The Price of Upwardness

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Abstract

Not every directed acyclic graph (DAG) whose underlying undirected graph is planar admits an upward planar drawing. We are interested in pushing the notion of upward drawings beyond planarity by considering upward k-planar drawings of DAGs in which the edges are monotonically increasing in a common direction and every edge is crossed at most k times for some integer $k \geq 1$. We show that the number of crossings per edge in a monotone drawing is in general unbounded for the class of bipartite outerplanar, cubic, or bounded pathwidth DAGs. However, it is at most two for outerpaths and it is at most quadratic in the bandwidth in general. From the computational point of view, we prove that upward-k-planarity testing is NP-complete already for $k = 1$ and even for restricted instances for which upward planarity testing is polynomial. On the positive side, we can decide in linear time whether a single-source DAG admits an upward 1-planar drawing in which all vertices are incident to the outer face.

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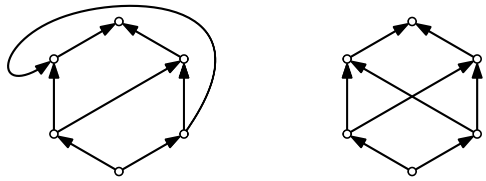
1 Introduction

Graph drawing "beyond planarity" studies the combinatorial and algorithmic questions related to representations of graphs where edges can cross but some crossing configurations are forbidden. Depending on the forbidden crossing configuration, different beyond-planar types of drawings can be defined including, for example, RAC, k-planar, fan planar, and quasi planar drawings. See [19, 30, 32] for surveys and books.

While most of the literature about beyond planar graph drawing has focused on undirected graphs (one of the few exceptions being [2,3] which studies RAC upward drawings), we study *upward k-planar drawings* of acyclic digraphs (DAGs), i.e., drawings of DAGs where the edges monotonically increase in y-direction and each edge can be crossed at most k times. The minimum k such that a DAG admits an upward k-planar drawing is called its *upward local crossing number*. We focus on values of $k = 1, 2$ and investigate both combinatorial properties and complexity questions. Our research is motivated by the observation that well-known DAGs that are not *upward-planar*, i.e., not upward 0-planar, do admit a drawing where every edge is crossed at most a constant number of times; see, e.g., Figure 1.



(a) (b)

Figure 1 A graph that is not upward planar but admits an upward 1-planar drawing.

Our contribution.

- A graph is an *outerpath* if it has a planar drawing in which each vertex is incident to the outer face and the internal faces induce a path in the dual graph. Papakostas [35] observed that there is a directed acyclic 8-vertex outerpath that is not upward-planar (see Figure 3a). We strengthen this observation by showing that there exists a directed acyclic fan (that is, a very specific outerpath) that has no upward-planar drawing (Proposition 1). On the other hand, we show that every directed acyclic outerpath is upward 2-planar (Theorem 9) and that the upward local crossing number is quadratic in the bandwidth (Theorem 6). However, the upward local crossing number of bipartite outerplanar DAGs (Theorem 2), bipartite DAGs with bounded pathwidth (Corollary 4), and cubic DAGs (Proposition 5) is in general unbounded.
- We show that upward 1-planarity testing is NP-complete, even for graph families where upward planarity testing can be solved in polynomial time. These include: single-source single-sink series-parallel DAGs with a fixed rotation system; single-source two-sink series-parallel DAGs where the rotation system is not fixed; and single-source single-sink DAGs without fixed rotation system that can be obtained from a K_4 by replacing the edges with series-parallel DAGs (Theorem 11).
- Finally, following a common trend in the study of beyond planar graph representations, we consider the *outer model*, in which all vertices are required to lie on a common face while maintaining the original requirements [19,30,32]. We prove that testing whether a single-source DAG admits an upward outer-1-planar drawing can be done in linear time (Theorem 13).

The details of omitted or sketched proofs can be found in the full version [1].

Related Work. A drawing of a graph is *monotone* if all edges are drawn monotone with respect to some direction, e.g., a drawing is *y-monotone* or *upward*, if each edge intersects each horizontal line at most once. The corresponding crossing number is introduced and studied in [24,37]. Schaefer [36] mentions the upward crossing number and the local crossing number but not their combination. Schaefer [36, p. 64] also showed that a drawing with the minimum number of crossings per edge can require incident edges that cross. The edges of the provided 4-planar example graph can be oriented such that the resulting directed graph admits an upward 4-planar drawing. Thus, also an upward drawing that achieves the minimum local crossing number can require incident edges that cross. Also, the so-called strong Hanani–Tutte theorem carries over to directed graphs: Fulek et al. [24, Theorem 3.1] showed that every undirected graph that has a monotone drawing where any pair of independent edges crosses an even number of times also has a planar monotone drawing with the same vertex positions. This implies that in any upward drawing of a graph that is not upward-planar there must be a pair of independent edges that crosses an odd number of times.

Upward drawings of directed acyclic graphs have been studied in the context of (upward) book embeddings. In that model the vertices are drawn on a vertical line (a spine) following a topological order of the graph, while all edges are pointing upwards. To reduce the edge crossings, edges are partitioned into the fewest number of crossing-free subsets (pages). Studying upward book embeddings is a popular topic, which is usually centered around determining the smallest number of pages for various graph classes [22,23,29,31,34] or deciding whether a graph admits an upward drawing with a given number of pages [7,8,10,11,12]. Our model is equivalent to *topological book embeddings* [28,33], which are a relaxed version of book embeddings in which edges are allowed to cross the spine. To the best of our knowledge, earlier papers considered only the problem of minimizing the number of spine crossings, whereas we want to bound the maximum number of edge crossings per edge (ignoring the spine).

2 Preliminaries

A *drawing* Γ of a graph G maps the vertices of G to distinct points in the plane and the edges of G to Jordan arcs. For a vertex v of G and a drawing Γ of G, let $x_\Gamma(v)$ and $y_\Gamma(v)$ denote the x- and y-coordinates of v in Γ, respectively; when Γ is clear from the context, we may omit it and simply use the notation $x(v)$ and $y(v)$. A *face* of Γ is a region of the plane delimited by maximal uncrossed arc portions of the edges of G. The unique unbounded face of Γ is its *outer face*, the other faces are its *internal faces*. An *outer edge* is one incident to the outer face; all other edges are *inner edges*. The *rotation* of a vertex v in Γ is the counterclockwise cyclic order of the edges incident to v. The *rotation system* of Γ is the set of rotations of its vertices.

The drawing Γ is *planar* if no two of its edges cross; it is *k-planar* if each edge is crossed at most k times. A graph is *(k-)planar* if it admits a (k-)planar drawing; it is *outer (k-)planar* if it admits a (k-)planar drawing where all vertices are incident to the outer face.

A *planar embedding* $\mathcal{E}$ of a planar graph G is an equivalence class of planar drawings of G, namely those that have the same set of faces. Each face can be described as a sequence of edges and vertices of G which bound the corresponding region in the plane; each such sequence is a face of G in the embedding $\mathcal{E}$. A planar embedding $\mathcal{E}$ of a connected graph can also be described by specifying the rotation system and the outer face associated with any drawing of $\mathcal{E}$.

Let Γ be a non-planar drawing of a graph G; the *planarization* of Γ is the planar drawing Γ' of the *planarized graph* G' obtained by replacing each crossing of Γ with a dummy vertex. If Γ is 1-planar, the planarization can be obtained as follows. Let uv and wz be any two edges that cross in Γ; they are replaced in Γ' by the edges ux, xv, wx and xz, where x is the dummy vertex. Two non-planar drawings of a graph G have the same *embedding* if their planarizations have the same planar embedding. An embedding $\mathcal{E}$ of G can also be described by specifying the planarized graph G' and one of its planar embeddings. A planar graph with a given planar embedding is also called *plane graph*. An *outerplane graph* is a plane graph whose vertices are all incident to the outer face. A *fan* is a maximal outerpath that contains a vertex c that is adjacent to all other vertices; we call c the *central vertex* of the fan. A *2-tree* is a planar graph that can be reduced to an edge by iteratively removing a degree-two vertex that closes a 3-cycle. A *series-parallel graph* is a graph that can be augmented to a 2-tree by adding edges (and no vertices).

