

Bounding the Treewidth of Outer k -Planar Graphs via Triangulations

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Abstract

The *treewidth* is a structural parameter that measures the tree-likeness of a graph. Many algorithmic and combinatorial results are expressed in terms of the treewidth. In this paper, we study the treewidth of *outer k -planar* graphs, that is, graphs that admit a straight-line drawing where all the vertices lie on a circle, and every edge is crossed by at most k other edges.

Wood and Telle [New York J. Math., 2007] showed that every outer k -planar graph has treewidth at most $3k + 11$ using so-called planar decompositions, and later, Auer et al. [Algorithmica, 2016] proved that the treewidth of outer 1-planar graphs is at most 3, which is tight.

In this paper, we improve the general upper bound to $1.5k + 2$ and give a tight bound of 4 for $k = 2$. We also establish a lower bound: we show that, for every even k , there is an outer k -planar graph with treewidth $k + 2$. Our new bound immediately implies a better bound on the *cop number*, which answers an open question of Durocher et al. [GD 2023] in the affirmative.

Our treewidth bound relies on a new and simple triangulation method for outer k -planar graphs that yields few crossings with graph edges per edge of the triangulation. Our method also enables us to obtain a tight upper bound of $k + 2$ for the *separation number* of outer k -planar graphs, improving an upper bound of $2k + 3$ by Chaplick et al. [GD 2017]. We also consider *outer min- k -planar* graphs, a generalization of outer k -planar graphs, where we achieve smaller improvements.

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1 Introduction

Treewidth measures the tree-likeness of a graph via so-called tree decompositions. A *tree decomposition* of a graph G covers the vertex set of G by bags such that every edge is in some bag and the bags form a tree such that, for every vertex v of G , the bags that contain v form a subtree. The width of a tree decomposition is the maximum size of a bag minus one. The treewidth of G is the minimum width over all tree decompositions of G .

Treewidth is an important structural parameter because the running time of many graph algorithms and algorithms for drawing graphs depend on the treewidth. Recently treewidth-based techniques have been successfully applied for *convex* drawings, that is, straight-line drawings where vertices lie on a circle. For example, Bannister and Eppstein [8] ([7]) showed that if a graph admits a convex drawing with at most k crossings in total, then its treewidth is bounded by a function of k . Furthermore, they used this fact to give, for a fixed k , a linear-time algorithm (via extended monadic second-order logic and Courcelle’s theorem [13]) to decide whether a convex drawing with at most k crossings exists for a given graph. Chaplick et al. [12] ([11]) generalized this to *bundled crossings* (a crossing of two bundles of edges such that in each bundle the edges travel in parallel, which is counted as one crossing). Another prominent class of graphs with convex drawings is the class of *outer k -planar* graphs, that is, graphs that admit a convex drawing where each edge is crossed at most k times. Unlike *k -planar* graphs (without the restriction on the placement of vertices), the treewidth of outer k -planar graphs can be bounded by a function of k only (see discussion below). Similarly to Bannister and Eppstein [8], Chaplick et al. [10] used this to test in linear time, for any fixed k , whether a given graph is *full outer k -planar*, i.e., whether it admits an outer k -planar drawing where no crossing appears on the boundary of the outer face.

For disambiguation, recall that *k -outerplanar* graphs are defined as follows. A drawing of a graph is 1-outerplanar if all vertices of the graph lie on the outer face. For $k > 1$, a drawing is k -outerplanar if deleting the vertices on the outer face yields a $(k - 1)$ -outerplanar drawing. For $k \geq 1$, a graph is k -outerplanar if it admits a k -outerplanar drawing.

In this paper, we are particularly interested in the treewidth of outer k -planar graphs. Note that every graph is outer k -planar for some value of k . For a graph G , let $\text{lcr}^\circ(G)$ denote the *convex local crossing number* of G – the smallest k such that G is outer k -planar. Consult Schaefer’s survey [19] for details on this and many other types of crossing numbers. Outer k -planar graphs admit balanced separators of size $O(k)$ (more below) associated with the drawing, which makes it possible to test, for any fixed k , outer k -planarity in quasi-polynomial time [10]. This implies that the recognition problem is not NP-hard unless the Exponential Time Hypothesis fails [10].

We also consider a generalization of outer k -planar graphs, namely *outer min- k -planar graphs*, which are graphs that admit a convex drawing where, for every pair of crossing edges, at least one edge is crossed at most k times. Wood and Telle [20, Proposition 8.5] showed that any (min-) outer k -planar graph has treewidth at most $3k + 11$. More precisely, they showed that, for every outer min- k -planar graph G , one obtains a tree decomposition of width at most $3k + 11$ from a *planar decomposition* of G , which is a generalization of a tree decomposition. The proof uses a result by Bodlaender [9] on k -outerplanar graphs.

For constant k , better treewidth bounds are known. A folklore result is that outerplanar graphs ($k = 0$) have treewidth at most 2. Auer et al. [5, 6] ([4]) showed that maximal outer 1-planar graphs are chordal graphs and, therefore, have treewidth at most 3, which is tight.

Another parameter linked to the treewidth is the separation number of a graph. A pair of vertex sets (A, B) is a *separation* of a graph G if $A \cup B = V(G)$ and there is no edge between $A \setminus B$ and $B \setminus A$. A separation (A, B) is said to be *balanced* if the sizes of both $A \setminus B$ and

$B \setminus A$ are at most $2n/3$, where n is $|V(G)|$. For a (balanced) separation (A, B) , the set $A \cap B$ is called (balanced) *separator*, and $|A \cap B|$ is the *order* of (A, B) . The *separation number* of G , $\text{sn}(G)$, is the minimum integer k such that every subgraph of G has a balanced separation of order k . Note that, for any subgraph G' of G , $\text{tw}(G') \leq \text{tw}(G)$. This implies that, for every graph G , $\text{sn}(G) \leq \text{tw}(G) + 1$ [15]. On the other hand, Dvořák and Norin [15] showed that, for every graph G , $\text{tw}(G) \leq 15 \text{sn}(G)$.

Our contribution. Given an outer k -planar drawing Γ of a graph G , let the *outer cycle* of G be the cycle that connects the vertices of G in the order along the circle on which they lie in Γ (even if G does not contain all edges of this cycle). We introduce two simple methods to construct, given an outer k -planar drawing Γ of a graph G , a triangulation of the outer cycle of G with the property that each edge of the triangulation is crossed by at most k edges of G ; see Section 3. The resulting triangulations yield the following bounds; see Section 4.

