Weakly Leveled Planarity with Bounded Span

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Abstract

This paper studies planar drawings of graphs in which each vertex is represented as a point along a sequence of horizontal lines, called levels, and each edge is either a horizontal segment or a strictly y-monotone curve. A graph is s-span weakly leveled planar if it admits such a drawing where the edges have span at most s; the span of an edge is the number of levels it touches minus one. We investigate the problem of computing s-span weakly leveled planar drawings from both the computational and the combinatorial perspectives. We prove the problem to be para-NP-hard with respect to its natural parameter s and investigate its complexity with respect to widely used structural parameters. We show the existence of a polynomial-size kernel with respect to vertex cover number and prove that the problem is FPT when parameterized by treedepth. We also present upper and lower bounds on the span for various graph classes. Notably, we show that cycle trees, a family of 2-outerplanar graphs generalizing Halin graphs, are $\Theta(\log n)$-span weakly leveled planar and 4-span weakly leveled planar when 3-connected. As a byproduct of these combinatorial results, we obtain improved bounds on the edge-length ratio of the graph families under consideration.

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1 Introduction

Computing crossing-free drawings of planar graphs is at the heart of Graph Drawing. Indeed, since the seminal papers by Fáry [43] and by Tutte [57] were published, a rich body of literature has been devoted to the study of crossing-free drawings of planar graphs that satisfy a variety of optimization criteria, including the area [30, 48], the angular resolution [41, 51], the face convexity [16, 17, 27], the total edge length [55], and the edge-length ratio [14, 18, 19]; see also [31, 56].

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In this paper, we focus on crossing-free drawings where the edges are represented as simple Jordan arcs and have the additional constraint of being *y-monotone*, that is, traversing each edge from one end-vertex to the other one, the y-coordinates never increase or never decrease. This leads to a generalization of the well-known *layered drawing* style [20, 21, 35], where vertices are assigned to horizontal lines, called *levels*, and edges only connect vertices on different levels. We also allow edges between vertices on the same level and seek for drawings of *bounded span*, i.e., in which the edges span few levels. In their seminal work [44], Heath and Rosenberg study *leveled planar drawings*, i.e., in which edges only connect vertices on consecutive levels and no two edges cross. We also mention the algorithmic framework by Sugiyama et al. [54], which yields layered drawings for the so-called hierarchical graphs. In this framework, edges that span more than one level are transformed into paths by inserting a dummy vertex for each level they cross. Hence minimizing the edge span (or equivalently, the number of dummy vertices along the edges) is a relevant optimization criterion.

Inspired by these works, we study *s-span weakly leveled planar drawings*, which are crossing-free y-monotone drawings in which each edge touches at most $s + 1$ levels; see Fig. 1. Note that 1-span weakly leveled planar drawings have been studied in different contexts; for example, Bannister et al. [3] prove that graphs that admit such drawings have layered pathwidth at most two[1]. Felsner et al. [38,39] show that every outerplanar graph has a 1-span weakly leveled planar drawing and use this to compute a 3D drawing of the graph in linear volume; a similar construction by Dujmović et al. [37] yields a 2-span leveled planar drawing for every outerplanar graph, which can be used to bound the queue number [44] of these graphs. In general, our work also relates to track layouts [37] and to the recently-introduced layered decompositions [36], but in contrast to these research works we insist on planarity.

Our contributions. We address the problem of computing a weakly leveled planar drawing with bounded span both from the complexity and from the combinatorial perspectives. Specifically, the s-SPAN (WEAKLY) LEVELED PLANARITY problem asks whether a graph admits a (weakly) leveled planar drawing where the span of every edge is at most s. The main contributions of this paper can be summarized as follows.

- In Section 3, we show that the s-SPAN WEAKLY LEVELED PLANARITY problem is NP-complete for any fixed $s \geq 1$ (Theorem 3). Our proof technique implies that s-SPAN LEVELED PLANARITY is also NP-complete. This generalizes the NP-completeness result by Heath and Rosenberg [44] which holds for $s = 1$.

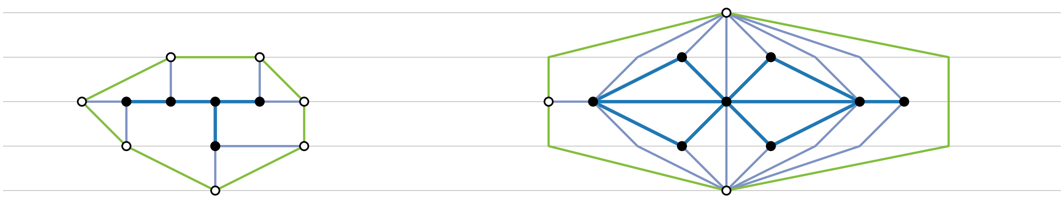


Figure 1 A 1-span weakly leveled planar drawing of the Frucht graph (left) and a 4-span weakly leveled planar drawing of the Goldner-Harary graph (right).

[1] Bannister et al. use the term weakly leveled planar drawing to mean 1-span weakly leveled planar drawing. We use a different terminology because we allow edges which can span more than one level.

- The para-NP-hardness of s-Span Weakly leveled planarity parameterized by the span s motivates the study of FPT approaches with respect to structural parameters of the input graph. In Section 4, we show that the s-Span Weakly leveled planarity problem has a kernel of polynomial size when parameterized by vertex cover number (Theorem 6) and has a (non-polynomial) kernel when parameterized by treedepth (Theorem 9). As also pointed out in [58], designing FPT algorithms parameterized by structural parameters bounded by the vertex cover number, such as the treedepth, pathwidth, and treewidth is a challenging research direction in the context of graph drawing (see, e.g., [1,5–10,24,45]). Again, our algorithms can also be adapted to work for s-Span Leveled Planarity.
- In Section 5, we give combinatorial bounds on the span of weakly leveled planar drawings of various graph classes. It is known that outerplanar graphs admit weakly leveled planar drawings with span 1 [39]. We extend the investigation by considering both graphs with outerplanarity 2 and graphs with treewidth 2. We prove that some 2-outerplanar graphs require a linear span (Theorem 10). Since Halin graphs (which have outerplanarity 2) admit weakly leveled planar drawings with span 1 [2,33], we consider 3-connected cycle-trees [25,29], which also have outerplanarity 2 and include Halin graphs as a subfamily. Indeed, while the Halin graphs are those graphs of polyhedra containing a face that shares an edge with every other face, the 3-connected cycle-trees are the graphs of polyhedra containing a face that shares a vertex with every other face. We show that 3-connected cycle-trees have weakly leveled planar drawings with span 4, which is necessary in the worst case (Theorem 11). For general cycle-trees, we prove $\Theta(\log n)$ span (Theorem 14); such a difference between the 3-connected and 2-connected case was somewhat surprising for us. Concerning graphs of treewidth 2, we prove an upper bound of $O(\sqrt{n})$ and a lower bound of $2^{\Omega(\sqrt{\log n})}$ on the span of their weakly leveled planar drawings (Theorem 15).