A *(simple, finite) directed graph* (*digraph* for short) G consists of a finite set $V(G)$ of *vertices* and a finite set $E(G) \subseteq \{(u, v) \mid u, v \in V(G), u \neq v\}$ of ordered pairs of vertices, which are called *edges*. A *source* (resp. *sink*) of G is a vertex with no incoming (resp. no outgoing) edges. A *single-source graph* is a digraph with a single source and, possibly, multiple sinks. A digraph G is an *st-graph* if: (i) it is acyclic and (ii) it has a single source s and a single sink t. An st-graph is a *planar st-graph* if it admits a planar embedding with s and t on the outer face. We say that a drawing of a digraph G is *upward* if every (directed) edge (u, v) of G is mapped to a y-monotone Jordan arc with $y(u) < y(v)$. Clearly, a digraph admits an upward drawing only if it does not contain a directed cycle. Therefore, we assume for the rest of the paper that the input graph is a *DAG*, a directed acyclic graph. Such a graph has a *linear extension*, i.e., a vertex order $v_1, \dots, v_n$ such that, for any directed edge (v_i, v_j), we have $i < j$. We say that a DAG is planar, outerplanar, or series-parallel if its underlying undirected graph is planar, outerplanar, or series-parallel, respectively.

Let Γ be an upward drawing of a DAG G. By the upwardness, the rotation system of Γ is such that for every vertex v of Γ the rotation of v has only one maximal subsequence of outgoing (incoming) edges. We call such a rotation system a *bimodal rotation system*. An *upward embedding* of a DAG G is an embedding of G with a bimodal rotation system. The minimum k such that a digraph G admits an upward k-planar drawing is called its *upward local crossing number* and denoted by $\mathrm{lcr}^{\uparrow}(G)$.

For any positive integer k, we use $[k]$ as shorthand for $\{1, 2, \dots, k\}$. A *path-decomposition* of a graph $G = (V, E)$ is a sequence $P = \langle X_1, \dots, X_\ell \rangle$ of subsets of V, called *buckets*, such that (1) for each edge $e \in E$ there is a bucket that contains both end vertices of e, and (2) the set of buckets that contain a particular vertex $v \in V$ forms a contiguous subsequence of P. The *width* of a path-decomposition is one less than the size of the largest bucket. The *pathwidth* of the graph G is the width of a path decomposition of G with the smallest width.

3 Lower Bounds

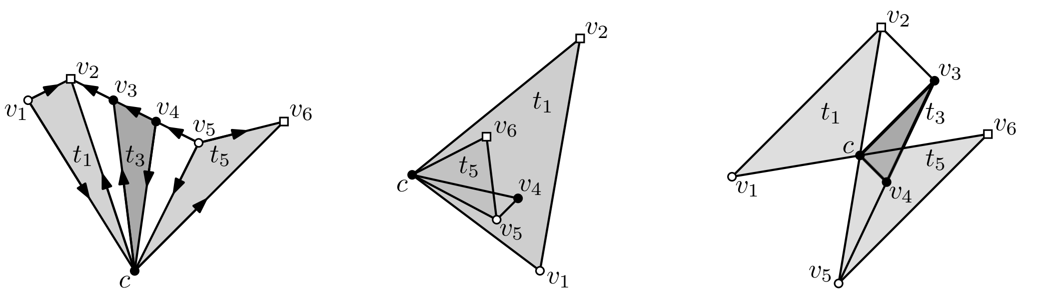
We start with a negative result that shows that even very special directed acyclic outerpaths may not admit upward-planar drawings, thus strengthening Papakostas' observation [35].

▶ **Proposition 1.** *Not every directed acyclic fan is upward-planar.*

Proof. Consider the 7-vertex fan F depicted in Figure 2a. Suppose for a contradiction that F is upward planar, that is, F admits an upward planar drawing Γ. Let c be the central vertex of F. We assume that c is placed at the origin. We say that a triangle of F is *positive* (*negative*, respectively) if the corresponding region of the plane in Γ contains the point $(\varepsilon, 0)$ ($(-\varepsilon, 0)$, respectively) for a sufficiently small value $\varepsilon > 0$. The triangles that have one vertex below c and one vertex above c (namely $t_1 = \triangle cv_1v_2$, $t_3 = \triangle cv_3v_4$, and $t_5 = \triangle cv_5v_6$) are either positive or negative.

If both t_1 and t_5 are positive, then one must contain the other in Γ, say, t_1 contains t_5; see Figure 2b. But then vertices v_3 and v_4 must also lie inside t_1. If both lie inside t_5, then the edge (v_3, v_2) intersects an edge of t_5. So both must lie outside t_5. But v_4 lies on one hand above v_5 and on the other hand below c and, thus, below v_6. So the edge (v_4, c) intersects the edge (v_5, v_6). (If t_5 is contained in t_1, the edge (c, v_3) intersects the edge (v_1, v_2).)

By symmetry, not both t_1 and t_5 can be negative, so exactly one of t_1 and t_5 must be negative, say, t_1; see Figure 2c. Now first assume that t_3 is positive. Due to edge (v_3, v_2), vertex v_3 must be outside t_5, so t_3 cannot be inside t_5. On the other hand, t_3 cannot contain t_5 because v_4 is above v_5. Hence t_3 intersects t_5. Finally, assume that t_3 is negative. Due to edge (v_5, v_4), vertex v_4 must be outside t_1, so t_3 cannot be inside t_1. On the other hand, t_3 cannot contain t_1 because v_3 is below v_2. Hence t_3 intersects t_1. ◀



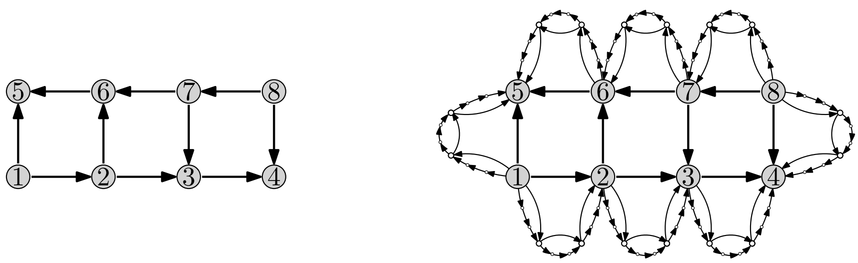
(a) The fan F. **(b)** Case: t_5 is contained in t_1. **(c)** A 1-planar upward drawing of F.

Figure 2 A directed acyclic fan F that does not admit a planar upward drawing.

By iteratively adding paths on every outer edge of an outerplanar but not upward-planar DAG, we can construct outerplanar DAGs with an unbounded upward local crossing number.

▶ **Theorem 2.** *For each $\ell \geq 0$, there is a bipartite outerplanar DAG G_ℓ with $n_\ell = 8 \cdot 3^\ell$ vertices, maximum degree $\Delta_\ell = 2\ell + 3$, and upward local crossing number greater than $\ell/6$, which is in $\Omega(\log n_\ell)$ and $\Omega(\Delta_\ell)$.*

Proof. The bipartite graph G_0 in Figure 3a is not upward planar [35]. For $\ell \geq 1$, we construct G_ℓ from $G_{\ell-1}$ by adding a 3-edge path on every outer edge of the graph. Figure 3b shows G_2. The maximum degree of G_ℓ is $\Delta_\ell = 2\ell + 3$. The number of vertices is $n_\ell = 8 + \sum_{i=1}^{\ell} 8 \cdot 3^{i-1} \cdot 2 = 8 \cdot 3^\ell$.



(a) G_0. **(b)** G_2.

Figure 3 There is a family $(G_\ell)_{\ell \geq 0}$ of bipartite outerplanar graphs such that G_ℓ has n_ℓ vertices, maximum degree Δ_ℓ, and upward local crossing number in $\Omega(\Delta_\ell) \cap \Omega(\log n_\ell)$.

