Brief Announcement: Solvability of Three-Process General Tasks

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Abstract

The topological view on distributed computing represents a task T as a relation Δ between the complex $\mathcal{I}$ of its inputs and the complex $\mathcal{O}$ of its outputs. A cornerstone result in the field is an elegant computability characterization of the solvability of colorless tasks in terms of $\mathcal{I}$, $\mathcal{O}$ and Δ. Essentially, *a colorless task is wait-free solvable if and only if there is a continuous map from the geometric realization of* $\mathcal{I}$ *to that of* $\mathcal{O}$ *that respects* Δ.

This paper makes headway towards providing an analogous characterization for general tasks, which are not necessarily colorless, by concentrating on the case of three-process inputless tasks. Our key contribution is identifying *local articulation points* as an obstacle for the solvability of general tasks, and defining a topological deformation on the output complex of a task T, which eliminates these points by splitting them, to obtain a new task T', with an adjusted relation Δ' between the input complex $\mathcal{I}$ and an output complex $\mathcal{O}'$ without articulation points. We obtain a new characterization of wait-free solvability of three-process general tasks: *T is wait-free solvable if and only if there is a continuous map from the geometric realization of* $\mathcal{I}$ *to that of* $\mathcal{O}'$ *that respects* Δ'.

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1 Introduction

For more than thirty years, distributed computing has been studied through the lens of topology, developing a deep understanding of the *solvability of tasks*. In this approach, a *simplex* represents a configuration as a set of vertexes, each representing the state of one process. In general, each vertex has an associated process *id*, sometimes referred to as its *color*. *Tasks* are triples $(\mathcal{I}, \mathcal{O}, \Delta)$, where $\mathcal{I}$ and $\mathcal{O}$ are *simplicial complexes* modelling the inputs and outputs of the task, and Δ is a relation specifying the possible valid outputs, $\Delta(\sigma)$, for each input simplex $\sigma \in \mathcal{I}$. For any initial configuration σ of $\mathcal{I}$, each process starts with an input vertex of σ colored by its ID, and must decide on an output vertex with its color, such that the vertices decided by the processes form a simplex τ of $\Delta(\sigma)$.

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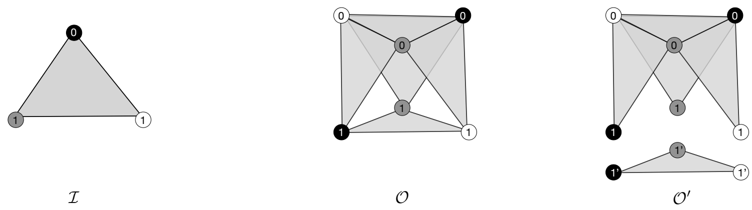

Figure 1 *The majority consensus task: input complex (left), output complex (center) and output complex after splitting (right).*

A major contribution of the topological approach is a set of novel impossibility results and algorithms for specific tasks, through fundamental characterizations of solvable tasks in various distributed computing models. Beyond telling us what is solvable and what is not, characterization results tell us *what makes tasks unsolvable*, indicating the obstructions to solvability, and sometimes pointing how these obstructions can be avoided.

The focus of this work is on *wait-free* protocols for solving a task in a read/write shared memory system. A cornerstone result for this model is the *asynchronous computability theorem* (ACT) of Herlihy and Shavit [18] (see also [14, Theorem 11.2.1]). The remarkable insight of this theorem is that a protocol solving a task in this model corresponds to a color-preserving simplicial mapping from a *chromatic subdivision* of the input complex into allowable outputs in the output complex.

As stated, however, the ACT does not provide us with a direct connection to standard topology notions relating the topology of the input and the output complexes. Such a characterization is so far known only for *colorless* tasks [7, 15]. A colorless task is defined only in terms of input and output values, regardless of the number of processes involved, and regardless of which process has a particular input or output value; accordingly, $\mathcal{I}$ and $\mathcal{O}$ consist of sets of values, without process IDs. Well-known examples are the consensus task [10] and its generalization to set consensus [8]. Colorless tasks are simpler to analyze and in particular, they have an elegant computability characterization in terms of their input and output complexes [14, 17]: *a colorless task is wait-free solvable if and only if there is a continuous map from* $|\mathcal{I}|$ *to* $|\mathcal{O}|$ *respecting* Δ. Recall that $|K|$ denotes the geometric realization of a simplicial complex.