- We improve the upper bound of Wood and Telle [20] regarding the treewidth of outer k -planar graphs from $3k + 11$ to $1.5k + 2$ (Theorem 13 in Section 4.1). Our proof is constructive and implies a practical algorithm; the tree decomposition that we obtain follows the weak dual of the triangulation. For outer 2-planar graphs, our methods yield an upper bound of 4 (Theorem 12), which is tight due to K_5 .
- Chaplick et al. [10] showed that, for every outer k -planar graph G , its separation number is at most $2k + 3$. We improve this upper bound to $k + 2$; see Section 4.2.
- We give new lower bounds of $k + 2$ for both the treewidth and the separation number of outer k -planar graphs; see Section 5. Note that the latter bound is tight (Theorem 17).
- Durocher et al. [14] recently proved that the cop number of general 1-planar graphs is not bounded, but the maximal 1-planar graphs have cop number at most 3. Since it is known [16] that for every graph G , $\text{cop}(G) \leq \text{tw}(G)/2 + 1$, Durocher et al. [14] observed that the treewidth bound of Wood and Telle yields, for every outer k -planar graph G , that $\text{cop}(G) \leq 1.5k + 6.5$. They asked explicitly whether the multiplicative factor of 1.5 can be improved. We answer this question in the affirmative since our new treewidth bound immediately yields the better bound $\text{cop}(G) \leq 0.75k + 1.75$.
- For outer min- k -planar graphs, our triangulation improves the bound of Wood and Telle $3k + 11$ slightly to $3k + 1$ and gives a bound $2k + 1$ on the separation number; see Section 4.

Related results. A structurally similar type of result is known for *2-layer k -planar* graphs. A 2-layer k -planar graph is a bipartite graph that admits a k -planar straight-line drawing with vertices placed on two parallel lines. The maximal 2-layer 0-planar graphs (called *caterpillars*) are the maximal pathwidth-1 graphs [17, 18]. Angelini et al. [2] ([3]) showed that 2-layer k -planar graphs have pathwidth at most $k + 1$ and gave a lower bound of $(k + 3)/2$ for every odd number k . Hence, their result can be considered as a smooth generalization of the caterpillar result. Similarly, our treewidth bound $1.5k + 2$ for outer k -planar graphs smoothly generalizes the treewidth-2 bound for maximal outerplanar graphs.

2 Preliminaries

A *convex drawing* Γ of a graph is a straight-line drawing where the vertices of the graph are placed on different points of a circle, which we call the *circle of Γ* . An *outer k -planar drawing* Γ of a graph is a convex drawing such that every edge crosses at most k other edges. A counterclockwise walk of the circle of Γ yields a cyclic order on the vertices of the graph. We say that two sets on distinct four vertices $\{v_1, w_1\}$, and $\{v_2, w_2\}$ are *intertwined* in Γ

if they are ordered $\langle v_1, v_2, w_1, w_2 \rangle$ or $\langle v_1, w_2, v_2, w_1 \rangle$ in this cyclic order. Observe that two edges $\{v_1, w_1\}$ and $\{v_2, w_2\}$ cross in Γ if and only if they span four different vertices and are intertwined. In the remainder of this section, we make some simple observations that will be helpful later; they can be skipped by more experienced readers.

► **Observation 1.** *Let G be a graph. If Γ is an outer k -planar drawing of G , then every convex drawing Γ' of G with the same cyclic vertex order as in Γ is also an outer k -planar drawing of G .*

Proof. As the cyclic orders on vertices of G defined by Γ and Γ' are the same, exactly the same pairs of edges cross in Γ and in Γ' . ◀

In the definition of convex drawings, we could have allowed placing vertices on arbitrary convex shapes (instead of a circle) and drawing edges as curves (instead of straight-line segments) – as long as every curve is drawn inside the circle and no two curves cross more than once. Using curves will be sometimes convenient in our proofs later.

► **Observation 2.** *Let G be a graph, let Γ be an outer k -planar drawing of G , and let c be a curve that joins vertices v and w of G and is contained inside the circle of Γ . If c crosses at most l edges of Γ , then the straight-line segment \overline{vw} also crosses at most l edges of Γ .*

Proof. As c must cross every edge $\{v', w'\}$ of G that is intertwined with $\{v, w\}$, we get that c must have at least one crossing with every edge crossed by the straight-line segment \overline{vw} . ◀

A graph G is *maximal outer k -planar* if G is outer k -planar and G does not contain any vertex pair $e \in V^2(G) \setminus E(G)$ such that the graph $(V(G), E(G) \cup \{e\})$ is still outer k -planar. Since removing edges increases neither treewidth nor separation number, we are interested in properties of maximal outer k -planar graphs.

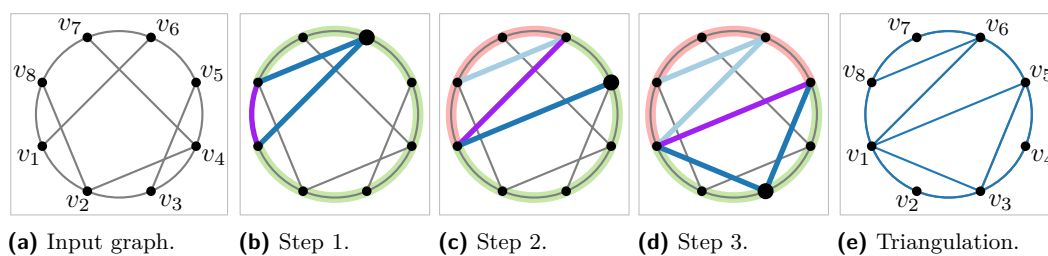
► **Observation 3.** *If G is a maximal outer k -planar graph with at least three vertices, then, in every outer k -planar drawing of G , the outer face is bounded by a simple cycle.*

Proof. Let v and w be two vertices that are consecutive in the cyclic order defined by some outer k -planar drawing Γ of G . For a contradiction, suppose that v and w are not adjacent. As v and w are consecutive on the circle of Γ , the set $\{v, w\}$ is not intertwined with any edge in Γ . Thus, Γ can be extended to an outer k -planar drawing of $G + \{v, w\}$, which contradicts the maximality of G . ◀

We remark that Observations 1–3 analogously hold for outer min- k -planar graphs.

3 Triangulations

In this section, we present two simple strategies to triangulate maximal outer k -planar graphs. These triangulations will serve as tools to construct tree decompositions and balanced separators in Section 4. We assume that some outer k -planar drawing Γ of a graph G on n vertices is given. During the triangulation procedures, we will modify the drawing, but we will never change the cyclic order of the vertices. Thus, let v_1, v_2, \dots, v_n be the vertices of G in the cyclic (counterclockwise) order defined by Γ . As we focus on maximal graphs, by Observation 3, we have that $\langle v_1, v_2, \dots, v_n, v_1 \rangle$ is a cycle in G . Our goal is to construct a triangulation \mathcal{T} of this cycle such that the edges of \mathcal{T} cross only a limited number of edges of G . We refer to the edges of \mathcal{T} as *links* in order to distinguish them from the edges of G . Every edge of the outer cycle is a link in \mathcal{T} ; we call these links *outer links*. We will select



■ **Figure 1** Steps of the splitting procedure. The active edge is purple, the (new/old) links are (dark/light) blue, the left/right side of the active edge is red/green. The split vertex is big.

a set of $n - 3$ other pairwise non-intertwined pairs of vertices as *inner* links that, together with the outer links, form a triangulated n -gon. Note that some of the inner links may coincide with edges of G . We say that a link is *pierced* by an edge of G if their endpoints are intertwined in the cyclic order. The *piercing number* of a link is the number of edges of G that pierce the link. In particular, the piercing number of outer links is 0, and if a link coincides with an edge, then its piercing number is at most k by the outer k -planarity of Γ . We define the *edge piercing number* of \mathcal{T} to be the maximum piercing number of any link of \mathcal{T} . In Section 4 we show how to use triangulations with small edge piercing number.