Remarks. Dujmović et al. [35] present an FPT algorithm to minimize the number of levels in a leveled planar graph drawing, where the parameter is the total number of levels. They claim that they can similarly get an FPT algorithm that minimizes the span in a leveled planar graph drawing, where the parameter is the span. Our algorithm differs from the one of Dujmović et al. [35] in three directions: (i) We optimize the span of a weakly level planar drawing, which is not necessarily optimized by minimizing the span of a leveled planar drawing; (ii) we consider structural parameters rather than a parameter of the drawing; one common point of our three algorithms is to derive a bound on the span *from* the bound on the structural parameter; and (iii) our algorithms perform conceptually simple kernelizations, while the one in [35] exploits a sophisticated dynamic programming on a path decomposition of the input graph.

Concerning the combinatorial contribution, a byproduct of our results implies new bounds on the planar edge-length ratio [46,49,50] of families of planar graphs. The planar edge-length ratio of a planar graph is the minimum edge-length ratio (that is, the ratio of the longest to the shortest edge) over all planar straight-line drawings of the graph. Borrazzo and Frati [19] have proven that the planar edge-length ratio of an n-vertex 2-tree is $O(n^{0.695})$. Theorem 15, together with a result relating the span of a weakly leveled planar drawing to its edge-length ratio [33, Lemma 4] lowers the upper bound of [19] to $O(\sqrt{n})$ (Corollary 16). We analogously get an upper bound of 9 on the edge-length ratio of 3-connected cycle-trees (Corollary 13).

Sketched or omitted proofs can be found in [4].

2 Preliminaries

In the paper, we only consider simple connected graphs, unless otherwise specified. We use standard terminology in the context of graph theory [34] and graph drawing [31].

A *plane graph* is a planar graph together with a *planar embedding*, which is an equivalence class of planar drawings, where two drawings are equivalent if they have the same clockwise order of the edges incident to each vertex and order of the vertices along the outer face.

A graph drawing is *y-monotone* if each edge is drawn as a strictly y-monotone curve and *weakly y-monotone* if each edge is drawn as a horizontal segment or as a strictly y-monotone curve. For a positive integer k, we denote by $[k]$ the set $\{1,\dots,k\}$. A *leveling* of a graph $G=(V,E)$ is a function $\ell\colon V\to[k]$. A leveling ℓ of G is *proper* if, for any edge $(u,v)\in E$, it holds $|\ell(u)-\ell(v)|=1$, and it is *weakly proper* if $|\ell(u)-\ell(v)|\leq 1$. For each $i\in[k]$, we define $V_i=\ell^{-1}(i)$ and call it the *i-th level* of ℓ. The *height* of ℓ is $h(\ell)=\max_{v\in V}\ell(v)-\min_{v\in V}\ell(v)$. A *level graph* is a pair (G,ℓ), where G is a graph and ℓ is a leveling of G. A *(weakly) level planar drawing* of a level graph (G,ℓ) is a planar (weakly) y-monotone drawing of G where each vertex is drawn with y-coordinate $\ell(v)$. A level graph (G,ℓ) is *(weakly) level planar* if it admits a (weakly) level planar drawing. A (weakly) leveled planar drawing of a graph G is a (weakly) level planar drawing of a level graph (G,ℓ), for some leveling ℓ of G.

The following observation rephrases a result of Di Battista and Nardelli [32, Lemma 1] in the weakly-level planar setting.

▶ **Observation 1.** *Let (G,ℓ) be a level graph such that ℓ is (weakly) proper. For each $i\in[k]$, let $\prec_i$ be a linear ordering on $\ell^{-1}(i)$. Then, there exists a (weakly) level planar drawing of (G,ℓ) that respects $\prec_i$ (i.e., in which the left-to-right ordering of the vertices in $\ell^{-1}(i)$ is $\prec_i$) if and only if:*

(i) *if $(u,v)\in E(G)$ with $\ell(u)=\ell(v)=i$, then u and v are consecutive in $\prec_i$; and*

(ii) *if (u,v) and (w,x) are two independent edges (i.e., $\{u,v\}\cap\{w,x\}=\emptyset$) with $\ell(u)=\ell(w)=i$, $\ell(v)=\ell(x)=i+1$, and $u\prec_i w$, then $v\prec_{i+1} x$.*

The *span* of an edge (u,v) of a level graph (G,ℓ) is $\text{span}_\ell(u,v)=|\ell(u)-\ell(v)|$. The *span* of a leveling ℓ of G is $\text{span}(\ell)=\max_{(u,v)\in E}\text{span}_\ell(u,v)$. Given a graph G, we consider the problem of finding a leveling ℓ that minimizes $\text{span}(\ell)$ among all levelings where (G,ℓ) is weakly level planar. Specifically, given a positive integer s, we call s-Span (Weakly) leveled planarity the problem of testing whether a graph G admits a leveling ℓ, with $\text{span}(\ell)\leq s$, such that (G,ℓ) is (weakly) level planar. The 1-Span Leveled Planarity problem has been studied under the name of Leveled Planar by Heath and Rosenberg [44].

A (weakly) y-monotone drawing Γ of a graph defines a leveling ℓ, called the *associated leveling* of Γ, where vertices with the same y-coordinate are assigned to the same level and the levels are ordered by increasing y-coordinates of the vertices they contain. Thus, the span of an edge (u,v) in Γ is $\text{span}_\ell(u,v)$, the span of Γ is $\text{span}(\ell)$, and the height of Γ is $h(\ell)$.

The following lemma appears implicitly in the proof of Lemma 4 in [33].

▶ **Lemma 2.** *Any graph that admits an s-span weakly leveled planar drawing with height h has an $(2s+1)$-span leveled planar drawing with height $2h+1$.*

A planar drawing of a graph is *outerplanar* if all the vertices are external, and *2-outerplanar* if removing the external vertices yields an outerplanar drawing. A graph is *outerplanar* (*2-outerplanar*) if it admits an outerplanar drawing (resp. 2-outerplanar drawing). A *2-outerplane* graph is a 2-outerplanar graph with an associated planar embedding which corresponds to 2-outerplanar drawings. A *cycle-tree* is a 2-outerplane graph such that removing the external vertices yields a tree. A *Halin graph* is a 3-connected plane graph G such that removing the external edges yields a tree whose leaves are exactly the external vertices of G (and whose internal vertices have degree at least 3). Note that Halin graphs form a subfamily of the cycle-trees.

3 NP-completeness

This section is devoted to the proof of the following result.

▶ **Theorem 3.** *For any fixed $s \geq 1$, s-Span Weakly leveled planarity is NP-complete.*

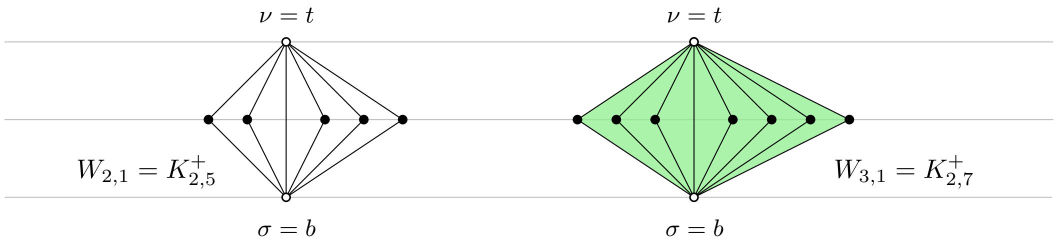
Proof sketch. The NP-membership is trivial. We prove the NP-hardness via a linear-time reduction from the 1-Span Leveled Planarity problem, which was proved NP-complete by Heath and Rosenberg [44]. We distinguish based on whether $s = 1$ or $s > 1$.

