, a) \in A(G)\}$ the set of incoming vertices of a in $V(G)$. A graph G has the *containment Property* if for all $a, b \in V(G)$, $In_G(a) \subset In_G(b)$ or $In_G(b) \subset In_G(a)$. We say that a graph G has the *Immediacy Property* if for all $a, b, c \in V(G)$, $(a, b), (b, c) \in A(G)$ implies that $(a, c) \in A(G)$.

▶ **Definition 2** ([14])**.** *We set $ImS_n = \{G \in \mathcal{G}_n \mid G$ has the Immediacy and Containment properties $\}$. The Iterated Immediate Snapshot message adversary for $n+1$ processes is the message adversary $IIS_n = ImS_n^\omega$.*

The Iterated Immediate Snapshot model was first introduced as a (shared) memory model and then has been shown to be equivalent to the above message adversary first as tournaments and iterated tournaments [6, 1], then as this message adversary [14, 16]. See also [26] for a survey of the reductions involved in these layered models.

2.3 Execution of a Distributed Algorithm

Given a message adversary $\mathcal{M}$ and a set of initial configurations $\mathcal{I}$, we define what is an execution of a given algorithm $\mathcal{A}$ subject to $\mathcal{M}$ with initialization $\mathcal{I}$. An execution is constituted of an initialization step, and a (possibly infinite) sequence of rounds of messages exchanges and corresponding local state updates. When the initialization is clear from the context, we will use *scenario* and *execution* interchangeably.

An execution of an algorithm $\mathcal{A}$ under scenario $w \in \mathcal{M}$ and initialization $\iota \in \mathcal{I}$ is the following. This execution is denoted $\iota.w$. First, ι affects the initial state to all processes of Π. Then the system progresses in rounds. A round is decomposed in 3 steps : sending, receiving, updating the local state. At round $r \in \mathbb{N}$, messages are sent by the processes using the `SendAll()` primitive. The fact that the corresponding receive actions, using the `Receive()` primitive, will be successful depends on $G = w(r)$, G is called the *instant graph* at round r.

Let $p, q \in \Pi$. The message sent by p is received by q on the condition that the arc $(p, q) \in A(G)$. Then, all processes update their state according to the received values and $\mathcal{A}$. Note that it is assumed that p always receives its own value, that is $(p, p) \in A(G)$ for all p and G. However, in examples, this might be implicit for clarity and brevity.

Let $w \in \mathcal{M}, \iota \in \mathcal{I}$. Given $u \in Pref(w)$, we denote by $\mathbf{s}_p(\iota.u)$ the state of process p at the $len(u)$-th round of the algorithm $\mathcal{A}$ under scenario w with initialization ι. This means that $\mathbf{s}_p(\iota.\varepsilon) = \iota(p)$ represents the initial state of p in ι, where ε denotes the empty word.

3 Task Definition

We start by restating some standard definitions of combinatorial topology.

▶ **Definition 3** (Abstract simplicial complex). *[14, Def 3.2.1] Let V be a set, and C a collection of finite subsets of V. C is an abstract simplicial complex on V if*

1. $\forall \sigma \in C, \forall \tau \subseteq \sigma$, *we have* $\tau \in C$;
2. $\forall v \in V, \{v\} \in C$.

An element of V is a *vertex* of C and $V(C)$ denotes the set of vertices of C. A set $\sigma \in C$ is a *simplex* where $dim\ \sigma$ is the number of vertices in σ minus one. We say that σ is a *facet* if there is no other simplex that contains σ. If $C_1 \subseteq C_2$ then we say that C_1 is a *subcomplex* of C_2, a complex is *pure* if all facets have the same dimension. The pair (C_1, χ_{C_1}) is a *chromatic complex* if C_1 is a complex and the function $\chi_{C_1} : V(C_1) \to \Pi$ has the property that $\forall \sigma \in C_1, \forall v_1, v_2 \in V(\sigma), v_1 \neq v_2 \Leftrightarrow \chi_{C_1}(v_1) \neq \chi_{C_1}(v_2)$.

The *border* of a simplex σ, is $\partial(\sigma) = \{\tau \in \sigma | dim(\tau) = \dim(\sigma) - 1\}$. A *$\ell$-skeleton* of C_1 is the collection of the simplices of dimension equal or less than ℓ, we write $skel^{\ell}(C_1)$. The *star* of a simplex $\sigma \in C_1$ is $St(\sigma, C_1) = \bigcup_{\tau \in C_1, \sigma \subseteq \tau} \tau$, the *extended star* is $St^*(\sigma, C_1) = \bigcup_{v \in \sigma} St(v, C_1)$.

▶ **Definition 4** (Simplical map). *[14, Def 3.2.2] Let C_1, C_2 be two simplicial complexes, a simplicial map is a map $\Phi : V(C_1) \to V(C_2)$ such that $\forall \sigma \in C_1, \Phi(\sigma) \in C_2$.*

▶ **Definition 5** (Carrier map). *[14, Def 3.4.1] Let C_1, C_2 be two simplicial complexes, a carrier map $\Phi : C_1 \to 2^{C_2}$ is a mapping such that $\forall \sigma, \tau \in C_1$, and $\sigma \subseteq \tau$ imply $\Phi(\sigma) \subseteq \Phi(\tau)$.*

In addition, a carrier map $\Phi : C_1 \to 2^{C_2}$ is *rigid* when $\forall \sigma \in C_1, dim(\sigma) = d, \Phi(\sigma)$ is a pure complex of dimension d. A simplicial map $\varphi : C_1 \to C_2$ is *carried* by Φ if $\forall \sigma \in C_1, \varphi(\sigma) \in \Phi(\sigma)$. A carrier map is *chromatic* if it is rigid and $\forall \sigma \in C_1, \chi_{C_1}(\sigma) = \chi_{C_2}(\varphi(\sigma))$ where $\chi_{C_2}(\varphi(\sigma)) = \{\chi_{C_2}(v) | v \in V(\varphi(\sigma))\}$. We say that V_{in} is the domain of input values and V_{out} the domain of output values.

▶ **Definition 6** (Colorless Task). *[14, Def 4.2.1] A* colorless task *is a triple* $(\mathcal{I}, \mathcal{O}, \Delta)$ *where :*

- $\mathcal{I}$ *is the input complex, where each simplex is a subset of* V_{in}*,*
- $\mathcal{O}$ *is the output complex, where each simplex is a subset of* V_{out}*,*
- $\Delta : \mathcal{I} \to 2^{\mathcal{O}}$ *is a carrier map that encodes the specification of the task.*

In [14, Chap 4.1], the notion of colorless protocol is presented both operationally and combinatorially. We will give the corresponding geometric version in Algorithm 1.

▶ **Definition 7.** *An algorithm* $\mathcal{A}$ *solves a colorless task* $(\mathcal{I}, \mathcal{O}, \Delta)$ *for the message adversary* $\mathcal{M}$ *if for any* $\iota \in \mathcal{I}$*, any scenario* $w \in \mathcal{M}$*, there exist* u *a prefix of* w *such that the state of the system* $\{\mathbf{s}_0(\iota.u), \ldots, \mathbf{s}_n(\iota.u)\} = out$ *satisfies the specification of the task, ie* $out \in \Delta(\iota)$*.*

4 Geometric Definition of Simplicial Complexes

4.1 Standard Definitions

In this paper, we actually handle simplicial complexes as geometric complexes, so we present the standard definitions of simplicial complexes in the geometric setting [24]. We fix $N \in \mathbb{N}$. We note $\mathbf{B}(x, r) = \{y \in X | d(x, y) \leq r\}$ with $x \in \mathbb{R}^N, r \in \mathbb{R}$ and $d(x, y)$ the Euclidean distance on $\mathbb{R}^N$.

▶ **Definition 8** (Geometric Simplex). *Let* $n \in \mathbb{N}$*. A finite set* $\sigma = \{x_0, \ldots, x_n\} \subset \mathbb{R}^N$ *is called a* simplex *of dimension* n *if the vectors* $\{x_1 - x_0, \ldots, x_n - x_0\}$ *are linearly independent.*

We denote by $|\sigma|$ the convex hull of σ and $Int(\sigma)$ is the interior of $|\sigma|$. We denote $\mathbb{S}^n$ "the" simplex of dimension n : through this paper we assume a fixed embedding in $\mathbb{R}^N$ for $\mathbb{S}^n = (x_0^*, \ldots, x_n^*)$. We will also assume that its diameter $diam(\mathbb{S}^n)$ is 1. We usually associate χ such that $\chi(x_i^*) = i$, to get the chromatic simplex $\mathbb{S}^n$.