Our *splitting procedure* starts by declaring the link $\{v_1, v_n\}$ active and considers all other vertices to lie on the *right* side of the link. See Figure 1 for illustration. Clearly, the link $\{v_1, v_n\}$ is not pierced. In each recursive step, the input is an active link $\{v_i, v_k\}$ with $1 \leq i < k \leq n$ (promised to be pierced at most some limited number of times), along with a distinguished right side v_{i+1}, \dots, v_{k-1} , where currently no inner links have been selected apart from $\{v_i, v_k\}$. (Vertices $v_1, \dots, v_{i-1}, v_{k+1}, \dots, v_n$ form the *left* side.) The goal is to pick, among the vertices on the right side, a *split vertex* v_j such that the new links $\{v_i, v_j\}$ and $\{v_j, v_k\}$ are pierced by at most some limited number of edges. The new links are added to \mathcal{T} (completing the triangle $v_i v_j v_k$) and become active. The link $\{v_i, v_k\}$ ceases to be active. The split gives rise to two new splitting instances $\{v_i, v_j\}$ and $\{v_j, v_k\}$, which are then solved recursively.

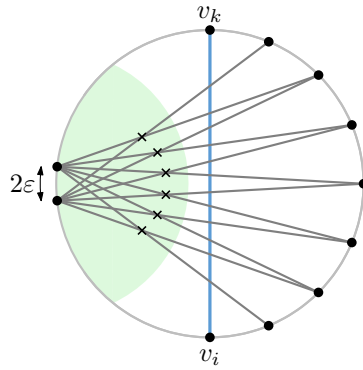
The set of edges of G that pierce the active link are the *piercing edges*. Note that every piercing edge has one left and one right endpoint. Intuitively, the edges with two left endpoints are of no concern to the splitting procedure. The next lemma allows us to push all crossings among piercing edges to the left of $\{v_i, v_k\}$.

► **Lemma 4.** *Given an outer k -planar drawing Γ of a graph G and an active link $\{v_i, v_k\}$, there exists an outer k -planar drawing Γ' with the same cyclic order as Γ , and all crossings among the edges piercing the active link are drawn to the left of the active link.*

Proof. We draw the vertices of G on the unit circle, keeping the cyclic order defined by Γ . We set $\varepsilon > 0$ to a sufficiently small value. Then we place v_i at $(0, -1)$, v_k at $(0, 1)$, all the vertices on the left side of the link within arc distance ε of $(-1, 0)$, and the vertices on the right side of the link evenly along the right semicircle, see Figure 2.

Now, Observation 1 yields that the new drawing is an outer k -planar drawing of G with the same cyclic order. As every pair of intertwined piercing edges has their left endpoints in arc distance at most ε from $(-1, 0)$ and their right endpoints in arc distance at least π/n from each other, the crossing of two such segments is to the left of the active link. ◀

Lemma 4 gives us an equivalent drawing of G in which piercing edges cross the active link in the same order from bottom to top as their right endpoints occur in the cyclic order. This allows us to draw the piercing edges without crossings on the right side of the active link.



■ **Figure 2** We can push crossings among the edges that pierce the active link $\{v_i, v_k\}$ to the left. We show only the top- and bottommost endpoints on the left. Crossings occur exclusively in the green area, which can be made arbitrarily small.

Now, we are ready to present our first and basic triangulation method. Asymptotically, it yields a worse bound than Lemma 6, but it serves as a gentle introduction into our techniques, and it will be used for small values of k and for outer min- k -planar graphs later.

► **Lemma 5.** *For $k \geq 1$, every outer k -planar drawing of a maximal outer k -planar graph admits a triangulation of the outer cycle with edge piercing number at most $2k - 1$.*

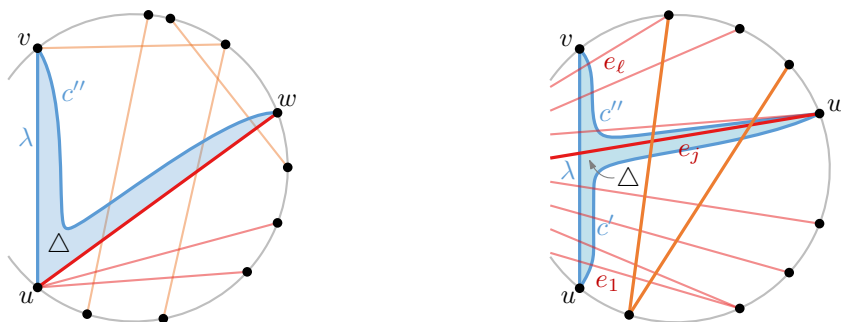
Proof. We prove the lemma by providing an appropriate splitting procedure that recursively constructs the triangulation. Let Γ be an outer k -planar drawing of the given maximal outer k -planar graph G . We prove by induction on the number of steps of the splitting procedure that every active link has piercing number at most $2k - 1$. The claim is true in the beginning, as we start with the link $\{v_1, v_n\}$, which is not pierced at all.

Now we consider a later step of the procedure. Let $\lambda = \{v_i, v_k\}$ denote the active link. For brevity, set $u = v_i$ and $v = v_k$. There are two cases.

In the first case, λ is not pierced by any edge of G . Observe that $\{v_i, v_{i+1}\}$ is an edge, so u has at least one neighbor on the right side of λ in G . We select the split vertex, w , to be the neighbor of u from the right side of λ that is closest to v in the circular order; see Figure 3a. In this case, the link $\{u, w\}$ coincides with an edge of G , and thus is pierced by at most k edges. To see that the link $\{w, v\}$ is also pierced by at most k edges, let c'' be a curve that starts in w , goes along $\{w, u\}$ towards u on the right side (in the direction of walking), then follows $\{u, v\}$ on the right side, and ends at v . This curve crosses only the edges that pierce the link $\{u, w\}$, so at most k edges pierce the link $\{w, v\}$ by Observation 2.

In the second case, λ is pierced by edges of G . Using Lemma 4, we modify the drawing Γ so that the piercing edges cross λ in the same bottom-to-top order as their right endpoints. Let e_1, e_2, \dots, e_ℓ denote the piercing edges, and let w_1, w_2, \dots, w_ℓ denote their respective, not necessarily different, right endpoints in the order from bottom to top. By induction, we have $\ell \leq 2k - 1$. Then we pick the middle edge e_j , where $j = \lceil (\ell + 1)/2 \rceil$, and set the split vertex $w = w_j$.

To bound the number of edges that pierce the link $\{u, w\}$, we draw the curve c' as in Figure 3b. We start the curve c' at u and follow λ until before it crosses e_j , then follow e_j until we arrive at w . The curve c' crosses at most $\lceil (\ell - 1)/2 \rceil$ edges that pierce λ and at most k edges that pierce e_j . Thus, there are at most $\lceil (\ell - 1)/2 \rceil + k \leq \lceil (2k - 2)/2 \rceil + k = 2k - 1$ edges that pierce the link $\{u, w\}$. We can argue symmetrically for the link $\{w, v\}$; see curve c'' in Figure 3b.