Consider now an upward k-planar drawing Γ of G_ℓ for some k. Since G_0 is not upward planar, there must be a pair of independent edges of G_0 that crosses an odd number of times in Γ. Observe that G_0 has no upward planar drawing in which only the two inner edges cross an odd number of times, for otherwise the two cycles $\langle 1,2,6,5\rangle$ and $\langle 3,4,8,7\rangle$ would intersect an odd number of times, which is impossible. Thus, in Γ there must be an outer edge e of G_0 that is crossed by an independent edge e' of G_0 an odd number of times. We choose e' to be an outer edge of G_0, if possible.

We now determine a cycle C of G_ℓ that is crossed by e an odd number of times and does also not contain any end vertex of e. If e' is an inner edge, then we take the outer path P of G_0 that connects the ends of e' and does not contain e; this is not intersected by e due to our choice of e'. Let C be the concatenation of P and e'. In this case C has length at most six.

If e' is an outer edge, we do the following: We start with the path P of length three that was added for e'. If P contains an edge that is crossed an odd number of times by e then we replace e' by such an edge and continue. More precisely, let $e_1 = e'$ and initialize $i = 1$. Let P_1 be the path of length three that was added for e_1. While P_i contains an edge that is crossed an odd number of times by e, let e_{i+1} be such an edge, let P_{i+1} be the path of length three that was added for e_i, and increase i by one. Since e is crossed at most k times, this process stops at some $i < k$. Let C be the cycle that is composed of P_i and e_i. In this case C has length four.

Cycle C might cross itself. However, it divides the plane into cells. Since e crosses C an odd number of times, it follows that the end vertices of e must be in different cells of the plane. This means that not only e but also the ℓ edge-disjoint paths that were added on top of e have to cross C. But C contains at most six edges, each of which can be crossed at most k times. This is impossible if $\ell \geq 6k$. Hence, if there is an upward k-planar drawing then $\ell < 6k$, which means that $k > \ell/6$. ◀

We now show that if we expand the graph class beyond outerplanar graphs, then we get a lower bound on the upward local crossing number that is even linear in the number of vertices. The graphs in our construction have pathwidth 2, as opposed to the graphs in Theorem 2 whose pathwidth is logarithmic. Observe that a *caterpillar*, i.e., a tree that can be reduced to a path by removing all degree-1 vertices, has pathwidth 1, and that the pathwidth can increase by at most 1 if we add a vertex with some incident edges or subdivide some edges.

▶ **Theorem 3.** *For every $k \geq 1$, there exists a DAG with $\Theta(k)$ vertices, maximum degree in $\Theta(k)$, and pathwidth 2 that does not admit an upward k-planar drawing.*

Proof. Let G_k be the graph consisting of the four vertices a, b_1, b_2, and c and the following set of edges and degree-2 vertices (see also Figure 4):

- edges (a, b_1) and (a, b_2);
- for $i \in [2]$ and $j \in [3k + 1]$, a *through-vertex at b_i*, i.e., a vertex $d_i^{(j)}$ and edges $(b_i, d_i^{(j)})$ and $(d_i^{(j)}, c)$;
- for $j \in [6k + 1]$, a *source below a*, i.e., a vertex $s^{(j)}$ and edges $(s^{(j)}, a)$ and $(s^{(j)}, c)$;
- for $i \in [2]$ and $j \in [4k + 1]$, a *sink above b_i*, i.e., a vertex $t_i^{(j)}$ and edges $(b_i, t_i^{(j)})$ and $(c, t_i^{(j)})$.

Clearly, G_k has $\mathcal{O}(k)$ vertices, and pathwidth 2, since $G - c$ is a caterpillar and has pathwidth 1.

Assume that there was an upward k-planar drawing Γ of G_k. Up to renaming, we may assume that $y(b_2) \leq y(b_1)$. Delete all but one of the through-vertices at b_1 from the drawing; in what follows, we write d_1 for the one that we keep (it does not matter which one).

Among the $3k + 1$ through-vertices $d_2^{(j)}$ at b_2, there exists at least one for which the path $\langle b_2, d_2^{(j)}, c\rangle$ crosses none of the three edges in the path $\langle a, b_1, d_1, c\rangle$, for otherwise there would be an edge with more than k crossings. Delete all other through-vertices at b_2; in what

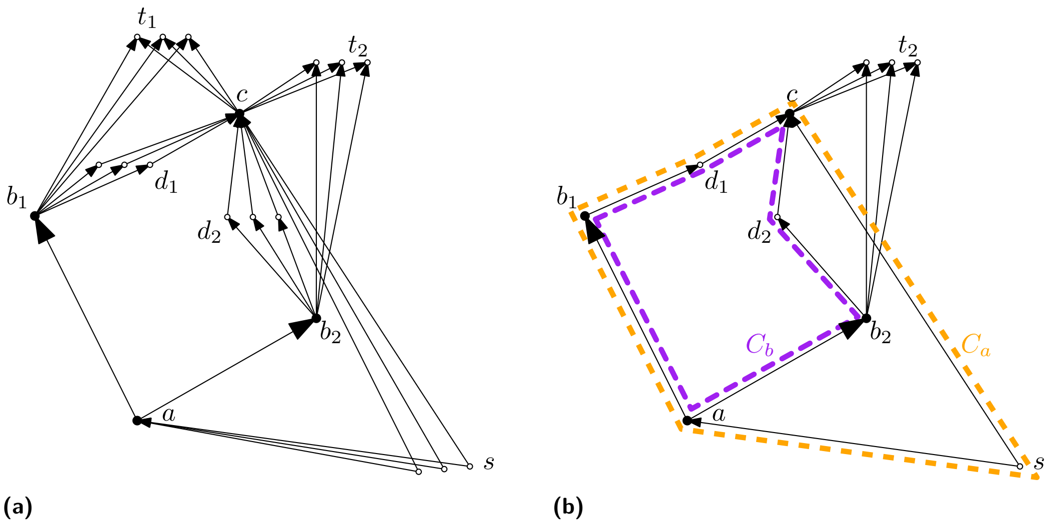


Figure 4 A graph of pathwidth 2 (drawn upward) that does not have an upward k-planar drawing. (a) We only show three of the $\Theta(k)$ vertices of each group. (b) Cycles C_a and C_b.

follows we write d_2 for the one that we keep. Let a' be the topmost intersection point of (a, b_1) and (a, b_2) (possibly $a' = a$). Since $y(a) \leq y(a') < y(b_2) \leq y(b_1)$ the curve C_b formed by the two directed paths $\langle a', b_i, d_i, c\rangle$ (for $i \in [2]$) is drawn without crossing in Γ.

Curve C_b uses six edges, therefore among the $6k + 1$ sources below a, there exists one, call it s, for which edge (s, c) crosses no edge of C_b. Since $y(s) < y(a)$, vertex s is outside C_b, and so the entire edge (s, c) is outside C_b, except at the endpoint c. In particular, among the three edges (d_1, c), (d_2, c), and (s, c) that are incoming at c, edge (s, c) is either leftmost or rightmost (but cannot be the middle one). We assume here that (s, c) is rightmost, the other case is symmetric. Write $\{p, q\} = \{1, 2\}$ such that the left-to-right order of incoming edges at c is (d_p, c), (d_q, c), (s, c). In Figure 4, we have $p = 1$ and $q = 2$.

Edge (s, a) is also outside C_b, except perhaps at endpoint a, since it uses smaller y-coordinates. Let s' be the topmost intersection point of (s, a) and (s, c). Then there are no crossings in the curve C_a formed by the directed paths $\langle s', a, b_p, d_p, c\rangle$ and $\langle s', c\rangle$. By our choice of p and q, vertex d_q is *inside* C_a, and so is the entire path $\langle a', b_q, d_q, c\rangle$, except at the ends since it is part of C_b. In particular, b_q is inside C_a, whereas, for $j \in [4k+1]$, $t_q^{(j)}$ is outside C_a due to $y(c) < y(t_q^{(j)})$. It follows that one of the four edges (a, b_p), (b_p, d_p), (d_p, c) and (s, c) must be crossed at least $k + 1$ times by edges from b_q to the sinks above it. Thus, the drawing was not k-planar, a contradiction. ◀

The graphs that we constructed in the proof of Theorem 3 are not bipartite, but one can make them bipartite by subdividing all edges once. This at best cuts the local crossing number in half, increases the pathwidth by at most 1, and yields the following result.