▶ **Definition 9** ([24]). *A* simplicial complex *is a collection* C *of* simplices *such that :*

(a) *If* $\sigma \in C$ *and* $\sigma' \subset \sigma$*, then* $\sigma' \in C$*,*

(b) *If* $\sigma, \tau \in C$ *and* $|\sigma| \cap |\tau| \neq \emptyset$ *then there exists* $\sigma' \in C$ *such that*
  - $|\sigma| \cap |\tau| = |\sigma'|$*,*
  - $\sigma' \subset \sigma, \sigma' \subset \tau$*.*

We denote $\wr C \wr = \bigcup_{S \in C} |S|$, this is the *geometrization of* C. Note that the geometrization here should not be confused with the standard *geometric realization*. They are the same at the set level but not at the topological level. A discussion in Appendix A providesx more information on this subject. Since the difference only appears for infinite complexes, we will still denote $|\sigma|$ the convex hull of a simplex σ, instead of $\wr \sigma \wr$.

We use the same terminology as for abstract complexes, with some additionals concepts. Let A and B be simplicial complexes. A map $f : V(A) \to V(B)$ defines a *simplicial map* if it preserves the simplices, *i.e.* for each simplex σ of A, the image $f(\sigma)$ is a simplex of B. By linear combination of the barycentric coordinates, f extends to the *barycentric map* $\wr f \wr : \wr A \wr \to \wr B \wr$. This can be done by taking any simplex $\sigma = \{x_0, \ldots, x_n\}$ of A. Since any $y \in |\sigma|$ is obtained as $y = \sum_{i=0}^n t_i.x_i$ with $t_i \in [0, 1]$ and $\sum_{i=0}^n t_i = 1$, we set $\wr f \wr(y) = \sum_{i=0}^n t_i.f(x_i)$.

For any geometric chromatic simplex $\sigma = (v_0, v_1, \ldots, v_n)$ (ie with a fixed order on the set of vertices), we have an unique affine map called the *characteristic map* $\varphi_\sigma : |\mathbb{S}^n| \to |\sigma|$ taking the i^{th} vertex of $\mathbb{S}^n$ to v_i. This is indeed the barycentric map of the simplicial map taking the i^{th} vertex of $\mathbb{S}^n$ to v_i.

Let $X \subset \mathbb{R}^N$, a function $f : X \to |C_2|$ respects a carrier map $\Delta : C_1 \to 2^{C_2}$ with $X \subseteq \wr C_1 \wr$, if $\forall \sigma \in C_1, f(|\sigma| \cap X) \subseteq \Delta(\sigma)$. The *open star* of $\sigma \in C_1 : St^\circ(\sigma, C_1) = \bigcup_{\tau \in C_1, \sigma \subseteq \tau} Int(\tau)$.

▶ **Definition 10** (Subdivision). *[14, Def 3.6.1] Let C_1, C_2 be two geometric simplicial complexes. We say that C_2 is a subdivision of C_1 if : $\wr C_1 \wr = \wr C_2 \wr$, and each simplex of C_1 is the union of finitely many simplices of C_2.*

4.2 Geometric Encoding of Iterated Immediate Snapshots Configurations

Here we present the mapping *geo* that links points of $\mathbb{R}^N$ and executions of the Iterated Immediate Snapshot model. Since this has been introduced in [8], this is only sketched here. The reader can refer to Appendix B for all the technical details in the setting of this paper.

There are two equivalent ways to define *geo*. It can be seen as the limit value of running a specific algorithm, called *the Chromatic Average algorithm*. Or, for a given execution w, it can be seen as the limit of iterating the Standard Chromatic Subdivision along the simplices corresponding to the successive instant graphs $w(r)$. The only difference with [8], is that we have to adapt to the setting of colorless algorithms by introducing the *Reduced Chromatic Average algorithm*. But the ideas and proof techniques are in essence the same as [8].

Algorithm 1 The reduced version of Chromatic Average Algorithm for process i with initial value $x_i^* \in \mathbb{R}^N$.

$x \leftarrow x_i^*$;
Loop *forever*
  `SendAll`(x);
  $V \leftarrow$`Receive()` // set of all received values including its own;
  $d \leftarrow sizeof(V) - 1$ // the process received d values, excluding its own ;
  $x = \frac{1}{2d+1}x + \sum_{y \in V \setminus \{x\}} \frac{2}{2d+1} y$;
EndLoop

We consider Algorithm 1, which is an adaptation of the Chromatic Average algorithm of [8]. As proved in [8] and in Appendix B, it is possible to show that the values x of all processes converge to the same limit $geo(w)$ for any execution $w \in IIS_n$. It is related to the known fact that the standard chromatic subdivision is *mesh-shrinking* [14].

4.3 A Topology for IIS_n

We present the *geometrization topology* on the set of execution of IIS as introduced in [8]. It is the topology induced by geo^{-1} from the standard topology in $\mathbb{R}^N$.

The *geometrization topology* is defined on IIS_n by considering as open sets the sets $geo^{-1}(\Omega)$ where Ω is an open set of $\mathbb{R}^N$. A collection of sets can define a topology when any union of sets of the collection is in the collection, and when any finite intersection of sets of the collection is in the collection. This is straightforward for a collection of inverse images of a collection that satisfies these properties. Note this also makes *geo* continuous by definition.

By considering the definition of *geo* from the iterations of the Standard Chromatic subdivision, we also have $geo(IIS_n) = |\mathbb{S}^n|$. Now, we want to associate a geometric point to any execution $w \in IIS_n$ with a specific initial configuration ι. Hence, we extend the construction on the simplex $\mathbb{S}^n$ to any simplicial complex $\mathcal{I}$ in the following way: $\forall \iota \in \mathcal{I}, \forall w \in \mathcal{M}$, $geo(\iota, w) = \varphi_\iota(geo(w))$, where φ_ι is the characteristic map of ι, mapping $\mathbb{S}^{dim(\iota)}$ to ι. We define $geo(\mathcal{I} \times \mathcal{M}) = \bigcup_{w \in \mathcal{M}, \sigma \in \mathcal{I}} \varphi_\sigma(geo(w))$. This construction into the set of execution allow us to associate to any message adversary $\mathcal{M} \subseteq IIS$ a topological space in $\mathbb{R}^N$.

5 A Generalisation of the Asynchronous Computability Theorem

The main result of this paper is an extension to any submodel of IIS of the result [14, Thm. 4.3.1] about computability of colorless tasks in IIS. Our proof follows the same line as [14] with adaptation of some key tools. We first express the main result.

▶ **Theorem 11** (Colorless-GACT). *Let $(\mathcal{I}, \mathcal{O}, \Delta)$ be a colorless task. This is solvable on $\mathcal{M} \subseteq IIS_n$ if and only if there is a continuous function $f : geo(skel^n\mathcal{I} \times \mathcal{M}) \to |\mathcal{O}|$ carried by Δ.*

The rest of this section is a long proof of the main result. We prove this equivalence in four inductive steps starting from the right hand side of the above theorem. We only give here the outline of the proof, that is the properties we want to prove equivalent. All proper definitions stated here will be introduced along the way :

1. A continuous function $f : geo(\mathcal{I}\times\mathcal{M}) \to |\mathcal{O}|$ satisfies an η-star condition for some function η.
2. From this η-star condition, we construct a IIS-terminating subdivision and a semi-simplicial approximation of f
3. This semi-simplicial approximation of f yield an algorithm solving the task $(I, \mathcal{O}, \Delta)$
4. An algorithmic solution for $\mathcal{M}$ implies the existence of a continuous map $geo(\mathcal{I}\times\mathcal{M}) \to |\mathcal{O}|$

For the rest of this section, n is fixed, we note $\mathcal{I}$ instead of $skel^n\mathcal{I}$. We also fix $X \subseteq \mathbb{R}^N$. We will set $X = geo(\mathcal{I} \times \mathcal{M})$ in the end. Let $\mathcal{O}$ be a finite simplicial complex.

5.1 From continuous function to η-star condition

We adapt the notion of *star-condition.*

▶ **Definition 12** (Star Condition for η). *Let $\eta : X \longrightarrow]0, +\infty[$ and let $f : X \to |\mathcal{O}|$, f satisfies the star condition for η if $\forall x \in X, \exists v \in V(\mathcal{O}), f(\mathbf{B}(x, \eta(x)) \cap X) \subseteq St^\circ(v)$.*

We also say f satisfies the $\eta-$star condition when we have a given η for the star condition above. See Figure 3 in Appendix D for an illustration.

