On the Uncrossed Number of Graphs

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Abstract

Visualizing a graph G in the plane nicely, for example, without crossings, is unfortunately not always possible. To address this problem, Masařík and Hliněný [GD 2023] recently asked for each edge of G to be drawn without crossings while allowing multiple different drawings of G. More formally, a collection $\mathcal{D}$ of drawings of G is *uncrossed* if, for each edge e of G, there is a drawing in $\mathcal{D}$ such that e is uncrossed. The *uncrossed number* $\mathrm{unc}(G)$ of G is then the minimum number of drawings in some uncrossed collection of G.

No exact values of the uncrossed numbers have been determined yet, not even for simple graph classes. In this paper, we provide the exact values for uncrossed numbers of complete and complete bipartite graphs, partly confirming and partly refuting a conjecture posed by Hliněný and Masařík [GD 2023]. We also present a strong general lower bound on $\mathrm{unc}(G)$ in terms of the number of vertices and edges of G. Moreover, we prove NP-hardness of the related problem of determining the *edge crossing number* of a graph G, which is the smallest number of edges of G taken over all drawings of G that participate in a crossing. This problem was posed as open by Schaefer in his book [Crossing Numbers of Graphs 2018].

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1 Introduction

In a *drawing* of a graph G, the vertices are represented by distinct points in the plane and each edge corresponds to a simple continuous arc connecting the images of its end-vertices. As usual, we identify the vertices and their images, as well as the edges and the line segments representing them. We require that the edges pass through no vertices other than their endpoints. We assume for simplicity that any two edges have only finitely many points in common, no two edges touch at an interior point and no three edges meet at a common interior point.

A *crossing* in a drawing D of G is a common interior point of two edges of D where they properly cross. For a drawing D of a graph G, we say that an edge e of D is *uncrossed* in D if it does not share a crossing with any other edge of D.

There are two staple problems in the graph drawing field that defined the past eighty years of development in the area. The first one, dating back to World War II times [2, 21], is the problem of determining the *crossing number* $\mathrm{cr}(G)$ of a graph G, defined as the smallest number of crossings required in any drawing of G in the plane. The crossing number problem has been intensively studied ever since, especially in the past thirty years. Computing the crossing number is NP-hard on general graphs [3], and one can find a thorough overview of the area in a recent book by Schaefer [19].

The second, only slightly newer problem, is that of determining the *thickness* $\theta(G)$ of a graph G, defined as the smallest integer k such that G can be edge-partitioned into k planar graphs. This problem was proposed by Harary [7] in 1961 and since then this concept has played an important role in graph drawing. Unlike for planarity, deciding whether a graph is *biplanar*, that is whether $\theta(G) \leq 2$, is NP-complete [14]. For an overview of the progress up to 1998, consult a survey by Mutzel, Odenthal, and Scharbrodt [16].

In this paper, we investigate a very recent notion inspired by a fusion of both concepts into one. We say that a collection $\mathcal{D}(G)$ of drawings of G is *uncrossed* if for each edge e of G there is at least one drawing in $\mathcal{D}(G)$ in which e is uncrossed; see Figure 1 for an example. Hliněný and Masařík [11], in relation to extensions of the traditional crossing number of a graph, defined the *uncrossed number* $\mathrm{unc}(G)$ of a graph G as the smallest size of an uncrossed collection of drawings of G.

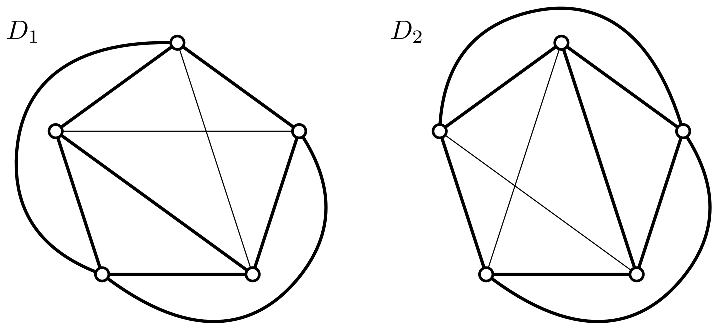


Figure 1 An uncrossed collection $\mathcal{D}(K_5) = \{D_1, D_2\}$ that shows $\mathrm{unc}(K_5) \leq 2$. The edges that are uncrossed are shown in thick lines. Since K_5 is not planar, we have $\mathrm{unc}(K_5) = 2$.