▶ **Corollary 4.** *There is a family of bipartite DAGs of constant pathwidth whose upward local crossing number is linear in the number of vertices.*

So far we needed graphs of unbounded maximum degree in order to enforce unbounded upward local crossing number. We now show that, intrinsically, this is not necessary.

▶ **Proposition 5.** *There are cubic DAGs whose upward local crossing number is at least linear in the number of vertices.*

Proof. The crossing number of a random cubic graph with n vertices is expected to be at least cn^2 for some absolute constant $c > 0$ [20], and thus there exist graphs yielding this bound. By the pigeon-hole principle, such a graph contains an edge with $\Omega(n)$ crossings among its $\Theta(n)$ edges. Impose arbitrary acyclic edge directions. ◀

4 Upper Bounds

The *bandwidth* $\mathrm{bw}(G)$ of an undirected graph G is the smallest positive integer k such that there is a labeling of the vertices by distinct numbers $1, \dots, n$ for which the labels of every pair of adjacent vertices differ by at most k.

▶ **Theorem 6.** *The upward local crossing number of a DAG G with maximum degree Δ is at most $\Delta \cdot (2\,\mathrm{bw}(G) - 2) \le 4\,\mathrm{bw}(G)(\mathrm{bw}(G) - 1)$, so it is in $\mathcal{O}(\Delta \cdot \mathrm{bw}(G)) \subseteq \mathcal{O}(\mathrm{bw}(G)^2)$.*

Proof. Observe that the maximum degree Δ of a graph G is bounded in terms of the bandwidth of G; namely, $\Delta \le 2\,\mathrm{bw}(G)$. Consider a linear extension of G. For every vertex v of G, let $y(v)$ be its index in the extension. Now consider a labeling of G corresponding to the bandwidth. For every vertex v of G, let $x(v)$ be label. Construct a drawing of G by first placing every vertex v at the point $(x(v), y(v))$ and by then perturbing vertices slightly so that the points are in general position. Adjacent vertices are connected via straight-line segments.

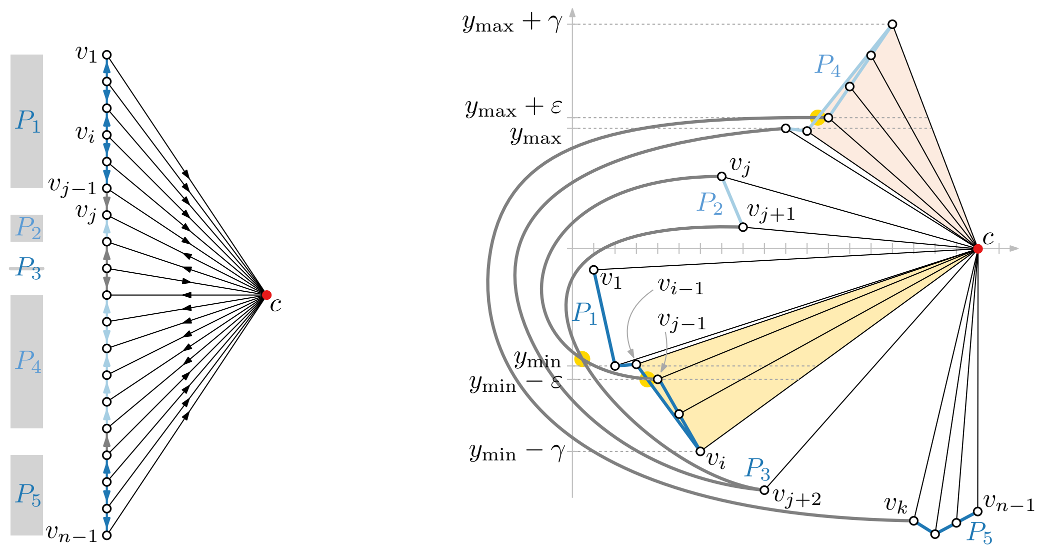
It is easy to see that the drawing is upward since it is consistent with the linear extension. Consider an arbitrary edge (u, v) with $x(u) < x(v)$. Every edge that crosses (u, v) must have its left endpoint in the interval $[x(u) - \mathrm{bw}(G) + 1, x(v) - 1]$. Since $x(v) - x(u) < \mathrm{bw}(G)$, there are at most $2\,\mathrm{bw}(G) - 2$ such vertices distinct from u, each of which is incident to at most Δ edges. Hence, (u, v) has at most $\Delta \cdot (2\,\mathrm{bw}(G) - 2)$ crossings. ◀

For some graphs, a sublinear bound on the bandwidth is known, see [13, 21, 38]. This gives upper bounds on the local crossing number of many graph classes (e.g., interval graphs, co-compoarability graphs, AT-free graphs, graphs of bounded treewidth); we list only a few:

▶ **Corollary 7.** *The following classes of DAGs have sublinear upward local crossing number:*
- *Square $k \times k$ grids have bandwidth $\Theta(k)$ and $\Delta = 4$, hence their upward local crossing number is in $\mathcal{O}(k) = \mathcal{O}(\sqrt{n})$.*
- *Directed planar graphs with maximum degree Δ have bandwidth $\mathcal{O}(\frac{n}{\log_\Delta n})$ [13], hence their upward local crossing number is in $\mathcal{O}(\frac{n \cdot \Delta}{\log_\Delta n})$.*

We complement the negative result in Proposition 1 by showing that every directed acyclic outerpath allows an upward 2-planar drawing. We start with a technical lemma on fans.

▶ **Lemma 8.** *Let c be the central vertex of a directed acyclic fan G, and let $P = \langle v_1, \dots, v_{n-1} \rangle$ be the path of the remaining vertices in G. Let $P_1, \dots, P_k$ be an ordered partition of P into maximal subpaths such that, for every $i \in [k]$, the edges between P_i and c either are all directed towards c or are all directed away from c. Then there is an upward 2-planar drawing of G with the following properties:*
1. *no edge incident to c is crossed;*
2. *vertex v_1 has x-coordinate 1, the central vertex c and v_{n-1} have x-coordinate $n - 1$, and the x-coordinates of $v_2, \dots, v_{n-2}$ are distinct values within $\{2, \dots, n - 2\}$;*
3. *for all edges all x-coordinates of the curves are at most $n - 1$; all edges incident to c and all edges of the subpaths $P_1, \dots, P_k$ are in the vertical strip between 1 and $n - 1$;*
4. *if P_1 is a directed path, then the edge between P_1 and P_2 is crossed at most once.*



(a) A directed fan G. **(b)** Upward drawing of G with at most two crossings per edge.

Figure 5 Upward 2-planar drawings of fans (Lemma 8). For $t \in [k]$, we add the path P_t below c (blue paths) or above c (green paths), going up and down as prescribed by the edge directions and such that no edge incident to c is crossed. We maintain the property that all vertices of P_t are on the outer face of the subgraph induced by P_t and c, except for possibly a last final part pointing upward if P_t is below c or pointing downward if P_t is above c. See the shaded areas, e.g., the final part $\langle v_i, \ldots, v_{j-1} \rangle$ of P_1. The edge connecting P_t and P_{t+1} (red edges) might either cross the last edge of P_t on the outer face (e.g., the edge of P_1 between v_{i-1} and v_i) or the edge connecting P_{t-1} to P_t in order to reach the outer face. The latter may have been crossed once before (as (v_{j-1}, v_j)).

Proof. We place c at $(n-1, 0)$; then we place $v_1, v_2, \ldots, v_{n-1}$ above or below c depending on the direction of the edges that connect them to c; see Figure 5 for an example.

For $i \in [n-2]$, we keep the invariant that, when we place v_i, the leftward ray from v_i reaches the outer face of the current drawing after crossing at most one other edge, and that this edge had been crossed at most once.