Case $s = 1$. Starting from a (bipartite) planar graph H, we construct a graph G that is a positive instance of 1-Span Weakly Leveled Planarity if and only if H is a positive instance of 1-Span Leveled Planarity, by replacing each edge (u, v) of H with a copy $K(u, v)$ of $K_{2,4}$, where u and v are identified with the two degree-4 vertices of $K(u, v)$.

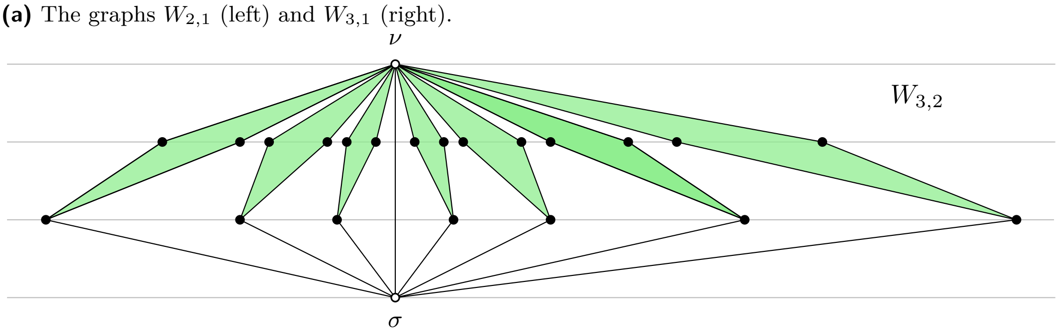
Suppose first that H admits a leveling ℓ_H in $k = h(\ell_H) + 1$ levels, with $\text{span}(\ell_H) \leq 1$, such that (H, ℓ_H) is level planar, and let Γ_H be a level planar drawing of (H, ℓ_H). Consider the leveling ℓ_G of G on $2k$ levels computed as follows. For each vertex $w \in V(H)$, we set $\ell_G(w) = 2 \cdot \ell_H(w)$. For each vertex $w \in V(G) \setminus V(H)$ in a graph $K(u, v)$, we set $\ell_G(w) = \min\{\ell_G(u), \ell_G(v)\} + 1$. By construction, ℓ_G is proper (and thus $\text{span}(\ell_G) \leq 1$), the vertices of $V(H)$ are assigned to even levels, and the vertices in $V(H) \setminus V(G)$ are assigned to odd levels. The graph $K(u, v)$ admits a leveled planar drawing with span 1 on three levels in which u and v lie strictly above and strictly below all other vertices of $K(u, v)$, respectively. This allows us to introduce a new level between any two consecutive levels in Γ_H and replace the drawing of each edge (u, v) of H with a drawing of $K(u, v)$ as the one described above. The resulting drawing is a level planar drawing Γ_G of (G, ℓ_G).

Suppose now that G admits a leveling ℓ_G, with $\text{span}(\ell_G) \leq 1$, such that (G, ℓ_G) is weakly level planar. We show that H admits a leveling ℓ_H, with $\text{span}(H) = 1$, such that (H, ℓ_H) is level planar. Note that any 1-span weakly leveled planar drawing of $K(u, v)$ is leveled planar and places u and v on different levels. Also, any edge of G belongs to $K(u, v)$ for some edge $(u, v) \in E(H)$. Thus, any 1-span weakly leveled planar drawing of G is leveled planar. Moreover, ℓ_G is proper. Let Γ_G be a level planar drawing of (G, ℓ_G). To construct a level planar drawing Γ_H of (H, ℓ_H), we simply set the ordering of the vertices on level i in Γ_H to be the ordering of these vertices on level $2i$ in Γ_G. We claim that such orderings satisfy Conditions (i) and (ii) of Observation 1, which proves that Γ_H is level planar.

Case $s > 1$. In our proof, we exploit special graphs $W_{i,h}$, with $1 \leq h < i$, having two designated vertices ν and σ, called *poles*; specifically, ν is the *north pole* and σ is the *south pole* of $W_{i,h}$; refer to Fig. 2. In the following, we denote by $K^+_{2,\alpha}$ the graph obtained from the complete bipartite graph $K_{2,\alpha}$ by adding an edge between the two vertices t and b of the size-2 bipartition class of the vertex set of $K_{2,\alpha}$. For any $i \geq 2$, the graphs $W_{i,h}$ are defined as follows. If $h = 1$, the graph $W_{i,1}$ coincides with $K^+_{2,2i+1}$; see Fig. 2a. If $h > 1$, the graph $W_{i,h}$ is obtained from $K^+_{2,2i+1}$ by removing each edge (t, x), with $x \neq b$, and by identifying t and b with the north and south pole of a copy of $W_{i,h-1}$, respectively; see Fig. 2b.



(a) The graphs $W_{2,1}$ (left) and $W_{3,1}$ (right).



(b) The graph $W_{3,2}$. The seven green shaded regions are copies of $W_{3,1}$.

Figure 2 Illustration for the construction of graphs $W_{i,h}$. Pole vertices are white filled.

The reduction for $s > 1$ is similar to the one for $s = 1$, but the role of $K_{2,4}$ is now played by $W_{s,s-1}$. For an edge $(u, v) \in E(H)$, we denote by $W_s(u, v)$ the copy of $W_{s,s-1}$ used to replace (u, v). The correctness of the reduction is based on the following claims.

▷ Claim 4. For any $i \geq 2$ and $h < i$, the graph $W_{i,h}$ admits a leveled planar drawing with span $h + 1$ in which the north pole of $W_{i,h}$ lies strictly above all the other vertices of $W_{i,h}$ and the south pole of $W_{i,h}$ lies strictly below all the other vertices of $W_{i,h}$.

▷ Claim 5. For any $i \geq 2$ and $h < i$, in any weakly leveled planar drawing of $W_{i,i-1}$ with span at most i, the edge connecting the poles of $W_{i,i-1}$ has span i.

We conclude the proof by observing that the construction of G can be done in polynomial time, for any fixed value of s; in particular, the number of vertices of G is bounded by the number of vertices of H times a computable function only depending on s. ◀

The proof of Theorem 3 also shows that, for any fixed $s \geq 1$, deciding whether a graph admits a (non-weakly) leveled planar drawing with span at most s is NP-complete, which generalizes the NP-completeness result by Heath and Rosenberg [44], which is limited to $s = 1$.