(a) Constructing the new triangle Δ if λ is not pierced by any other edge of G . (b) Constructing the new triangle Δ bounded by λ , c' , and c'' ; the orange edges that pierce both c' and c'' also cross the edge e_j .

■ **Figure 3** Constructing a triangulation \mathcal{T} with edge piercing number at most $2k - 1$.

As we have bounded the piercing number of every active link during the triangulation procedure by $2k - 1$, we get the desired bound of $2k - 1$ for the edge piercing number of the constructed triangulation. ◀

Next, we present our refined triangulation method, where we consider an additional edge to get a better bound for the edge piercing number.

► **Lemma 6.** *For every $k \geq 1$, every outer k -planar drawing of a maximal outer k -planar graph admits a triangulation of the outer cycle with edge piercing number at most k .*

Proof. The proof is similar to the proof of Lemma 5, and we use the same terminology. Let again Γ be an outer k -planar drawing of the given maximal outer k -planar graph G . We present a splitting procedure and show that every active link is pierced by at most k edges. For the base of the induction, observe that the link $\{v_1, v_n\}$ has no piercing edges. Now, let $\lambda = \{v_i, v_k\}$ be the active link and set $u = v_i$ and $v = v_k$. If λ has no piercing edges in Γ , we proceed as in the proof of Lemma 5 and pick the split vertex to be the neighbor of u from the right side of λ that is closest to v . Otherwise, λ is pierced by edges of G . Using Lemma 4, we assume that the piercing edges cross the active link in the same bottom-to-top order as their right endpoints. Now, let e_1, e_2, \dots, e_ℓ be the piercing edges, and let w_1, w_2, \dots, w_ℓ be their respective, not necessarily different, right endpoints in the order from bottom to top. We pick the middle edge e_j with $j = \lceil (\ell + 1)/2 \rceil$. If e_j has no crossings on the right side of the active link, then we pick w_j to be the split vertex. The two curves that start in w_j , follow e_j to e , and then follow λ to either u or v have at most $\lceil (\ell - 1)/2 \rceil \leq k$ crossings each, so we can proceed. Now, let x denote the first crossing on the edge e_j that occurs on the right side of λ , and let \hat{e} denote the edge that crosses e_j at x ; see Figure 4a.

We first consider the case where \hat{e} and λ are disjoint. Let a and b be the numbers of crossings on \hat{e} on the two sides of x . As $a + b + 1 \leq k$, we get that $\min\{a, b\} \leq \lfloor (k - 1)/2 \rfloor$. We select the split vertex w to be the endpoint of \hat{e} on the side that has fewer, i.e., $\min\{a, b\}$, crossings along \hat{e} . We now assume $w_j < w < v$. The case $u < w < w_j$ is symmetric.

We bound the piercing number of the link $\{v, w\}$. To this end, we start tracing a curve c'' at v and follow the link λ until just before it crosses e_j , then we follow e_j until just before it crosses \hat{e} , and then we follow \hat{e} until we arrive at w ; see Figure 4a. In the first part, the curve crosses at most $\lceil (\ell - 1)/2 \rceil \leq \lceil (k - 1)/2 \rceil$ edges. The curve does not cross any edge in the middle part, as the crossing between e_j and \hat{e} is the first crossing along e_j starting from λ . In the last part, the curve crosses at most $\min\{a, b\} \leq \lfloor (k - 1)/2 \rfloor$ edges. Thus, by Observation 2, the piercing number of the link $\{v, w\}$ is at most $\lceil (k - 1)/2 \rceil + \lfloor (k - 1)/2 \rfloor \leq k - 1$.



(a) λ and \hat{e} do not share an endpoint; the new triangle Δ is bounded by λ , c' , and c'' .

(b) λ and \hat{e} share the endpoint u ; the new triangle Δ is bounded by λ , \hat{e} , and c'' .

■ **Figure 4** Constructing a triangulation \mathcal{T} with edge piercing number k .

To bound the piercing number of the link $\{u, w\}$, we start tracing a curve c' at u , follow the link λ until just after it crosses e_j and then follow the curve c'' to w , see Figure 4a. The curve c' crosses the same set of edges as the curve c'' plus the edge e_j , so c'' has at most k crossings. Thus, the piercing number of the link $\{u, w\}$ is at most k .

Now we consider the last case, where \hat{e} and λ have a common endpoint. In this case, we choose the other endpoint of \hat{e} as w (see Figure 4b) and analyse the two subcases.

We first assume that u is the common endpoint. Then the link $\{u, w\} = \hat{e}$ is an edge of G , so it has at most k crossings. To bound the piercing number of the link $\{v, w\}$, we argue with the curve c'' that goes from v to w as defined in the previous case. Observe that the edge $\{u, w\}$ has exactly j crossings on the part from u to x (including x). Therefore, we can argue as above and conclude that the piercing number of the edge $\{v, w\}$ is at most $(\ell - j) + (k - j) = k - (2j - \ell)$. This is bounded from above by $k - 1$ since $2j - \ell = 2\lceil(\ell + 1)/2\rceil - \ell$ is 1 if ℓ is odd and 2 otherwise.

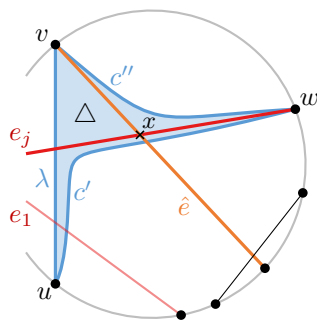
Similarly, if v is the common endpoint, then the link $\{v, w\} = \hat{e}$ is an edge of G , so it is crossed by at most k edges, and the piercing number of the link $\{u, w\}$ is at most $(j - 1) + (k - (\ell - j + 1)) = k + (2j - \ell) - 2 \leq k$. ◀

The *piercing number* of a triangle of \mathcal{T} is the sum of the piercing numbers of the links that form the sides of the triangle. We define the *triangle piercing number* of \mathcal{T} to be the maximum piercing number of any triangle of \mathcal{T} . Note that the triangle piercing number of a triangulation is at most three times its edge piercing number, but it can be smaller. If k is odd, we obtain a slightly stronger result; if k is odd, for every triangle t that we create in the proof of Lemma 6, at least one edge of t has at most $k - 1$ crossings.

► **Lemma 7.** *For every odd $k \geq 1$, every outer k -planar drawing of a maximal outer k -planar graph admits a triangulation of the outer cycle with edge piercing number at most k and triangle piercing number at most $3k - 1$.*

Proof. Following the proof of Lemma 6, in all cases, there is an edge with piercing number at most $k - 1$. If λ has no piercing edges, its piercing number is $0 \leq k - 1$. Otherwise, if \hat{e} and $\{u, v\}$ are disjoint, $\{v, w\}$ is pierced by at most $k - 1$ edges. In the last case, where \hat{e} and $\{u, v\}$ have a common endpoint, $\{v, w\}$ is pierced by at most $k - 1$ edges if ℓ is odd. If ℓ is even, it implies $\ell \neq k$ and therefore the link λ is pierced by $\ell \leq k - 1$ edges. ◀

With a slight modification of the strategy described in the proof of Lemma 6 and a careful analysis, we also obtain a better bound of the triangle piercing number for the case $k = 2$.