The motivation for the uncrossed number [11] is that finding a handful of different drawings of a graph G instead of just one "flawless" drawing shall highlight different aspects of G and thus could be useful for the visualization of G, besides the theoretical interest. The requirement that each edge is uncrossed in at least one drawing is a natural way to enforce that the drawings will highlight each aspect of the graph as a whole.

Let us also formulate the decision version of the problem of determining $\mathrm{unc}(G)$ of a given graph G.

> UNCROSSEDNUMBER
> **Input:** A graph G and a positive integer k.
> **Question:** Are there drawings $D_1, \dots, D_k$ of G such that, for each edge $e \in E(G)$, there exists an $i \in [k]$ such that e is uncrossed in the drawing D_i?

Clearly, for every graph G, we have

$$\mathrm{unc}(G) \geq \theta(G), \tag{1}$$

because the uncrossed edges in each drawing of an uncrossed collection of G induce an edge-partition of G into planar graphs. However, this new concept significantly differs from thickness (which just partitions the edges of G) in the sense that all edges of G have to be present along with the uncrossed subdrawing in each drawing of our uncrossed collection. In fact, the requirements of an uncrossed collection bring us close to the related notion of the *outerthickness* $\theta_o(G)$ of G, which is the minimum number of outerplanar graphs into which we can decompose G.

> OUTERTHICKNESS
> **Input:** A graph G and a positive integer k.
> **Question:** Can G be decomposed into k outerplanar graphs?

As noted by Hliněný and Masařík [11], given a decomposition $\{G_1, \dots, G_k\}$ of G into outerplanar graphs, we can let D_i be an outerplanar drawing of G_i with all remaining edges of G being drawn in the outer face. This gives us

$$\mathrm{unc}(G) \leq \theta_o(G) \tag{2}$$

for every graph G. Combining this with a result of Gonçalves [4], which implies $\theta_o(G) \leq 2\theta(G)$, we actually obtain the following chain of inequalities

$$\frac{1}{2}\theta_o(G) \leq \theta(G) \leq \mathrm{unc}(G) \leq \theta_o(G) \leq 2\theta(G). \tag{3}$$

So far, the exact values of uncrossed numbers are not very well understood. Masařík and Hliněný [11] exactly determined $\mathrm{unc}(G)$ of only a few sporadic examples of graphs G, such as $\mathrm{unc}(K_7) = 3$.

Our Results

We determine the exact values of uncrossed numbers for specific and natural graph classes. First, we derive the formula for the uncrossed number of complete graphs.

▶ **Theorem 1.** *For every positive integer n, it holds that*

$$\mathrm{unc}(K_n) = \begin{cases} \lceil \frac{n+1}{4} \rceil, & \text{for } n \notin \{4, 7\} \\ 3, & \text{for } n = 7 \\ 1, & \text{for } n = 4. \end{cases}$$

We also find the exact formula for the uncrossed number of complete bipartite graphs.

▶ **Theorem 2.** *For all positive integers m and n with $m \leq n$, it holds that*

$$\mathrm{unc}(K_{m,n}) = \begin{cases} \lceil \frac{mn}{2m+n-2} \rceil, & \text{for } m \leq n \leq 2m-2 \\ \lceil \frac{mn}{2m+n-1} \rceil, & \text{for } n = 2m-1 \\ \lceil \frac{mn}{2m+n} \rceil, & \text{for } 6 \leq 2m \leq n \\ 1, & \text{for } m \leq 2. \end{cases}$$

Let us mention that the exact values of the thickness $\theta(K_{m,n})$ of complete bipartite graphs are not known for all values of m and n; see [17] for further discussion.

We compare our formulas on $\mathrm{unc}(K_n)$ and $\mathrm{unc}(K_{m,n})$ with known formulas on the outerthickness of K_n and $K_{m,n}$. Hliněný and Masařík [11, Section 6] conjectured the uncrossed numbers and outerthickness to be the same for both complete and complete bipartite graphs except in the planar but not outerplanar cases. Guy and Nowakowski [5, 6] showed that

$$\theta_o(K_n) = \begin{cases} \left\lceil \frac{n+1}{4} \right\rceil, & \text{for } n \neq 7 \\ 3, & \text{for } n = 7 \end{cases} \tag{4}$$

and

$$\theta_o(K_{m,n}) = \left\lceil \frac{mn}{2m+n-2} \right\rceil \tag{5}$$

for all positive integers m and n with $m \leq n$. Note that it follows from Theorem 1 and Equation (4) that $\mathrm{unc}(K_n) = \theta_o(K_n)$ for every $n \neq 4$. For $n = 4$, we have $\mathrm{unc}(K_4) = 1$ while $\theta_o(K_4) = 2$. This confirms the conjecture of Hliněný and Masařík [11] in the case of complete graphs.