A similar characterization for *general* tasks, which are not necessarily colorless, has eluded researchers. General tasks that cannot be stated as colorless tasks are sometimes called *chromatic*. Several such tasks have been studied, notably renaming [3]. A simple example is the *majority consensus* task. In this weaker form of binary consensus, it is allowed to decide different values (when processes do not all start with the same input), but only if more processes decide 0 than 1. Figure 1 illustrates this task for a single three-process input configuration: two processes start with 1 and the other with 0. Tasks whose input complex contains a single facet are called *inputless* in the literature. Impossibility results are usually achieved using inputless versions of tasks, which is also the case in our examples, and we concentrate on such tasks in this work. In our figures, processes are identified by black, grey and white colors, and their respective inputs are inside the vertices, and with the analogous convention for output values, the output complex, where the respective output values are inside vertices.

Researchers have tried to characterize the solvability of general tasks in terms of continuous maps, analogous to the colorless characterization. The value of such a characterization would not only be due to its direct nature from the input complex to the output complex, but also due to the direct connection to topology: continuous maps between spaces.

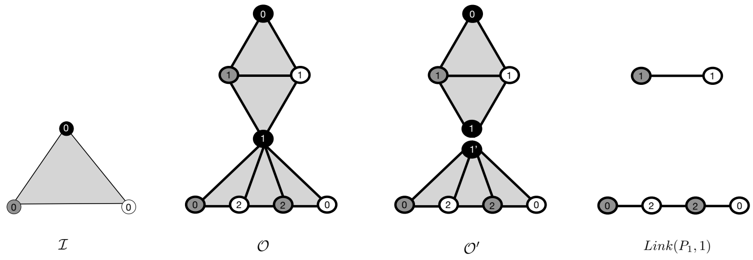

Figure 2 *The hourglass task: input complex (left), output complex (center left), output complex after splitting (center right) and the link of the local articulation point (right).*

The quest for such a characterization has failed, because *it is not true* that a chromatic task is solvable if and only if there is a continuous map from the input to the output complex. This has been demonstrated with the *hourglass* task (Figure 2) in [14, Section 11.1]. The hourglass task is an inputless task, where each process running solo decides on 0. Process P_0 (black) running concurrently with either P_1 or P_2 can also (in addition to their solo values) decide on their respective vertexes, with output 1. While P_1 and P_2 running concurrently can additionally decide their respective vertices with value 2. When all three run concurrently, any triangle is a valid output simplex. Despite there being a continuous map from the input to the output complex respecting the input/output relation for the hourglass task, the task is unsolvable. The same holds for the majority consensus task, as proved in our paper.

2 Our Results

This paper proves a *necessary and sufficient* condition for the solvability of colored *three-process* inputless tasks, $(\mathcal{I}, \mathcal{O}, \Delta)$, directly from the input complex, not to the output complex, $\mathcal{O}$, since this is impossible, but to an output complex $\mathcal{O}'$, easily derived from $\mathcal{O}$.

Our study goes through identifying a notion of a *local articulation point* (LAP) in the output complex: a vertex whose neighborhood, its *link* in the topological parlance, is disconnected. Figure 2(right) depicts the link of a vertex in the hourglass task, which is a graph consisting of two connected components: the output edges compatible with the vertex of P_1 deciding 1.

We show a novel method for dealing with each local articulation point by *splitting* the output complex around it, eventually creating a *link-connected* output complex $\mathcal{O}'$. See the right-hand side of Figure 1, for splitting the majority consensus task, and Figure 2, for splitting the hourglass task. This yields a task T', with the same input complex $\mathcal{I}$ as the original task and an adjusted relation Δ' between $\mathcal{I}$ and $\mathcal{O}'$. We show that the solvability of the original general task is equivalent to the solvability of the task derived from T by eliminating all the LAPs.