■ **Figure 5** λ and \hat{e} share the endpoint v ; the new triangle Δ is bounded by λ , c' , and c'' .

► **Lemma 8.** *Every outer 2-planar drawing of a maximal outer 2-planar graph admits a triangulation of the outer cycle with edge piercing number at most 2 and triangle piercing number at most 4.*

Proof. We roughly follow the strategy of the proof of Lemma 6. Recall that λ is the active link and that ℓ is the number of edges that pierce λ . If $\ell > 0$, then e_j with $j = \lceil (\ell + 1)/2 \rceil$ is the “middle” edge among those that pierce λ . If e_j has crossings on the right side of λ , then, among these, x is the crossing point of e_j closest to λ . The edge \hat{e} is the edge that crosses e_j in x . As in the proof of Lemma 6, we distinguish the following cases.

1. λ has no piercing edges.
2. e_j has no crossings to the right of λ .
3. λ and \hat{e} are disjoint (Figure 4a).
4. λ and \hat{e} meet in u (Figure 4b).
5. λ and \hat{e} meet in v .

In cases 1–4, we choose the split vertex w as in the proof of Lemma 6, which means that the new active links are pierced at most $k = 2$ times.

In case 5, we now choose w to be w_j (see Figure 5). Let c' be a curve that starts in u , follows λ until just before e_j and then follows e_j to w . Let c'' be a curve that starts in v , follows \hat{e} to x , and then follows e_j to w . In order to show that each of the two curves is crossed by at most two edges, we need to do some preparations. We say that an edge lies to the *left* of the active link if both of its endpoints lie on the left side or one endpoint lies on the left and the other is an endpoint of the active link. We further say that an edge e piercing a link λ is *anchored* with respect to λ if there exists another edge f that crosses e and either $f = \lambda$ or f lies to the left of λ . Observe that an anchored edge can have at most one crossing to the right of λ .

We claim that, during the splitting procedure described above, every edge piercing an active link is anchored (with respect to the active link). The claim is true in the beginning as $\lambda = \{v_1, v_n\}$ is not pierced. In case 1, every edge that pierces $\{u, w\}$ or $\{v, w\}$ also crosses the edge $\{u, w\}$, so it is anchored. In case 2, every edge that pierces $\{u, w\}$ or $\{v, w\}$ also pierces $\lambda = \{u, v\}$ and is anchored by induction. In case 3, observe that no edge crosses e_j between λ and x (as x is the first crossing on e_j to the right of λ), and no edge crosses \hat{e} between x and w (by the choice of w and since $\lfloor (k - 1)/2 \rfloor = 0$ for $k = 2$). Therefore, every edge that pierces $\{u, w\}$ or $\{v, w\}$ also pierces λ and is anchored by induction. In case 4, $\{u, w\}$ is an edge, so every edge piercing it is anchored. Every edge that pierces $\{v, w\}$ either crosses $\{u, w\}$ or pierces λ and is anchored by induction. Similarly, in case 5, every edge that pierces $\{u, w\}$ or $\{v, w\}$ either pierces λ or crosses e_j (which lies to the left of both new active links).

14:10 Bounding the Treewidth of Outer k -Planar Graphs via Triangulations

We now return to case 5 and bound the piercing numbers of the new active links $\{u, w\}$ and $\{v, w\}$. To do this, we count the edges that cross the curves c' and c'' , respectively. As e_j is anchored with respect to λ , we get that x is the only crossing on e_j to the right of λ . Now, it is easy to see that c' is pierced only by \hat{e} and possibly by e_1 in case $j = 2$. For c'' , we can show that there are no crossings along the curve. First, observe that any edge that crosses \hat{e} between v and x has to pierce λ , which is not possible. Second, no edge crosses e_j between x and w , as e_j is anchored and x is the only crossing on e_j to the right of λ . Combining these two observations, we see that there is no place for c'' to cross with an edge.

It remains to show that the triangle piercing number of the new triangle Δ formed by u , v , and w is at most 4. In case 1, λ has no piercing edges. In cases 2, 4, and 5, $\{v, w\}$ has no piercing edges. In case 3, we distinguish two subcases depending on the relative position of w . When $w_j < w$, then $\{v, w\}$ has no piercing edges. When $w < w_j$, then each of $\{u, w\}$ and $\{v, w\}$ is pierced at most once. ◀

Lastly, we combine the two triangulation strategies from the proofs of Lemmas 5 and 6 to obtain a triangulation method for outer min- k -planar graphs.

► **Lemma 9.** *For every $k \geq 1$, every outer min- k -planar drawing of a maximal outer min- k -planar graph admits a triangulation of the outer cycle with edge piercing number at most $2k - 1$ and triangle piercing number at most $6k - 3$.*

Proof. We proceed as in the proof of Lemma 5. We describe a splitting procedure such that every active link has piercing number at most $2k - 1$. Consult Figure 3 for a reminder of the notation and the strategy. As the given drawing is outer min- k -planar, edges may now have more than k crossings. We call such edges *heavy* and the other edges *light*. By definition, heavy edges cross only light edges.

We first explain why the previous proof breaks in both cases if the curves c' and c'' are routed along heavy edges. Recall that in the first case when λ has no piercing edges, we selected the split vertex w to be the neighbor of u from the right side of λ that is closest to v . If, however, the edge $\{u, w\}$ is heavy, then we cannot bound the number of crossings along the curve c'' . In the second case, we selected the split vertex to be the right endpoint w_j of the middle piercing edge e_j . If the edge e_j is heavy, then we cannot bound the numbers of crossings along c' and c'' .

To resolve the issue with the second case, we apply the strategy from the proof of Lemma 6. Observe that if e_j is heavy, then for the first crossing x with an edge \hat{e} on the right side of λ , the edge \hat{e} is a light. Similarly, as in the proof of Lemma 6 (see Figure 4), we select the split vertex w to be the endpoint of \hat{e} that is different from u and v and in case both endpoints are, the one on the side of x with fewer crossings. The piercing numbers of the two new links are at most $\lceil (2k - 1)/2 \rceil + k - 1 \leq 2k - 1$.

To resolve the issue with the first case, we use a similar strategy. If the edge $\{u, w\}$ is heavy, let x be the crossing on $\{u, w\}$ that is closest to u and let \hat{e} be the crossing edge. Note that \hat{e} is light. We select the split vertex w to be any right endpoint of \hat{e} . Now consider the two curves that start at w , go along \hat{e} to x , then along $\{u, w\}$ to u , and optionally to v along λ . Both curves have at most k crossings, only with edges that also cross \hat{e} . ◀

► **Remark 10.** Given the intersection graph of the edges of G , the triangulations in Lemmas 5–9 can be constructed in $O(nk)$ time. Each step of the splitting procedures presented in Lemmas 5–9 can be implemented to run in $O(k)$ time, as the edges that pierce the new active links also pierce at least one of the edges λ , e_j , and \hat{e} .