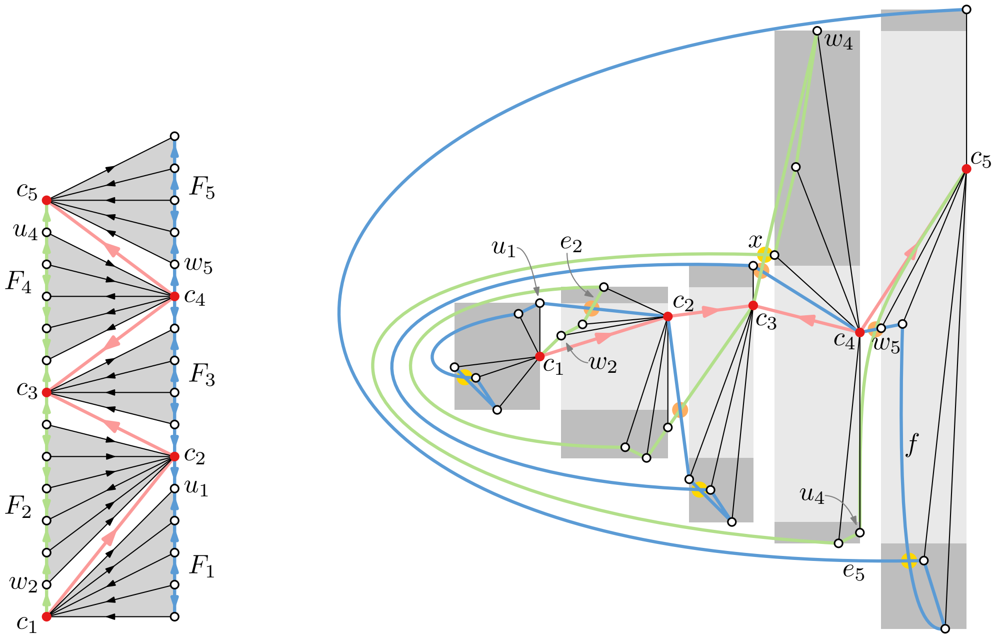
In order to choose appropriate y-coordinates, we maintain two values $y_{\min}$ and $y_{\max}$ indicating the minimum and maximum y-coordinate of any so far drawn vertex. Consider now a subpath $P' \in \{P_1, \ldots, P_k\}$. Let v_h be the first and let v_{j-1} be the last vertex of P', i.e., $P' = \langle v_h, v_{h+1}, \ldots, v_{j-1} \rangle$. We describe in detail the case that the edge from $v_h, \ldots, v_{j-1}$ to c are directed towards c that is, v_h must lie below c. The other case is symmetric. We place v_h at x-coordinate h and with a y-coordinate sufficiently below $y_{\min}$. If $h = j-1$ we are done.

We now consider the cases $j = n$ or $(v_{j-1}, v_{j-2}) \in E$. In that case, we place $v_{h+1}, \ldots, v_{j-1}$ using x-coordinates $h+1, \ldots, j-1$, going up and down as needed but remaining below the x-axis. The edges are drawn such that all vertices of P' remain on the outer face of the drawing. I.e., if we use straight-line edges, then, for $i \in [n-2]$, the slope of $v_i v_{i+1}$ must be less than the slope of $v_i c$. Since we go towards c, we can draw P' and the edges that connect $v_1, v_2, \ldots, v_{n-1}$ to c without any crossings.

If $j \neq n$ and $(v_{j-2}, v_{j-1}) \in E$, then let $i \in \{h, \ldots, j-1\}$ be the smallest index such that the subpath $\langle v_i, v_{i+1}, \ldots, v_{j-1} \rangle$ is directed. In that case, we place $v_{h+1}, \ldots, v_{i-1}$ at x-coordinates $h+1, \ldots, i-1$, going slightly up and down as in the case described above. Let $y_{\min}$ be the smallest among the y-coordinates of all points placed so far.

Then we place $v_i, v_{i+1}, \ldots, v_{j-1}$ in reverse order, i.e., at x-coordinates $j-1, j-2, \ldots, i$. Set $y(v_i) = y_{\min} - \gamma$ and $y(v_{j-1}) = y_{\min} - \varepsilon$ for some (large) $\gamma > 0$ and (small) $\varepsilon > 0$ such that v_{j-1} lies inside the triangle $\triangle v_{i-1}v_i c$ (pale yellow in Figure 5b) if $i > h$ and within the triangle $\triangle o v_i c$, with $o = (0,0)$ otherwise. (Observe that in the case $i = h$, we already required that v_i is sufficiently below $y_{\min}$; this is now further specified here.) Draw $v_{i+1}, \ldots, v_{j-2}$ on the segment $\overline{v_i v_{j-1}}$. Now, if $i > h$ then the vertex v_{j-1} can reach the outer face via the edge (v_i, v_{i-1}) which was not crossed so far. If $i = h$ then v_{j-1} is on the outer face if $P' = P_1$, otherwise it can reach the current outer face by crossing the edge (v_h, v_{h-1}). This edge might have crossed one edge when it was initially drawn but so far no other edge.

Observe that when we draw the next maximal subpath, we place v_j at $(j, y_{\max} + 1)$, i.e., in particular in the outer face of the current drawing. The edge from v_{j-1} to v_j must be directed towards v_j since the orientation is acylic. Thus, we can draw the edge between v_{j-1} and v_j upward with at most one crossing, causing at most a second crossing on (v_h, v_{h-1}). ◀



(a) A directed outerpath G'. **(b)** Upward drawing of G' with at most two crossings per edge.

Figure 6 Example in- and output of our drawing algorithm (edge crossings due to Lemma 8 are highlighted in yellow; other edge crossings are highlighted in orange).

Now we describe our construction for general outerpaths; see Figure 6.

▶ **Theorem 9.** *Every directed acyclic outerpath admits an upward 2-planar drawing.*

Proof. Without loss of generality, we can assume that the given outerpath is maximal: if the outerpath has interior faces that are not triangles, we temporarily triangulate them using additional edges, which we direct such that they do not induce directed cycles and which we remove after drawing the maximal outerpath.

Let G' be such a graph; see Figure 6a. Let $c_1, c_2, \ldots, c_k$ be the vertices of degree at least 4 in G' (marked red in Figure 6). These vertices form a path (light red in Figure 6); let them be numbered along this path, which we call the *backbone* of G'. We assign every vertex v that does not lie on the backbone to a neighboring backbone vertex; if v is incident to an inner edge, we assign v to the other endpoint of that edge. Otherwise v has degree 2 and is incident to a unique backbone vertex via an outer edge, and we assign v to this backbone vertex. For $i \in [k]$, backbone vertex c_i induces, together with the vertices assigned to it, a fan F_i.

We draw the backbone in an x-monotone fashion. We start by drawing F_1 with the algorithm for drawing a fan as detailed in the proof of Lemma 8; see the leftmost gray box in Figure 6b. Then, for $i \in \{2, \dots, k\}$, we set $x(c_i)$ to $x(c_{i-1})$ plus the number of inner edges incident to c_i and we set $y(c_i)$ depending (i) on the y-coordinates of the two neighbors of c_i that have already been drawn (c_{i-1} and the common neighbor u_{i-1} of c_{i-1} and c_i in F_{i-1}) and (ii) on the directions of the edges that connect these vertices to c_i; see, for example, the placement of c_5 in Figure 6b. Then we draw F_i with respect to the position of c_i, again using the algorithm from the proof of Lemma 8 with the following modifications. In general, vertices in F_i that are adjacent to c_i via an edge directed towards c_i (resp. from c_i) are placed below (resp. above) all vertices in the drawings of $F_1, \dots, F_i$; see the dark gray boxes below (resp. above) $c_2, \dots, c_5$ in Figure 6b. If an edge of F_i connects two neighbors of c_i one of which lies above c_i and one of which lies below c_i, then we route this edge to the left of all drawings of $F_1, \dots, F_{i-1}$.

An exception to this rule occurs if c_i and the common neighbor w_i of c_{i-1} and c_i in F_i must be both above or both below c_{i-1} due to the directions of the corresponding edges. Let u_{i-1} be the common neighbor of c_{i-1} and c_i in F_{i-1}. We assume, without loss of generality, that c_i is above c_{i-1}. Let P_1 and P_2 be the first and second maximal subpath from Lemma 8 applied to F_i, and let e_i be the edge connecting P_1 and P_2. We distinguish two subcases.