▶ **Proposition 13.** *Let $f : X \to |\mathcal{O}|$ a continuous function. Then there is $\eta : X \longrightarrow]0, +\infty[$ such that f satisfies the η-star condition.*

Proof. We recall the standard definition of continuity : $\forall x \in X, \forall\epsilon > 0, \exists\delta_\epsilon(x) > 0$ such that $\forall x_0, x_0 \in \mathbf{B}(x, \delta_\epsilon(x)) \Rightarrow f(x_0) \in \mathbf{B}(f(x), \epsilon)$. Let $y \in |\mathcal{O}|$ and $\sigma_y \in \mathcal{O}$ the simplex of minimal dimension such that $y \in |\sigma_y|$. Let $\epsilon(y) = d(y, |\mathcal{O}| \setminus St^\circ(\sigma_y))$. We know that $\epsilon(y) \neq 0$ because $y \in St^\circ(\sigma_y)$, which is an open space. From there, the η-star condition is obtained with $\eta(x) = \delta_{\epsilon(f(x))}(x)$ since $f(\mathbf{B}(x, \eta(x)) \cap X) \subseteq \mathbf{B}(f(x), \epsilon(f(x))) \subseteq St^\circ(\sigma_{f(x)})$. ◀

5.2 From η-star condition to semi-simplicial approximation

We say that a simplicial complex is compatible with a subspace X if it covers X entirely with every simplex needed for such a cover.

▶ **Definition 14** (Complex compatible with a subspace). *Let $X \subseteq \mathbb{R}^N$ and C a simplicial complex. We say that C is compatible with X if $X \subseteq \wr C \wr$, and for all facet σ of C, $|\sigma| \cap X \neq \emptyset$.*

We use the notion of terminating subdivisions, that were introduced in [10, 3]. Here we present a more complete and explicit definition, this is needed since we are in the geometric context. The intuition behind this construction is that we want to associate a complex with the set of executions of a given algorithm on $\mathcal{M}$. As the termination of this algorithm could be not uniform (even with a fixed initial configuration), the associated complex may be of infinite size. Since a subdivision of a simplex cannot be infinite, we have to define an adapted construction from the iterated application of the Standard Chromatic subdivision.

Given a complex C, let $C(T) = \bigcup_{\sigma \in C, V(\sigma) \subseteq T} \sigma$ with $T \subseteq V(C)$ to represent the subcomplex of C formed by the vertices in T. Moreover, $JOIN(C_1, C_2) = \{|\sigma \cup \tau| | \sigma \in C_1, \tau \in C_2\}$ is the usual join of simplices [19]. We define $EChr$ as the following operator, given C and $T \subseteq V(C)$. Intuitively, the vertex marked as terminated are in T. We note $U = V(C) \setminus T$. The operator $EChr$ subdivides with the standard chromatic subdivision the facets that are fully in U, does not modify the ones that are fully in T and subdivides in an adequate way the facets in between.

$$EChr(T, C) = (\bigcup_{\sigma \in C} Chr\, \sigma(U)) \cup (\bigcup_{\sigma \in C} JOIN(Chr\, \sigma(U), \sigma(T))) \tag{1}$$

▶ **Definition 15** (IIS-Terminating subdivision)**.** *Let $\mathcal{I}$ a simplicial complex. The sequences $C_0, C_1, \dots$ (collection of simplices) and $T_0, T_1, \dots$ (collection of increasing set of vertices) form a IIS-*terminating subdivision of $\mathcal{I}$*, if we have for all $i \in \mathbb{N}$:*

1. $C_0 = \mathcal{I}, T_0 = \emptyset$
2. $C_{i+1} \subseteq EChr(T_i, C_i)$
3. $T_i \subseteq V(C_i)$

We say that $\bigcup C_i(T_i)$ is an IIS-terminating subdivision complex. This is actually a simplicial complex, as proved in Appendix C.

A simplicial approximation is a standard topological construct that is the basis of the proof technique of the similar computability theorem in [14, Th 4.3.1]. A simplicial approximation is a simplicial function that approximates (in some sense) a function where the domain and the co-domain are simplicial complexes. We have to adapt this definition since here we only have that the co-domain as a simplicial complex.

▶ **Definition 16** (semi-simplicial approximation)**.** *Let $f : X \to |\mathcal{O}|$ a function. The function $\psi : V(C) \to V(\mathcal{O})$ is a semi-simplicial approximation for f if C a IIS-terminating subdivision compatible with X, and ψ is a simplicial map such that $\forall \sigma \in C, f(St^\circ(\sigma) \cap X) \subseteq St^\circ(\psi(\sigma))$.*

In order to construct an IIS-terminating subdivision of $\mathcal{I}$ compatible with X, we need a predicate to "set in a terminating state". Let $\eta : X \longrightarrow]0, +\infty[$, C a simplicial complex and $v \in V(C)$, we define $\mathbf{P}_\eta(v, C) = \{\exists x \in X, | \; |St(v, C)| \subseteq \mathbf{B}(x, \eta(x))\}$. Let $C_0 = \mathcal{I}$ the simplicial complex to subdivide, $U_0 = V(C_0)$, $T_0 = \emptyset$. For all $i \in \mathbb{N}$ we set :

1. $D_{i+1} = EChr(T_i, C_i)$
2. $C_{i+1} = \{\sigma \in D_{i+1} | \; |\sigma| \cap X \neq \emptyset$ and σ is a facet of $D_{i+1}\}$
3. $T_{i+1} = \{v \in V(C_{i+1}) | \mathbf{P}_\eta(v, C_{i+1})\}$

The final complex is $C_\eta = \bigcup_{i \in \mathbb{N}} C_i(T_i)$. In Figure 1 the vertices marked in red are the ones in T_i. On the left, the subdivided simplex is the one without vertices in red. On the right, some simplices are added with a $JOIN$ operation. Then the simplices in blue and in dotted lines are the ones that will be removed at step 2, since they do not intersect X.

▶ **Proposition 17.** *C_η is a simplicial complex.*

Proof. We have D_i is a simplicial complex. In C_i, removing facets of D_i still yields a simplicial complex. ◀

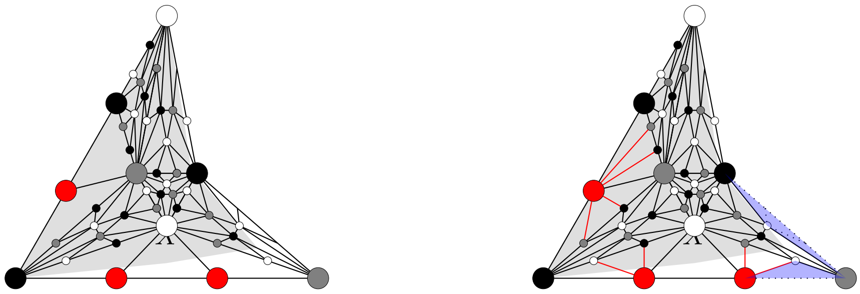

Figure 1 Construction of the IIS-Terminating Subdivision compatible with a space X.

▶ **Proposition 18.** *The subdivision C_η is compatible with X.*

Proof. For the inclusion property, $\forall x \in X$, since $x \in |C_0|, \forall r \in \mathbb{N}, \exists \sigma_r \in C_r$ such that $x \in |\sigma_r|$. The subdivision operator Chr is mesh-shrinking, this means that $\exists r_0 \in \mathbb{N}, \forall v \in V(\sigma_{r_0}), mesh(St(v, C_{r_0})) < \eta(x)$. Then $St(v, C_{r_0}) \subseteq B(x, \eta(x))$, which means that all vertex of σ_{r_0} are in T_{r_0} hence σ_{r_0} is in C_η so $x \in \wr C_\eta \wr$ and $X \subseteq \wr C_\eta \wr$. Since we only remove facet $\sigma \in C_\eta$ such that $|\sigma| \cap X = \emptyset$ in the second step of the construction we have the compatibility of C_η with X ◀

We can now construct a semi-simplicial approximation with C_η.

▶ **Proposition 19.** *Let $\eta : X \longrightarrow]0, +\infty[$ and let $f : X \to |\mathcal{O}|$ a function that satisfies the η-star condition, then f has a semi-simplicial approximation $\psi_\eta : V(C_\eta) \to V(\mathcal{O})$.*

Proof. Let C_η be the IIS-terminating subdivision of $\mathcal{I}$ defined above from η. Let σ a simplex of C_η, v a vertex of $V(\sigma)$, Since $\mathbf{P}_\eta(v, C_\eta)$ is satisfied, $\exists x_v \in X$ such that $|St(v, C_\eta)| \subseteq \mathbf{B}(x_v, \eta(x))$. By the η-star property we have that $\exists y_v \in V(\mathcal{O}), f(\mathbf{B}(x_v, \eta(x_v)) \cap X) \subseteq St^\circ(y_v, \mathcal{O})$. Let $\psi_\eta(v) = y_v$. Let's prove that ψ_η is indeed a semi-simplicial approximation.