4 Applications

In this section, we show several applications of our triangulations from Section 3. Among others, we improve the bound of Wood and Telle [20] on the treewidth of outer k -planar graphs from $3k + 11$ to $1.5k + 2$ (see Section 4.1) and improve the bound of Chaplick et al. [10] on the separation number of outer k -planar graphs from $2k + 3$ to $k + 2$; see Section 4.2. In addition, we improve the bound of Wood and Telle of $3k + 11$ on the treewidth of outer min- k -planar graphs to $3k + 1$; see Theorem 14.

4.1 Treewidth

Now we show our main tool to obtain a better upper bound on the treewidth. Note that we do not refer specifically to outer k -planar graphs since our tool can be used for any drawing of a graph whose vertices lie on the outer face.

Consider a graph G with a drawing Γ whose outer face is bounded by a simple cycle C that contains all vertices of G . Let \mathcal{T} be a planar triangulation of C and f be some triangular face of \mathcal{T} . We call an edge of $E(G) \setminus E(\mathcal{T})$ *short* with respect to f if one of its endpoints is on f and it pierces a link of f . We call an edge of $E(G) \setminus E(\mathcal{T})$ *long* with respect to f if neither of its endpoints is on f and it pierces two links of f .

► **Lemma 11.** *If G is a graph with a convex drawing whose outer cycle admits a triangulation \mathcal{T} with triangle piercing number at most c , then $\text{tw}(G) \leq (c + 5)/2$.*

Proof. Let T be a tree obtained by taking the weak dual graph of \mathcal{T} and associating each node of T with a bag that contains the vertices of G that are incident to the corresponding face. We modify this tree to consider the piercing edges, ensuring that the sizes of bags do not exceed $(c + 7)/2$. (Recall that the width of a tree decomposition is the size of the largest bag minus 1.)

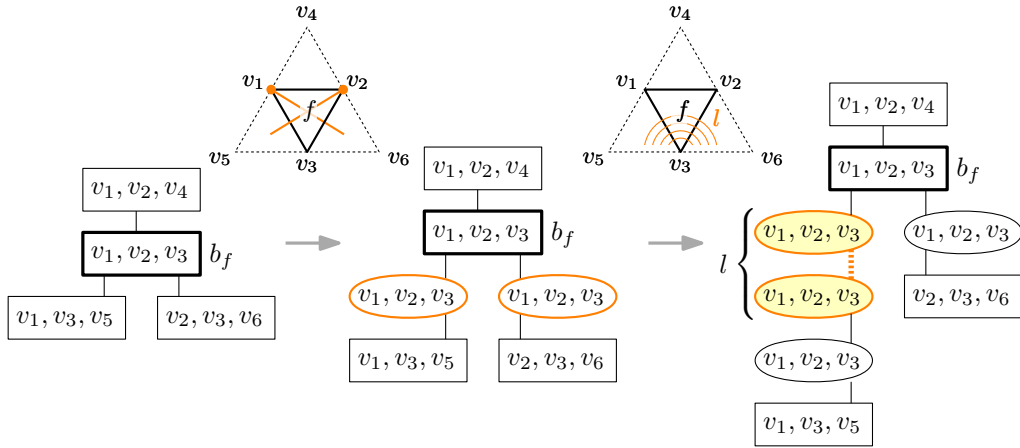
If \mathcal{T} has an unpierced inner link $\lambda = \{u, v\}$, we split \mathcal{T} along λ into two triangulations and compute tree decompositions for them. Since each of them has a bag that contains λ , we can connect them and obtain a tree decomposition of G .

Hence, from now on, we assume that every inner link of \mathcal{T} is pierced at least once. We root T at an arbitrary leaf. The basic idea is as follows. Consider an edge $\{u, v\}$ that pierces inner links of \mathcal{T} . Then we *lift* both u and v to the bag b that is the lowest common ancestor of the highest bags b_u and b_v that contain u and v , respectively. That is, we place a copy of u into each bag on the way from b_u to b and symmetrically for v . If we apply this strategy naively, then there may be a bag with $c + 3$ vertices in the worst case. In the following, we describe how to improve this bound.

Consider a face f of the triangulation \mathcal{T} . Let b_f be the bag in T that corresponds to f . Let v_1, v_2 , and v_3 be the vertices of f such that f shares the link $\{v_1, v_2\}$ with the face that corresponds to the parent of f in T (if any). An edge of $E(G) \setminus E(\mathcal{T})$ is *lineal* if one of its endpoints is in the bag b_f or above and the other endpoint is in a bag below b_f in the tree T . A long edge of $E(G) \setminus E(\mathcal{T})$ is *bent* if its endpoints are in different subtrees of T .

We perform the following two modifications of T for each face f ; see Figure 6.

- We subdivide each edge of T between b_f and its children (if any) by a new *copy bag* that is a copy of b_f . We call these bags *primary* copy bags.
- If f has two children in T , assume that piercing number of $\lambda = \{v_1, v_3\}$ is less or equal to that of $\lambda' = \{v_2, v_3\}$ and that λ is the common link of f and its left child face. If there are $l > 0$ bent edges $\{u_1, w_1\}, \dots, \{u_l, w_l\}$ w.r.t. f , then we subdivide the edge of T between b_f and its left child by additional l copies of b_f . These *secondary* copy bags are numbered $b_{f,1}, \dots, b_{f,l}$ from top to bottom; see Figure 6.



■ **Figure 6** Steps 1 and 2 of the modification. Copy bags are oval; secondary copy bags are yellow.

Next, we lift vertices in bottom-up order, e.g., in preorder. More precisely, at each *original* bag b_f (that is, not a copy bag) in our preorder traversal of T , we process each edge e of G that pierces f , according to its type as follows.

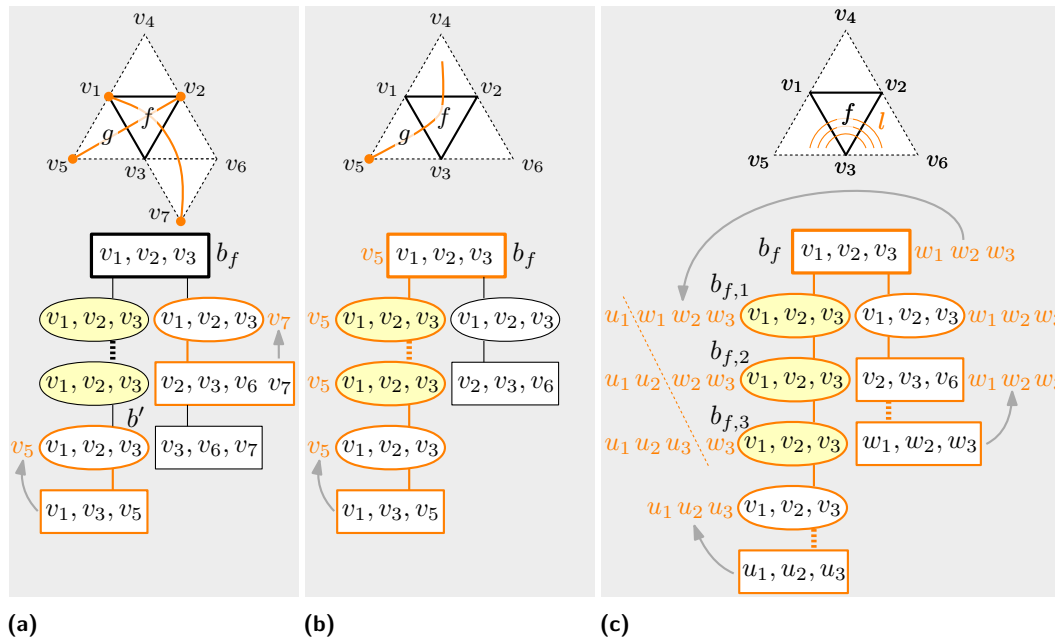
- If e is a short edge, that is, it is incident to a vertex of f , namely to v_1 or v_2 , then we lift the highest occurrence of the other endpoint u of e by one bag (as v_5 and v_7 in Figure 7a) to the primary copy bag of b_f .
- If e is a long edge that pierces the link $\{v_1, v_2\}$, then we lift the highest occurrence of the other endpoint u of e all the way to b_f ; see Figure 7b.
- If $e = \{u_i, w_i\}$ for some $i \in \{1, \dots, l\}$, then we lift both endpoints of e . First, we lift the highest occurrence of u_i from the left subtree to $b_{f,i}$. Second, we lift the highest occurrence of w_i in the right subtree of b_f to b_f . Third, we copy w_i into each bag from b_f down to $b_{f,i}$; see Figure 7c. In this way, the two endpoints of e meet in $b_{f,i}$.