Since

$$\left\lceil \frac{mn}{2m+n} \right\rceil \leq \left\lceil \frac{mn}{2m+n-2} \right\rceil = \left\lceil \frac{mn}{2m+n} + \frac{2mn}{(2m+n-2)(2m+n)} \right\rceil \leq \left\lceil \frac{mn}{2m+n} \right\rceil + 1$$

for $n \geq 2m-1 > 1$, it follows from Theorem 2 and Equation (5) that the uncrossed number $\mathrm{unc}(K_{m,n})$ differs from the outerthickness $\theta_o(K_{m,n})$ of $K_{m,n}$ by at most 1. In particular, we have $\mathrm{unc}(K_{n,n}) = \theta_o(K_{n,n})$ for every positive integer n. However, Theorem 2 and (5) give, for example, $\mathrm{unc}(K_{4,7}) = 2$ and $\theta_o(K_{4,7}) = 3$. Since $K_{4,7}$ is not planar, this refutes the conjecture of Hliněný and Masařík [11] in the case of complete bipartite graphs.

Second, we turn our attention to general graphs and their uncrossed number. We improve the trivial lower bound of $\mathrm{unc}(G) \geq \lceil m/(3n-6) \rceil$ for any graph G with n vertices and m edges. By carefully balancing between the numbers of edges in uncrossed subdrawings of G and the numbers of edges that can be drawn within faces of uncrossed subdrawings, we derive the following estimate.

▶ **Theorem 3.** *Every connected graph G with $n \geq 3$ vertices and $m \geq 0$ edges satisfies*

$$\mathrm{unc}(G) \geq \left\lceil \frac{m}{f(n,m)} \right\rceil$$

where $f(n,m) = (3n-5+\sqrt{(3n-5)^2-4m})/2$.

The bound from Theorem 3 becomes interesting for $m \geq 3n-6$. This is because we then have $f(n,m) \leq 3n-6$ for all integers $n \geq 2$ as

$$\sqrt{(3n-5)^2-4m} \leq \sqrt{9n^2-42n+49} = 3n-7$$

for any $m \geq 3n-6 \geq 0$. It follows that the lower bound from Theorem 3 is at least as good as $\mathrm{unc}(G) \geq \lceil m/(3n-6) \rceil$ for any connected G with $n \geq 2$ vertices and $m \geq 3n-6$ edges.

The lower bound from Theorem 3 gets stronger as the graph G gets denser. For example, if G contains n vertices and εn^2 edges for n sufficiently large and $\varepsilon \in (0, 1/2)$, we get

$$f(n,m) = (3n-5+\sqrt{(9-4\varepsilon)n^2-30n+25})/2 \leq (3+\sqrt{9-4\varepsilon})n/2.$$

Since $(3+\sqrt{9-4\varepsilon})/2 < 3$ for $\varepsilon > 0$, we obtain $\mathrm{unc}(G) \geq \left\lceil \frac{m}{c_\varepsilon n} \right\rceil$ for any $\varepsilon > 0$ and some constant $c_\varepsilon < 3$, instead of trivial $\mathrm{unc}(G) \geq \left\lceil \frac{m}{3n-6} \right\rceil$. We note that the best constant c_ε obtainable from Theorem 3 is $(3+\sqrt{7})/2 \sim 2.82$ as $\varepsilon \leq 1/2$.

We also consider computational complexity aspects related to the UncrossedNumber problem. As we will see later, a closely related problem is the one of determining the *edge crossing number* of a given graph G, which is the smallest number of edges involved in crossings taken over all drawings of G. The notion of the edge crossing number is based on results by Ringel [18], Harborth and Mengersen [8, 9], and Harborth and Thürmann [10].

EdgeCrossingNumber	
Input:	A graph G and a positive integer k.
Question:	Is there a drawing D of G with at most k edges involved in crossings?

The complementary problem to EdgeCrossingNumber is the following one.

MaximumUncrossedSubgraph	
Input:	A graph G and a positive integer k.
Question:	Is there a drawing D of G with at least k edges not involved in any crossings?