▶ **Theorem 1** (informal). *T is solvable if and only if T' is solvable.*

When T is inputless, this implies that it is wait-free solvable if and only if there is a continuous map from the geometric realization of $\mathcal{I}$ to that of $\mathcal{O}'$, which respects Δ'.

▶ **Theorem 2** (informal). *A general inputless task $T = (\mathcal{I}, \mathcal{O}, \Delta)$ is solvable if and only if there is a continuous map from $|\mathcal{I}|$ to $|\mathcal{O}'|$ carried by Δ'.*

The following necessary condition is useful when considering specific tasks:

▶ **Corollary 3.** *A general task* $(\mathcal{I}, \mathcal{O}, \Delta)$ *is not solvable if there is an edge* $\{x, x'\} \in \mathcal{I}$ *such that for every* $y \in \Delta(x)$ *and* $y' \in \Delta(x')$*, any path from* y *to* y' *in* $\Delta(\{x, x'\})$ *goes through a LAP.*

This corollary can be used to prove the impossibility of solving the majority consensus task, the hourglass task, and of the *pinwheel* task, which we define in the full paper.

Three-Process Tasks as a Stepping-Stone for Future Investigation. The restriction to three processes could be seen as a limitation of our results, and indeed it is a stepping-stone for future investigation. However, there are several reasons for concentrating on this case.

First, in this case there is no need to use algebraic topology, making the paper accessible to a wider audience.

Second, the case of three processes has played an important role in past research, because it is the smallest dimension where topological properties beyond graph connectivity appear. In this case, two types of obstructions – local articulation points and contractibility – are neatly identifiable. In smaller dimensions, e.g., for two-process tasks, if the output complex has a local articulation point then it is also not connected (in the graph-theoretic sense), and hence, it is not solvable. Indeed, for two-process general tasks, there is a characterization based on continuous maps (a consequence of [14, Theorem 2.5.2], analogous to the seminal one for message passing models [5]). On the other hand, in dimensions higher than 2, i.e., with four or more processes, a disconnected link may be connected (in the graph-theoretic sense).

Finally, undecidability results such as [11, 16] are proved using colorless *loop agreement* tasks [15, 16, 21], defined using an output complex $\mathcal{O}$, and a *loop* in it. Roughly, each process starts on one of *three* distinguished vertexes of the loop; if they start on the same vertex, they decide on this vertex; if they start on two distinct distinguished vertexes, then they decide vertices belonging to the same edge along the path linking their starting vertexes; finally, if they start on all three distinguished vertices, then they can decide vertexes belonging to any simplex of $\mathcal{O}$. Like all colorless tasks, loop agreement is defined independently of the number of processes. Notice however that it is defined on *two-dimensional* input and output complexes, and hence, all the arguments are essentially in the *three-process* case. Moreover, some results [15, Theorem 5.4] do not generalize to more than three processes. Note that our approach is different from the one used in [11], where the undecidability of three-process tasks was proved by a reduction from the *contractibility problem* to the task-solvability problem. The contractibility problem asks whether a given loop of a simplicial complex can be continuously transformed into a point, a problem which is known to be undecidable even for 2-dimensional simplicial complexes (see, e.g., [23]). Gafni and Koutsoupias prove the reduction by showing that contractibility is undecidable for the special case of chromatic complexes and loops of length 3. To do so, they first show that the contractibility problem is undecidable for link-connected two-dimensional complexes. We instead transform the output complex to be link-connected, and can then argue directly about colorless tasks.

3 Discussion and Related Work

This paper provides a new characterization for wait-free solvability of general three-process tasks, which is based on splitting *local articulation points*. We prove that the task T is wait-free solvable if and only if there is a continuous map from the geometric realization of the input complex of T to the geometric realization of the deformed output complex, which respects the deformed task mapping.

Our characterization exposes two types of obstructions to solvability: The first, which exists only in chromatic tasks, are local articulation points; these obstructions can be effectively detected (and removed). The second is identical to the one that exists for colorless tasks, namely, the existence of a continuous map from the geometric realization of $\mathcal{I}$ to that of $\mathcal{O}'$ carried by Δ'. The locality of the former obstruction makes it an ideal target for extension-based impossibility proofs [2, 4]. The latter obstruction is known to be undecidable [11, 15], as it is closely related to the topological notion of loop contractibility.