The modifications described above make sure that the resulting tree T' is indeed a tree decomposition of G , i.e., for each vertex v of G , the subtree induced by the bags that contain v is connected, and for each edge e of G , there is a bag in T' that contains both endpoints of e .

In the following, we show that each bag of T' contains at most $(c+7)/2$ vertices, i.e., we have added at most $(c+1)/2$ to the initial three vertices. For each face f , the number of vertices added to the original bag b_f is the number of long edges w.r.t. f , which is at most $c/2$. It remains to bound the number of vertices that are added to each copy bag of b_f .

First, we show the bound for secondary copy bags. Recall that, for $i \in \{1, \dots, l\}$, we have added to $b_{f,i}$ from below i endpoints of the bent long edges w.r.t. f and from above $l+1-i$ endpoints of bent long edges; see Figure 7c. In addition, we have added a vertex for each lineal long edge. Therefore, the number of added vertices is at most one plus the number of long edges w.r.t. f that pierce the link $\{v_1, v_3\}$. Recall that the link $\{v_1, v_3\}$ was chosen to have at most as many piercing edges as $\{v_2, v_3\}$. Because each inner link of \mathcal{T} , including $\{v_1, v_2\}$, is pierced at least once, $\{v_1, v_3\}$ is pierced at most $(c-1)/2$ times. Thus, we have added at most $(c+1)/2$ vertices to $b_{f,i}$, and so $|b_{f,i}| \leq (c+7)/2$.

Now, we bound the number of vertices that are added to the primary (the bottommost) copy bag b' of b_f . Note that such a bag is a parent bag of some original bag b_g such that the faces f and g are adjacent. There are two types of vertices that are added to b' . The first type consists of the endpoints of the lineal long edges of b_g (at most $c/2$). The second one is just one vertex w of the face g that is not shared with f (for example, the endpoint v_5 of g



■ **Figure 7** Lifting up endpoints of the edges that pierce links of the face f .

in Figure 7a). We have added w only if there are edges in $E(G) \setminus E(\mathcal{T})$ with lower endpoint w (lower w.r.t. T). Each such edge contributes to the piercing number of g . Hence, the number of lineal long edges w.r.t. g is at most $(c - 1)/2$. In both cases ($w \in b'$ and $w \notin b'$), we have added at most $(c + 1)/2$ vertices to b' , so $|b'| \leq (c + 7)/2$. ◀

Lemmas 8 and 11 give us an upper bound 4 for outer 2-planar graphs, extending the known tight bound $k + 2$ for $k = \{0, 1\}$ [5, 6]. Note that K_5 is outer 2-planar and $\text{tw}(K_5) = 4$.

► **Theorem 12.** *Every outer 2-planar graph has treewidth at most 4, which is tight.*

► **Theorem 13.** *Every outer k -planar graph has treewidth at most $1.5k + 2$.*

Proof. We already know better bounds for $k \leq 2$. For $k \geq 3$, by Lemmas 6, 7, and 11, we obtain a bound $1.5k + 2.5$ for even k and $1.5k + 2$ for odd k . As treewidth is an integer, we obtain $1.5k + 2$ for general k . ◀

By Lemmas 9 and 11, we also obtain the following bound for outer min- k -planar graphs, which improves the previously known bound $3k + 11$ by Wood and Telle [20]. Note that outer min-0-planar graphs are outerplanar graphs.

► **Theorem 14.** *For $k \geq 1$, every outer min- k -planar graph has treewidth at most $3k + 1$.*

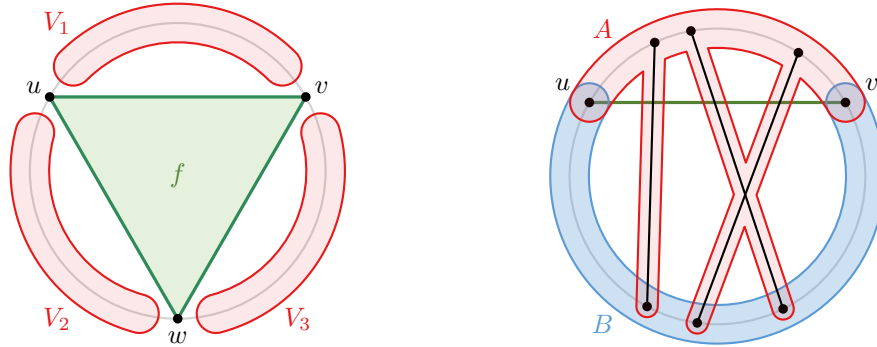
4.2 Separation Number

Theorems 13 and 14 immediately imply upper bounds on the separation number of outer k -planar and outer min- k -planar graphs. However, using our triangulations directly, we obtain even better bounds, namely $k + 2$ for outer k -planar graphs and $2k + 1$ for outer min- k -planar graphs. The first bound improves the bound of $2k + 3$ by Chaplick et al. [10].

► **Lemma 15.** *If G is a graph with a convex drawing whose outer cycle admits a triangulation \mathcal{T} with edge piercing number at most c , then G has a balanced separator of size at most $c + 2$.*

Proof. We construct a balanced separation of order at most $c + 2$ using \mathcal{T} . In short, we select a link λ of \mathcal{T} that is a balanced separator for \mathcal{T} and put its endpoints into the separator. Then, we add at most c vertices to the separator according to the edges piercing λ .

First, we find a “centroid” triangle of \mathcal{T} as follows. Let T be the tree that is the weak dual of \mathcal{T} . It is well known that every tree contains a vertex such that, after removing it, the number of vertices in each subtree is at most half of the original tree. Let $f = \{u, v, w\}$ be the triangle corresponding to such a vertex of T . We partition $V \setminus f$ into three disjoint sets V_1, V_2, V_3 ; see Figure 8a. We may assume that V_1 is the largest among the three sets.