4 Parameterized Complexity

Motivated by the NP-hardness of the s-Span Weakly leveled planarity problem (Theorem 3), we consider the parameterized complexity of the problem. Recall that a problem $\mathcal{P}$ whose input is an n-vertex graph G is *fixed-parameter tractable* (for short, *FPT*) with respect to some parameter k if it can be solved via an algorithm with running time $O(f(k) \cdot p(n))$, where f is a computable function and p is a polynomial function. A *kernelization* for $\mathcal{P}$ is an algorithm that constructs in polynomial time (in n) an instance (G', k'), called *kernel*, such that: (i) the *size* of the kernel, i.e., the number of vertices in G', is some computable function of k; (ii) (G', k') and (G, k) are equivalent instances; and (iii) k' is some computable function of k. If $\mathcal{P}$ admits a kernel w.r.t. some parameter k, then it is FPT w.r.t. k.

▶ **Theorem 6.** *Let (G, s) be an instance of s-Span Weakly leveled planarity with a vertex cover C of size k. There exists a kernelization that applied to (G, s) constructs a kernel of size $O(k^2)$. Hence, the problem is FPT with respect to the size of a vertex cover.*

Proof sketch. First, we give a kernel (with respect to a parameterization by k and s) of size $O(k \cdot s)$. Second, we show that any planar graph G with vertex cover number k admits a weakly leveled planar drawing with span at most $3k$, which allows us to assume $s \leq 3k$.

For the kernel with respect to $k+s$, we follow a classical reduction approach. By planarity, the number of vertices of $G - C$ with three or more neighbors in C can be bounded by $2k$ (e.g., using [40, Lemma 13.3]), and the number of pairs from C with a degree-2 neighbor in $G - C$ is at most $3k$. For each vertex $c \in C$ with more than three degree-1 neighbors in $G - C$, we only keep three of such neighbors. Then in any drawing of the reduced instance, a neighbor v_c of c is not on the same level as c, and thus we can reinsert the removed vertices next to v_c. Also, for each pair of vertices $\{c, d\} \in C$ that are common neighbors of more than $4s + 5$ degree-2 vertices in $G - C$, we only keep $4s + 5$ of such degree-2 vertices. Then in any drawing of the reduced instance with span at most s, a neighbor v_{cd} of c and d lies strictly between the levels of c and d, and thus we can reinsert the removed vertices next to v_{cd}. As these reductions can be performed in polynomial time, this yields a kernel of size $O(k \cdot s)$.

To bound the span, we consider a more strict trimming operation that removes all degree-1 vertices of $G - C$ and replaces all degree-2 vertices of $G - C$ with the same two neighbors $u, v \in C$ by a single edge (u, v). As above, the size of this trimmed graph is $O(k)$. It therefore admits a planar leveled drawing of height (and thus also span) $O(k)$, into which the removed vertices can be inserted without asymptotically increasing the height. ◀

Let (G, s) be an instance of s-Span Weakly leveled planarity and b be a parameter. A set M of vertices of G is a *modulator to components of size b* (*b-modulator* for short) if every connected component of $G - M$ has size at most b. Note that a 1-modulator is a vertex cover. We show that testing whether a graph with a b-modulator of size k admits a weakly leveled planar drawing with span s is FPT w.r.t. $b+k$. The neighbors in M of a component C of $G - M$ are its *attachments*, and are denoted by $\mathrm{att}(C)$. We also denote by $\mathrm{bridge}(C)$ the graph consisting of C, $\mathrm{att}(C)$, and the edges between C and its attachments.

We generalize the technique for vertex cover and give a kernel with respect to $b + k + s$ and further show that any planar graph admits a leveled planar drawing with height (and hence span) bounded by $(5b + 1)bk$. Hence, s can be bounded by a function of $b + k$, which yields the result. Differently from vertex cover, the components of $G - M$ are not single vertices but have up to b vertices. Nevertheless, we use planarity to bound the number of components with three or more attachments by $2k$ and the number of pairs $\{u, v\} \in M$ that are the attachments of components of $G - M$ by $3k$. The most challenging part is again dealing with the components of $G - M$ with one or two attachments, as their number might not be bounded by any function of $b+k$. Here, the key insight is that, since these components have size at most b, there is only a bounded number of “types” of these components. More precisely, two components C_1, C_2 of $G - M$ that have the same attachments are called *equivalent* if there is an isomorphism between $\mathrm{bridge}(C_1)$ and $\mathrm{bridge}(C_2)$ that leaves the attachments fixed. We use the fact that there is a computable (in fact exponential, see e.g. [26]) function $f : \mathbb{N} \to \mathbb{N}$ such that the number of equivalence classes is bounded by $f(b)$.

We can show that if an equivalence class of components with one attachment (with two attachments) is sufficiently large, that is, it is at least as large as a suitable function of s and b, then in any leveled planar drawing with bounded span, one such component must be drawn on levels that are all strictly above or all strictly below its attachment (resp. on levels that are strictly between its two attachments). Then arbitrarily many equivalent components can be inserted into a drawing without increasing its span. This justifies the following reduction rules.

▶ **Rule 1.** *For every vertex $v \in M$, let $\mathcal{C}_v$ be a set containing all and only the equivalent components C of $G - M$ such that $\mathrm{att}(C) = \{v\}$. If $|\mathcal{C}_v| > (4s + 4)b$, then remove all but $(4s + 4)b$ of these components from G.*

▶ **Rule 2.** *For every pair of vertices $\{u, v\} \in M$, let $\mathcal{C}_{uv}$ be a set containing all and only the equivalent components C of $G - M$ with $\mathrm{att}(C) = \{u, v\}$. If $|\mathcal{C}_{uv}| > (8s + 8)(b + 1)$, then remove all but $(8s + 8)(b + 1)$ such components from G.*

We can now sketch a proof of the following theorem.

▶ **Theorem 7.** *Let (G, s) be an instance of s-Span Weakly leveled planarity with a b-modulator of size k. There exists a kernelization that applied to (G, s) constructs a kernel of size $O(f(b) \cdot k^2 \cdot b^4)$. Hence, the problem is FPT with respect to $k + b$.*

Proof sketch. As mentioned above, the number of components of $G - M$ with three or more attachments is at most $2k$. Rules 1 and 2 bound the number of equivalent components with one and two attachments. The fact that the number of equivalence classes is bounded by $f(b)$ then yields a kernel of size $O(f(b) \cdot k \cdot b^2 \cdot s)$. Note that testing whether two components are equivalent can be reduced to an ordinary planar graph isomorphism problem [47], by connecting each attachment to a sufficiently long path, which forces the isomorphism to leave the attachments fixed. Hence, the described reduction can be performed in polynomial time.

To drop the dependence on s of the kernel size, we prove that every planar graph admits a leveled planar drawing whose height (and hence span) is at most $(5b + 1)bk$. To this end, we use a trim operation that removes all components of $G - M$ with a single attachment and replaces multiple components of $G - M$ with attachments $\{u, v\}$ by a single edge (u, v). The size of the trimmed graph is at most $(5b + 1)k$, therefore there exists a leveled planar drawing of the trimmed graph whose span is bounded by the same function. Then the removed components, which have size at most b, can be reinserted by introducing b new levels below each level, i.e., the span increases by a linear factor in b. This yields the desired leveled planar drawing with span at most $(5b + 1)bk$ and thus a kernel of the claimed size. ◀

We now move to treedepth. A treedepth decomposition of a graph $G = (V, E)$ is a tree T on vertex set V with the property that every edge of G connects a pair of vertices that have an ancestor-descendant relationship in T. The *treedepth* of G is the minimum depth (i.e., maximum number of vertices in any root-to-leaf path) of a treedepth decomposition T of G.