In his monograph on crossing numbers, Schaefer [19] mentions that the problem of determining the computational complexity of EdgeCrossingNumber is open. Here, we resolve this open question by showing that the problem is NP-complete.

▶ **Theorem 4.** *The* EdgeCrossingNumber *problem is* NP*-complete.*

By the complementarity of the problems MaximumUncrossedSubgraph and EdgeCrossingNumber, we obtain the following result.

▶ **Corollary 5.** *The* MaximumUncrossedSubgraph *problem is* NP*-complete.*

As a consequence of our reduction, we also obtain the following relative result.

▶ **Theorem 6.** *If the* Outerthickness *problem is* NP*-hard, then also the* UncrossedNumber *problem is* NP*-hard.*

However, in contrast to the complexity of the thickness problem, which was shown to be NP-hard already in 1983 by Mansfield [14], the complexity of the Outerthickness problem remains open.

2 Preliminaries

We may, without loss of generality, restrict to only simple graphs in the whole paper. This is since, in each of the formulated problems, whenever an edge e is a part of an uncrossed subdrawing (as discussed next), any other edge parallel to e can be drawn uncrossed closely along e, too.

Let D' be a subdrawing of D consisting of only uncrossed edges of D. Note that we do not require D' to contain all such edges. In this situation, we call D' an *uncrossed subdrawing* of G and we say that it *represents* the subgraph of G formed by edges that are drawn in D'. Precisely, D' is an uncrossed subdrawing of G if there exists a drawing D of a graph G such that D' is formed by a subset of the uncrossed edges of D.

▶ **Lemma 7.** *Let D' be an uncrossed subdrawing of a connected graph G. Then D' is a planar drawing and, for every edge $\{u, v\}$ of G, the vertices u and v are incident to a common face of D'. Moreover, there is an uncrossed subdrawing D'' of G such that D'' represents a connected supergraph of the graph represented by D'.*

Proof. The drawing D' is clearly planar, as, by the definition of D', each edge of D' is uncrossed in a drawing D of G and thus also in D'. Moreover, it is a folklore fact that two vertices u and v in a planar drawing, here in D', are *not* incident to a common face if and only if there exists a cycle $C \subseteq D'$ such that u and v are drawn on different sides of C. In the latter case, however, the edge $\{u, v\}$ would cross some edge of C in D, which is impossible since no edge of D' is crossed.

We prove the second part by induction on the number of connected components represented by D'. The case of one component is trivial, as $D'' = D'$. Otherwise, since G is connected, there exists an edge $e = \{u, v\}$ of G that is not drawn in D' and such that u and v belong to different components represented by D'. By the first part of the lemma, the vertices u and v are incident to the same face of D'. So, let D^+ be the planar drawing obtained from D' by adding a crossing-free arc representing the edge e. Clearly, every edge of G is still incident to a common face of D^+, and so D^+ can be completed into a drawing of G such that D^+ stays uncrossed. The subgraph of G represented by D^+ has fewer components than we started with, and so we find the desired D'' by induction. ◀

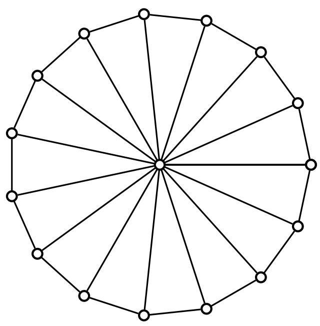


Figure 2 The wheel graph W_{15}.

For a graph G, let $h(G)$ be the maximum number of uncrossed edges in some drawing D of G. Let DW_n be a planar drawing of the wheel graph W_n on n vertices; see Figure 2. Note that DW_n is unique up to homeomorphism of the sphere and reflection as W_n is 3-connected. The following result by Ringel [18] gives a formula for $h(K_n)$ for every integer $n \geq 4$, and additionally claims that drawings of K_n with the maximum number of uncrossed edges have a unique structure.

▶ **Theorem 8** ([18]). *For every integer $n \geq 4$, we have $h(K_n) = 2n - 2$. Moreover, if D is a drawing of K_n with $2n - 2$ uncrossed edges, then D contains the drawing DW_n with all edges from $D \setminus DW_n$ being drawn in the outer face of DW_n.*

We also mention an analogous result for the complete bipartite graphs $K_{m,n}$, derived by Mengersen [15].