(a) “Centroid” triangle f and vertex sets V_1, V_2, V_3 . (b) Edges piercing $\{u, v\}$ and separation (A, B) .

■ **Figure 8** A triangulation with edge piercing number c yields a balanced separator of size at most $c + 2$.

Now, we construct the desired separation with the help of the sets V_1, V_2, V_3 . As $\lambda = \{u, v\}$ is a link of \mathcal{T} , at most c edges pierce λ , connecting vertices in V_1 with vertices in $V_2 \cup \{w\} \cup V_3$. Let S be the set of endpoints of the piercing edges on the latter side. Let $A = V_1 \cup \{u, v\} \cup S$ and $B = V_2 \cup \{u, v, w\} \cup V_3$; see Figure 8b. We claim that (A, B) is a balanced separation of order at most $c + 2$.

Clearly, the order of (A, B) is $|A \cap B| = |S \cup \{u, v\}|$, which is at most $c + 2$. Hence, it suffices to show that the sizes of $A \setminus B$ and $B \setminus A$ are at most $2n/3$. To this end, we first bound the size of V_1 as follows:

$$\frac{n}{3} - 1 \leq |V_1| \leq \frac{n}{2} - 1.$$

The lower bound holds as $|V_1| + |V_2| + |V_3| = n - 3$ and $|V_1|$ is the largest. The upper bound can be shown by the fact that a triangulated outerplanar graph has exactly $F + 2$ vertices, where F is the number of triangles. By the way of choosing f , the vertex set $V_1 \cup \{u, v\}$ induces at most $n/2 - 1$ triangles. Therefore, $|V_1| = |V_1 \cup \{u, v\}| - 2 \leq (n/2 - 1 + 2) - 2 = n/2 - 1$.

Now we can confirm that the sizes of $A \setminus B$ and $B \setminus A$ are at most $2n/3$ as follows:

$$|A \setminus B| = |V_1| \leq \frac{n}{2} - 1 \leq \frac{2}{3}n \quad \text{and}$$

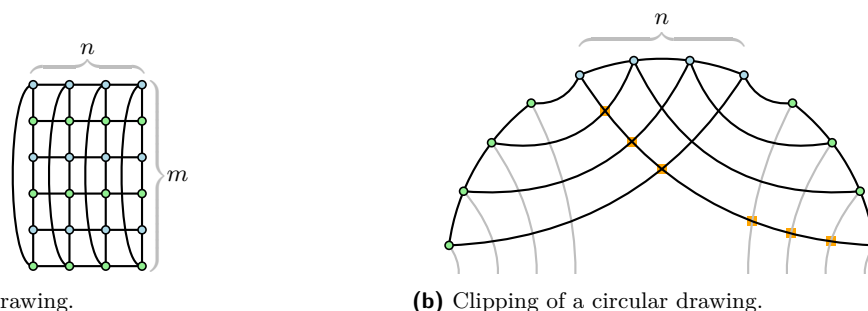
$$|B \setminus A| \leq |V_2 \cup \{w\} \cup V_3| = |V \setminus (V_1 \cup \{u, v\})| \leq n - \left(\frac{n}{3} - 1\right) - 2 = \frac{2}{3}n - 1. \quad \blacktriangleleft$$

For every k , the classes of outer k -planar graphs and outer min- k -planar graphs are closed under taking subgraphs. Hence, by Lemmas 6, 9, and 15, we obtain the following bounds.

► **Theorem 16.** *Every outer k -planar graph has separation number at most $k + 2$, and for $k \geq 1$, every outer min- k -planar graph has separation number at most $2k + 1$.*

5 Lower Bounds

In this section, we complement the results in Section 4, by giving lower bounds for the separation number and treewidth of outer k -planar graphs. To show them, we use grid-like graphs called *stacked prisms*. The $m \times n$ stacked prism $Y_{m,n}$ is the (planar) graph obtained by connecting all pairs vertices in the topmost and bottommost row of the $m \times n$ grid that are in the same column; see Figure 9a.



(a) Grid-like drawing.

(b) Clipping of a circular drawing.

■ **Figure 9** Two drawings of the stacked prism $Y_{m,n}$.

► **Theorem 17.** *For every even number $k \geq 0$, there exists a graph G such that $\text{lcr}^\circ(G) = k$ and $\text{sn}(G) = k + 2$.*

Proof. We first observe that, if m is even, $\text{lcr}^\circ(Y_{m,n})$ is at most $2n - 2$. If m is even, we can place the m rows of $Y_{m,n}$, each of length n , one after the other, alternating in their direction, around a circle; see Figure 9b. The grid has two types of edges: the *row edges* and the *column edges*, which connect vertices in the same row or column, respectively. As Figure 9b shows, the row edges have no crossing. For each $i \in [n]$, there is a column edge e_i that spans $2i - 2$ other vertices along the perimeter of the circle. Each of these vertices is incident to exactly one column edge that crosses e_i . Hence, every column edge has at most $2n - 2$ crossings, and $\text{lcr}^\circ(Y_{m,n}) \leq 2n - 2$.

Let $n = k/2 + 1$, and let m be a sufficiently large even number. Then we claim that $Y_{m,n}$ fulfills the conditions. As we discussed $\text{lcr}^\circ(Y_{m,n}) \leq k$ holds and therefore $\text{sn}(Y_{m,n}) \leq k + 2$ follows from Theorem 16. Hence, it suffices to show $\text{sn}(Y_{m,n}) \geq k + 2 = 2n$, which also implies $\text{lcr}^\circ(Y_{m,n}) \geq 2n - 2 = k$ by Theorem 16.

Suppose that $Y_{m,n}$ has a balanced separator S of size less than $2n$. As there are n columns, there is a column that contains at most one vertex of S . This column contains a path P of length $m - 1$ that does not intersect S . Now observe that at least $m - 2n$ rows do not contain any vertex of S , and therefore, all vertices in these rows are connected to P . Hence, after removing S , the size of the connected component C that contains P is at least $n(m - 2n)$. The ratio $|V(C)|/|V(Y_{m,n})|$ is $1 - (2n/m)$, which is greater than $2/3$ if m is sufficiently large. ◀

Aidun et al. [1] showed that $\text{tw}(Y_{m,n}) = 2n$ if $m > 2n$. With the same stacked prism, we obtain the following lower bound.

► **Theorem 18.** *For every even number $k \geq 0$, there exists a graph G such that $\text{lcr}^\circ(G) = k$ and $\text{tw}(G) = k + 2$.*

6 Conclusion and Open Problems

We have introduced methods for triangulating drawings of outer k -planar graphs such that the triangulation edges cross few graph edges. These triangulations yield better bounds on treewidth and separation number of outer k -planar graphs. Our method is constructive; the corresponding treewidth decomposition and balanced separation can be computed efficiently. Via our triangulations, we improved the multiplicative constant in the upper bound on the treewidth of outer k -planar graphs from 3 to 1.5; we showed a lower bound of 1. What is the correct multiplicative constant? Finding triangulations of outer k -planar graphs with lower triangle piercing number could be a step in this direction. It would also be interesting to find other graph classes that admit triangulations with low triangle piercing number.

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