Let td be the treedepth of G and T be a treedepth decomposition of G with depth td. Let r be the root of T. For a vertex $u \in V$, we denote by T_u the subtree of T rooted at u, by V_u the vertex set of T_u, by $d(u)$ the depth of u (where $d(u) = 1$ if u is a leaf and $d(u) = td$ if $u = r$), and by $R(u)$ the set of vertices on the path from u to r (end-vertices included).

As for Theorems 6 and 7, we can show that the instance (G, s) is positive if s is sufficiently large, namely larger than $((5td)^{td} + 1)^{td}$. Hence, we can assume that s is bounded w.r.t. td.

▶ **Theorem 8.** *Every planar graph with treedepth td has a leveled planar drawing of height at most $((5td)^{td} + 1)^{td}$.*

Proof sketch. We apply a trimming operation similar to the one described in Theorem 7. The effect of this operation, when applied to a vertex v, is to bound the number of children of v in T by $5td$. Assume that every vertex in $V_v \setminus \{v\}$ has at most $5td$ children. Then $R(v)$

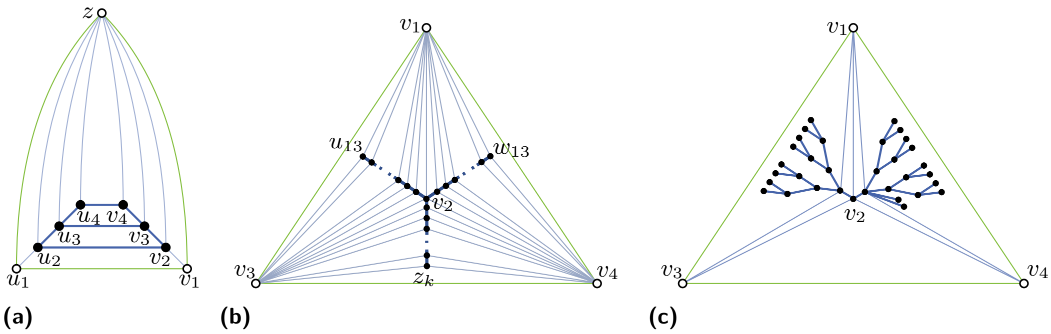


Figure 3 (a) A n-vertex 2-outerplanar graph requiring $\Omega(n)$ span in every weakly leveled planar drawing. (b) An n-vertex 3-connected cycle-tree requiring span 4 in every weakly leveled planar drawing. (c) An n-vertex cycle-tree requiring $\Omega(\log n)$ span in every weakly leveled planar drawing. The graph resulting from the removal of the vertices incident to the outer face is drawn bold.

is a vertex set of size at most td and, due to the degree bound, any connected component of $G - R(v)$ whose vertices are in V_v has size bounded by $(5td)^{td}$. Similarly as for modulators, we can remove such components with a single attachment and replace multiple components with the same two attachments by an edge. This bounds the degree of v in T to $5td$ ($2td$ for components with three or more attachments and $3td$ for components with two attachments).

We apply this trimming operation in batches. The first batch processes all leaves (which has no effect); each next batch consists of all the vertices whose children are already processed. After at most td batches the whole tree is processed (and in fact reduced to a single vertex by processing the root). We then undo these steps while maintaining a leveled planar drawing. The key point here is that, due to the degree bound, each of the removed components contains at most $(5td)^{td}$ vertices, and we can simultaneously reinsert all removed components at the cost of inserting $(5td)^{td}$ levels below each existing level in the current drawing. Therefore, the number of levels multiplies by $(5td)^{td} + 1$ per batch. ◀

We thus obtain the following.

▶ **Theorem 9.** *Let (G, s) be an instance of s-Span Weakly leveled planarity with treedepth td. There exists a kernelization that applied to (G, s) constructs a kernel whose size is a computable function of td. Hence, the problem is FPT with respect to the treedepth.*

Proof sketch. We perform a kernelization with respect to $td + s$. We then apply Theorem 8 to bound the span in terms of treedepth. We use a strategy similar to the one in Theorem 8 for processing the vertices in td batches. However, instead of the trimming operation used there, we use Rules 1 and 2 to bound, for each vertex v, the number of components of $G - R(v)$ whose vertices are in V_v by a function $g(td, s)$. Eventually, we obtain an equivalent instance in which the vertices have their degree bounded by $g(td, s)$ in T. This is the desired kernel. ◀

5 Upper and Lower Bounds

In this section, we establish upper and lower bounds on the span of weakly leveled planar drawings of certain graph classes.

▶ **Theorem 10.** *There exists an n-vertex 2-outerplanar graph such that every weakly leveled planar drawing of it has span in $\Omega(n)$.*

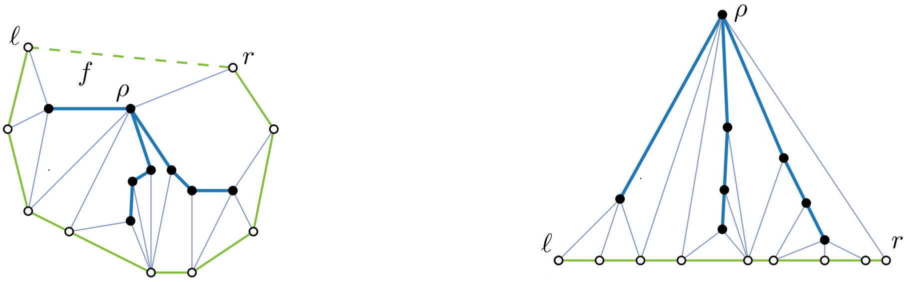


Figure 4 (a) An almost-3-connected path-tree G, drawn with solid edges. The path-vertices are white and the tree-vertices are black. (b) The path-tree G with root ρ.

Proof sketch. The lower bound is provided by the graph G_k of Fig. 3a, which is composed of $k := \lfloor (n-1)/2 \rfloor$ "1-fused stacked cycles" and introduced by Biedl [11]. It is easy to observe that each cycle has an edge that spans two more levels than any edge of a cycle stacked inside it, from which the linear lower bound follows. ◀

There is however a well-studied graph family, the Halin graphs, which have outerplanarity 2 and admit 1-span weakly leveled planar drawings [2, 33]. This motivates the study of cycle-trees [29], a superclass of Halin graphs still having outerplanarity 2. We first consider 3-connected cycle-trees showing a constant span and then extend the study to general cycle-trees. The approach for 3-connected cycle-trees relies on removing an edge from the external face so to obtain a graph for which we construct a suitable decomposition tree. We conclude by discussing the span of weakly leveled planar drawings of planar graphs with treewidth 2.