We know that $\forall \sigma \in C_\eta, \forall v \in V(\sigma), \mathbf{P}_\eta(v, C_\eta)$ is true, then we have that : $\bigcap_{v \in V(\sigma)} |St(v, C_\eta)| \subseteq \bigcap_{v \in V(\sigma)} \mathbf{B}(x_v, \eta(x_v))$. The η-star condition gives that $\bigcap_{v \in V(\sigma)} f(\mathbf{B}(x_v, \eta(x_v))) \subseteq St^\circ(y_v, \mathcal{O})$. By noticing that $St^\circ(v, C_\eta) \subseteq |St(v, C_\eta)|$, we can combine theses inclusions and obtain that : $\bigcap_{v \in V(\sigma)} f(St^\circ(v, C_\eta)) \subseteq \bigcap_{v \in V(\sigma)} St^\circ(y_v, \mathcal{O})$ since $\bigcap_{v \in V(\sigma)} St(v) = St(\sigma)$.

This can be rewritten as : $f(St^\circ(\sigma, C_\eta)) \subseteq \bigcap_{v \in V(\sigma)} St^\circ(\psi_\eta(v), \mathcal{O})$, which is the property of the definition 16. Furthermore, because C_η is compatible with X we have that $\exists x \in X, x \in |\sigma|$. Since $|\sigma| \subseteq |St(\sigma)|$ which implies $f(x) \in |\psi_\eta(\sigma)|$ then $\bigcap_{v \in V(\sigma)} St^\circ(\psi_\eta(v), \mathcal{O})$ is non-empty therefore $\psi_\eta(\sigma)$ is a simplex, the function ψ_η is simplicial. ◀

We need to prove that the semi-simplicial approximation $\psi_\eta : V(C_\eta) \to V(O)$ of f is carried by the carrier map of f.

▶ **Lemma 20** (semi-simplicial approximation and carrier map)**.** *Let $\eta : X \longrightarrow]0, +\infty[$ and let $f : X \to |\mathcal{O}|$ a continuous function that respects $\Delta : \mathcal{I} \to 2^{\mathcal{O}}$ a carrier map. Then the semi-simplicial approximation $\psi_\eta : C_\eta \to \mathcal{O}$ of f respects also Δ.*

Proof. Let $\sigma \in C_\eta$, with $\sigma = \{v_0, v_1 \dots v_k\}$, $\psi_\eta(\sigma) = \{y_0, y_1, \dots y_k\}$ and $\{x_0, x_1, \dots x_k\}$ a points of X such that $|St(v_i)| \subseteq \mathbf{B}(x_i, \eta(x_i))$. Also, we have that $f(|\sigma|) \subseteq |\Delta(\sigma)|$ because f respects Δ and by construction of $C_\eta, f(\mathbf{B}(x_i, \eta(x_i))) \subseteq St^\circ(y_i)$. By way of contradiction, assume that $\psi_\eta(v_i) \notin \Delta(\sigma)$, by the $\mathbf{P}_\eta$ property we have that x_i covers $|St(v_i)|$. When we apply the function f, we obtain that $f(|\sigma|) \subseteq f(\mathbf{B}(x_i, \eta(x_i)) \cap X) \subseteq St^\circ(\psi_\eta(v_i))$. We can remark that $\psi_\eta(v_i) \notin \Delta(\sigma) \Rightarrow St^\circ(\psi_\eta(v_i)) \nsubseteq \Delta(\sigma)$. We can conclude that $f(\sigma) \nsubseteq \Delta(\sigma)$, which contradicts our hypothesis. ◀

5.3 From semi-simplicial approximation to an algorithm

Now we show that a semi-simplicial approximation can be used to define an algorithm.

▶ **Proposition 21.** *Let* $f : geo(\mathcal{I} \times \mathcal{M}) \to |\mathcal{O}|$ *a continuous function which respects a carrier* Δ *then the task* $(\mathcal{I}, \mathcal{O}, \Delta)$ *is solvable by an algorithm in* $\mathcal{M}$*.*

We define the following Algorithm 2 from the Chromatic Averaging Algorithm and using C_η and ψ_η, x_p is the initial position of p in the complex $\mathcal{I}$.

Algorithm 2 $\mathcal{A}_{\psi_\eta}$: algorithm derived from $\psi_\eta : C_\eta \to \mathcal{O}$.

$x \leftarrow x_p$;
$i \leftarrow 0$;
while $x \notin T_i$ **do**
  $i \leftarrow i + 1$;
  `SendAll`(x);
  $V \leftarrow$`Receive()` // set of all received values including its own;
  $d \leftarrow sizeof(V) - 1$ // the process received d values, excluding its own ;
  $x = \frac{1}{2d+1} x + \sum_{y \in V \setminus \{x\}} \frac{2}{2d+1} y$;
end
return $\psi_\eta(x)$;

▶ **Proposition 22.** *The Algorithm* $\mathcal{A}_{\psi_\eta}$ *terminates for all executions in* $\mathcal{I} \times \mathcal{M}$*.*

Proof. We set $X = geo(\mathcal{I} \times \mathcal{M})$ and the corresponding C_η is compatible with X which implies that $X \subseteq \wr C_\eta \wr$. Then $\forall w \in \mathcal{I} \times \mathcal{M},\ geo(w) \in X$. We can deduce that $\exists \sigma \in C_\eta,\ geo(w) \in |\sigma|$. By construction of C_η, $\exists i \in \mathbb{N}$ such that $x \in T_i$, hence every processes terminate. ◀

▶ **Proposition 23.** *Algorithm* $\mathcal{A}_{\psi_\eta}$ *respects the specification described by* Δ

Proof. The decision is given by ψ_η, a semi-simplicial approximation of f that respects Δ. ◀

5.4 From an algorithm to a continuous function

We conclude this proof by constructing a continuous function from a given algorithm. We will need to normalize this algorithm first. In the colorless setting, a normalized algorithm is an algorithm where when a process sees any decision value, it decides instantly one of these values. If an algorithm is correct, its normalized version is also correct for this colorless task.

▶ **Proposition 24.** *Let* $\mathcal{A}$ *a normalized algorithm for solving the task* $(\mathcal{I}, \mathcal{O}, \Delta)$ *in the submodel* $\mathcal{M}$ *then there exists a continuous function* $f : geo(skel^n(\mathcal{I}) \times \mathcal{M}) \to |\mathcal{O}|$ *that respects* Δ*.*

Proof. An algorithm solving a task $(\mathcal{I}, \mathcal{O}, \Delta)$ provides a decision function $\varphi_\mathcal{A} : Y \to V(\mathcal{O})$, where $Y \subset |skel^n(\mathcal{I}|$ (Y is the set of vertices of $Chr^r(\mathcal{I})$, for all r). We use this decision function $\varphi_\mathcal{A}$ to construct a IIS-terminating subdivision. Let $C_0, C_1, C_2, \ldots$ be a sequence of complexes and $T_0, T_1, T_2, \ldots$ a sequence of vertices of these complexes. We fix $T_0 = \emptyset$ and $C_0 = \mathcal{I}$ which immediately satisfies the condition 1). A vertex is added in T_i if the algorithm decide on the couple $(Process, View)$ at the round i. Since every decision of process are permanent we have the properties 3) and 4) of IIS-terminating subdivision. At the round i of the algorithm we construct the complex C_i using the $EChr$ operator, this operation corresponds to one round of IIS while allowing non-uniform termination and compatibility with $\mathcal{M}$. If we take $C_\mathcal{A} = \bigcup_{i\in\mathbb{N}} C_i(T_i)$, this correspond to the set of processes that will terminate in our algorithm and yields a IIS-terminating subdivision. Furthermore, $C_\mathcal{A}$ is compatible with X, because $\varphi_\mathcal{A}$ is a decision function on every execution of $\mathcal{I} \times \mathcal{M}$ then we have that $X \subseteq C_\mathcal{A}$. Also, if $\sigma \in C_\mathcal{A}, |\sigma| \cap X = \emptyset$ then this means that we decide on a execution that is not in $\mathcal{M}$, which is outside of our algorithm scope. Hence $C_\mathcal{A}$ is compatible with X. Since every vertex in $V(C_\mathcal{A})$ correspond to an execution in $\mathcal{I} \times \mathcal{M}$, from $\varphi_\mathcal{A}$ we obtain a simplicial function $\varphi : V(C_\mathcal{A}) \to V(\mathcal{O})$ that respect Δ.