▶ **Theorem 9** ([15]). *For all positive integers m and n with $m \leq n$, we have*

$$h(K_{m,n}) = \begin{cases} 2m + n - 2, & \text{for } m = n \\ 2m + n - 1, & \text{for } m < n < 2m \\ 2m + n, & \text{for } 2m \leq n. \end{cases}$$

The parameter $h(G)$ can be used to estimate the uncrossed number of G. Let $\{D_1, \dots, D_k\}$ be an uncrossed collection of drawings of a graph G that has m edges. Since every drawing D_i contains at most $h(G)$ edges that are uncrossed by any other edge in D_i, we immediately obtain the following lower bound

$$\mathrm{unc}(G) \geq \left\lceil \frac{m}{h(G)} \right\rceil. \tag{6}$$

This bound together with Theorems 8 and 9 give us quite close estimates for $\mathrm{unc}(K_n)$ and $\mathrm{unc}(K_{m,n})$, respectively. In particular, for $n \geq 2$ we have

$$\mathrm{unc}(K_n) \geq \left\lceil \frac{\binom{n}{2}}{2n-2} \right\rceil. \tag{7}$$

On the other hand, we recall the upper bound (2) on the uncrossed number of an arbitrary graph G using the notion of outerthickness of G.

3 Proof of Theorem 1

In this section, we prove Theorem 1 by providing the exact formula for the uncrossed number of complete graphs. That is, we show

$$\mathrm{unc}(K_n) = \begin{cases} \lceil \frac{n+1}{4} \rceil, & \text{for } n \notin \{4, 7\} \\ 3, & \text{for } n = 7 \\ 1, & \text{for } n = 4 \end{cases}$$

for every positive integer n.

We start with the upper bound, which is easier to prove. For $n \notin \{4, 7\}$, the upper bound follows from (2) and (4) as we have

$$\mathrm{unc}(K_n) \leq \theta_o(K_n) = \left\lceil \frac{n+1}{4} \right\rceil.$$

For $n = 4$, we obviously have $\mathrm{unc}(K_4) = 1$ as K_4 is planar. Finally, $\mathrm{unc}(K_7) = 3$ was proved by Hliněný and Masařík [11, Proposition 3.1].

It remains to prove the lower bound. Since we already know that $\mathrm{unc}(K_7) = 3$ and $\mathrm{unc}(K_4) = 1$ and the statement is trivial for $n \leq 3$, it suffices to consider the case $n \geq 5$ with $n \neq 7$. Let $\{D_1, \dots, D_k\}$ be an uncrossed collection of drawings of K_n and let $D'_1, \dots, D'_k$ be corresponding uncrossed subdrawings of K_n such that $D'_1 \cup \dots \cup D'_k$ covers $E(K_n)$. By (7),

$$\mathrm{unc}(K_n) \geq \left\lceil \frac{\binom{n}{2}}{2n-2} \right\rceil.$$

By Theorem 8, we get that if any uncrossed subdrawing D'_i contains $2n-2$ edges, then D'_i (as a wheel) contains a *universal vertex*, that is, a vertex that is adjacent to all remaining vertices in D'_i. If every drawing D'_i contains at most $2n-3$ edges, then

$$\mathrm{unc}(K_n) \geq \left\lceil \frac{\binom{n}{2}}{2n-3} \right\rceil = \left\lceil \frac{n}{4} + \frac{n}{4(2n-3)} \right\rceil = \left\lceil \frac{n+1}{4} \right\rceil \tag{8}$$

and we are done.

Thus, suppose that some drawing D'_i contains $2n-2$ edges. Without loss of generality, we can assume $i=1$. We then know that D'_1 contains a universal vertex v. In every drawing D'_j with $j>1$, the edges incident to v are already counted for D'_1, thus we can consider the drawings $D'_2, \ldots, D'_k$ to be uncrossed drawings for K_{n-1} obtained from K_n by removing v. Note that these uncrossed drawings of K_{n-1} cover $E(K_{n-1})$. Then, each D'_j contributes at most $2n-4$ new uncrossed edges of K_{n-1} as $h(K_{n-1}) = 2(n-1)-2 = 2n-4$ by Theorem 8. So the number k of drawings satisfies

$$\binom{n}{2} \leq 2n-2+(k-1)(2n-4) = (2n-4)k+2. \tag{9}$$

However, $(2n-4)k+2 \leq (2n-3)k$ when $k \geq 2$, which is satisfied for $n \geq 5$ by (9). Hence, for k' being the smallest positive integer that satisfies $\binom{n}{2} \leq k'(2n-3)$ we obtain $k \geq k'$. Thus, we again have the inequality (8). ◀

A proof of Theorem 2 follows a similar path as that of Theorem 1, but there are several complications on the way. The upper bound requires a construction for case $n \geq 2m-1$ besides using bounds (2) and (5). The lower bound is handled by Theorem 9 except for cases $m < n \leq 2m-2$ that require a detailed lengthy analysis. Hence, we left the full proof for the arXiv version [1].