Path-Trees. A *path-tree* is a plane graph G that can be augmented to a cycle-tree G' by adding the edge $e = (\ell, r)$ in its outer face; see Fig. 4a. W.l.o.g., let ℓ occur right before r in clockwise order along the outer face of G'; then ℓ and r are the *leftmost* and *rightmost* *path-vertex* of G, respectively. The external (internal) vertices of G' are *path-vertices* (*tree-vertices*). The tree-vertices induce a tree in G. We can select any tree-vertex ρ incident to the unique internal face of G' incident to e as the *root* of G. Then G is *almost-3-connected* if it becomes 3-connected by adding the edges (ρ, ℓ), (ρ, r), and (ℓ, r), if they are not already part of G. If G is almost-3-connected, the path-vertices induce a path in G.

SPQ-decomposition of path-trees. Let G be an almost-3-connected path-tree with root ρ, leftmost path-vertex λ, and rightmost path-vertex r. We define the *SPQ-decomposition* of G, introduced in [29], which constructs a tree $\mathcal{T}$, called the *SPQ-tree* of G. The nodes of $\mathcal{T}$ are of three types: *S*-, *P*-, and *Q-nodes*. Each node μ of $\mathcal{T}$ corresponds to a subgraph G_μ of G, called the *pertinent graph* of μ, which is an almost-3-connected rooted path-tree. We denote by ρ_μ the root of G_μ (a tree-vertex), by λ_μ the leftmost path-vertex of G_μ, and by r_μ the rightmost path-vertex of G_μ. To handle the base case, we consider as a path-tree also a graph whose path is the single edge (λ, r) and whose tree consists of a single vertex ρ, possibly adjacent to only one of λ and r. Also, we consider as almost-3-connected a path-tree such that adding (ρ, r), (ρ, λ), and (λ, r), if missing, yields a 3-cycle.

We now describe the decomposition.

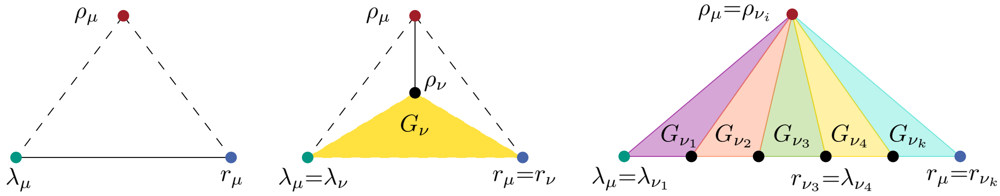


Figure 5 Path-trees corresponding to a Q-node (left), an S-node (middle), and a P-node (right). Dashed edges may or may not belong to G_μ. Shaded triangles represent smaller path-trees G_{ν_i}.

- Q-NODE: the pertinent graph G_μ of a *Q-node* μ is an almost-3-connected rooted path-tree which consists of ρ_μ, λ_μ, and r_μ. The edge (λ_μ, r_μ) belongs to G_μ, while the edges (ρ_μ, λ_μ) and (ρ_μ, r_μ) may not exist; see Fig. 5(left).
- S-NODE: the pertinent graph G_μ of an *S-node* μ is an almost-3-connected rooted path-tree which consists of ρ_μ and of an almost-3-connected path-tree G_ν, where ρ_μ is adjacent to ρ_ν and, possibly, to λ_ν and r_ν. We have that μ has a unique child in $\mathcal{T}$, namely a node ν whose pertinent graph is G_ν. Further, we have $\lambda_\mu = \lambda_\nu$ and $r_\mu = r_\nu$; see Fig. 5(middle).
- P-NODE: the pertinent graph G_μ of a *P-node* μ is an almost-3-connected rooted path-tree which consists of almost-3-connected rooted path-trees $G_{\nu_1}, \dots, G_{\nu_k}$, with $k > 1$. This composition is defined as follows. First, we have $\rho_\mu = \rho_{\nu_1} = \cdots = \rho_{\nu_k}$. Second, we have $\lambda_{\nu_i} = r_{\nu_{i-1}}$, for $i = 2, \dots, k$. Third, μ has children $\nu_1, \dots, \nu_k$ (in this left-to-right order) in $\mathcal{T}$, where G_{ν_i} is the pertinent graph of ν_i, for $i = 1, \dots, k$. Finally, we have $\lambda_\mu = \lambda_{\nu_1}$ and $r_\mu = r_{\nu_k}$; see Fig. 5(right).

In the following, all considered SPQ-trees are *canonical*, that is, the child of every P-node is an S- or Q-node. For a given path-tree, a canonical SPQ-tree always exists [23].

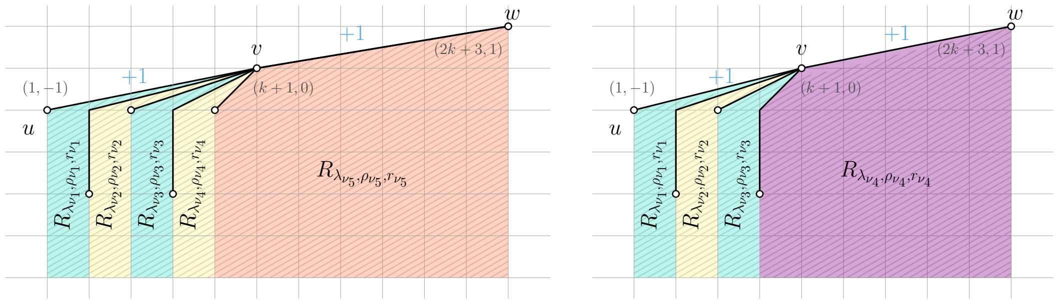
3-connected Cycle-Trees. Let G be a plane graph with three consecutive vertices u, v, w encountered in this order when walking in clockwise direction along the boundary of the outer face of G. A leveling of G is *single-sink with respect to* (u, v, w) if all vertices of G have a neighbor on a higher level, except for exactly one of $\{u, v, w\}$. A single-sink leveling ℓ with respect to (u, v, w) is *flat* if $\ell(u) < \ell(v) < \ell(w)$ or $\ell(w) < \ell(v) < \ell(u)$; ℓ is a *roof* if $\ell(v) > \ell(u)$ and $\ell(v) > \ell(w)$. Note that a single-sink leveling is necessarily either roof or flat.

Given a single-sink leveling ℓ of G with respect to (u, v, w), a *good* weakly leveled planar drawing Γ of (G, ℓ) is one with the following properties:

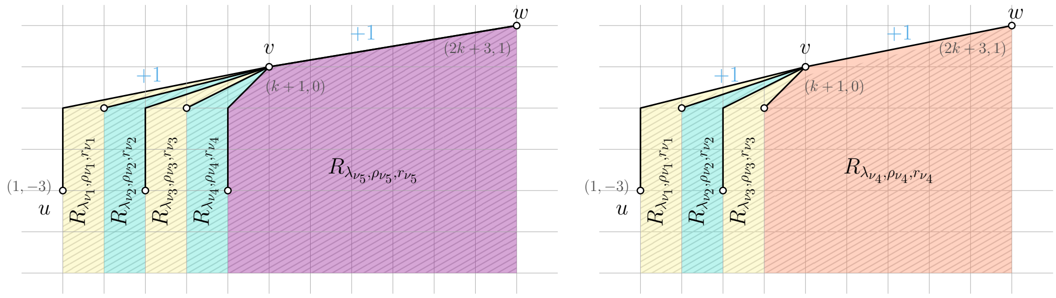
1. Γ respects the planar embedding of G;
2. it holds that $x(u) < x(w)$ in Γ; and
3. all vertices of $V(G) \setminus \{u, v, w\}$ are contained in the interior of the bounded region R_{uvw} defined by the path (u, v, w), by the vertical rays starting at u and w, and by the horizontal line $y := \min_{z \in V(G)} \ell(z)$.