Then we have that $\varphi(St(v, C_\mathcal{A})) \subseteq St(\varphi(v), \mathcal{O})$. We can now classically extend the simplicial function φ to a function $\varphi_C : \wr C_\mathcal{A} \wr \to |\mathcal{O}|$ by linear extension on the barycentric coordinates. This extension guaranties that φ_C respects Δ : $\varphi_C(|\sigma|) \subseteq |\Delta(\sigma)|$ since $\varphi(\sigma) \in \Delta(\sigma)$. We need to prove the continuity[1] of φ_C : $\forall x \in \wr C_\mathcal{A} \wr, \exists \sigma \in C_\mathcal{A}, x \in |\sigma|, \exists r \in \mathbb{N}, V(\sigma) \subseteq T_r$ and $V(\sigma) \nsubseteq T_{r-1}$. Then if $x \in Int(\sigma)$ the continuity can be obtained directly because the barycentric extension is always continuous on a given simplex. If $x \in \partial(\sigma)$ then because the algorithm is normalized we have that $\forall v \in V(\sigma), St(v, C_{r+1}(T_{r+1})) = St(v, C_{r+1})$. Since this a finite simplicial complex we obtain the continuity with a barycentric extension. ◀

6 Application to Set-Agreement

We give here a direct, and therefore simpler than [8], proof for general set-agreement computability. For all $n \in \mathbb{N}$, the *set-agreement problem* is defined by the following properties [23]. Given initial *init* values in $[0, n]$, each process outputs a value such that

Agreement the size of the set of output values is at most n,
Validity the output values are initial values of some processes,
Termination All processes terminates.

▶ **Theorem 25** ([8]). *It is possible to solve Set-Agreement on $\mathcal{M} \subset IIS_n$ iff $geo(\mathcal{M}) \neq |\mathbb{S}^n|$.*

Proof. We denote by $(\mathcal{I}_{sa}, \mathcal{O}_{sa}, \Delta_{sa})$ the colorless task for set-agreement. We have $skel^n\mathcal{I}_{sa} = \mathbb{S}^n$. For the necessary condition, we first get from Thm. 11 that there exists a continuous function from $geo(\mathbb{S}^n \times \mathcal{M})$, *i.e.* a continuous function from $geo(\mathcal{M})$ to $|\mathcal{O}_{sa}|$. $\mathcal{O}_{sa}$ is $\partial\mathbb{S}^n$, the boundary of $\mathbb{S}^n$. The No Retraction theorem [13, Cor. 2.15] states that there is no continuous function from $|S^n|$ to $|\partial S^n|$. This means that $geo(\mathcal{M})$ cannot be equal to $|\mathbb{S}^n|$.

In the reverse direction, if $geo(\mathcal{M}) \subsetneq |\mathbb{S}^n|$, we note x_0 a point in $|\S^n|$ that is not in $geo(\mathcal{M})$. We can construct in a standard way a continuous function from $geo(\mathcal{M})$ to $|\partial S^n|$ by using x_0 as a base point to "project" points x of $geo(\mathcal{M})$ onto $|\partial\mathbb{S}^n|$: the image of x is the intersection of the half-line x_0x with $|\partial\mathbb{S}^n|$ (in the special case where $x_0 \in |\partial\mathbb{S}^n|$, we project on the complex obtained by removing from ∂S^n the simplexes that contains x_0) . ◀

This characterization is quite expected, it is known the No-Retraction theorem is the topological obstruction for solving set-agreement in models such as the Iterated Immediate Snapshot model. We underline that this proof is way simpler that the proof of Coutouly and Godard in [8], that used Sperner and König lemmas in very involved ways. We underline that having at least one missing point from $|\mathbb{S}^n|$, ie a hole in $geo(\mathcal{I}_{sa} \times \mathcal{M})$, does not mean that $\mathcal{M}$ is $\mathcal{G}_n^\omega$ minus one execution. Since geo is not injective, many executions could be removed, that is all executions that maps to some x_0. These pre-images are called $geo-$classes, they are fully described in [8]. Some $geo-$classes are of infinite size when $n \geq 2$.

[1] we emphasize that the underlying topology here is not that of standard geometric realizations of complexes, therefore being simplicial does not imply the linear extension to be continuous in the sense we have to prove here.

7 Application to Adversaries Submodels

An adversary, in the sense of [14, Chap. 5.4], is a message adversary where the executions are exactly defined by the set of possible simultaneous failures.

Formally, we say that a process p is influencing a time t a process q, if there is a sequence of messages starting at time t from p that eventually reaches q. Finally, given $w \in IIS_n$, we denote by $Q(w)$ the set of processes that are influencing infinitely many times in w all the other processes. In IIS_n, this set is always non empty. In the message adversary setting, the set of "failed" processes is the set $\Pi \setminus Q(w)$. An adversary A is defined by a set $F(A)$ of subsets of Π that is inclusion-closed. The set of corresponding executions is denoted as $\mathcal{M}_A = \{w \in IIS_n | \exists P \in F(A), Q(w) = \Pi \setminus P\}$. A well investigated case is the adversary R_t, given $t \leq n$, where $F(R_t)$ is the set of subsets of size at most t. It is the $t-$resilient layered immediate snapshot protocol submodel.

As in [14, Chap. 5.4], we define a core as a minimal set of processes that will not all fail in any execution. For the $t-$resilient layered snapshot protocol model, a core is any subset of size $t + 1$. Even if processes are independent of the set of input values in the colorless setting, we will be able to assign a set of input value to any core C. Hence, we choose a core $C = \{p_0, \ldots p_c\}$ of size $c + 1$ and we will construct an application $\pi^*_c : \mathcal{I} \times \mathcal{M}_A \to |skel^c\mathcal{I}|$.

Let G a graph of ImS_n, given a set of vertices C, we denote by $G[C]$ the subgraph induced by C, that is $V(G[C]) = C$ and $E(G[C]) = E(G) \cap (C \times C)$. We extend this notation to executions, $\forall w \in IIS_n,$, with $w = G_1, G_2, \ldots,$ we set $w[C] = G_1[C], G_2[C], \ldots.$

The function π^*_c is constructed by applying this reduction for a chosen C_w to every execution of $\mathcal{M}_A$. We fix an order on the processes. We set C_w to be the set $\Pi \setminus Q(w)$ together with the q lowest processes of $Q(w)$, where $q = c + 1 - |\Pi \setminus Q(w)|$. The set C_w is always of size $c + 1$ and is therefore not in $F(A)$. Finally, we set $\pi^*(w) = geo(w[C_w])$.

▶ **Proposition 26.** *The function $\pi^*_c : \mathcal{I} \times \mathcal{M}_A \to |skel^c\mathcal{I}|$ has the following properties :*
1. *it is continuous and surjective,*
2. *$\forall w, w' \in \mathcal{M}_A$, $geo(w) = geo(w') \Rightarrow \pi^*_c(w) = \pi^*_c(w')$*

Proof. The property (1) is directly obtained by construction. For the second property, first we remark that $geo(w) = geo(w')$ implies $Q(w) = Q(w')$. Indeed, consider $p \in \Pi$, such that $p \in Q(w)$. The process p cannot distinguish w from w' otherwise all other processes will eventually distinguish the executions. It means that the set of processes that influence p infinitely many times is the same in both executions. By definition, this set includes $Q(w')$ in the execution w'. Since p is influencing infinitely many times all processes in w, and influence is transitive, we have $Q(w') \subset Q(w)$. Symmetrically, we get $Q(w) \subset Q(w')$.

Therefore $C_w = C_{w'}$. We conclude by a simple case by case analysis from the different cases where $geo(w) = geo(w')$ as given in [8, Th. 25]. ◀

So we can also define a function $\pi : geo(\mathcal{I} \times \mathcal{M}_A) \to |skel^c\mathcal{I}|$ by setting $\pi(x) = \pi^*_c(w)$, where w is any element of $geo^{-1}(x)$. We will now show that a restriction of π actually enjoys a very interesting topological property. First, we give a standard definition.

▶ **Definition 27** (Fiber Bundle [13]). *Let E, B, F topological spaces. (E, B, π, F) is a fiber bundle with base B and fiber F if $\pi : E \to B$ is a continuous surjection such that for every $x \in B$, there is an open neighborhood $U \subseteq B$ of x such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$, and $U \times F$ is the product space in such a way that π agrees with the projection proj onto the first factor,* i.e. $\pi_{|\pi^{-1}(U)} = \varphi \circ proj$.

One example of a classic example of fiber bundle is in Appendix E. We say that a run $w \in \mathcal{M}_A$ is *special* if w has a suffix w' (after step j) where all the instant graphs have their sources in $Q(w)$ and for all $i \geq j$ $G(i)$ is such that the arcs between $Q(w) \cap C_w$ and its complement $Q(w) \setminus C_w$ are all from $Q(w) \cap C_w$ when i is even, and to $Q(w) \cap C_w$ when i is odd. We denote Spe_A the set of special runs of $\mathcal{M}_A$.