4 Proof of Theorem 3

Here, we show that every connected graph G with $n \geq 3$ vertices and $m \geq 0$ edges satisfies

$$\mathrm{unc}(G) \geq \left\lceil \frac{m}{f(n,m)} \right\rceil$$

where $f(n,m) = (3n-5+\sqrt{(3n-5)^2-4m})/2$.

Let $\mathcal{D}(G) = \{D_1, \ldots, D_k\}$ be an uncrossed collection of drawings of G. For every $i \in [k]$, let D'_i be a subdrawing of D_i containing only edges of D_i that are uncrossed in D_i. By Lemma 7, each drawing D'_i is then a plane graph with the property that every edge of G that is not an edge of D'_i is contained in a single face of D'_i. Moreover, since G is connected, we can assume without loss of generality by this lemma that each D'_i represents a connected subgraph of G as to bound $\mathrm{unc}(G)$ from below it suffices to consider drawings D'_i with as many edges as possible.

Fix an arbitrary $i \in [k]$. The number of vertices of D'_i equals n. We use m_i to denote the number of edges of D'_i and we will show that $m_i \leq f(n,m)$.

We set $\mathcal{F}_i$ to be the set of faces of D'_i and $f_i = |\mathcal{F}_i|$. For a face F of D'_i, we use $v(F)$ for the number of vertices of D'_i that are contained in the boundary of F and we write $|F|$ for the number of times we meet an interior of an edge of D'_i when traversing F along its boundary. That is, $|F|$ is the length of the facial walk. Note that each edge can be counted once or twice in $|F|$ and so we have $v(F) \leq |F|$ as D'_i represents a connected subgraph of G. We assume that at least one edge of F is counted once in $|F|$ and that $v(F) \geq 3$ for every face F as otherwise there is only a single face in $\mathcal{F}_i$ and D'_i is a tree with $m_i \leq n-1 \leq f(n,m)$ for $n \geq 3$. Also, observe that

$$\sum_{F \in \mathcal{F}_i} |F| = 2m_i \tag{10}$$

as every edge is incident to a face of D'_i from the left and from the right.

Since every edge of G that is not an edge of D'_i is contained in a single face of D'_i, we have

$$\sum_{F \in \mathcal{F}_i} \left(\binom{v(F)}{2} - v(F) \right) \geq m - m_i. \tag{11}$$

This is because vertices of each face F can span up to $\binom{v(F)}{2}$ edges of D_i and at least $v(F)$ pairs of vertices of D_i are already used for edges of D'_i as each face F contains an edge that is counted only once in $|F|$. The left hand side of (11) can be rewritten as

$$\frac{1}{2} \sum_{F \in \mathcal{F}_i} v(F)(v(F)-3).$$

Since $v(F) \geq 3$ and $|F| \geq v(F)$ for every face F from $\mathcal{F}_i$, we obtain

$$\frac{1}{2} \sum_{F \in \mathcal{F}_i} |F|(|F|-3) \geq m - m_i.$$

Since $|F| - 3 \geq 0$, the left-hand side can be bounded from above by

$$\frac{1}{2} \left(\sum_{F \in \mathcal{F}_i} |F| \right) \left(\sum_{F \in \mathcal{F}_i} (|F|-3) \right) = m_i(2m_i - 3f_i)$$

where we used (10) twice. Altogether, we obtain $m_i(2m_i - 3f_i) \geq m - m_i$, which can be rewritten as

$$f_i \leq \frac{2m_i}{3} - \frac{m - m_i}{3m_i}.$$

Plugging this estimate into Euler's formula $n - m_i + f_i = 2$, we get

$$m_i \leq 3n - 5 - \frac{m}{m_i},$$

which after solving the corresponding quadratic inequality for m_i gives the final estimate

$$m_i \leq (3n - 5 + \sqrt{(3n-5)^2 - 4m})/2 = f(n,m).$$

Since i was arbitrary, we see that each drawing D'_i contains at most $f(n,m)$ edges of G and therefore, we indeed have

$$k \geq \mathrm{unc}(G) \geq \left\lceil \frac{m}{f(n,m)} \right\rceil. \qquad \blacktriangleleft$$