Let a and b be two non-zero integers. A good weakly leveled planar drawing Γ of (G, ℓ) is an *(a,b)-flat drawing* if ℓ is flat, $a = \ell(v) - \ell(u)$, and $b = \ell(w) - \ell(v)$; it is an *(a,b)-roof drawing* if ℓ is roof, $a = \ell(v) - \ell(u)$, and $b = \ell(w) - \ell(v)$. Note that, by definition, in an (a, b)-flat drawing we have that a and b are either both positive or both negative, while in an (a, b)-roof drawing a is positive and b is negative.

▶ **Theorem 11.** *Every 3-connected cycle-tree admits a 4-span weakly leveled planar drawing. Also, for all $n \geq 43$, there exists an n-vertex 3-connected cycle-tree G such that every weakly leveled planar drawing of G has span greater than or equal to 4.*



(a) A $(1,1)$-flat weakly leveled planar drawing when μ has an odd (left) or even (right) number of children.



(b) A $(3,1)$-flat weakly leveled planar drawing when μ has an odd (left) or even (right) number of children.

Figure 6 Illustrations for the proof of Theorem 11, when μ is a P-node.

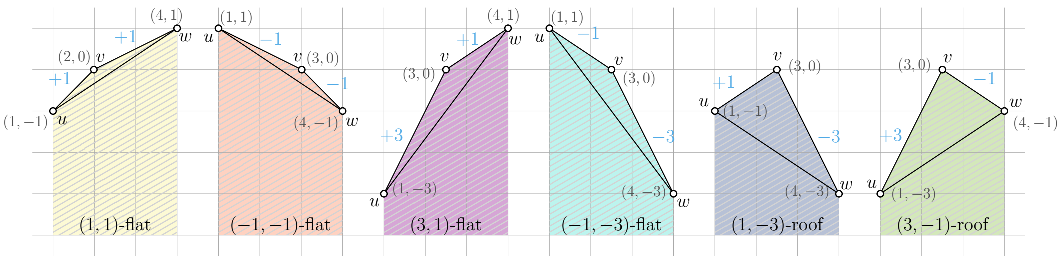
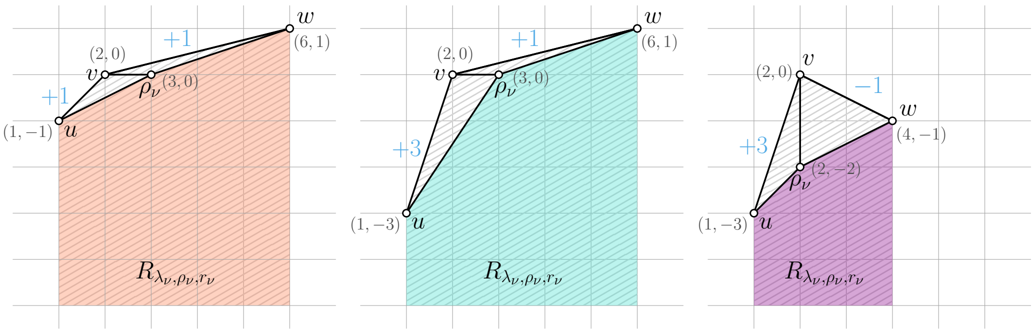


Figure 7 Illustrations for the proof of Theorem 11, when μ is a Q-node and the height of $\mathcal{T}$ is 0.

Proof sketch. We first prove the statement for almost 3-connected path trees. Let G be such a graph and $\mathcal{T}$ be its SPQ-tree with root μ. Let $u = \lambda_\mu$, $v = \rho_\mu$, and $w = r_\mu$. Since removing edges does not increase the span of a weakly leveled planar drawing, we can assume that the edges (u, v) and (v, w) belong to G and that G is internally triangulated. That is, we prove the statement when G is a maximal almost-3-connected path-tree. The proof is based on recursively constructing a drawing of $G = G_\mu$, where the recursion is on the SPQ-tree $\mathcal{T}$ of G, according to the following case distinction (for details, see [4]).

If μ is a **P-node**, then G_μ has flat levelings ℓ^i_μ for $i = 1, \dots, 4$, with $\text{span}(\ell^i_\mu) \leq 4$, such that (G_μ, ℓ^i_μ) admits a p_i-flat weakly leveled planar drawing with $p_1 = (-1, -1)$, $p_2 = (1, 1)$, $p_3 = (-1, -3)$, and $p_4 = (3, 1)$, see Fig. 6. Let k be the number of children of μ in $\mathcal{T}$. Each flat leveling ℓ^i_μ is obtained by combining roof levelings for the pertinent graphs of the leftmost (or the righmost) $k-1$ children of μ with a flat leveling of the pertinent graph of the rightmost (resp. leftmost) child of μ. In particular, the $k-1$ children of μ for which flat levelings are used alternate, in left-to-right order, between $(1, -3)$-roof drawings and $(3, -1)$-roof drawings. If μ is a **Q-node**, then G_μ has flat levelings ℓ^i_μ for $i = 1, \dots, 4$, with $\text{span}(\ell^i_\mu) \leq 4$, such



(a) $(1,1)$-flat **(b)** $(3,1)$-flat **(c)** $(3,-1)$-roof

Figure 8 Illustrations for the proof of Theorem 11, when μ is an S-node.

that (G_μ, ℓ^i_μ) admits a q_i-flat weakly leveled planar drawing with $q_1 = (1,1)$, $q_2 = (-1,-1)$, $q_3 = (3,1)$, and $q_4 = (-1,-3)$. Also, G_μ has roof levelings ℓ^j_μ for $j = 5,6$, with $\text{span}(\ell^j_\mu) \leq 4$, such that (G_μ, ℓ^j_μ) admits a q_j-roof weakly leveled planar drawing with $q_5 = (1,-3)$ and $q_6 = (3,-1)$, see Fig. 7. Finally, if μ is am **S-node**, then G_μ has the same type of levelings and weakly leveled planar drawings as in the case in which it is a Q-node, see Fig. 8. Each of such levelings is obtained from a flat leveling of the pertinent graph of the unique child of μ.

For triconnected cycle-trees, we remove an edge e on the outer face, after an augmentation we obtain a $(1,1)$-flat weakly leveled planar drawing, and insert back e with span 2.

The proof of the theorem is completed by observing that some 3-connected cycle-trees, like the one in Fig. 3b, require span at least 4. ◀

The approach in the proof of Theorem 11 can be implemented in quadratic time. To get linear time, we can maintain only the order of the vertices on their levels and calculate the exact coordinates at the end of the algorithm.

Similar to [37, Lemma 14], one can prove that s-span weakly leveled planar graphs have queue number at most $s+1$; see [4] for details. Thus, we have the following.