If P_1 is a directed path leaving w_i, then we draw P_1 above the edge $c_{i-1}c_i$ and we draw the edge e_i straight, without going around all drawings of $F_1, \dots, F_{i-1}$. In this case e_i is directed from P_1 to P_2. Hence, e_i crosses the edge $u_{i-1}c_i$ if $u_{i-1}c_i$ is directed from c_i to u_{i-1}; see the situation for c_2 in Figure 6b. Note that e_i may receive a second crossing when we draw the remainder of F_i in the usual way.

Otherwise, that is, if P_1 contains an edge directed towards the left endpoint w_i of P_1, let f be the first such edge. We then place the part of P_1 up to the first endpoint of f below the edge $c_{i-1}c_i$; see w_5 and f in Figure 6b. If the edge $u_{i-1}c_i$ is directed towards c_i, we draw it between w_i and the edge $c_{i-1}c_i$. Then it crosses the edge $c_{i-1}w_i$ but no other edge. We place the second endpoint of f below all vertices in $V(F_1) \cup \dots \cup V(F_{i-1})$ and continue with the remainder of F_i as usual.

In any case, if $1 < i < k$, then the last vertex u_{i-1} of F_{i-1} is connected to c_i and c_{i-1} is connected to the first vertex w_i in F_i. These two edges may cross each other; see the crossings highlighted in orange in Figure 6b. If the edge $c_{i-1}w_i$ goes, say, up but the following outer edges go down until a vertex v_k below c_i is reached, then the edge $c_{i-1}w_i$ may be crossed a second time by the edge $v_{k-1}v_k$; see the crossing labeled x on the edge c_3w_4 in Figure 6b. But due to property 4 of Lemma 8, edge $v_{k-1}v_k$ had been crossed at most once within its fan. Also $c_{i-1}w_i$ cannot have a third crossing. Thus, all edges are crossed at most twice. ◀

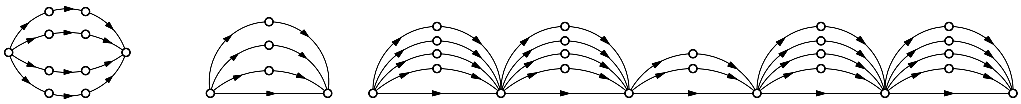
One can argue that every maximal pathwidth-2 graph can be generated from a maximal outerpath by connecting some pairs of adjacent vertices using an arbitrary number of (new) paths of length 2. In spite of the simplicity of this operation, we cannot hope to generalize the above result to pathwidth-2 graphs; see the linear lower bound on the upward local crossing number for such graphs stated in Theorem 3.

5 Complexity of Testing

Here we prove that upward 1-planarity testing is NP-complete even for structurally simple DAGs, both when a bimodal rotation system is fixed and when it is not fixed. We also show that testing upward outer-1-planarity for single-source DAGs can be solved in linear time.

5.1 Testing Upward 1-Planarity

We first define a few gadgets; all of them are planar st-graphs. For positive integers b and q, let a *(b, q)-parallel* be the parallel composition of b oriented paths each consisting of q edges; see Figure 7a. For a positive integer p, let a *(p)-gate* be the parallel composition of an oriented edge and a $(p-1, 2)$-parallel; see Figure 7b. For positive integers h, q, and a, let an *(h, q, a)-chain* consist of a series of h (q)-gates, followed by exactly one (a)-gate, followed again by h (q)-gates; see Figure 7c.



(a) A $(4, 3)$-parallel. **(b)** A (4)-gate. **(c)** A $(2, 5, 3)$-chain.

Figure 7 Illustrations for the gadgets used in the construction of G_A and of G_B.

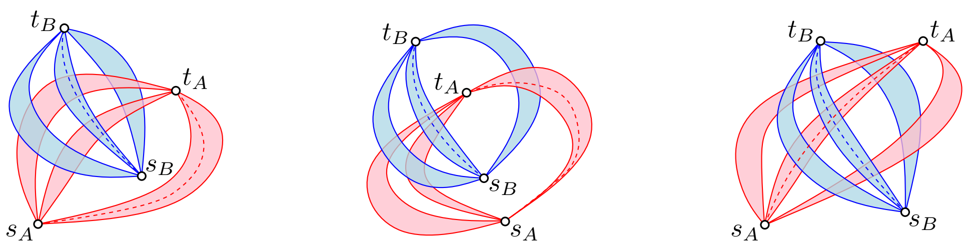
An instance of 3-Partition is a multiset $I = \{a_1, a_2, a_3, \dots, a_k\}$ of positive integers such that $b = k/3$ is an integer and $\sum_{i=1}^{k} a_i = W \cdot b$, with W integer. The 3-Partition problem asks if there exists a partition of the set I into b 3-element subsets such that the sum of the elements of each subset is exactly W. Since 3-Partition is strongly NP-hard [25], we may assume that W is bounded by a polynomial in b.

We associate with a given instance I of 3-Partition two planar st-graphs G_A and G_B defined as follows. Digraph G_A is the parallel composition of $(b-1, W+1, a_i)$-chains, one for every $i \in \{1, \dots, k\}$. Digraph G_B is an (b, q)-parallel, with $q = W + (k-3)(W+1)$. Note that the underlying undirected graphs of both G_A and G_B are series-parallel.

Let G be any digraph that contains the two subgraphs G_A and G_B defined above. Let s_A and t_A (resp. s_B and t_B) be the two vertices of G that are the source and the sink of G_A (resp. G_B). Let Γ be a 1-planar drawing of G and let Γ_{AB} be the 1-planar drawing obtained by restricting Γ to the nodes and edges of G_A and G_B. We say that G_A and G_B *cross* in Γ if in Γ_{AB} every s_At_A-path (i.e., a path directed from s_A to t_A) crosses every s_Bt_B-path. See Figures 8a and 8b for examples of graph G_A and G_B that do not cross or cross in a drawing of Γ_{AB}, respectively.

▶ **Lemma 10.** *Let I be an instance of 3-Partition and let G_A and G_B be the two planar st-graphs associated with I. Let G be a digraph containing G_A and G_B as subgraphs such that G has an upward 1-planar drawing if and only if G_A crosses G_B. There exists an upward 1-planar drawing Γ of G if and only if I admits a solution.*

Proof sketch. We prove that if G admits an upward 1-planar drawing Γ, then Γ provides a solution of instance I of 3-Partition. By hypothesis, G_A and G_B cross in Γ; see Figure 9. Observe that only one path among the b paths of G_B can traverse one (a_i)-gate G_A. Also, every path of G_B crosses all the $(W+1, b-1, a_i)$-chains of G_A. In particular, every path of G_B must cross at least three (a_i)-gates since it has not enough edges to cross more than $k-2$ $(W+1)$-gates. Also, if one path of G_B crossed more than three (a_i)-gates, then some other path of G_B that would cross at most two (a_i)-gates. Therefore, every path π of G_B



(a) **(b)** **(c)**

Figure 8 Illustrations for the definition of crossing st-subgraphs. (a) and (b) Two planar st-graphs G_A and G_B that do not cross, as witnessed by the two non-crossing dashed paths. (c) Two planar st-graphs G_A and G_B that cross.

Table 1 A comparison between results in the literature about the complexity of testing upward planarity and the results discussed in this paper about the complexity of testing upward 1-planarity.