▶ **Proposition 28.** *The function $\pi : E \longrightarrow skel^c(\mathcal{I})$ is a fiber bundle with $E = geo(Spe_A)$, $B = skel^c(\mathcal{I})$ and $F = \mathbb{S}^{n-c-1}$.*

Proof. For any point $x \in |skel^c\mathcal{I}|$, there is a special run w in $\mathcal{M}_A$ such that $geo(w) = x$. Indeed, given C a core of size $c+1$ with $c+1$ different initial values, it is possible to complement the execution $w^* \in IIS_C$ such that $\pi(geo(w^*)) = x$ in a special way : after the step j^* where only processes in $Q(w^*)$ influence all others in C, in instant graph G_i, $i \geq j^*$, processes from $\Pi \setminus C$ have an arc to processes in $C \setminus Q(w^*)$ and arcs between $\Pi \setminus C$ and $Q(w^*)$ alternate direction if i is even or odd. The arcs between processes of $\Pi \setminus C$ can be any pattern from $ImS_{\Pi \setminus C}$. This means that the restriction of π on E is surjective.

Now we focus on the neighborhood condition. As previously, we consider $x \in |skel^c\mathcal{I}|$, and the corresponding w^* and j^*. We set U to be the neighbourhood of x where executions share the prefix of w^* up to step j^*. In the previous section, we have said that we can define w by complementing w^* choosing any pattern in $ImS_{\Pi \setminus C}$. We remark that there is actually no other way to complement w^* to get a special execution. So the fiber $\pi^{-1}(x)$ is homeomorphic to $geo(IIS_{\Pi \setminus C})$, that is exactly $\mathbb{S}^{n-c-1}$. ◀

Concluding, from the main theorem, a colorless task $(\mathcal{I}, \mathcal{O}, \Delta)$ is solvable on $\mathcal{M}_A$ if and only if there exists a continuous function $f : geo(\mathcal{I} \times \mathcal{M}_A) \longrightarrow |\mathcal{O}|$ carried by Δ. We will show that this is equivalent to the existence of a continuous function $g : |skel^c\mathcal{I}| \longrightarrow |\mathcal{O}|$ carried by Δ so we can get an alternative and fully topological proof of the following.

▶ **Theorem 29** ([14, Th.5.4.3]). *A colorless task $(\mathcal{I}, \mathcal{O}, \Delta)$ is solvable on $\mathcal{M}_A$ for an adversary A with a core of size c if and only if there exists a continuous function $g : |skel^c\mathcal{I}| \longrightarrow |\mathcal{O}|$ carried by Δ.*

Proof. We show that the existence of f is equivalent to the existence of g. We assume $c < n$ otherwise the statement are equal and g is f. We start with the easy direction, assuming there exists g a continuous function $g : |skel^c\mathcal{I}| \longrightarrow |\mathcal{O}|$ carried by Δ. For a given facet S of $skel^n\mathcal{I}$, since $c < n$, there exists $x_1 \in |S|$ such that $x_1 \notin geo(S \times \mathcal{M}_A)$, it is therefore possible to have a retract from $|S| \setminus \{x_1\}$ onto $|Skel^{n-1}(S)|$. We can repeat this until reaching $|Skel^c(S)|$. We consider μ the composition of this sequence of retracts of $|Skel^n(\mathcal{I})| \setminus \{x_1, x_2, ...\}$ onto $|skel^cS|$. We set $f = g \circ \mu$. Such f is continuous by composition. Since this is a retract, μ is the identity on $|skel^cS|$ and f is carried by Δ.

Now, we assume that we have a continuous function $f : geo(\mathcal{I} \times \mathcal{M}_A) \longrightarrow |\mathcal{O}|$ carried by Δ. We would like to define $g = f \circ s$ where s would be a kind of right inverse for π as defined above. In order to show that, we will use the fact that π is a fiber bundle for E and $B = |skel^c\mathcal{I}|$. In the context of fiber bundles, what we are looking for is called a (cross) section s, that is, a continuous function $s : B \longrightarrow E$ such that $\pi \circ s = Id_B$. Cross-sections do not always exist, however since the fiber F is $\mathbb{S}^d$, we get that there is indeed a section s, see e.g. [7, Cor. 7.13], as a corollary of Whitehead Obstruction theorem. Since f is continuous, g is also continuous. By construction of π, g is also carried by Δ on $skel^c\mathcal{I}$ since f also is. ◀

We have this immediate corollary for the $t-$resilient layered snapshot protocol model R_t.

▶ **Corollary 30.** *Let $t \leq n$. A colorless task $(\mathcal{I}, \mathcal{O}, \Delta)$ is solvable on R_t if and only if there exists a continuous function $g : |skel^t\mathcal{I}| \longrightarrow |\mathcal{O}|$ carried by Δ.*

8 Conclusion

In this work, we have presented a simple characterization of computability of colorless tasks for any submodels of the IIS model. We believe that this theorem will have many applications, from simpler proof of known results to new characterisation of some colorless tasks. Note also that it is possible to extend the presented technique to submodels of models corresponding to mesh-shrinking subdivisions (like the barycentric subdivision), we underline it would change the definition of *geo*, therefore this would not mean that a colorless task would be solvable for the same submodels. Together with the kind of classical topology approaches that we have shown to be effective in the two applications suggest that this work opens many perpespective to investigate computability in more general distributed models.

Since we are actually using the geometrization topology in this paper, we complement the remarks from [8] by some important points about this topology. In a topological space, a neighbourhood for point x is an open set containing x. The set of neighbourhoods of x is denoted N_x. A topological space is said to satisfy the T_0 separation axiom if $x \neq y \Longrightarrow N_x \neq N_y$. When $N_x = N_y$, we say that x and y are not (topology) distinguishable.

Since the topology we are building upon for $\wr IIS_n \wr$ is the one induced by the standard space $\mathbb{R}^N$, which satisfies T_0, via the geo^{-1} mapping, it is straightforward to see that non-distinguishable sets are exactly the geo-equivalence classes that are not singletons, since any neighbourhood of w in the geometrization topology will be a neighbourhood of w', when $geo(w) = geo(w')$. A description of theses geo-equivalence classes can be found in [8], and it is shown that there always exists non-singleton classes. By construction, the topology on IIS_n is therefore not T_0. However, if we quotient this space by the classes of indistinguishability, which is called the Kolmogorov quotient, we obtain a topological space homeomorphic to $|\mathbb{S}^n|$. So up to Kolmogorov quotient, the topology introduced here for investigating colorless tasks on IIS_n can be considered *classical*.

We are also looking forward to address colored tasks by an extension of these results. Since it is known that a statement like Thm. 11 is not strong enough for some non-coloured task, there needs to have some additional conditions in the theorem statement. Another line of research would be to characterize, in a topological way, the colored tasks that admit a characterization à la Thm. 11, related to a better understanding of the relationship between the set of executions seen as a topological space with the geometrization topology, which is quite simple, and seen as a topological space with the general topology of [3].

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A A Counter-Example about Geometric Realizations

We remind the reader that the *geometrization* of C, denoted $\wr C \wr$, that is the union of the convex hulls $|\sigma|$ of the simplices σ of C, is endowed with the standard topology from $\mathbb{R}^N$.

This should not be confused with the geometric realization, that is endowed of what is called a weak topology.

In this section, we provide an example of a simplicial complex whodse topology as a geometric realization is different from the topology it has as geometrization, that is in the ambient R^N space. That means that there exists infinite complex for which the topological spaces $\wr C \wr$ and $|C|$ are not necessarily homeomorphic. This is actually quite well known, see e.g. [19]. This example can actually be translated exactly to the distributed executions that exhibit an error from [10] in [11, Sect. 5.1].

The example is given with $N = 1$ but that can be generalized to any N. We consider $C = \{0\} \cup \{[\frac{1}{r+1}, \frac{1}{r}] \mid r \in \mathbb{N}^*\}$.

We denote $|C|$ the topological space of C defined as a geometric realization. The closed sets of $|C|$ are the sets F such that $F \cap S$ is closed (in $\mathbb{R}$) for all $S \in C$, see [24]. Therefore $|C|$ has two connected components. We have $F =]0, 1]$ is closed in $|C|$ since $F \cap [\frac{1}{r+1}, \frac{1}{r}] = [\frac{1}{r+1}, \frac{1}{r}]$, hence is closed for all r. Moreover, $F \cap \{0\} = \emptyset$ which is also closed in $\mathbb{R}$. We also have that $\{0\}$ is closed in $|C|$, so C can be covered by two disjoint closed sets, it is therefore not connected.