▶ **Corollary 12.** *The queue number of 3-connected cycle-trees is at most 5.*

The *edge-length ratio* of a straight-line graph drawing is the maximum ratio between the Euclidean lengths of e_1 and e_2, over all edge pairs (e_1, e_2). The *planar edge-length ratio* of a planar graph G is the infimum edge-length ratio of Γ, over all planar straight-line drawings Γ of G. Constant upper bounds on the planar edge-length ratio are known for outerplanar graphs [50] and for Halin graphs [33]. We exploit the property that graphs that admit s-span weakly leveled planar drawings have planar edge-length ratio at most $2s+1$ [33, Lemma 4] to obtain a constant upper bound on the edge-length ratio of 3-connected cycle trees.

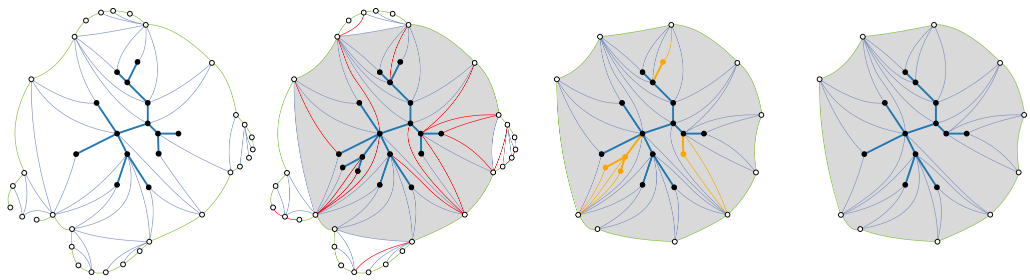
▶ **Corollary 13.** *The planar edge-length ratio of 3-connected cycle-trees is at most 9.*

General Cycle-Trees. We now discuss general cycle-trees, for which we can prove a $\Theta(\log n)$ bound on the span of their weakly leveled planar drawings.

▶ **Theorem 14.** *Every n-vertex cycle-tree has an s-span weakly leveled planar drawing such that $s \in O(\log n)$. Also, there exists an n-vertex cycle-tree such that every weakly leveled planar drawing of G has span in $\Omega(\log n)$.*

Proof sketch. For the lower bound, we observe that some cycle-trees require span $\Omega(\log n)$. Indeed, in any planar drawing of the graph in Fig. 3c, a cycle with 3 vertices contains a complete binary tree with $\Omega(n)$ vertices in its interior. Then the lower bound on the span follows from the fact that any weakly leveled planar drawing Γ of a complete binary tree with $\Omega(n)$ vertices has height $\Omega(\log n)$ (because it has $\Omega(\log n)$ pathwidth [15,52] and the height of Γ is lower-bounded by a linear lower function of the pathwidth of the tree [35]).

For the upper bound, let G be a connected n-vertex cycle-tree. Let $\mathcal{E}$ be a plane embedding of G in which the outer face is delimited by a walk W, so that removing the vertices of W from G one gets a tree T; see Fig. 9a. We add the maximum number of edges connecting vertices of W with vertices of W and of T, while preserving planarity, simplicity, and the property that every vertex of W is incident to the outer face; see Fig. 9b.



(a) **(b)** **(c)** **(d)**

Figure 9 (a) An embedding $\mathcal{E}$ of a cycle-tree G, where the tree T is represented by bold lines. (b) The augmentation of G; the added edges are red. The face f of $\mathcal{E}_{G_W}$ is gray. (c) Removing the components of G outside C. (d) Removing the components of G inside C.

We now remove some parts of the graph, so that it turns into a 3-connected cycle-tree H. Let G_W be the subgraph of G induced by the vertices of W and let $\mathcal{E}_{G_W}$ be the restriction of $\mathcal{E}$ to G_W. There is a unique face f of $\mathcal{E}_{G_W}$ that contains T in its interior; let C be the cycle delimiting f. We remove from G the vertices of G_W not in C. The removed vertices induce connected subgraphs of G, called *components of G outside C*; see Fig. 9c. Also, we remove from G all the vertices of T that have at most one neighbor in C. This results in the removal of subtrees of T, which we call *components of G inside C*; see Fig. 9d.

We next apply Theorem 11 to construct a weakly leveled planar drawing Λ of H with $O(1)$ span and insert $O(\log n)$ levels between any two consecutive levels of Λ. We use such levels to re-introduce the components of G inside and outside C, thus obtaining a weakly leveled planar drawing of G with $O(\log n)$ span. The components of G inside C are trees that can be drawn inside the internal faces of H with $O(\log n)$ height, while ensuring the required vertex visibilities, via an algorithm similar to well-known tree drawing algorithms [22,28,53]. The components of G outside C are outerplanar graphs that can be drawn in the outer face of H with $O(\log n)$ height via a suitable combination of results by Biedl [11,12]. ◀

Planar Graphs with Treewidth 2. In this section, we show that sub-linear span can be achieved for planar graphs with treewidth 2. Note that this is not possible for planar graphs of larger treewidth, as the graph in Fig. 3a has treewidth three and requires span $\Omega(n)$.

▶ **Theorem 15.** *Every n-vertex planar graph with treewidth 2 has an s-span weakly leveled planar drawing such that $s \in O(\sqrt{n})$. Also, there exists an n-vertex planar graph with treewidth 2 such that every weakly leveled planar drawing of G has span in $2^{\Omega(\sqrt{\log n})}$.*

Proof sketch. Biedl [11] proved that every n-vertex planar graph G with treewidth 2 admits a planar y-monotone grid drawing Γ with $O(\sqrt{n})$ *height*, that is, the drawing touches $O(\sqrt{n})$ horizontal grid lines. Interpreting the placement of the vertices along these lines as a leveling shows that G admits a leveled planar drawing Γ with height, and hence span, $O(\sqrt{n})$.

The lower bound uses a construction by Frati [42]. Note that $2^{\Omega(\sqrt{\log n})}$ is larger than any poly-logarithmic function of n, but smaller than any polynomial function of n. ◀

Since graphs that admit s-span weakly leveled planar drawings have planar edge-length ratio at most $2s+1$ [33, Lemma 4], we obtain the following result a corollary of Theorem 15, improving upon a previous $O(n^{0.695})$ bound by Borrazzo and Frati [19]

▶ **Corollary 16.** *Treewidth-2 graphs with n vertices have planar edge-length ratio $O(\sqrt{n})$.*

6 Open Problems

We studied s-span weakly leveled planar drawings from an algorithmic and a combinatorial perspective. We conclude by listing natural open problems arising from our research:

- Does s-Span Weakly leveled planarity have a kernel of polynomial size when parameterized by the treedepth? Is the problem FPT with respect to the treewidth?
- Theorem 15 shows a gap between the lower and upper bounds in the span for the family of 2-trees. It would be interesting to reduce and possibly close this gap.
- It would also be interesting to close the gap between the lower bound of $\Omega(\log n)$ [13, 14] and the upper bound of $O(\sqrt{n})$ of Corollary 16 on the edge-length ratio of 2-trees.

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