Underlying planar graph	**Acyclic orientation**	**Upward planarity**		**Upward 1-planarity**	
		fixed embedding	variable embedding	fixed rotation system	variable rotation system
Series-parallel	**multi-source multi-sink**	Polynomial [15, 18]		NP-complete Theorem 11 Case 1	NP-complete Theorem 11 Case 3
	single-source single-sink	Trivially polynomial			Trivially polynomial
General graph	**multi-source multi-sink**	Polynomial [9]	NP-complete [27]	NP-complete Corollary 12	NP-complete Theorem 11 Case 2
	single-source single-sink	Polynomial [14]			

must cross exactly three (a_i)-gates and $k-3$ $(W+1)$-gates in Γ. Note that the number of crossings of π with the three (a_i)-gates is exactly W. It follows that if G has an upward 1-planar drawing then the instance I of 3-PARTITION admits a solution. Conversely, if the instance I of 3-PARTITION admits a solution it is easy to construct an upward 1-planar drawing Γ_{AB} of G_A and G_B where G_A and G_B cross. ◀

▶ **Theorem 11.** *Testing upward 1-planarity is* NP*-complete even in the following restricted cases:*

1. *The bimodal rotation system is fixed, the DAG has exactly one source and exactly one sink, the underlying graph is series-parallel.*
2. *The bimodal rotation system is not fixed, the DAG has exactly one source and exactly one sink, the underlying planar graph is obtained by replacing the edges of a K_4 with series-parallel graphs.*
3. *The bimodal rotation system is not fixed, the underlying graph is series-parallel, there is one source and two sinks.*

Proof sketch. It is immediate to observe that upward 1-planarity testing is in NP, as one can guess an upward 1-planar embedding and test it in polynomial time. In order to show that the problem is NP-hard for the cases in the statement it suffices, by Lemma 10, to exhibit digraphs that contain G_A and G_B as subgraphs and that admit upward 1-planar

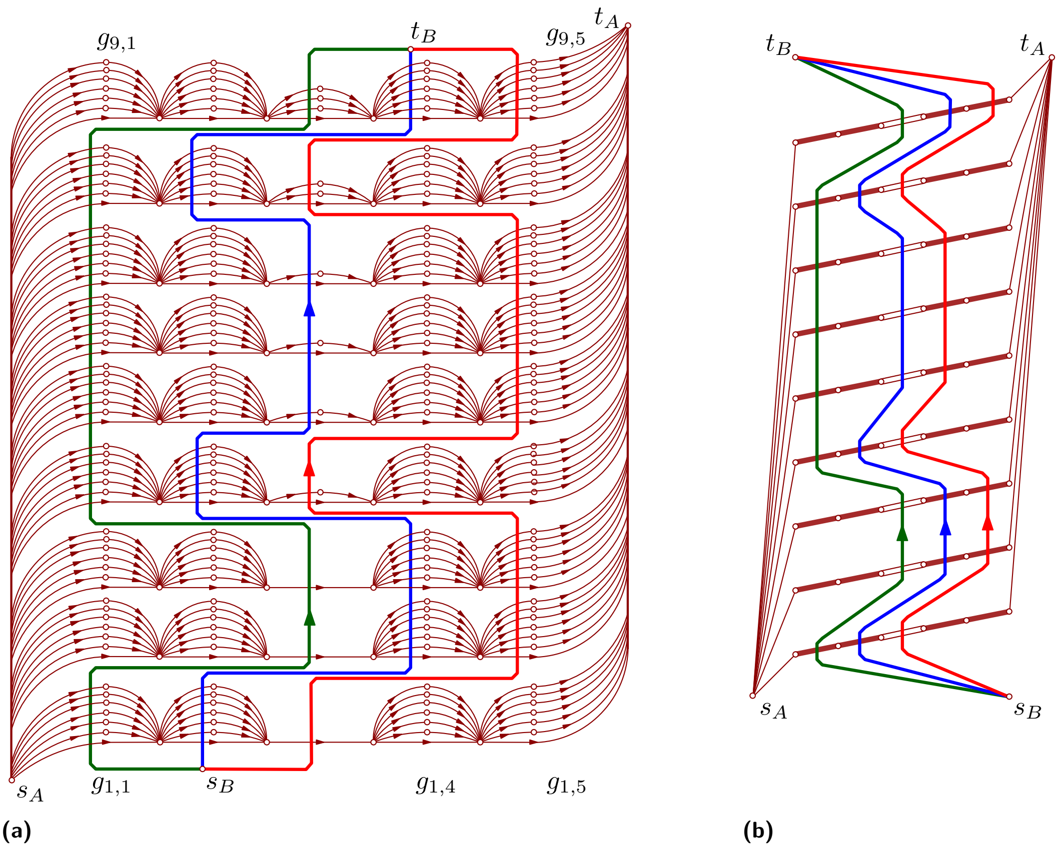


Figure 9 (a) Digraph G_A (dark red) and a schematic representation of digraph G_B where each colored curve represents a directed path with $W + (k-3)(W+1)$ edges. The corresponding instance of 3-PARTITION is $I = \{1, 1, 1, 2, 2, 2, 2, 3, 4\}$, with $b = 3$ and $W = 6$. The 1-planar drawing corresponds to the solution $\{1, 1, 4\}$ (green path), $\{2, 2, 2\}$ (blue path), and $\{1, 2, 3\}$ (red path). The drawing in (a) is not upward but it can be made upward by stretching it vertically as shown in (b), where thick edges represent (q)-gates and the central white-filled edges represent (a)-gates.

drawings if and only if G_A and G_B cross in them. Let m_A and m_B be the number of edges of G_A and G_B, respectively. Let a *barrier* be a planar st-graph consisting of a $(d, 2)$-parallel, where $d = m_A + m_B + 1$. Note that neither G_A nor G_B can cross a barrier in such a way that every edge is crossed at most once. The instances that we use for the cases listed in the statement are depicted in Figure 10a (Case 1), Figure 10c (Case 2) and Figure 10e (Case 3), where the thick edges represent barriers and G_A and G_B can be identified by their poles. As shown in Figure 10b, Figure 10d, and Figure 10f an upward 1-planar drawing of such graphs forces G_A and G_B to cross, hence, implies the hardness of computing such drawings. ◀

The following corollary is an immediate consequence of the argument used to prove the second case in the statement of Theorem 11.

▶ **Corollary 12.** *Testing upward 1-planarity is* NP*-complete for single source-single sink DAGs with a fixed bimodal rotation system, whose underlying planar graph is obtained by replacing the edges of a K_4 with series-parallel graphs.*

We conclude this section by remarking some differences between the complexity of upward planarity testing and upward 1-planarity testing. When the bimodal rotation system is fixed, upward planarity testing can be solved in polynomial time [9], whereas upward 1-planarity

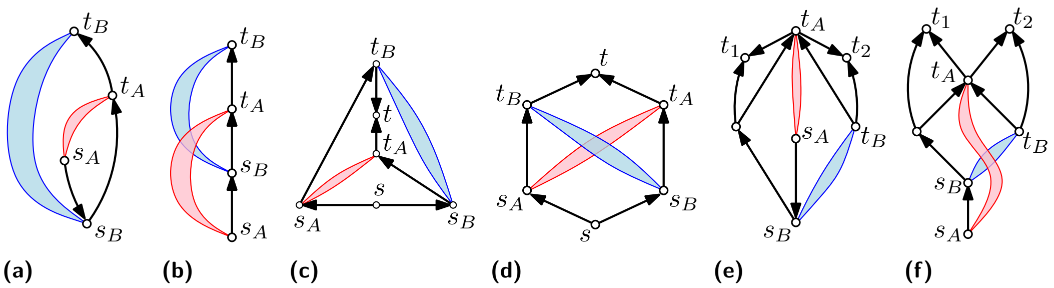


Figure 10 Some digraphs for the proof of Theorem 11. Thick black edges represent barriers.

testing is NP-hard (Theorem 11). Also, when the bimodal rotation system is not fixed and the digraph has a constant number of sources and sinks, differently from upward 1-planarity testing, upward planarity testing can again be solved in polynomial time [14]. On the other hand, any digraph whose bimodal rotation system is not fixed, whose underlying graph is series-parallel, and that has only one source and only one sink is always upward planar and thus upward 1-planar. Indeed, adding an edge between any two vertices of the undirected underlying series-parallel graph yields a planar graph (see, e.g., [17]). It follows that G can be turned into a planar st-graph by connecting its source to its sink by an edge and hence it is upward planar [26]. This discussion is summarized in Table 1.

5.2 Testing Upward Outer-1-Planarity

To complement the results of Section 5.1, we consider a restricted setting that has often been studied in the "beyond planarity" literature to show the tractability of an otherwise NP-hard problem. Namely, we describe how to test whether a single-source DAG admits an *upward outer-1-planar drawing*, i.e., one that is both upward and outer-1-planar.