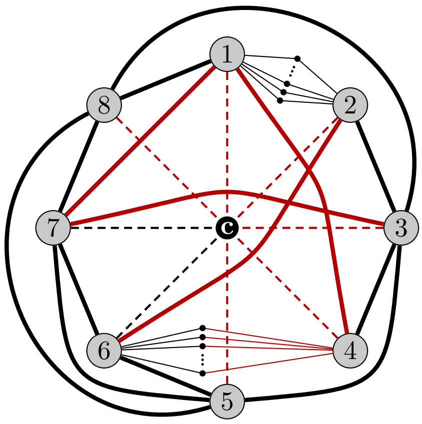


Figure 3 EdgeCrossingNumber instance for proof of Theorem 4 in Section 5. Red edges are crossed. Thick edges represent M-bundles corresponding to edges from the Maximum Outerplanar Subgraph instance G, see in detail the edges between 1 and 2 and between 6 and 4. The dashed edges and c form the central star.

5 Proof of Theorem 4

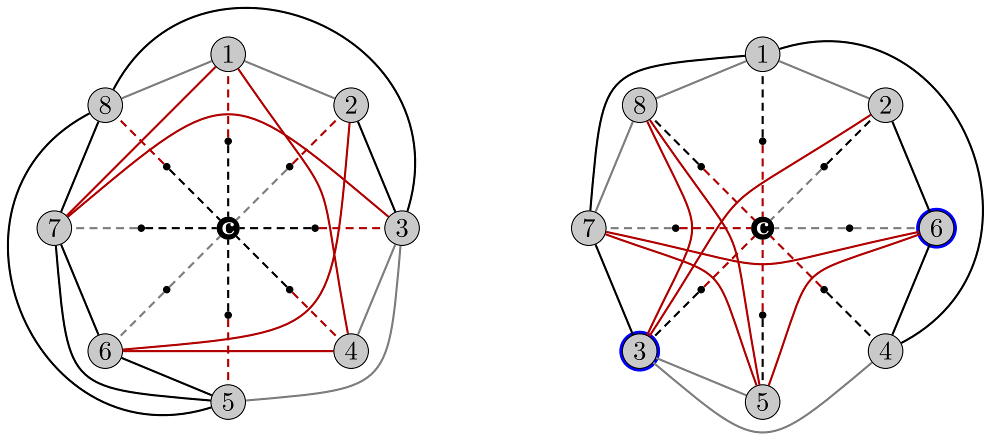
In this section, we prove that EdgeCrossingNumber is NP-complete. Membership of this problem in the class NP is trivial. To show NP-hardness, we reduce from the following NP-complete problem [13, 20].

Maximum Outerplanar Subgraph	
Input:	A graph $G = (V, E)$ and a positive integer k.
Question:	Is there an outerplanar subgraph of G with at least k edges?

Assume an instance of Maximum Outerplanar Subgraph. Let $M > |V|$, say $M = 2|V|$, and $k' = |E| - k$. We augment G into a graph G', and show that G' can be drawn with at most $Mk' + |V|$ crossed edges, if and only if G admits an outerplanar subgraph with at least k edges. The graph G' is obtained via two augmenting steps: We add a *central star*, i.e., a vertex with an edge to each original vertex of G. Then, we replace each original edge in G by M parallel paths of length two, which we call an *M-bundle*. An example of this transformation can be seen in Figure 3.

Suppose there is a drawing of G' with at most $Mk' + |V|$ crossed edges. We want to modify this drawing into a drawing of G. To this end, we first remove every path belonging to an M-bundle, if either of its two edges is crossed. We also remove the central vertex and all of its incident edges. All remaining edges are uncrossed and belong to an M-bundle path. As there are at most $Mk' + |V| < M(k' + 1)$ crossed edges in the drawing, for at least $|E| - k' = k$ edges from G there is at least one path of its corresponding M-bundle that is not removed. We contract for each edge of G one edge of one of the remaining paths of its M-bundle and remove all other M-bundle paths. The vertices from G all share the face created by removing the central vertex and all vertices from M-bundles are either contracted or removed. Thus, we have an outerplanar drawing of a subgraph of G with at least k edges.