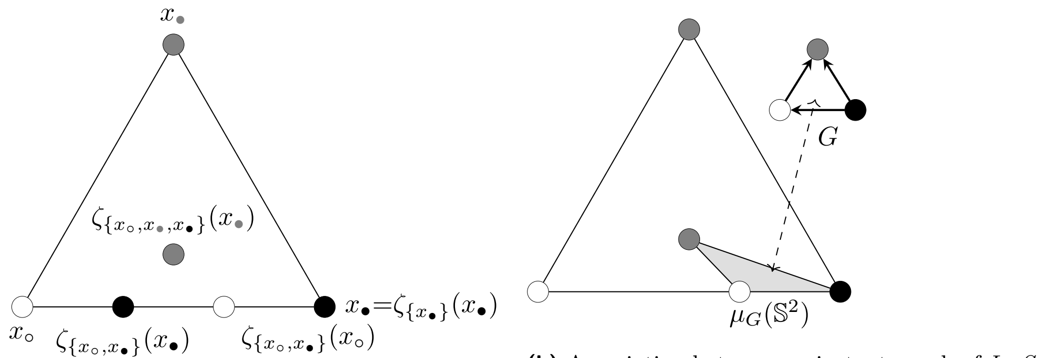
On the other end, at the set level, $\wr C \wr$ is exactly $[0, 1]$. So within the standard ambient topology of $\mathbb{R}$, $\wr C \wr$ is connected.

Since they do not have the same number of connected components, the two spaces $\wr C \wr$ and $|C|$ cannot be homeomorphic.

This type of problem can happend in many distributed situation, as in [11] :let $\mathcal{M}_1 = IIS \setminus \{\{\circ\!\!\leftrightarrow\!\!\bullet, \circ\!\!\leftarrow\!\!\bullet^\omega\}, \{\circ\!\!\rightarrow\!\!\bullet, \circ\!\!\leftarrow\!\!\bullet^\omega\}\}$ and $\mathcal{M}_2 = IIS \setminus \{\circ\!\!\rightarrow\!\!\bullet, \circ\!\!\leftarrow\!\!\bullet^\omega\}$ to remark that only $\mathcal{M}_1$ can solve the binary consensus task.

B The Standard Chromatic Subdivision

Here we present the standard chromatic subdivision, [14] and [19], as a geometric complex. We start with chromatic subdivisions.



(a) Encoding of the pair (process,view) to a point.

(b) Association between an instant graph of ImS_2 (top) and a simplex of $Chr(\mathbb{S}^2)$ is illustrated.

Figure 2 Construction of $Chr(\mathbb{S}^2)$ as a geometric encoding for IIS_2.

▶ **Definition 31** (Chromatic Subdivision). *Given $(S, \mathcal{P})$ a chromatic simplex, a chromatic subdivision of S is a chromatic simplicial complex $(C, \mathcal{P}_C)$ such that*

- *C is a subdivision of S (*i.e. $\wr C \wr = |S|$*),*
- *$\forall x \in V(S), \mathcal{P}_C(x) = \mathcal{P}(x)$.*

Note that it is not necessary to assume $V(S) \subset V(C)$ here, since the vertices of the simplex S being extremal points, they are necessarily in $V(C)$.

We start by defining some geometric transformations of simplices (here seen as sets of points). The choice of the coefficients will be justified later.

▶ **Definition 32.** *Consider a simplex $V = (y_0, \dots, y_d)$ of size $d+1$ in $\mathbb{R}^N$. We define the function $\zeta_V : V \longrightarrow \mathbb{R}^N$ by, for all $i \in [0, d]$*

$$\zeta_V(y_i) = \frac{1}{2d+1} y_i + \sum_{j \neq i} \frac{2}{2d+1} y_j$$

We now define directly in a geometric way the *standard chromatic subdivision* of simplex $(S, \mathcal{P})$, where $S = (x_0, x_1, \dots, x_n)$ and $\mathcal{P}(x_i) = i$.

The chromatic subdivision $Chr(S)$ for the chromatic simplex $S = (x_0, \dots, x_n)$ is a simplicial complex defined by the set of vertices $V(Chr(S)) = \{\zeta_V(x_i) \mid i \in [0, n], V \subset V(S), x_i \in V\}$.

From the previous definition, for each pair (i, V), $i \in [0, n]$ and $V \subset V(S)$ with $i \in V$, there is an associated vertex $x = \zeta_V(x_i)$ of $Chr(S)$, and conversely each vertex has an associated pair. The *color* of (i, V) is i. The set V is called the *view*. We define Φ the following *presentation* of a vertex x, $\Phi(x) = (\mathcal{P}(x), V_x)$ where $\mathcal{P}(x) = i$ and $V_x = V$.

The simplices of $Chr(S)$ are the set of $d+1$ points $\{\zeta_{V_0}(x_{i_0}), \cdots, \zeta_{V_d}(x_{i_d})\}$ where

- there exists a permutation π on $[0, d]$ such that $V_{\pi(0)} \subseteq \cdots \subseteq V_{\pi(d)}$,
- If $i_j \in \mathcal{P}(V_\ell)$ then $V_j \subset V_\ell$.

In Fig. 2, we present the construction for $Chr(\mathbb{S}^2)$. For convenience, we associate $\circ, {\color{gray}\bullet}, \bullet$ to the processes $0, 1, 2$ respectively. In Fig. 2a, we consider the triangle $x_\circ, x_{\color{gray}\bullet}, x_\bullet$ in $\mathbb{R}^2$, with $x_\circ = (0,0)$, $x_\bullet = (1,0)$, $x_{\color{gray}\bullet} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. We have that $\zeta_{\{x_\circ,x_\bullet\}}(x_\bullet) = (\frac{1}{3}, 0)$, $\zeta_{\{x_\circ,x_\bullet\}}(x_\circ) = (\frac{2}{3}, 0)$ and $\zeta_{\{x_\circ,x_{\color{gray}\bullet},x_\bullet\}}(x_{\color{gray}\bullet}) = (\frac{1}{2}, \frac{\sqrt{3}}{10})$. The relation between instant graph G (top) and simplex $\left\{(\frac{2}{3}, 0), (1, 0), (\frac{1}{2}, \frac{\sqrt{3}}{10})\right\}$ (grey area in Fig. 2b) is detailed in the section 4.2.

In the following, we will be interested in iterations of $Chr(\mathbb{S}^n, \mathcal{P})$. The last property of the definition of chromatic subdivision means with we can drop the C index in the coloring of complex C and use $\mathcal{P}$ to denote the coloring at all steps. From its special role, it is called the *process color* and we sometimes drop $\mathcal{P}$ in $Chr(S, \mathcal{P})$ using in the following $Chr(S)$ for all simplices S of iterations of $Chr(\mathbb{S}^n)$.

In [20], Kozlov showed how the standard chromatic subdivision complex relates to Schlegel diagrams (special projections of cross-polytopes), and used this relation to prove the standard chromatic subdivision was actually a subdivision. In [14, section 3.6.3], a general embedding in $\mathbb{R}^n$ parameterized by $\epsilon \in \mathbb{R}$ is given for the standard chromatic subdivision. The geometrization here is done choosing $\epsilon = \frac{d}{2d+1}$ in order to have “well balanced” drawings.

B.1 Colorless Algorithms in the Iterated Immediate Snapshots Model

It is well known, see [14, Chap. 3&4, Def. 3.6.3], that each maximal simplex $S = \{\zeta_{V_0}(x_{i_0}), \cdots, \zeta_{V_n}(x_{i_n})\}$ from the chromatic subdivision of $\mathbb{S}^n$ can be associated with a graph of ImS_n denoted $\Theta(S)$. In [8], a suitable geometric encoding of the standard chromatic subdivision has been given, this is also detailed here.We can transpose the previous geometric presentation with an averaging algorithm called the *Chromatic Average* Algorithm, presented in Algorithm 1, in a way that encode the IIS model. It was first introduced in [8], here we present the colorless adaptation where only the set of values that is received is taken into account. That is, if two processes send the same value (*i.e.* they are associated to the same point in $\mathbb{R}^N$), this is considered only once in the averaging. Since it still use the formula of 31 this yield again the standard chromatic subdivision.

Executing one round of the loop in Chromatic Average for instant graph G, the state of process i is $x_i' = \zeta_{V_i}(x_i^*)$, where V_i is the view of i on this round, that is the set of (j, x_j) it has received. It use the instant graph of the IIS model that are encoding in the following way : We have $V(\Theta(S)) = \Pi_n = [0, n]$ and set $\Theta(\zeta_{V_j}(x_{i_j})) = \mathcal{P}(x_{i_j})$. The arcs are defined using the representation Φ of points, $A(\Theta(S)) = \{(i, j) \mid i \neq j, V_i \subseteq V_j\}$. The mapping θ will denote Θ^{-1}. Then $\Theta(\{\zeta_{V_0}(x_0^*), \cdots, \zeta_{V_n}(x_n^*)\}) = G$. See eg. in Fig. 2a in the Appendix B. Adjacency for a given i corresponds to the smallest subset containing x_i. This one round transformation for the canonical $\mathbb{S}^n$ can actually be done for any simplex S of dimension n of $\mathbb{R}^N$.