▶ **Theorem 13.** *For single-source DAGs, upward outer-1-planarity can be tested in linear time.*

This section provides the main ideas behind this result; recall that all details can be found in the full version [1]. In the following, let G be a single-source DAG. As a first step, we characterize the single-source DAGs that admit an upward outer-1-planar drawing as those that admit an outer-1-planar embedding whose planarization is acyclic. In particular, this implies that we may treat the biconnected components of G independently, and we therefore assume in the following that G is biconnected. We assume familiarity with the SPQR-tree [17]. Note that, in the version of the SPQR-tree that we use, there are no Q-nodes. Instead, skeletons contain both real and virtual edges.

Our testing algorithm builds on the results of Auer et al. [4, 5, 6] for testing outer-1-planarity. A necessary condition is that the skeleton of each R-node is a K_4 and the skeleton of each P-node contains at most four virtual edges plus, possibly, one real edge. In a nutshell, Auer et al. [5] show that there is a bijection between the outer-1-planar embeddings of a biconnected graph G and certain (non-planar) embeddings of all skeletons of the SPQR-tree $\mathcal{T}$ of G. These non-planar embeddings need to satisfy *local conditions* which state that every virtual edge must have at least a part of it incident to the outer face, a virtual edge may only receive a crossing if it belongs to a P-node and corresponds to an S-node, and if a virtual edge receives a crossing, then the end of it that is not incident to the outer face, if any, must correspond to a real edge of the graph. As an example consider the embeddings of the skeletons of node ρ and λ in Figure 11. The shown embedding, where the segment of edge e_ρ

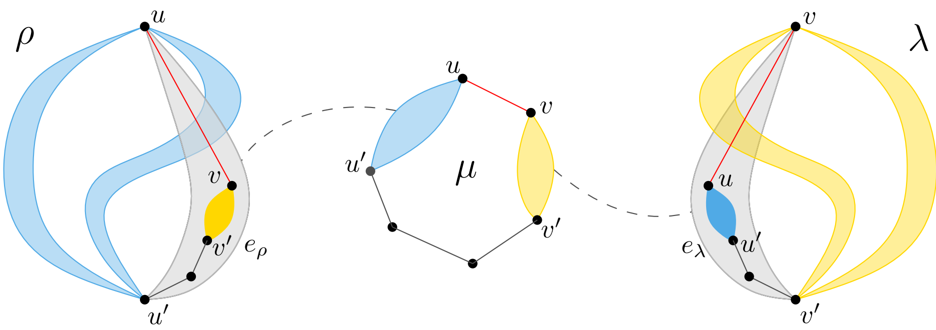


Figure 11 Illustration for the necessary conditions that allow for an outer-1-planar embedding.

incident to u is not on the outer face, requires that the expansion graph of e_ρ starts with a real edge (here the edge uv) at u. We note that in this case, the edge uv is in fact a real edge in a neighboring S-node. Unlike the case of SPQR-trees and planar embeddings, the embeddings of different skeleton cannot be combined independently; instead, there is also a *global condition* that requires that no real edge receives crossings from two P-nodes. For example, the embeddings of skel(ρ) and skel(λ) shown in Figure 11 both imply a crossing on the edge uv and are therefore not compatible.

We want to restrict our attention to outer-1-planar embeddings whose planarization is acyclic. For this, we orient the edges in our skeletons, where real edges are endowed with their orientation in G and a virtual edge $\{u, v\}$ is directed as (u, v) if its expansion graph contains a directed path from u to v. Note that if the expansion graph of $\{u, v\}$ contains no directed paths between its poles, the virtual edge remains undirected. We then show that, in order to obtain an outer-1-planar embedding whose planarization is acyclic, we may only combine what we call *acyclic* embeddings of the skeletons, which do not already locally produce cycles; see Figure 12 for an example. Conversely, we prove that if we choose for each oriented skeleton an embedding that satisfies the local conditions and is acyclic, and moreover, these choices also satisfy the global condition, then they together define an outer-1-planar embedding of G whose planarization is acyclic, and hence admits an upward drawing.

The algorithm therefore works as follows. Since each skeleton admits at most 12 embeddings that satisfy the local conditions [5], we can enumerate them and test for each of them whether it is acyclic in total linear time. After this step, we have for each node μ of $\mathcal{T}$ a set $\mathcal{F}_\mu$

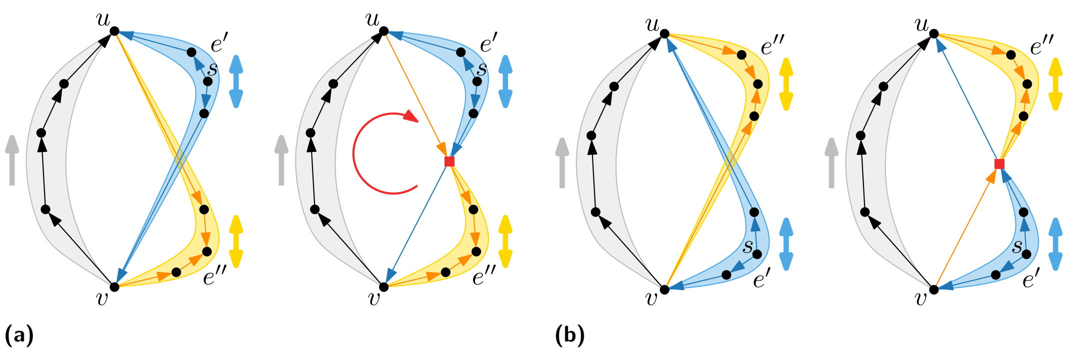


Figure 12 Two embeddings of the skeleton of a P-node μ and the corresponding planarizations. The planarization in (a) contains a directed cycle, the one in (b) does not. Thick arrowed edges show the direction of the virtual edges; a double arrow indicates an undirected virtual edge.

of candidate embeddings. It remains to choose for each node μ one embedding $\mathcal{E}_\mu \in \mathcal{F}_\mu$ such that the global condition is satisfied, i.e., no two P-nodes put crossings on the same real edge. We say that such a choice of embeddings is *consistent*. To decide whether such a choice exists, we construct a conflict graph H whose vertices are the embeddings of the skeletons, each $\mathcal{F}_\mu$ forms a clique and two embeddings of different P-nodes are connected by an edge if and only if they put a crossing on the same real edge. Then a consistent choice corresponds to an independent set in H whose size is the number of nodes of $\mathcal{T}$. Since the size of H is linear in the size of G and we can show that the treewidth of H is at most 36, the existence of such an independent set can be tested in linear time [16]. We note that, if the test succeeds, we can also construct the upward outer-1-planar embedding of G in the same running time.

6 Conclusion

In this paper we initiated the study of upward k-planar drawings, that is, upward drawings of directed acyclic graphs such that every edge is crossed at most k times for a given constant k. We first gave upper and lower bounds for the upward local crossing number of various graph families, i.e., the minimum k such that every graph from the respective family admits an upward k-planar drawing. We strengthen these combinatorial results by proving that testing a DAG for upward k-planarity is NP-complete even for $k = 1$. On the positive side, testing upward outer-1-planarity for single source digraphs can be done in linear time. We conclude the paper by listing some open problems that may stimulate further research.

1. Is there a directed outerpath that does not admit an upward 1-planar drawing?
2. Consider the class $\mathcal{O}_\Delta$ of outerplanar graphs (or even 2-trees) of maximum degree Δ. Is there a function f such that every graph in $\mathcal{O}_\Delta$ admits an $f(\Delta)$-planar upward drawing?
3. In light of the lower bounds in Section 3, it is natural to consider graphs with a special structure, in order to prove sublinear upper bounds on their (upward) local crossing number. For example, Wood and Telle [39, Corollary 8.3] show that every (undirected) graph of maximum degree Δ and treewidth τ admits a (straight-line) drawing in which every edge crosses $\mathcal{O}(\Delta^2\tau)$ other edges. Can the *upward* local crossing number be bounded similarly by a function in Δ and τ?
4. Do planar graphs of maximum degree Δ have upward local crossing number $\mathcal{O}(f(\Delta)n^{1-\epsilon})$ for some function f and some constant $\epsilon > 0$?
5. Can upward outer-1-planarity be efficiently tested for multi-source and multi-sink DAGs?
6. Investigate parameterized approaches to testing upward 1-planarity.

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