Link connectivity has showed up in previous papers about chromatic tasks, starting with the work of Herlihy and Shavit [18]. Nevertheless, our paper is the first to identify the precise role of link connectivity, by concentrating on the special case of three process.

There are two previous approximations to continuous characterization. First, when assuming link connectivity, there are sufficiency results for general tasks ([22] and [14, Section 11.5]), without a matching necessary condition. Another related notion are *continuous tasks* [12], which have an input/output specification that is a continuous function between the geometric realizations of the input and output complex. The characterization is that a task is solvable if and only if there exists a *chromatic function* (a notion introduced in this paper) from the input complex $\mathcal{I}$ to the output complex $\mathcal{O}$ respecting the task specification. Our characterization is in contrast more explicit about the obstructions (since it exposes the role of local articulation points), and establishes a direct connection with colorless solvability (after removing articulation points, colored and colorless solvability are the same).

Havlicek [13] have studied the goal of identifying *computable obstructions* to wait-free solvability, taking into account that any such mechanism must be necessarily incomplete, due to the known undecidability of this problem [11, 15]. The mechanism presented by Havlicek can nevertheless find obstructions for consensus and other tasks, but only those related to homology groups. It would be interesting to extend it to deal with the type of link connectivity obstructions studied here.

The full paper presents new chromatic versions of consensus, *majority consensus*, approximate agreement, *majority approximate agreement*, and set agreement, *pinwheel task*, that may be of interest on their own. They are obtained by removing some *output* triangles. Notice that removing output simplexes is the opposite of what is done in the condition-based approach [20], where instead *input* simplexes are removed to obtain an easier task, instead of a harder one.

Removing output simplexes to obtain our new tasks perturbs the symmetry of the original tasks, so that the chromatic version can no longer be defined without referring to ids. It would be interesting to derive a systematic way of transforming any colorless task in this way. It would be also interesting to consider the same idea but for any number of processes, n, and investigate the simplexes than need to be removed to obtain a decidable obstruction. That is, removing some output simplexes of $k \leq n$ processes, while otherwise leaving the task unchanged. For what value of k the obstruction becomes decidable?

Pseudospheres [1] are a succinct mathematical notation that was used in the topological approach of distributed computing to state that any process can take any value. Our examples leave intact all the pseudospheres of dimension 1, while destroying those of dimension 2. It would be nice to generalize the examples to higher dimensions (and more than three processes).

Our splitting deformation draws upon work on modelling of real-world objects, used in computer-aided design (CAD) [19]. There is a long line of papers studying splitting deformations mostly of two and three-dimensional simplicial complexes, because these are the dimensions of most graphics applications (and for technical reasons, as discussed in,

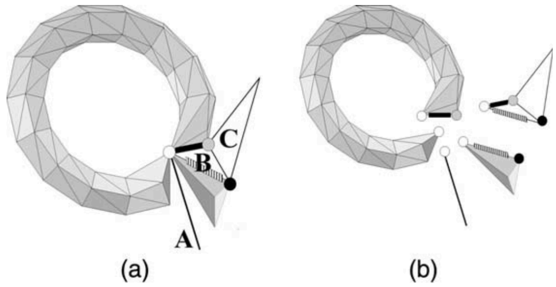

Figure 3 *[9, Figure 1] An example (a) of a non-manifold object (described by a 3D simplicial complex made of tetrahedra, triangles, and edges) with a dangling edge (A) and a dangling surface formed by two triangles (B) and (C) and its decomposition (b) into "simple" components.*

e.g. [6]), although there is also work on higher dimensional complexes. The same splitting we do has been used (e.g. [9, Fig.1], replicated in Figure 3), but not to fix a disconnected link; instead, the interest has been in doing additional splittings, even of edges, because the goal in this research line is to decompose a non-manifold complex into an assembly of manifolds, or at least into components that belong to simpler, well-understood class of complexes where efficient data structures are known.

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