Similarly, for every outerplanar subgraph H of G with at least k edges we can construct a drawing of G' with at most $Mk' + |V|$ crossed edges. First, we draw H in an outerplanar embedding, then we draw the central star into the outer face. Next, we draw the at most



(a) First drawing.

(b) Second drawing. Vertices 3 and 6 swapped places.

Figure 4 An instance of the reduction from OUTERTHICKNESS to UNCROSSEDNUMBER. The original graph G in this instance is drawn with solid edges and has outerthickness 2 (as the two subdrawings in solid black and gray edges certify). The dashed edges and black vertices form the central star around c added to G in the reduction. In each drawing, all crossed edges are red and uncrossed edges of the particular drawing are black, and gray edges are uncrossed in both drawings.

$|E| - k = k'$ remaining edges of G in such a way that they only cross one another and the edges of the central star. Finally, we replace every edge of G with an M-bundle. The newly added vertices are positioned in such a way that at most one of the edges of each path is crossed. Therefore, there are at most Mk' crossed edges from the M-bundles and at most $|V|$ crossed edges from the central star, for a total of at most $Mk' + |V|$ crossed edges. ◀

6 Proof of Theorem 6

We show that if OUTERTHICKNESS is NP-hard, then UNCROSSEDNUMBER is NP-hard as well using a reduction from OUTERTHICKNESS to UNCROSSEDNUMBER.

The reduction employs similar arguments as used in Section 5. Let (G, k) be an instance of the problem OUTERTHICKNESS. We augment the input graph G into a graph G' by adding a vertex and connecting each vertex of G to it with a path of length two. We call the added structure the *central star*. See Figure 4 for an example of this transformation.

Let D be a drawing of G'. Consider the uncrossed subdrawing D'_G consisting of the vertices and all uncrossed edges from G. As there is a path in $D \setminus D'_G$ between each two vertices from G, we know that D'_G is outerplanar. Thus, if $\mathrm{unc}(G') \leq k$ and $D_1, \ldots, D_k$ is an uncrossed collection of G', then the respective subdrawings restricted to G decompose G into k outerplanar graphs.

Conversely, if G can be decomposed into $k \geq 2$ outerplanar subgraphs $G_1, \ldots, G_k$, then we can construct an uncrossed collection $D_1, \ldots, D_k$ of G' in the following way: In every drawing D_i, we first draw G_i as an outerplanar graph and we embed the central star in the outer face. Then, we draw the remaining original edges in such a way that they only cross each other and edges from the central star. In D_1, all crossings on the central star lie on edges incident to vertices of G, and in all other drawings, the crossings on the central star involve only edges incident to the universal vertex. This way we assure that also every edge of the central star is uncrossed in some drawing. ◀

7 Conclusions and Open Problems

We provided exact values of the uncrossed number for complete and complete bipartite graphs. The hypercube graphs form another natural graph class to consider as their outerthickness and thickness were determined exactly; see [5, 12]. However, we are not aware of any formula for the uncrossed number for the hypercube graphs.

▶ **Question 10.** *Determine the exact value of the uncrossed number for the hypercube graphs.*

In Theorem 3, we determined a general lower bound on $\mathrm{unc}(G)$ in terms of the number of the edges and vertices of G by showing $\mathrm{unc}(G) \geq \lceil \frac{m}{cn} \rceil - O(n) - O(m)$ for some constant c with $0 < c \leq 3$. In particular, we argued that the smallest obtainable constant c is approximately 2.82 for the case of dense n-vertex graphs with εn^2 edges where $\varepsilon \in (0, 1/2)$ is a fixed constant. Can one obtain a better leading constant in the general lower bound on $\mathrm{unc}(G)$ for such dense graphs G?

We also propose investigating other properties of the uncrossed number. We conjecture that the uncrossed number can be arbitrarily far apart from the outerthickness despite them being quite similar on the graph classes we mainly investigated in this paper. In fact, it follows from our results that the difference between the outerthickness and the uncrossed number of complete and complete bipartite graphs is never larger than one.

▶ **Conjecture 11.** *For every positive integer k, there is a graph G such that*

$$\theta_o(G) - \mathrm{unc}(G) \geq k.$$

Lastly, it would be interesting to finally settle the computational complexity of the outerthickness problem. We conjecture that the OUTERTHICKNESS problem is NP-hard. Note that if true, this would also settle the computational complexity of UNCROSSEDNUMBER by Theorem 6.

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