By iterating, the chromatic subdivisions $Chr^r(\mathbb{S}^n)$ are given by the global state under all possible r rounds of the Chromatic Average Algorithm. Finite rounds give the Iterated Chromatic Subdivision (hence the name). This is an algorithm that is not meant to terminate (like the full information protocol). The executions of this algorithm are used below to define a topology on IIS_n.

For $G \in ImS_n$, we denote $\mu_G(S)$ the geometric simplex that is the image of S by one round the Chromatic average algorithm under instant graph G.

To start defining this topology we need to define the function *geo*. Let $w \in IIS_n$, $w = G_1 G_2 \cdots$. For the prefix of w of size r, S a simplex of dimension n, we define $geo(w_{|r})(S) = \mu_{G_r} \circ \mu_{G_{r-1}} \circ \cdots \circ \mu_{G_1}(S)$. Finally, we set $geo(w) = \lim_{r \longrightarrow \infty} geo(w_{|r})$

The Chromatic Average algorithm is therefore the geometric counterpart to the Full Information Protocol that is associated with Chr [14]. In particular, any algorithm can be presented as the Chromatic Average together with a terminating condition and a decision function of x.

C Proof that an IIS-terminating subdivision is a simplicial complex

We will use the following lemma to prove that an IIS-terminating subdivision is a simplicial complex. Note that since we are in the geometric setting, this is not as straightforward as in the abstract setting. We need to carefully check that everything "glues" nicely.

▶ **Lemma 33.** *Let σ a simplex with vertices partitioned in two disjoints set U and T. Then the collection $JOIN(Chr(\sigma(U)), \sigma(T))$ is a simplicial complex.*

Proof. Let $\tau = Chr\,\sigma(U)$ it's a simplicial complex. We have that $|\sigma(U)| = |\tau| \subseteq |\sigma|$. Let $\alpha = JOIN(\tau, \sigma(T))$, the facets of α are the facets of τ in union with the facets of $\sigma(T)$. All off these simplices are closed by inclusion which implies the first property of Def 9. For the intersection property of 9, we take $\beta_1, \beta_2 \in \alpha$ such that $|\beta_1| \cap |\beta_2| = |\beta_3|$ and $|\beta_3| \neq \emptyset$, $\tau_i = \{v \in V(\beta_i) | v \in |\tau|\}$. If $V(\beta_3) \subseteq T$ then β_3 remain unchanged. If $V(\beta_3) = V(\tau_i)$ then because Chr is a subdivision β_3 is a simplicial complex. Else $V(\beta_3)$ is partitioned in $V(\tau)$ and $T(\sigma)$, since τ_i is a subdivision, $JOIN(\tau_i, \sigma(T))$ is a simplicial complex. Moreover, $|\tau_i| \subseteq |\sigma|$ and $|\sigma(T)| \subseteq |\sigma|$, hence β_3 is a simplicial complex. All of this gives that $JOIN(\tau, \sigma(T))$ is indeed a simplicial complex. ◀

▶ **Proposition 34.** *$C = \bigcup C_i(T_i)$ is a simplicial complex.*

For convenience of the reader, we rewrite here the definition of $EChr$:
$EChr(T_i, C_i) = (\bigcup_{\sigma \in C_i} Chr\,\sigma(U_i)) \cup (\bigcup_{\sigma \in C_i} JOIN(Chr\,\sigma(U_i), \sigma(T_i))$.

Proof. We start be proving that for all $i \in \mathbb{N}$, the objects C_i and $C_i(T_i)$ are simplicial complexes.

The first step constructs C_{i+1}, it is a union of two operations. The first one $(\bigcup_{\sigma \in C_i} Chr(\sigma(U_i)))$ takes simplices and apply a mesh-shrinking subdivision, which by definition yields a simplicial complex. The second one $(\bigcup_{\sigma \in C_i} JOIN(V(Chr\,\sigma(U_i), T_\sigma)$ is an union of $JOIN$ on a partition of vertices of a simplex, which by lemma 33 yield again a simplicial complex. We have to prove now that all of this simplices "glues back together nicely". Let $\sigma_1, \sigma_2 \in C_i$ such that $|\sigma_1| \cap |\sigma_2| \neq \emptyset$, then by induction we know that C_i is a simplicial complex then $\exists |\sigma_3| \in C_i, |\sigma_1| \cap |\sigma_2| = |\sigma_3|$. We can make a disjunction of case the vertices of σ_3 to prove that the simplices are intersecting correctly.

1. If $V(\sigma_3) \subseteq T_i$ then the simplex σ_3 is not modified in C_{i+1}
2. If $V(\sigma_3) \subseteq U_i$, the subdivision Chr restricted to α is the same if we look from σ or τ, hence we keep the property of simplicial complexes in C_{i+1}.
3. if $V(\sigma_3) = V(\sigma_3(T_i)) \cup V(\sigma_3(U_i))$ with $V(\sigma_3) \cap T_i \neq \emptyset$ and $V(\sigma_3) \cap U_i \neq \emptyset$. Then by the two later cases, we know that $\sigma_3(T_i)$ and $\sigma_3(U_i)$ preserve the simplicial complex. After that we are doing a $JOIN$ between vertices in the same simplex σ_3 which by lemma 33 yield a simplicial complex.

We can deduce from those 3 cases that C_{i+1} is indeed a simplicial complex, which means that C is also a simplicial complex. ◀

D Additional figure

In Fig. 3 we have $x \in X$, $\mathbf{B}(x, \eta(x))$ is in green. We apply the function f and because it satisfies the η-star condition we can exhibit $v_x \in \mathcal{O}$ such that $f(\mathbf{B}(x, \eta(x) \cap X) \subseteq St^\circ(v_x)$, $St^\circ(v_x))$ is colored in light blue.

The figure 4 outline the relation between the main object of the proof of the proposition 28.

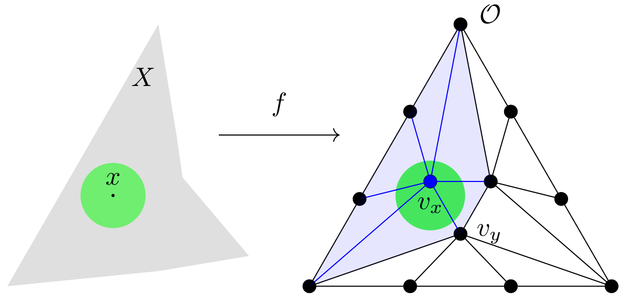


Figure 3 An η-star condition representation.

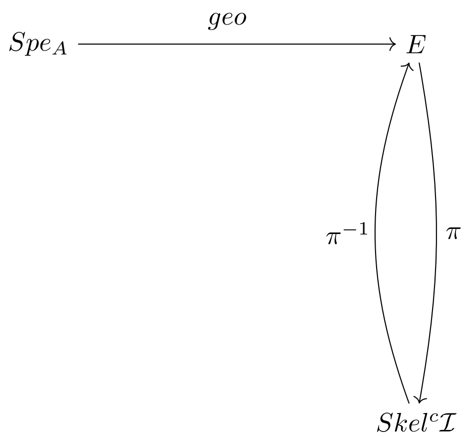


Figure 4 Illustration of proof 28.

E Example of simple fiber bundle and link to distributed system

We acknoledge that fiber bundle might not be a well know mathematical object for some reader, in this section we attempt to adress this difficulty.

One good example of fiber bundle is a Möbius strip. It can be seen as a fiber bundle with a cirlce as B and segment as fiber F. With E the möbius strip, the function $\pi : E \to B$ is a projection of the segment into the base. It is easy to check that for small portion of the circle there is an homeomorphism to a slice of the möbius strip.

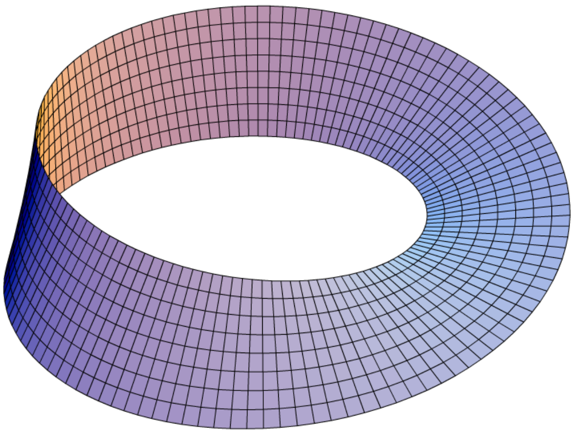


Figure 5 A Möbius Strip.